Math Club Team Round

Meeting 2 - October 5, 2018

1 Problems

Problem 1. Find 1+3+5+...+105.

Problem 2. Bob would like to eat at Waysub. Waysub offers 3 breads, 4 cheeses, and 5 vegetables. Bob must choose exactly one type of bread and cheese, but may choose as many types of vegetables as he wants. Find the number of sandwiches that Bob can order.

Problem 3. Triangle ABC has centroid O (the point where the medians meet). If ABO is an equilateral triangle of side length 4, what is the area of triangle ABC?

Problem 4. A number is called funny if it is a five digit number, its hundreds and tens digits are the same, and no other digits repeat. How many funny numbers are there?

Problem 5. 5374_n has the value of 15673_8. What is n?

Problem 6. Let T be the sum of all positive integers of the form $2^r * 3^s$, where r and s are nonnegative integers that do not exceed 4. Find the remainder when T is divided by 1000.

Problem 7. There are 8 lily pads in a row. A frog starts on the third lily pad and can either stay where it is, hop forwards 1, forwards 2, or backwards 1 lily pad. How many ways can the frog end up on the sixth lily pad after 5 steps?

Problem 8. Johnny likes to eat sugar, but Papa doesnt like it. Everytime Johnny starts a bag of sugar he finishes it. The refrigerator starts out with 2000 bags of sugar and every day Johnny can secretly eat 10% or less of it without Papa noticing. Unfortunately Papa is smart and will notice if the amount of sugar bags has dropped below 1000. How many days worth of sugar does Johnny have assuming he eats his maximum share every day and can not let Papa find out?

Problem 9. Point P is a random point in regular hexagon with side length 4. What is the probability that a circle of radius 1 with center P touches or intersects the hexagon?

Problem 10. How many factors of 12! are divisible by 12?

Problem 11. In a public forum debate tournament, a debate team debates three rounds in prelims. Each debate team needs to pay \$50 as an entry fee. A certain debate team has a 2/5 chance of winning each debate round, regardless of the previous results in the tournament. For each round the debate team wins, they win \$30. How much money is this particular debate team expected to lose in the tournament?

Problem 12. How many ways are there to place 4 distinct boys and 4 distinct girls in an ordered line such that there are no more than 2 people of the same gender next to each other?

Problem 13. Find the remainder when 3^{83} is divided by 100.

Problem 14. A circle of radius 13 has chord AB with the length of 24. If C is the midpoint of major arc AB, what is the length of AC?

Problem 15. Novak Djokovic and Roger Federer have played 15 professional matches against each other at the Grand Slam level. Djokovic has won 9 of those matches while Federer has won 6. Assume this means the next match they play, Djokovic has a 9/15 chance of winning while Federer has a 6/15 chance of winning. Also assume that for all future matches they play, the chance each player has of winning is the number of previous matches won out of the total number of matches played against each other. If Djokovic and Federer play 5 more matches, what is the probability that Djokovic will still lead the rivalry after those 5 matches?

Problem 16. One fair die is rolled; let a denote the number that comes up. We then roll a dice; let the sum of the resulting a numbers be b. Finally, we roll b dice, and let c be the sum of the resulting b numbers. Find the expected (average) value of c.

Problem 17. An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point (5,5)?

Problem 18. Given a 9×9 chess board, we consider all the rectangles whose edges lie along grid lines (the board consists of 81 unit squares, and the grid lines lie on the borders of the unit squares). For each such rectangle, we put a mark in every one of the unit squares inside it. When this process is completed, how many unit squares will contain an even number of marks?

2 Solutions

Problem 1. Find 1+3+5+...+105.

Answer: 2809

The sum of the first consecutive odd integers is equal to the number of terms squared. There are 53 terms, so the sum is 53^2 or 2809.

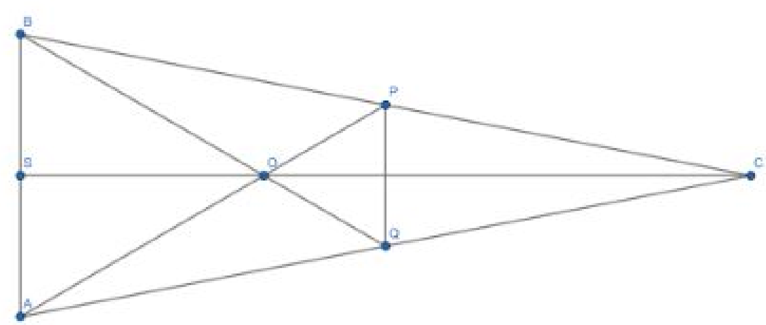
Problem 2. Bob would like to eat at Waysub. Waysub offers 3 breads, 4 cheeses, and 5 vegetables. Bob must choose exactly one type of bread and cheese, but may choose as many types of vegetables as he wants. Find the number of sandwiches that Bob can order.

Answer: 384

The number of possible subsets of vegetables is 2^5= 32 possibilities of vegetables. Therefore there are 3*4*32=384 possible sandwiches.

Problem 3. Triangle ABC has centroid O (the point where the medians meet). If ABO is an equilateral triangle of side length 4, what is the area of triangle ABC?

Answer: $12\sqrt{3}$



Extend the medians from A, B, and C, to points P, Q, and S, respectively. Then connect $\overline{PQ}$ to form two similar isosceles triangles in $\triangle$ABC and $\triangle$QPC. $\overline{AC}$ is twice the length of $\overline{QC}$ and therefore all the other side lengths and the height of $\triangle$ABC are double of that in $\triangle$QPC. $\overline{\text{AB}}$ is length 4 so $\overline{\text{QP}}$ must be length 2. Because they are both equilateral triangles you can find that $\overline{\text{SO}}$ is length $3\sqrt{3}$ and $\overline{\text{SA}}$ is double that, length $6\sqrt{3}$. Finally using $A = bh/2$ you get $6\sqrt{3}{\cdot}4/2$ which is equal to $12\sqrt{3}$.

Problem 4. A number is called funny if it is a five digit number, its hundreds and tens digits are the same, and no other digits repeat. How many funny numbers are there?

Answer: 4536

Start by picking the hundreds and tens digits first, then the ten thousands digit, then the thousands digit, and finally the ones digit. If the hundreds and tens digits are not 0, there are $9{\cdot}1{\cdot}8{\cdot}8{\cdot}7$ numbers while if the hundreds and tens digit is 0, there are $1{\cdot}1{\cdot}9{\cdot}8{\cdot}7$. Adding the previous calculations gives a total of 4536 numbers.

Problem 5. 5374_n has the value of 15673_8. What is n?

Answer: 11

$5n^3 + 3n^2 + 7n + 4 = 4096 + 512 * 5 + 64 * 6 + 8 * 7 + 3 = 7099$
$5n^3 + 3n^2 + 7n - 7095 = 0$
Any integer solution must be a factor of 7095 The prime factors of 7095 include 3, 5, 11, and 43. Because the value must be slightly more than 10, you can try 11 and find that it satisfies the equation thus being the solution.

Problem 6. Let T be the sum of all positive integers of the form $2^r * 3^s$, where r and s are nonnegative integers that do not exceed 4. Find the remainder when T is divided by 1000.

Answer: 751

This isn't the best solution by far.
1+3+9+27+81+2+6+18+54+162+4+12+36+108+324+8+24+72+
216+648+16+48+144+432+1296 = 3751.
$3751 \mod 1000 \equiv 751$

Problem 7. There are 8 lily pads in a row. A frog starts on the third lily pad and can either stay where it is, hop forwards 1, forwards 2, or backwards 1 lily pad. How many ways can the frog end up on the sixth lily pad after 5 steps?

Answer: 155

To get from the third to the sixth lily pad, we must move a total of 3 steps forward, or +3. There are a total of 5 steps, and each step can be either -1, 0, +1, or +2. Casework time:
-1 -1 1 2 2 $\frac{5!}{2!*2!} = 30$
-1 0 0 2 2 $\frac{5!}{2!*2!} = 30$
-1 0 1 1 2 $\frac{5!}{2!} = 60$
-1 1 1 1 1 $\frac{5!}{4!} = 5$
0 0 0 1 2 $\frac{5!}{3!} = 20$
0 0 1 1 1 $\frac{5!}{3!*2!} = 10$
In this scenario, there are no combinations that need to be eliminated because there are 2 lily pads behind the third and 2 lily pads in front of the sixth lily pad, so our answer is $30 + 30 + 60 + 5 + 20 + 10 = 155$

Problem 8. Johnny likes to eat sugar, but Papa doesnt like it. Everytime Johnny starts a bag of sugar he finishes it. The refrigerator starts out with 2000 bags of sugar and every day Johnny can secretly eat 10% or less of it without Papa noticing. Unfortunately Papa is smart and will notice if the amount of sugar bags has dropped below 1000. How many days worth of sugar does Johnny have assuming he eats his maximum share every day and can not let Papa find out?

Answer: 6

Using the information you get the equation:
$2000 \cdot (0.9)^x > 1000$
$0.9^x > 0.5$
$(0.9)^6 = 0.531441$
$(0.9)^7 = 0.4782969$
On the 7th day Johnny would not be able to eat as Papa would find out therefore the answer is 6.

Problem 9. Point P is a random point in regular hexagon with side length 4. What is the probability that a circle of radius 1 with center P touches or intersects the hexagon?

Answer: $\frac{4\sqrt{3}-1}{12}$

The area of the larger hexagon with side length 4 is $(3\sqrt{3})*(s^2)/2 = 24\sqrt{3}$. For the circle to touch the hexagon, the center must be at most 1 unit away from the hexagons sides. Draw 6 lines, each of which is 1 unit inside one of the hexagons sides. Now, we have a smaller hexagon inside the larger hexagon, which has height $4\sqrt{3} - 2$ and side length $4 - 2/\sqrt{3}$. The area of this smaller hexagon is $3\sqrt{3} * (s^2)/2 = 26\sqrt{3} - 24$.
The probability of a point being inside the region between the smaller and larger hexagons is our answer, which is:
$1 - \frac{26\sqrt{3}-24}{24\sqrt{3}}$
$= 1 - \frac{13}{12} + \frac{1}{\sqrt{3}}$
$= \frac{1}{\sqrt{3}} - \frac{1}{12} = \frac{4\sqrt{3}-1}{12}$.

Problem 10. How many factors of 12! are divisible by 12?

Answer: 540

The number of factors of 12! divisible by 12 is the same thing as the number of factors of 11! divisible by 1, which is the number of factors of 11!
$11! = 2^8 * 3^4 * 5^2 * 7 * 11$
Add 1 to each of the powers to find the number of factors
$9 * 5 * 3 * 2 * 2 = 540$

Problem 11. In a public forum debate tournament, a debate team debates three rounds in prelims. Each debate team needs to pay \$50 as an entry fee. A certain debate team has a 2/5 chance of winning each debate round, regardless of the previous results in the tournament. For each round the debate team wins, they win \$30. How much money is this particular debate team expected to lose in the tournament?

Answer: \$14

The expected value is the sum of the probabilities of all outcomes multiplied by their respective values. The probability the debate team wins exactly 1 round is $3 * \frac{2}{5} * \frac{3}{5} * \frac{3}{5}$, which is then multiplied by the \$30 they win for the one round. The probability the debate team wins exactly 2 rounds is $3 * \frac{2}{5} * \frac{2}{5} * \frac{3}{5}$, which is then multiplied by the \$60 they win. The probability the debate team wins all 3 rounds is $3 * \frac{2}{5} * \frac{2}{5} * \frac{2}{5}$, which is then multiplied by the \$90 they win. The sum of these three values is 36, but since they have to pay a \$50 entry fee, their expected loss is \$14.

Problem 12. How many ways are there to place 4 distinct boys and 4 distinct girls in an ordered line such that there are no more than 2 people of the same gender next to each other?

Answer: 19584

BBGBGBGG BBGBGGBG BBGGBBGG BBGGBGBG BBGGBGGB BGBBGBGG
BGBBGGBG BGBGBBGG BGBGBGBG BGBGBGGB BGBGGBBG BGBGGBGB
BGGBBGBG BGGBBGGB BGGBGBBG BGGBGBGB BGGBGGBB

GBBGBBGG GBBGBGBG GBBGBGGB GBBGGBBG GBBGGBGB GBGBBGBG
GBGBBGGB GBGBGBBG GBGBGBGB GBGBGGBB GBGGBBGB GBGGBGBB
GGBBGBBG GGBBGBGB GGBBGGBB GGBGBBGB GGBGBGBB

With casework, we find that there are 34 possibilities of lines where girls are indistinguishable with each other and likewise for boys. Since there are 4! ways to arrange the girls and 4! ways to arrange the boys, the total combinations is therefore:
$4! \cdot 4! \cdot 34 = 19584$

Problem 13. Find the remainder when 3^{83} is divided by 100.

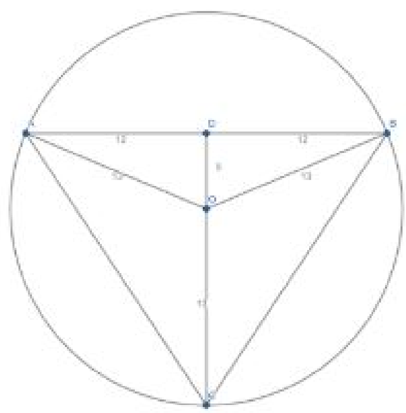
Answer: 27

Eulers Totient Theorem states that if a is relatively prime to p, then: $a^{\phi(p)} \equiv 1$, where ϕ(p) is the number of numbers less than p and relatively prime to p (including 1).
$\phi(100) = 40$
$3^{40} \equiv 1 \mod 100$

$3^{80} \equiv 1 \mod 100$
$3^{83} \equiv 27 \mod 100$

Problem 14. A circle of radius 13 has chord AB with the length of 24. If C is the midpoint of major arc AB, what is the length of AC?

Answer: $6\sqrt{13}$



Using all the information given and the pythagorean theorem a model like the one shown can be constructed. Furthermore, using the pythagorean theorem one more, you get $\sqrt{18^2 + 12^2}$ which simplifies to $6\sqrt{13}$.

Problem 15. Novak Djokovic and Roger Federer have played 15 professional matches against each other at the Grand Slam level. Djokovic has won 9 of those matches while Federer has won 6. Assume this means the next match they play, Djokovic has a 9/15 chance of winning while Federer has a 6/15 chance of winning. Also assume that for all future matches they play, the chance each player has of winning is the number of previous matches won out of the total number of matches played against each other. If Djokovic and Federer play 5 more matches, what is the probability that Djokovic will still lead the rivalry after those 5 matches?

Answer: $\frac{569}{646}$

In order for Djokovic to keep the lead in the rivalry, he needs to win at least two of the next 5 matches they play. If he only wins 0 or 1 of the next 5 matches, he either loses the lead or he ties with Federer in the rivalry. The probability of him winning 0 of the next 5 matches is:
$\frac{6}{15} * \frac{7}{16} * \frac{8}{17} * \frac{9}{18} * \frac{10}{19}$ while the probability of him winning 1 out of the next 5 matches is:
$5 * \frac{9}{15} * \frac{6}{15} * \frac{7}{15} * \frac{8}{15} * \frac{9}{15}$. The sum of the previous calculations is $\frac{77}{646}$, but since the fraction is the probability Djokovic doesnt keep his lead in the rivalry, the probability he does keep his lead in the rivalry is:
$1 - \frac{77}{646} = \frac{569}{646}$.

Problem 16. One fair die is rolled; let a denote the number that comes up. We then roll a dice; let the sum of the resulting a numbers be b. Finally, we roll b dice, and let c be the sum of the resulting b numbers. Find the expected (average) value of c.

Answer: $\frac{343}{8}$

The expected result of an individual die roll is (1+2+3+4+5+6)/6 = 7/2. For any particular value of b, if b dice are rolled independently, then the expected sum is $\frac{7}{2}^b$. Likewise, when we roll a dice, the expected value of their sum b is $\frac{7}{2}^a$, so the expected value of c is $\frac{7}{2}^a$. Similar reasoning again shows us that the expected value of a is 7/2 and so the expected value of c overall is $(\frac{7}{2})^3 = \frac{343}{8}$. (Source: HMMT 2002)

Problem 17. An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point (5,5)?

Answer: 84

We can change the ants sequence of moves to a sequence $a_1, a_2, ..., a_{10}$, with $a_i = 0$ if the i-th step is up, and $a_i = 1$ if the i-th step is right. We define a subsequence of moves a_i , a_i+1, . . . , a_j , (i $\leq$ j) as an up run if all terms of the subsequence are equal to 0, and $a_i - 1$ and $a_j + 1$ either do not exist or are not equal to 0, and define a right run similarly. In a sequence of moves, up runs and right runs alternate, so the number of up rights can

differ from the number of right runs by at most one.
Now let f(n) denote the number of sequences $a_1, a_2, ..., a_n$ where $a_i \in \{1, 2\}$ for 1 $\leq$ i $\leq$ n, and $a_1 + a_2 + \ + a_n = 5$. (In essence, we are splitting the possible 5 up moves into up runs, and we are doing the same with the right moves). We can easily compute that f(3) = 3, f(4) = 4, f(5) = 1, and f(n) = 0 otherwise. For each possible pair of numbers of up runs and right runs, we have two choices of which type of run is first. Our answer is then $2(f(3)^2 + f(3)f(4) + f(4)^2 + f(4)f(5) + f(5)^2)$ = 2(9 + 12 + 16 + 4 + 1) = 84. (Source: HMMT 2010)

Problem 18. Given a 9 $\times$ 9 chess board, we consider all the rectangles whose edges lie along grid lines (the board consists of 81 unit squares, and the grid lines lie on the borders of the unit squares). For each such rectangle, we put a mark in every one of the unit squares inside it. When this process is completed, how many unit squares will contain an even number of marks?

Answer: 56

Consider the rectangles which contain the square in the ith row and jth column. There are i possible positions for the upper edge of such a rectangle, 10 i for the lower edge, j for the left edge, and 10 j for the right edge; thus we have $i \cdot (10 - i) \cdot j \cdot (10 - j)$ rectangles altogether, which is odd iff i, j are both odd, i.e. iff i, j $\in \{1, 3, 5, 7, 9\}$. There are thus 25 unit squares which lie in an odd number of rectangles, so the answer is 81 - 25 = 56. (Source: HMMT 2002)