1 Problems

1. Evaluate $(2^0 + 1^8)(8^1 + 0^2)$.

   (A) 8 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 2018

2. A square and a rectangle have equal area. The longer side of the rectangle is four times as long as the shorter side of the rectangle. What is the ratio of the square's perimeter to the rectangle's perimeter?

   (A) $\frac{1}{3}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{4}{5}$

3. Derek's coin machine takes in two pennies and gives out a nickel and a penny. If Derek starts with $1.00 worth of pennies, what is the maximum amount of money he can end with by repeatedly using his machine?

   (A) $3.92 **(B)** $3.96 **(C)** $4.92 **(D)** $4.96 **(E)** $5.00

4. How many times does "MOO" appear vertically, horizontally, forwards, or backwards (but not diagonally) in the following word search?

   M O O M O O
   O M O O M O
   O O M O O M
   M O O M O O
   O M O O M O
   O O M O O M

   (A) 28 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 42

5. Two ants start at opposite vertices of a square. Each second, both ants randomly move along an edge to an adjacent vertex. What is the probability that the ants meet in 3 seconds or less?

   (A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{8}$ **(E)** $\frac{15}{16}$

6. M. Hamel of Alsace has a small garden. For every three petunias, there is one rose. For every two roses, there is one daisy. If Hamel has a total of 36 flowers, how many roses does he have?

   (A) 4 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 24

7. Jono starts with some number of red marbles. He may exchange a red marble for 4 blue marbles, or he may exchange 2 red marbles and 2 blue marbles for a green marble. After 12 exchanges, Jono only has green marbles. How many marbles did he start with?

   (A) 12 **(B)** 14 **(C)** 15 **(D)** 20 **(E)** 24

8. Let $f(x)$ be a function with inverse $g(x)$. Which of the following functions is the inverse of $f(x - 2018)$?

   (A) $g(x - 2018)$
   (B) $g(x + 2018)$
   (C) $g(x)$
   (D) $g(x) - 2018$
   (E) $g(x) + 2018$

9. Let n be a two-digit number, and let m be the number obtained by reversing the digits of n. If $m + n$ and $m - n$ are both perfect squares, find mn.

(A) 3640 **(B)** 4356 **(C)** 4606 **(D)** 5040 **(E)** 6786

10. A robot can build a house in 100 hours. A single robot starts building a house, and after every 30 minutes of work it is joined by another robot. How many robots are present when the house is complete?

(A) 10 **(B)** 14 **(C)** 18 **(D)** 20 **(E)** 25

11. Each side of a regular hexagon is extended in both directions. These six lines intersect at six new points that form a larger regular hexagon. What is the ratio of the area of the larger hexagon to the area of the smaller hexagon?

(A) $\sqrt{3}$ **(B)** $\frac{3\sqrt{3}}{2}$ **(C)** 2 **(D)** $\frac{4\sqrt{3}}{3}$ **(E)** 3

12. The speed of a bike is inversely proportional to the weight of its rider(s). Starting at Jasper, Ishaan bikes to pick up his friend Roy and then bikes back to Jasper. If this trip takes four times as long as it would have if Roy had just biked to Jasper himself, find the ratio of Ishaan's weight to Roy's weight.

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{3}$ **(D)** $\frac{3}{2}$ **(E)** 2

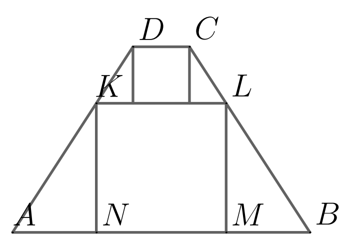
13. Yash has two identical wooden blocks in the shape of rectangular prisms. There are three ways for him to glue the two blocks together along a common face so that they form a single, large rectangular prism. The surface areas of these three possibilities are 132, 112, and 136. What is the volume of one of the wooden blocks?

(A) 40 **(B)** 45 **(C)** 54 **(D)** 64 **(E)** 70

14. Eight points are spaced equally around a circle. How many trapezoids (not rectangles) have all four vertices among these eight points?

(A) 16 **(B)** 24 **(C)** 32 **(D)** 40 **(E)** 48

15. Isosceles trapezoid $ABCD$ has two squares inscribed, as shown in the diagram. If $CD = 1$ and $AB = 5$, find the area of square $KLMN$.



(A) 3 **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 5

16. Two concentric circles have radii $\sqrt{2}$ and 2. Three points A, B, C are chosen on the larger circle such that $\triangle ABC$ is equilateral. Find the area of the region in the smaller circle that is also inside the triangle.

(A) 4 **(B)** $\frac{3}{2}(1 + \pi)$ **(C)** $\frac{14}{3}$ **(D)** $\frac{3\sqrt{3}}{2} + \pi$ **(E)** $3 + \frac{\pi}{2}$

17. Bryan and Kihan play a game. Both players start with a collection of marbles, and they take turns making moves by stealing enough marbles from the other player to double his own collection. Bryan starts with B marbles, Kihan starts with 10 marbles, and Kihan moves first. After three moves, the game ends because a fourth move is not possible. Find the sum of the possible values of B.

(A) 120 **(B)** 136 **(C)** 200 **(D)** 212 **(E)** 234

18. The following 6×6 grid is numbered, left to right and top to bottom. An ant starts at the square labeled 1 and, moving only right or down, makes its way to the square labeled 36. The *weight* of its path is the sum of all the numbers it walks on. Let S be the sum of the weights of all the possible paths the ant can take. Find the number of distinct prime factors of S.

1	2	3	4	5	6
7	8	9	10	11	12

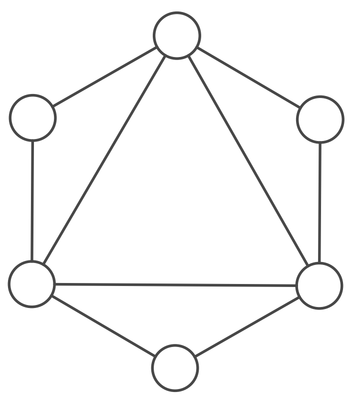
$\vdots$

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

19. Let C_1 be a circle with center O and radius 18, and let C_2 be a circle internally tangent to C_1 that passes through O. Let P be the point on C_1 diametrically opposite the point of tangency. The tangents from P to C_2 intersect C_1 at A and B. Compute the length of segment AB.

(A) $8\sqrt{2}$ **(B)** 10 **(C)** $12\sqrt{3}$ **(D)** $16\sqrt{2}$ **(E)** $18\sqrt{2}$

20. How many ways are there to fill the six vertices of the hexagon below with distinct, positive, one-digit integers so that the average of the three vertices of every triangle is an integer?



(A) 729 **(B)** 1296 **(C)** 1440 **(D)** 1728 **(E)** 2304

21. Find the number of ordered pairs of positive integers (a, b) with $a < b < 2017$ such that $10a$ is divisible by b and $10b$ is divisible by a.

(A) 201 **(B)** 806 **(C)** 1008 **(D)** 2015 **(E)** 4034

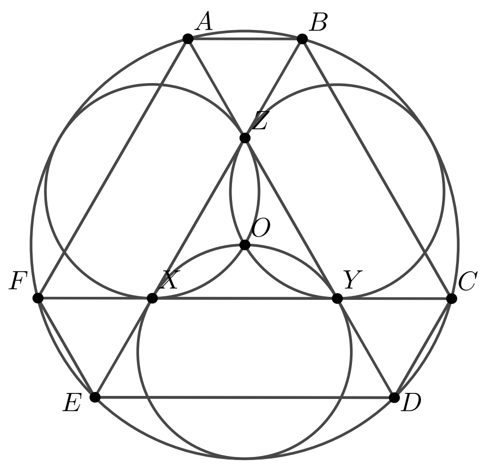
22. Let $s(n)$ denote the sum of the digits of a positive integer n. Find the number of three-digit positive integers $k \leq 500$ satisfying $s(k) = s(1000 - k)$.

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

23. Kayso, Kihan, and Kim each randomly choose a (not necessarily different) vertex of a cube and each paint the three faces that share that vertex. What is the probability that every face of the cube receives at least one layer of paint?

(A) $\frac{23}{64}$ **(B)** $\frac{25}{64}$ **(C)** $\frac{27}{64}$ **(D)** $\frac{29}{64}$ **(E)** $\frac{31}{64}$

24. Six points A, B, C, D, E, F are selected on a circle with center O such that $ABCDEF$ is an equiangular hexagon with $AB = CD = EF < BC = DE = FA$. Diagonals AD, BE, FC are drawn, intersecting at points X, Y, Z. The three circles passing through O and two of X, Y, Z are each internally tangent to O. Find BC/AB.



(A) $3\sqrt{3} - 3$ **(B)** $2\sqrt{5} - 2$ **(C)** $\frac{5}{2}$ **(D)** $\frac{3+\sqrt{5}}{2}$ **(E)** 3

25. Hubert throws 7 frogs onto the number line such that each frog lands on a distinct integer. Let d denote the distance between the farthest two frogs. Given that the average distance between any two frogs is 200, compute the minimum possible value of d.

(A) 350 **(B)** 352 **(C)** 354 **(D)** 356 **(E)** 358

2 Solutions

1. Evaluate $(2^0 + 1^8)(8^1 + 0^2)$.

   (A) 8 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 2018

   Answer: $\boxed{\textbf{(B) } 16}$

   $(2^0 + 1^8)(8^1 + 0^2) = (2)(8) = 16$.

2. A square and a rectangle have equal area. The longer side of the rectangle is four times as long as the shorter side of the rectangle. What is the ratio of the square's perimeter to the rectangle's perimeter?

   (A) $\frac{1}{3}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{4}{5}$

   Answer: $\boxed{\textbf{(E) } \frac{4}{5}}$

   Let the sides of the rectangle be ℓ and 4ℓ. Equal areas mean the square has side length 2ℓ. Thus, the ratio of the perimeters is $\frac{8\ell}{10\ell} = \frac{4}{5}$.

3. Derek's coin machine takes in two pennies and gives out a nickel and a penny. If Derek starts with \$1.00 worth of pennies, what is the maximum amount of money he can end with by repeatedly using his machine?

   (A) \$3.92 **(B)** \$3.96 **(C)** \$4.92 **(D)** \$4.96 **(E)** \$5.00

   Answer: $\boxed{\textbf{(D) } \$4.96}$

   Note that every use of the machine gets Derek a net monetary gain of \$0.04, with a net loss of one penny. He has 100 pennies, so after using it 99 times, he will be left with 1 penny, after which he cannot use his machine anymore. Thus his total balance will be $\$1.00 + 99(\$0.04) = \$4.96$.

4. How many times does "MOO" appear vertically, horizontally, forwards, or backwards (but not diagonally) in the following word search?

   M O O M O O
   O M O O M O
   O O M O O M
   M O O M O O
   O M O O M O
   O O M O O M

   (A) 28 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 42

   Answer: $\boxed{\textbf{(B) } 32}$

   Note that, by symmetry, "MOO" occurs the same number of times in each of the four directions. Parsing through the word search, we count 8 horizontal, rightward "MOO"s so the answer is $4 \cdot 8 = 32$.

5. Two ants start at opposite vertices of a square. Each second, both ants randomly move along an edge to an adjacent vertex. What is the probability that the ants meet in 3 seconds or less?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{8}$ **(E)** $\frac{15}{16}$

Answer: $\boxed{\textbf{(D) } \frac{7}{8}}$

We just need to find the probability that the ants do not meet in 3 seconds. Every second, the ants do not meet with probability of $\frac{1}{2}$, so in 3 seconds they do not meet with probability $\left(\frac{1}{2}\right)^3$. Thus the answer is $1 - \frac{1}{8} = \frac{7}{8}$.

6. M. Hamel of Alsace has a small garden. For every three petunias, there is one rose. For every two roses, there is one daisy. If Hamel has a total of 36 flowers, how many roses does he have?

(A) 4 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 24

Answer: $\boxed{\textbf{(B) } 8}$

Let the number of daisies be d. There are $2d$ roses and $6d$ petunias, for a total of $9d = 36$ flowers. Thus $d = 4$ and $2d = 8$.

7. Jono starts with some number of red marbles. He may exchange a red marble for 4 blue marbles, or he may exchange 2 red marbles and 2 blue marbles for a green marble. After 12 exchanges, Jono only has green marbles. How many marbles did he start with?

(A) 12 **(B)** 14 **(C)** 15 **(D)** 20 **(E)** 24

Answer: $\boxed{\textbf{(D) } 20}$

Let the two possible exchanges be Move 1 and Move 2. Let x be the number of times Jono made Move 1, so $12 - x$ is the number of times he used Move 2. He started with no blue marbles and ended with no blue marbles, gaining 4 blues from each Move 1 and losing 2 from each Move 2, so we have

$$4x = 2(12 - x)$$

We can solve $x = 4$. However, we also know that Jono ended with no red marbles. Thus if, Jono made Move 1 four times (each time losing one red) and Move 2 eight times (each time losing two reds), he must have started with $4 + 2(8) = 20$ red marbles.

8. Let $f(x)$ be a function with inverse $g(x)$. Which of the following functions is the inverse of $f(x - 2018)$?

(A) $g(x - 2018)$
(B) $g(x + 2018)$
(C) $g(x)$
(D) $g(x) - 2018$
(E) $g(x) + 2018$

Answer: $\boxed{\textbf{(E) } \text{g(x)+2018}}$

For any function $h(x)$ and its inverse $h^{-1}(x)$, we have $h^{-1}(h(x)) = x$. We seek the function that, when applied to $f(x-2018)$, gives x. We see that first applying $g(x)$ gives $x - 2018$, and then adding 2018 gives x, so the inverse is $g(x) + 2018$.

An alternate way to see this is by noting that the inverse of a function is its reflection over the line $y = x$. Thus, translating $f(x)$ 2018 units to the right must translate its inverse 2018 units upwards, which corresponds to $g(x) + 2018$.

9. Let n be a two-digit number, and let m be the number obtained by reversing the digits of n. If $m + n$ and $m - n$ are both perfect squares, find mn.

   (A) 3640 **(B)** 4356 **(C)** 4606 **(D)** 5040 **(E)** 6786

   Answer: $\boxed{\textbf{(A) } 3640}$

   Let the two-digit number be $10a + b$, where a and b are less than 10. Reversing the digits of n gives $10b + a$. Thus, $11(b + a)$ and $9(b - a)$ are both perfect squares. From the former we see that $b + a = 11$, and from the latter we see that $b - a = 1$ ($b - a$ must be an odd because $b + a = 11$). This gives $b = 6$ and $a = 5$, so the answer is $56 \cdot 65 = 3640$.

10. A robot can build a house in 100 hours. A single robot starts building a house, and after every 30 minutes of work it is joined by another robot. How many robots are present when the house is complete?

    (A) 10 **(B)** 14 **(C)** 18 **(D)** 20 **(E)** 25

    Answer: $\boxed{\textbf{(D) } 20}$

    Note that a robot works at a rate of $\frac{1}{100}$ houses per hour, or $\frac{1}{200}$ houses per half hour. Thus, we seek the smallest k so that

    $$\frac{1}{200} + \frac{2}{200} + \frac{3}{200} + \cdots + \frac{k}{200} \geq 1$$

    Which is $k = 20$.

11. Each side of a regular hexagon is extended in both directions. These six lines intersect at six new points that form a larger regular hexagon. What is the ratio of the area of the larger hexagon to the area of the smaller hexagon?

    (A) $\sqrt{3}$ **(B)** $\frac{3\sqrt{3}}{2}$ **(C)** 2 **(D)** $\frac{4\sqrt{3}}{3}$ **(E)** 3

    Answer: $\boxed{\textbf{(E) } 3}$

    The ratio of the areas is the square of the ratio of their side lengths. Doing some quick $30° - 60° - 90°$ geometry, we see that this ratio is $\sqrt{3}$. Thus the answer is $\sqrt{3}^2 = 3$.

12. The speed of a bike is inversely proportional to the weight of its rider(s). Starting at Jasper, Ishaan bikes to pick up his friend Roy and then bikes back to Jasper. If this trip takes four times as long as it would have if Roy had just biked to Jasper himself, find the ratio of Ishaan's weight to Roy's weight.

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{3}$ **(D)** $\frac{3}{2}$ **(E)** 2

Answer: $\boxed{\textbf{(D) } 3/2}$

Let Ishaan's weight be I, let Roy's weight be R, let Ishaan's speed alone on his bike be v_i, and let the distance from Jasper to Roy be d. Because speed is inversely proportional to weight, Roy's speed is $\frac{Iv_i}{R}$ and their speed together is $\frac{Iv_i}{I+R}$. From the given information we can write

$$\frac{d}{v_i} + \frac{d}{Iv_i/(I+R)} = 4 \cdot \frac{d}{Iv_i/R}$$

Canceling d and v_i and multiplying by I, we simplify and get $I + I + R = 4R$, so the answer is $I/R = 3/2$.

13. Yash has two identical wooden blocks in the shape of rectangular prisms. There are three ways for him to glue the two blocks together along a common face so that they form a single, large rectangular prism. The surface areas of these three possibilities are 132, 112, and 136. What is the volume of one of the wooden blocks?

(A) 40 **(B)** 45 **(C)** 54 **(D)** 64 **(E)** 70

Answer: $\boxed{\textbf{(A) } 40}$

Let the side lengths be a, b, c and let the surface area be $S = 2(ab + bc + ac)$. By gluing two blocks together, the resulting surface area must be twice the surface area of one block minus the covered common faces. Thus, we have

$$2S - 2ab = 132$$

$$2S - 2bc = 112$$

$$2S - 2ac = 136$$

Adding these equations, we have

$$6S - 2(ab + bc + ac) = 380$$

Or $5S = 380$, so $S = 76$. Plugging this in to each of our equations, we obtain $ab = 10$, $bc = 20$, and $ac = 8$. Volume $abc = \sqrt{ab \cdot bc \cdot ac} = 40$.

14. Eight points are spaced equally around a circle. How many trapezoids (not rectangles) have all four vertices among these eight points?

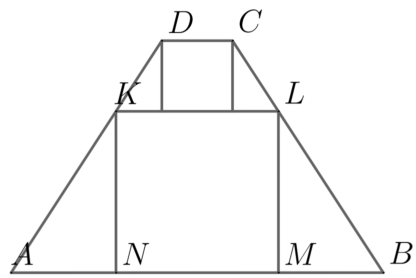
(A) 16 **(B)** 24 **(C)** 32 **(D)** 40 **(E)** 48

Answer: $\boxed{\textbf{(B) } 24}$

First, define the "length" of a segment as the number of points contained on the smaller arc determined by those two endpoints (this is just notation to make this write-up easier). For example, a segment connecting consecutive points has "length" 2.

A trapezoid has exactly one pair of parallel sides. We do some quick casework. If one of its bases has length 2, there are 8 ways to choose that segment and 2 ways to choose a parallel segment that does not form a rectangle. If the trapezoid does not have a base with length 2, we see that it must have bases of lengths 3 and 5. There are 8 more of the latter, for a total of 24.

15. Isosceles trapezoid $ABCD$ has two squares inscribed, as shown in the diagram. If $CD = 1$ and $AB = 5$, find the area of square $KLMN$.



(A) 3 **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 5

Answer: $\boxed{\textbf{(E) } 5}$

Let $NM = x$. Let the point where the perpendicular from D meets KL be P. Then $AN = \frac{5-x}{2}$ and $KP = \frac{x-1}{2}$. Note that DKP is similar to KAN, so $DP/KP = KN/NA$.

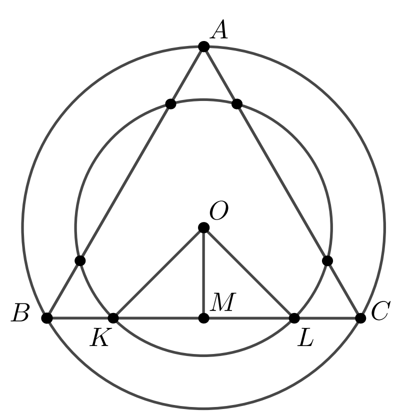
$$\frac{1}{\frac{x-1}{2}} = \frac{x}{\frac{5-x}{2}}$$

This gives $5 - x = x^2 - x$ or $x^2 = 5$, which is what we seek.

16. Two concentric circles have radii $\sqrt{2}$ and 2. Three points A, B, C are chosen on the larger circle such that $\triangle ABC$ is equilateral. Find the area of the region in the smaller circle that is also inside the triangle.

(A) 4 **(B)** $\frac{3}{2}(1+\pi)$ **(C)** $\frac{14}{3}$ **(D)** $\frac{3\sqrt{3}}{2}+\pi$ **(E)** $3+\frac{\pi}{2}$

Answer: $\boxed{\textbf{(E) } 3+\frac{\pi}{2}}$



Let M be the midpoint of BC and let K and L be the intersections denoted in the diagram above. ABC is equilateral with radius 2, so $OM = 1$. However, we are given that $OL = \sqrt{2}$, so it is clear that $\triangle OML$ and $\triangle OMK$ are isosceles right triangles. It follows that $\angle KOL = 90°$.

The area we seek is equal to the area of the small circle (which is 2π) minus the areas of the three "slivers" in the small circle but not in the triangle. The area of one of these slivers is $\frac{1}{4}(2\pi) - 1$, computed by subtracting the area of $\triangle KOL$ from the area of sector KOL. Thus the answer is $2\pi - 3\left(\frac{1}{4}(2\pi) - 1\right) = 3 + \frac{\pi}{2}$.

17. Bryan and Kihan play a game. Both players start with a collection of marbles, and they take turns making moves by stealing enough marbles from the other player to double his own collection. Bryan starts with B marbles, Kihan starts with 10 marbles, and Kihan moves first. After three moves, the game ends because a fourth move is not possible. Find the sum of the possible values of B.

(A) 120 **(B)** 136 **(C)** 200 **(D)** 212 **(E)** 234

Answer: $\boxed{\textbf{(D) } 212}$

Consider the following chart showing the number of marbles each player has (B, E) after each round.

$$(B - 10, 20)$$

$$(2B - 20, 30 - B)$$

$$(3B - 50, 60 - 2B)$$

At this point, it should be Bryan's turn, but the game ends because a move is impossible. This means that Kihan doesn't have enough marbles for Bryan to take, so we have

$$3B - 50 > 60 - 2B$$

or $B > 22$. However, we also know that because the game lasted three turns, a move was always possible before this. Thus we have the inequalities $20 \geq B - 10$ and $2B - 20 \geq 30 - B$. Only the former gives us new information: $B \leq 30$. Indeed, all values of B in $23, 24, \ldots, 30$ work. They sum to $\frac{1}{2} \cdot 53 \cdot 8 = 212$.

18. The following 6×6 grid is numbered, left to right and top to bottom. An ant starts at the square labeled 1 and, moving only right or down, makes its way to the square labeled 36. The *weight* of its path is the sum of all the numbers it walks on. Let S be the sum of the weights of all the possible paths the ant can take. Find the number of distinct prime factors of S.

1	2	3	4	5	6
7	8	9	10	11	12

⋮

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer: $\boxed{\textbf{(C) } 5}$

Consider any arbitrary path. Note that its reflection over the long diagonal from top-left to bottom-right produces another path. The sum of the weights of the path and its reflection is equal to $2 + 9 + 16 + ... + 72$, regardless of what the path is (try it for yourself - do you see why?) This sum is an arithmetic series and evaluates to $11 \cdot 37$. There are $\binom{10}{5}$ total paths, so there are half as many path-reflection pairs. Thus the sum of all the weights is

$$S = \frac{\binom{10}{5}}{2} \cdot 11 \cdot 37$$

This has 5 distinct prime factors.

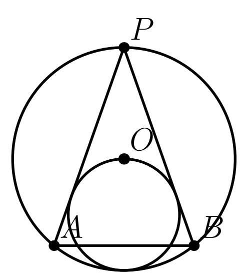
An alternate solution (found by Jay Li!), essentially the same but with less rigor, is to note that every path must pass through 11 total squares and that, on average, each square has weight $\frac{1 + 2 + \cdots + 36}{36} = \frac{37}{2}$. Thus each path, on average, has weight $11 \cdot \frac{37}{2}$, and there are $\binom{10}{5}$ paths, so we obtain the same answer as before.

Remark: If you think about it, this problem can actually be considered a variant of Gauss's elementary but well-known $1 + 2 + \cdots + 100$ problem.

19. Let C_1 be a circle with center O and radius 18, and let C_2 be a circle internally tangent to C_1 that passes through O. Let P be the point on C_1 diametrically opposite the point of tangency. The tangents from P to C_2 intersect C_1 at A and B. Compute the length of segment AB.

(A) $8\sqrt{2}$ **(B)** 10 **(C)** $12\sqrt{3}$ **(D)** $16\sqrt{2}$ **(E)** $18\sqrt{2}$

Answer: $\boxed{\textbf{(D) } 16\sqrt{2}}$



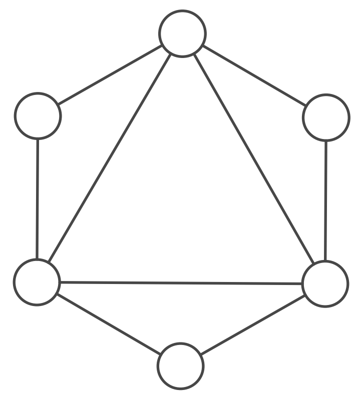
Let M be the midpoint of AB, let K be the center of C_2, and let Q be the point of tangency of PA on C_2. Note that PKQ is a right triangle whose hypotenuse is three times its shorter leg. Because of the shared angles, triangle PAM is similar to PKQ. Let $AM = x$. Thus, $PA = 3x$ and $PM = x\sqrt{8}$.

Consider triangle OAM. Its hypotenuse is 18 and its two legs have length x and $x\sqrt{8} - 18$. Pythagorean theorem gives

$$x^2 + (x\sqrt{8} - 18)^2 = 18^2$$

From which we can solve $x = 8\sqrt{2}$. We seek $2x = 16\sqrt{2}$.

20. How many ways are there to fill the six vertices of the hexagon below with distinct, positive, one-digit integers so that the average of the three vertices of every triangle is an integer?



(A) 729 **(B)** 1296 **(C)** 1440 **(D)** 1728 **(E)** 2304

Answer: $\boxed{\textbf{(B)}\ 1296}$

The sum of the vertices of any triangle must be a multiple of 3. We see that, for, any two triangles that share two vertices, their unshared vertices must be congruent modulo 3. Thus, for our hexagon, each pair of opposite vertices must be congruent mod 3. Furthermore, each pair must be unique modulo 3 because we are limited to $1, 2, \ldots, 9$ (otherwise we would need at least 4 numbers congruent mod 3, which is not possible).

There are $3! = 6$ ways to assign $0, 1, 2$ to be the values of the pairs mod 3, and since there are 3 numbers from $1, 2, \ldots, 9$ that are congruent to each of $0, 1, 2$, there are $3 \cdot 2$ ways to put in the numbers for each pair. Thus there are a total of $3! \cdot (3 \cdot 2)^3 = 1296$ ways.

21. Find the number of ordered pairs of positive integers (a, b) with $a < b < 2017$ such that $10a$ is divisible by b and $10b$ is divisible by a.

(A) 201 **(B)** 806 **(C)** 1008 **(D)** 2015 **(E)** 4034

Answer: $\boxed{\textbf{(D)}\ 2015}$

Let $d = \gcd(a, b)$ and let $a_1 = a/d$ and $b_1 = b/d$. It's clear that a_1 and b_1 are relatively prime.

The statement that $10a$ is divisible by b is equivalent to $10a_1$ is divisible by b_1 after canceling the factor of d. Because a_1 and b_1 are relatively prime, 10 must be divisible by b_1 and likewise for a_1. Thus the only possible pairs (a_1, b_1) are $(1, 10)$, $(1, 5)$, $(1, 2)$, and $(2, 5)$. These can all be scaled by a factor of d, which has no restriction as to what values it can have other than $a < b < 2017$. There are thus $201 + 403 + 1008 + 403 = 2015$ total solutions (a, b).

22. Let $s(n)$ denote the sum of the digits of a positive integer n. Find the number of three-digit positive integers $k \leq 500$ satisfying $s(k) = s(1000 - k)$.

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

Answer: $\boxed{\textbf{(A) } 31}$

Let k be abc, where a, b, and c are digits. Then $1000 - k$ has digits $9 - a$, $9 - b$, and $10 - c$. Equating the sum of the digits and simplifying, we have

$$a + b + c = 14$$

We have that $a \leq 5$ and $b, c \leq 9$. Doing some quick casework on a, we see that $a = 1$ gives 6 solutions, $a = 2$ gives 7 solutions, $a = 3$ gives 8 solutions, and $a = 4$ gives 9 solutions. The only solution when $a = 5$ is $k = 500$. The total is 31.

23. Kayso, Kihan, and Kim each randomly choose a (not necessarily different) vertex of a cube and each paint the three faces that share that vertex. What is the probability that every face of the cube receives at least one layer of paint?

(A) $\frac{23}{64}$ **(B)** $\frac{25}{64}$ **(C)** $\frac{27}{64}$ **(D)** $\frac{29}{64}$ **(E)** $\frac{31}{64}$

Answer: $\boxed{\textbf{(C) } \frac{27}{64}}$

We complementary count. There are $8^3 = 512$ total ways for the three to choose their vertices. Now we do casework on how many faces are left unpainted.

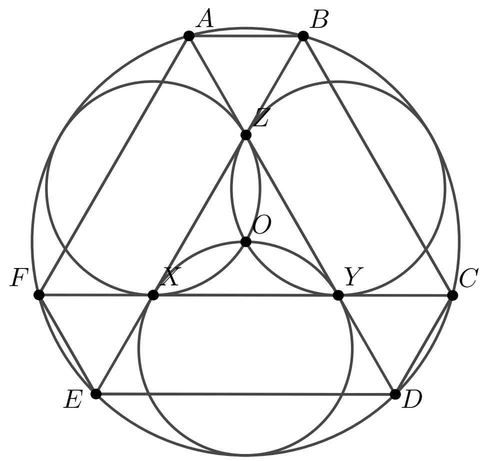
If exactly one face goes unpainted, the three vertices must all be on the same face. However, they could either occupy three of the vertices of the face OR they could collectively occupy only two vertices of one of the face's diagonals. There are 6 ways to choose the face, $4 \cdot 3 \cdot 2$ ways to assign them each a vertex in the first subcase, and $4 \cdot 3$ ways to do so in the second subcase, for a total of $6(12 + 24) = 216$ in this case.

If exactly two faces are unpainted, the three chosen vertices must occupy the two vertices of an edge. There are 12 edges to choose from, and 6 ways to to assign Kayso, Kihan, and Kim their choices, making 72.

If three faces are unpainted, all three chosen vertices must be the same. There are 8 ways in this case.

Finally, we have $\dfrac{512 - (216 + 72 + 8)}{512} = \dfrac{27}{64}$.

24. Six points A, B, C, D, E, F are selected on a circle with center O such that $ABCDEF$ is an equiangular hexagon with $AB = CD = EF < BC = DE = FA$. Diagonals AD, BE, FC are drawn, intersecting at points X, Y, Z. The three circles passing through O and two of X, Y, Z are each internally tangent to O. Find BC/AB.



(A) $3\sqrt{3}-3$ **(B)** $2\sqrt{5}-2$ **(C)** $\frac{5}{2}$ **(D)** $\frac{3+\sqrt{5}}{2}$ **(E)** 3

Answer: $\boxed{\textbf{(D) } \frac{3+\sqrt{5}}{2}}$

Let $AB = CD = EF = 1$ and $BC = DE = EA = x$. First note that every triangle here is equilateral, so we have some equal lengths: $DE = FY = x$ and $FE = FX = 1$, so $XY = x - 1$. $\triangle XYZ$ is equilateral, so we can quickly compute $OZ = \dfrac{x-1}{\sqrt{3}}$.

The radius of the large circle is twice the radius of each of the smaller circles, which each have radius equal to $OZ = OX = OY = \dfrac{x-1}{\sqrt{3}}$. (Can you prove this?)

Drop the perpendicular from Z to ED (passing through O) and let the foot of the perpendicular be M. We have that $ZM = \dfrac{x\sqrt{3}}{2}$ and $OM = ZM - OZ = \dfrac{x\sqrt{3}}{2} - \dfrac{x-1}{\sqrt{3}}$. We already know that the radius of the large circle $OE = \dfrac{2(x-1)}{\sqrt{3}}$. Pythagorean Theorem on $\triangle OEM$ gives

$$\left(\frac{x\sqrt{3}}{2}\right)^2 + \left(\frac{x\sqrt{3}}{2} - \frac{x-1}{\sqrt{3}}\right)^2 = \left(\frac{2(x-1)}{\sqrt{3}}\right)^2$$

Though it looks messy at first, a little bit of conservative algebra magically simplifies this to $x^2 - 3x + 1 = 0$, which has one positive root $\dfrac{3+\sqrt{5}}{2}$.

25. Hubert throws 7 frogs onto the number line such that each frog lands on a distinct integer. Let d denote the distance between the farthest two frogs. Given that the average distance between any two frogs is 200, compute the minimum possible value of d.

(A) 350 **(B)** 352 **(C)** 354 **(D)** 356 **(E)** 358

Answer: $\boxed{\textbf{(B) } 352}$

Let the six distances between consecutive frogs be $x_1, x_2, \ldots, x_6$, in that order. We would like an expression for the average distance. There are $\binom{7}{2}$ total distances. In order to sum all possible distances, we can consider the number of times each x_i is "used." There are i frogs to the left of x_i and $7-i$ frogs to its right, so our sum ends up looking like

$$\frac{6x_1 + 10x_2 + 12x_3 + 12x_4 + 10x_5 + 6x_6}{\binom{7}{2}} = 200$$

Simplifying, we have

$$3x_1 + 5x_2 + 6x_3 + 6x_4 + 5x_5 + 3x_6 = 2100$$

We would like to minimize $d = x_1 + x_2 + \cdots + x_6$. Let $a = x_2 + x_5$ and $b = x_3 + x_4$. Thus the expression becomes $3d + 2a + 3b = 2100$. We see that, to minimize d, we must maximize $2a + 3b$, and furthermore, a must be a multiple of 3. Let $2a + 3b = k$. Splitting up d and rewriting our expression once again, we have

$$3(x_1 + x_6) + a + 2k = 2100$$

We see that $3(x_1+x_6)+a$ must be even with a being a multiple of 3 and x_1, x_6 being positive integers, so its minimum value is 12. This corresponds to $k = 1044$ and $d = \dfrac{2100-k}{3} = 352$.