Math Club Relay

Meeting 1 - September 21, 2018

1 Problems

1.1 Round 1

Problem 1. A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly $\frac{1}{2}$. During the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than .503. What's the largest number of matches she could've won before the weekend began?

Problem 2. Find the sum of every even positive integer less than Answer 1 not divisible by 10.

Problem 3. How many positive integers less than Answer 2 are relatively prime to 1547? (Two integers are relatively prime if they have no common factors besides 1.)

Problem 4. Find the least positive integer n for which $1149^n - 338^n$ is divisible by Answer 3.

1.2 Round 2

Problem 1. In equilateral triangle ABC with side length 2, 3 segments are drawn to form segments AD, BE, and CF, with segments D, E, F on the midpoints of BC, AC, AB, respectively. The sum of the areas of the triangles that have one edge on BC is of the form $\frac{a\sqrt{b}}{c}$. Find $a - b - c$.

Problem 2. Let A be the answer to problem 1. Bob flips 10 fair coins. How many combinations of heads and tails are possible such that there are at least A heads in a row?

Problem 3. Let B be the answer to problem 2. Isosceles triangle XYZ has base XY and height equal to its base. The figure is rotated one full rotation around a line passing through X and perpendicular to XY. If the volume of the resulting solid is $\pi(B - 5)$, find the length of segment XY.

Problem 4. Let C be the answer to problem 3. Find the area of the shape

$$\frac{(x-y)^2}{18} + \frac{(x+y)^2}{C} = 1$$

Hint: The area of an eclipse can be found by the equation $A = \pi \cdot r_1 \cdot r_2$, where r_1 and r_2 are the axes of the eclipse.

1.3 Round 3

Problem 1. The name Uvuvwevwevwe has 12 letters.There are n different ways to rewrite this name using the same letters in different orders. For example, Vwevweuvuvwe would be one such permutation. The positive integer n can be denoted as $a.bcd \cdot 10^e$.

What is a+b+c+d+e?

Problem 2. Let A be the answer to the previous problem. J.R. Smith looks up at the scoreboard and tries to see the number of points for each team. Despite squinting J.R. still cant quite make out what the numbers are so his teammates help him.
Lebron James yells "Both teams have points between 1 and A inclusive!"
Tristan Thompson says "The numbers are not consecutive or equal!"
Kevin Love says "If one team has more than 10 points, the other is less than or equal to 10 points!"

How many different pairs of numbers could J.R. have guessed (assuming that all his teammates were telling the truth and that he somehow knows how to follow their instructions)?

Problem 3. Let B be the answer to problem 2. How many terms are there in the equation

$$(x+y+z)^B + (x-y-z)^B$$

after expanding and combining like terms?

Problem 4. Let C be the answer to problem 3. A circle of radius r is concentric with and outside a regular hexagon of side length $\sqrt{C}$. The probability that three entire sides of hexagon are visible from a randomly chosen point on the circle is $\frac{1}{2}$. What is r?

2 Solutions

2.1 Round 1

Problem 1. A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly $\frac{1}{2}$. During the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than .503. What's the largest number of matches she could've won before the weekend began?

Answer: 164

Let n be the number of matches won, so that $\frac{n}{2n} = \frac{1}{2}$, and $\frac{n+3}{2n+4} > \frac{503}{1000}$. Cross multiplying, 1000n + 3000 > 1006n + 2012, and n < $\frac{988}{6}$. Thus, the answer is 164.

Problem 2. Find the sum of every even positive integer less than Answer 1 not divisible by 10.

Answer: 5446

$(2 + 4 + ... + 162) - (10 + 20 + 30 + \ + 160)$
$= 2(1 + 2 + ... + 81) - 10(1 + 2 + ... + 16)$
$= 2 * 12 * 81 * (1 + 81) - 10 * 12 * 16 * (1 + 16)$
$= 5446$

Problem 3. How many positive integers less than Answer 2 are relatively prime to 1547? (Two integers are relatively prime if they have no common factors besides 1.)

Answer: 4055

The prime factorization of 1547 is 7*13*17. Thus, by the Principal of Inclusion Exclusion, the number of positive integers less than 5446 that are relatively prime to 1547 is

$$5445 - \lfloor\frac{5445}{7}\rfloor - \lfloor\frac{5445}{13}\rfloor - \lfloor\frac{5445}{17}\rfloor + \lfloor\frac{5445}{7\cdot 13}\rfloor + \lfloor\frac{5445}{7\cdot 17}\rfloor + \lfloor\frac{5445}{13\cdot 17}\rfloor - \lfloor\frac{5445}{7\cdot 13\cdot 17}\rfloor = 4055$$

Problem 4. Find the least positive integer n for which $1149^n - 338^n$ is divisible by Answer 3.

Answer: 4

If 1149n - 338n is divisible by 4055, then

$$1149^n \equiv 338^n \mod 4055$$

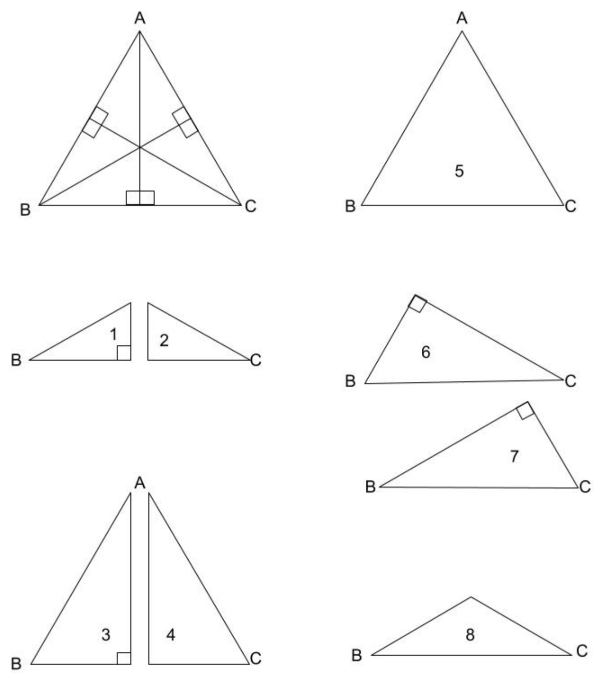
$$1149^n \cdot 12^n \equiv 338^n \cdot 12^n \mod 4055$$
$$13788^n \equiv 4056^n \mod 4055$$
$$13788^n \equiv 1 \mod 4055$$

The prime factorization of 4055 is $5 \cdot 811$, but since $13788^n \equiv 1 \mod 811$, then all thats needed to be found is an integer n for which $13788^n \equiv 1 \mod 5$. All integers divisible by 5 have ones digit 0 or 5, which means all integers congruent to 1 mod 5 have ones digit 1 or 6. However, all integers with ones digit 1 are odd, so the lowest n for which 13788^n has a ones digit 6. 8^4 has a ones digit 6, so $13788^4 \equiv 1 \mod 4055$, giving the value of 4 as n.

2.2 Round 2

Problem 1. In equilateral triangle ABC with side length 2, 3 segments are drawn to form segments AD, BE, and CF, with segments D, E, F on the midpoints of BC, AC, AB, respectively. The sum of the areas of the triangles that have one edge on BC is of the form $\frac{a\sqrt{b}}{c}$. Find $a - b - c$.

Answer: 5



Call the point of intersection of the 3 medians P. Then we have 8 triangles that meet the requirements: BPD, CPD, BAD, CAD, ABC, BFC, CEB, BPC.

$$\begin{cases} [BPD] = [CPD] = \frac{1}{2} * 1 * \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6} \\ [CAD] = [BAD] = \frac{1}{2} * 1 * \sqrt{3} = \frac{\sqrt{3}}{2} \\ [ABC] = \frac{1}{2} * 2 * \sqrt{3} = \sqrt{3} \\ [BFC] = [CEB] = \frac{1}{2} * 1 * \sqrt{3} = \frac{\sqrt{3}}{2} \\ [BPC] = \frac{1}{2} * 2 * \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{3} \end{cases}$$

$[BPD] + [CPD] + [CAD] + [BAD] + [ABC] + [BFC] + [CEB] + [BPC] =$
$\sqrt{3} * (\frac{1}{6} + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3}) = \frac{11*\sqrt{3}}{3}$
$a = 11$, $b = 3$, $c = 3$, so $a - b - c = 5$

Problem 2. Let A be the answer to problem 1. Bob flips 10 fair coins. How many combinations of heads and tails are possible such that there are at least A heads in a row?

Answer: 112

We will solve this by casework. A '[r]' denotes the case of a reflection, which is why the combinations will be doubled. Example: HHT can be reflected to form THH

Case 1: Exactly 5 heads in a row

HHHHHT _ _ _ _ $\longrightarrow 2^4 \cdot 2[r] = 32$
THHHHHT _ _ _ $\longrightarrow 2^3 \cdot 2[r] = 16$
_ THHHHHT _ _ $\longrightarrow 2^3 \cdot 2[r] = 16$

Case 2: Exactly 6 heads in a row -

HHHHHHT _ _ _ $\longrightarrow 2^3 \cdot 2[r] = 16$
THHHHHHT _ _ $\longrightarrow 2^2 \cdot 2[r] = 8$
_ THHHHHHT _ $\longrightarrow 2^2 \cdot 2[r] = 8$

Case 3: Exactly 7 heads in a row -

HHHHHHHT _ _ $\longrightarrow 2^2 \cdot 2[r] = 8$
THHHHHHHT _ $\longrightarrow 2 \cdot 2[r] = 4$

Case 4: Exactly 8 heads in a row -

HHHHHHHHT _ $\longrightarrow 2 \cdot 2[r] = 4$
THHHHHHHHT $\longrightarrow 1$

Case 5: Exactly 9 heads in a row -
THHHHHHHHH $\longrightarrow 1 \cdot 2[r] = 2$

Case 6: Exactly 10 heads in a row -
HHHHHHHHHH $\longrightarrow 1$

$$32 + 16 + 16 + 16 + 8 + 4 + 8 + 4 + 4 + 1 + 2 + 1 = 112$$

Problem 3. Let B be the answer to problem 2. Isosceles triangle XYZ has base XY and height equal to its base. The figure is rotated one full rotation around a line passing through X and perpendicular to XY. If the volume of the resulting solid is $\pi(B - 5)$, find the length of segment XY.

Answer: 6

The solid is equal to the volume of a big cone with radius length AB and height $2 \cdot$ AB minus 2 smaller cones with radius AB/2 and height AB. If we let x = AB, then the volume of the solid in terms of x is:

$$\frac{1}{3} \cdot \pi \cdot x^2 \cdot 2x - 2 \cdot \frac{1}{3} \cdot \pi \cdot (\frac{x}{2})^2 \cdot x$$

$$= \frac{2\pi x^3}{3} - \frac{\pi x^3}{6}$$

$$= \frac{\pi x^3}{2}$$

Using our answer from the last problem, we know that the volume is 108π. Therefore,

$$\frac{\pi x^3}{2} = 108\pi$$

$$x^3 = 216$$

$$x = 6$$

Problem 4. Let C be the answer to problem 3. Find the area of the shape

$$\frac{(x-y)^2}{18} + \frac{(x+y)^2}{C} = 1$$

Hint: The area of an eclipse can be found by the equation $A = \pi \cdot r_1 \cdot r_2$, where r_1 and r_2 are the axes of the eclipse.

Answer: $3\sqrt{3}\pi$

The shape is an ellipse rotated 45 degrees. The major axis is $\frac{(x-y)^2}{18}$. To find the radius of the major axis, we set x=-y and find x:

$$(x-y)^2 = 18$$

$$(2x)^2 = 18$$

$$x^2 = \frac{9}{2}$$

Because the radius is rotated 45 degrees, the radius is

$$\sqrt{(x^2) + (x^2)} = 3$$

We do the same with minor axis $\frac{(x+y)^2}{6}$ except set x=y, and we get

$$\sqrt{(x^2) + (x^2)} = \sqrt{3}.$$

The area of an ellipse is $\pi \cdot r_1 \cdot r_2$, so the area of the eclipse is

$$\pi \cdot 3 \cdot \sqrt{3} = 3\sqrt{3}\pi$$

2.3 Round 3

Problem 1. The name Uvuvwevwevwe has 12 letters.There are n different ways to rewrite this name using the same letters in different orders. For example, Vwevweuvuvwe would be one such permutation. The positive integer n can be denoted as $a.bcd \cdot 10^e$.

What is a+b+c+d+e?

Answer: 23

Permuting 2 us, 4 vs, 3 ws, and 3 es gives

$$\frac{12!}{2! \cdot 3! \cdot 3! \cdot 4!} = 277200 = 2.772 \cdot 10^5$$

This gives a = 2, b = 7, c = 7, d = 2, e = 5. Therefore

$$a + b + c + d + e = 23$$

Problem 2. Let A be the answer to the previous problem. J.R. Smith looks up at the scoreboard and tries to see the number of points for each team. Despite squinting J.R. still cant quite make out what the numbers are so his teammates help him.

Lebron James yells “Both teams have points between 1 and A inclusive!”

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How many different pairs of numbers could J.R. have guessed (assuming that all his teammates were telling the truth and that he somehow knows how to follow their instructions)?

Answer: 330

We proceed with casework

Case 1: Both teams scored less than or equal to 10 points

Without loss of generality, let Team A score less points than Team B. Following are the possibilities for the scores with (Team A score, Team B score):

(1,3), (1,4), (1,5) ... (1,9), (1,10) $\longrightarrow$ 8

(2,4), (2,5), ... (2,9), (2,10) $\longrightarrow$ 7

.

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.
(7,9), (7,10) $\longrightarrow$ 2
(8,10) $\longrightarrow$ 1
This gives 1+2+...+7+8 = 36 combinations.
To account for when Team A scores more than Team B, we double this number to get 72 possible scores.

Case 2: One team scored above 10 points
Without loss of generality, let Team B be the one that scored more than 10 points. There are 10 possible scores for Team A and 13 possible scores for Team B, giving a total of $10 \cdot 13 = 130$ scores. However, because the scores cant be consecutive, (10,11) is not a feasible score, giving 129 scores. We multiply this by 2 to account for when Team A is the one that scores more than 10 points, giving $129 \cdot 2 = 258$ possible scores.

To find the answer, we add the number of possible scores from both cases to get 72 + 258 = 330 possible scores.

Problem 3. Let B be the answer to problem 2. How many terms are there in the equation

$$(x + y + z)^B + (x - y - z)^B$$

after expanding and combining like terms?

Answer: 27556

Let, a = y + z, which leaves

$$(x + a)^{330} + (x - a)^{330}$$

Expanding the first part and removing the constants leaves

$$x^{330} + x^{329} \cdot a + x^{328} \cdot a^2 + ... + x^2 \cdot a^{328} + x \cdot a^{329} + a^{330}$$

Expanding the second part and removing the constants leaves

$$x^{330} - x^{329} \cdot a + x^{328} \cdot a^2 + ... + x^2 \cdot a^{328} - x \cdot a^{329} + a^{330}$$

Adding them together and cancelling like terms leaves

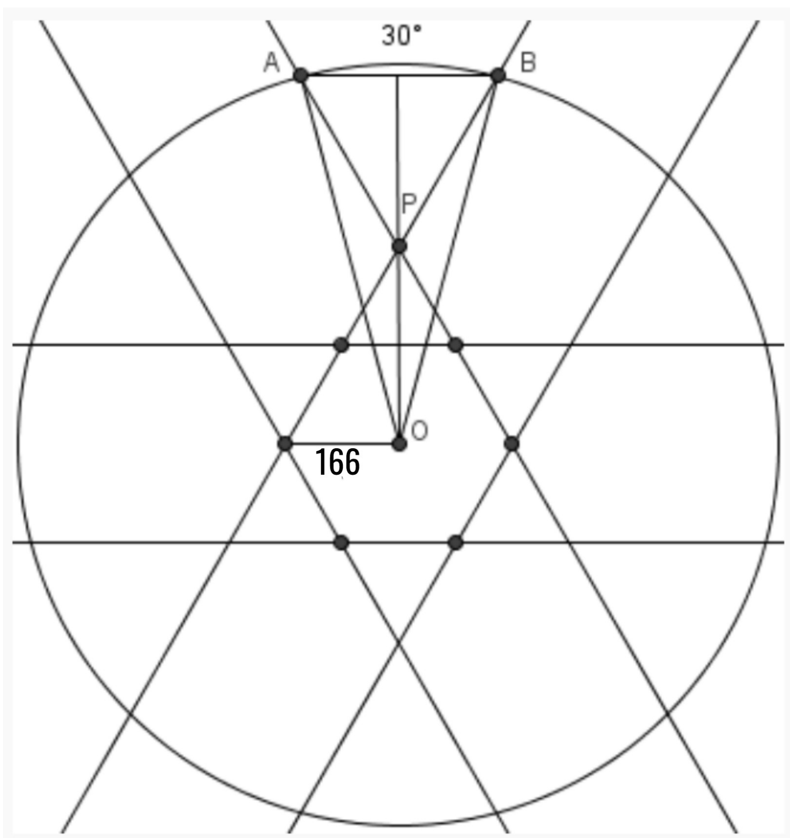
$$2(x^{330} + x^{328} \cdot a^2 + x^{326} \cdot a^4 + ... + x^4 \cdot a^{326} + x^2 \cdot a^{329} + a^{330})$$

As $(y+z)^n$ yields n+1 terms, substitute y+z back in for a and get term count of

$$1 + 3 + 5 + ... + 327 + 329 + 331 = 166^2 = 27556$$

Problem 4. Let C be the answer to problem 3. A circle of radius r is concentric with and outside a regular hexagon of side length $\sqrt{C}$. The probability that three entire sides of hexagon are visible from a randomly chosen point on the circle is $\frac{1}{2}$. What is r?

Answer: 836 + 2492



In order for the probability to be $\frac{1}{2}$, the degree of the arc must be 30 degrees. Triangle APO is isosceles as angle POA and angle PAO are 15 degrees. We know that PO is length $166\sqrt{3}$. Then knowing that $\cos(15°) = \frac{\sqrt{2}+\sqrt{6}}{4}$, we find that

$$\frac{\overline{\text{AO}}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4} \cdot 166\sqrt{3}$$

$$\overline{\text{AO}} = \frac{\sqrt{2}+\sqrt{6}}{4} \cdot 166\sqrt{3} \cdot 2$$

$$= 83\sqrt{6} + 249\sqrt{2}$$