Recursion and Induction

Inductive and recursive ideas involve using smaller, easier cases to address the problem at hand. They are often useful for solving problems that are difficult to directly attack. Note that there may be other ways to solve a few of the following problems, but keep in mind that induction and recursion will work for them as well.

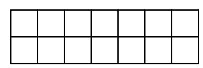
1. Show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

2. Show that $7^n - 1$ is a multiple of 6 for positive integer n.

3. Show that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

4. Show that an even number of people can be put into pairs in an odd number of ways. Compute this number for any given $2n$ people.

5. Compute the number of ways to tile the following shape with dominoes.



6. (MATHCOUNTS) The doctor gave Max ten vitamins, with instructions to take one or two each day until he runs out of vitamins. For example, Max could take a vitamin a day for ten days, or he could take two the first day and one a day for the next eight days. In how many different ways can Max take the ten vitamins?

7. (AMC 12) A subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is called ”spacy” if no two consecutive numbers are in the subset. For example, $\{1\}$ and $\{4, 6, 8\}$ are spacy. How many spacy subsets are there?

8. Aman has been tasked with tiling a 32 by 32 square grid with non-overlapping tiles consisting of three 1 by 1 unit squares arranged in a ”L” shape. There is a hole somewhere in the grid. Prove that he can tile the grid regardless of where the hole is located.

9. Let $a_1 = \sqrt{2}$, and for every $n \geq 1$ let $a_n = (\sqrt{2})^{a_{n-1}}$. Show that, a) this sequence remains strictly less than 2, and b) this sequence is strictly increasing.

10. (Mildorf) Shenai delivers newspapers to 10 houses along Main Street. Wishing to save effort, she doesn't always deliver to every house, but to avoid being fired she never misses three consecutive houses. Compute the number of ways Shenai could deliver papers in this manner.

11. Show that an isosceles triangle with one angle of 120 can be partitioned into k similar triangles for every integer $k \geq 4$.

12. (AIME) A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

    - Any cube may be the bottom cube in the tower.
    - The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

Solutions

1. Note that $1 = \frac{1(1+1)}{2}$, so the statement holds for $n = 1$. Assume that the statement is true for some arbitrary k. Thus, $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$. We want to show that the statement will also be true for $k + 1$. Adding $k + 1$ to both sides and doing some algebra,

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned} \tag{1}$$

   This is the statement we wanted to show true for $k + 1$, so we are done.

2. Note that $7^1 - 1 = 6$ is a multiple of 6. Assume that $7^k - 1$ is a multiple of 6 for some positive integer k. We want to show that $7^{k+1} - 1$ is also a multiple of 6. Note that $7^{k+1} - 1$ can be written as $7(7^k - 1) + 6$, which is a multiple of 6 because of our assumption. We conclude the induction.

3. Proceed as is shown in problem 1: show the base case, assume true for some k, add $(k+1)^3$ to both sides, and do some algebra to manipulate the terms into the desired form.

4. Let a_n denote the number of ways to pair $2n$ people. Note that a_1=1, which is odd. Assume that a_k is odd for some positive integer k. To calculate a_{k+1}, pick one of the $2k + 2$ people and pair him with one of the remaining $2k + 1$ people. This leaves $2k$ people to be paired, which happens in a_k ways. Thus, $a_{k+1} = (2k + 1)a_k$, which is clearly odd. It turns out that $a_k = (2k - 1)(2k - 3) \ldots (3)(1)$.

5. Let a_n be number of ways to tile a $2 \times n$ strip with dominoes. We want to find a_7.

   To calculate a_n in general, note that every valid tiling of the strip must end either with a single vertically oriented domino or two horizontally oriented dominoes. There are a_{n-1} ways that the former can happen and a_{n-2} ways that the latter can happen. Thus, we have the recursive formula $a_n = a_{n-1} + a_{n-2}$. It is clear that $a_1 = 1$ and $a_2 = 2$, so we compute these Fibonacci numbers up to $a_7 = 21$.

6. Let a_n be the number of ways Max can take n vitamins according to these rules. We want to find a_{10}.

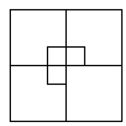
   To calculate a_n in general, note that to eat n vitamins Max must either 1) eat $n - 1$ vitamins and then finish by eating one more the next day, or 2) eat $n - 2$ vitamins and then finish by eating two more the next day. Again, we have the recursive formula $a_n = a_{n-1} + a_{n-2}$. It is clear that $a_1 = 1$ and $a_2 = 2$, so we compute these Fibonacci numbers up to $a_{10} = 89$.

7. Let a_n be the number of spacy subsets of $\{1, 2, \ldots, n\}$.We want to find a_8.

   To calculate a_n in general, note that every spacy subset either contains n or does not contain n. If the former is true, then $n-1$ cannot be in the subset, and thus there are a_{n-2} such spacy subsets. If the latter is true, then there is no restriction on the first $n-1$ numbers so there are a_{n-1} such spacy subsets. Again, we have the recursive formula $a_n = a_{n-1} + a_{n-2}$. It is clear that $a_1 = 2$ and $a_2 = 3$, so we can compute $a_8 = 55$.

   Note: The actual AMC 12 problem was slightly different.

8. We claim that Aman can tile every such $2^n \times 2^n$ grid. As a base case, consider a 2×2 square. It is clear that the hole can be in any of the four corners. Now assume that this is true for some k - we wish to show that it's true for $k+1$ as well. Note that a $2^{k+1} \times 2^{k+1}$ square can be partitioned into four $2^k \times 2^k$ squares, one of which must contain the hole. By our assumption, this square can be tiled wherever the hole is. Whichever quadrant this is, the other three can each be tiled to leave a hole in the corner, again by our assumption.

   This leaves an empty "L" shape in the center, which can be filled with one more tile. Thus, the $2^{k+1} \times 2^{k+1}$ can be successfully tiled wherever the hole is, and this completes the induction.

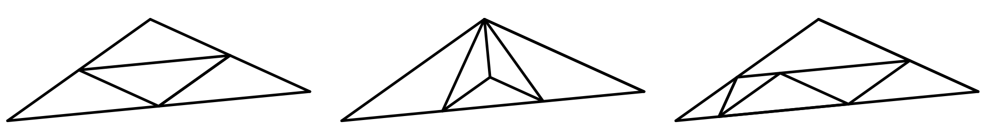
9. a) Note that the first term $\sqrt{2} < 2$. Now assume that $a_k < 2$ for some k. It follows that $a_{k+1} = (\sqrt{2})^{a_k} < \sqrt{2}^2 = 2$.

   b) Note that $\sqrt{2}^{\sqrt{2}} > \sqrt{2}$. Now assume that $a_k > a_{k-1}$ for some k. It follows that $a_{k+1} = (\sqrt{2})^{a_k} > (\sqrt{2})^{a_{k-1}} = a_k$.

10. Let a_n be the number of ways Shenai could deliver papers to n houses. We want to find a_{10}. We work out the small cases $a_1 = 2$, $a_2 = 4$, and $a_3 = 7$. Now consider the case $n = 4$. Either Shenai delivers to the first house, after which there are a_{n-1} possible routes, or she skips the first house. If she skips the first house she may deliver to the second house, after which there are a_{n-2} routes, or she may skip the second house. If she skips the first and second houses, she must deliver to the third house, which leaves a_{n-3} possible routes. Hence, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. We can use this recursive formula to compute $a_{10} = 504$.

    This is taken straight from Mildorf's solutions.

11. This is my favorite problem from this set. I claim not only that the triangle can be partitioned into similar triangles but actually into 120-30-30 isosceles triangles. The following diagrams illustrate the cases $n = 4, 5, 6$.

A little more rigor might be needed to show that they exist, but intuitively it's clear that they do.

Now assume that there exists a partitioning of the triangle into k smaller triangles. To obtain one for $k+3$, simply partition the triangle into four (as shown in the first diagram) and partition one of the smaller triangles into k. This inductive argument covers all integers greater than 6 and finishes the proof.

12. Let a_n be the numbers of ways to make such a tower with blocks of sizes $1, 2, \ldots, n$. We want to find a_8.

    We can compute $a_1 = 1$ and $a_2 = 2$. Note that for $n \geq 3$, to make every tower of size n we take a tower of size $n-1$ and insert block n in one of three possible locations: at the bottom, right above block $n-1$, or right above block $n-2$. From this we have the recursive formula $a_n = 3a_{n-1}$, and we can easily compute the last three digits of a_8 to be 458.