Financial Time Series

Lecture 3

ARMA models

- The model defined by

$$r_t - \varphi_1 r_{t-1} - \cdots - \varphi_p r_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q},$$

where $a_t \sim WN(0, \sigma^2{}_a)$ is called an ARMA(p, q) model if it is stationary.

- So, not so surprisingly ,ARMA means AR and MA

ARMA models

- Using the back shift operator B defined by $B^j X_t = X_{t-j}$, we may write,

$$\varphi(B) r_t = \theta(B) a_t,$$

where

$$\varphi(z) = 1 - \varphi_1 z - \cdots - \varphi_p z^p$$

and

$$\theta(z) = 1 - \theta_1 z - \cdots - \theta_q z^q$$

AR and MA representations

- Note that by polynomial division we may write

$$\frac{\theta(z)}{\varphi(z)} = 1 + \psi_1 z + \psi_2 z^2 + \cdots = \psi(z)$$

or

$$\frac{\varphi(z)}{\theta(z)} = 1 + \pi_1 z + \pi_2 z^2 + \cdots = \pi(z)$$

AR and MA representations

- This means that we may write an ARMA(p, q) as an MA

$$r_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$$

or as an AR

$$r_t = \frac{\varphi_0}{1 - \theta_1 - \cdots - \theta_q} + \pi_1 r_{t-1} + \pi_2 r_{t-2} + \cdots + a_t$$

Invertibility

- Considering the AR representation we say that the ARMA model is invertible if π_i decays to zero as i increases

- Note that for a pure AR(p) model we have that $\theta(z) = 1$ so that $\pi(z) = \varphi(z)$ and hence $\pi_i = 0$ for $i > p$

- A sufficient condition for invertibility is that all the roots of $\theta(z)$ are greater than one in modulus

ARMA(1,1)

- This model is highly related to the most common volatility models occuring later in the course

- The ARMA(1,1) is given by

$$r_t - \varphi_1 r_{t-1} = \varphi_0 + a_t - \theta_1 a_{t-1}$$

- This makes sense only if $\varphi_1 \neq \theta_1$, since otherwise there would be cancellation and we would have just a white noise series

Properties of the ARMA(1,1)

- If we take expectations in the above equation under the assumption of weak stationarity, we have that

$$E(r_t) = \frac{\varphi_0}{1 - \varphi_1}$$

- Multiplying the above equation by a_t, we see that

$$E(r_t a_t) = {\sigma^2}_a$$

Properties of the ARMA(1,1)

- So that

$$Var(r_t) = (\varphi_1)^2 Var(r_{t-1}) + \sigma^2{}_a + (\theta_1)^2\sigma^2{}_a - 2\varphi_1\theta_1 E(r_{t-1}a_{t-1})$$

gives

$$Var(r_t) = \frac{\sigma^2{}_a(1 + (\theta_1)^2 - 2\varphi_1\theta_1)}{1 - (\varphi_1)^2}$$

So we need $|\varphi_1| < 1$ in order for the variance to be positive

Properties of the ARMA(1,1)

- To find the autocovariance we assume that $\varphi_0 = 0$ and multiply the above equation by r_{t-l} and get

$$r_t r_{t-l} - \varphi_1 r_{t-1} r_{t-l} = a_t r_{t-l} - \theta_1 a_{t-1} r_{t-l}$$

- Taking expectations if $l = 1$ gives

$$\gamma_1 - \varphi_1 \gamma_0 = -\theta_1 {\sigma^2}_a$$

Properties of the ARMA(1,1)

- Taking expectations if $l = 2$ gives

$$\gamma_2 - \varphi_1 \gamma_1 = 0$$

- And if $l > 1$ we have

$$\gamma_l - \varphi_1 \gamma_{l-1} = 0$$

Properties of the ARMA(1,1)

- So the ACF of a weakly stationary ARMA(1,1) is given by

$$\rho_1 = \varphi_1 - \frac{\theta_1 {\sigma^2}_a}{\gamma_0}, \qquad \rho_l = \varphi_1 \rho_{l-1}, \qquad l > 1$$

Some nonstationary models

- In som contexts nonstationary models are more appropriate than stationary ones
- For instance prices of assets and foreign exchange rates tend to be nonstationary
- Series for which there is no fixed level are called unit-root nonstationary models

Unit-root nonstationary model

- A random walk is probably the most common example of a unit-root nonstationary model

- It may be used to describe log-prices of a stock

- A random walk $\{p_t\}$ may be defined by

$$p_t = p_{t-1} + a_t$$

- If a_t has a symmetric distribution there is a 50/50 chance of p_t to move up or down, conditional on p_{t-1}

Random Walk

- We may think of the RW as an AR(1) model with $\varphi_1 = 1$ so the clearly the model is not weakly stationary (and hence not (strictly) stationary)

- Note that h-step ahead predictions are given by

$$\hat{p}_t(h) = E\left(p_{t+h} | p_{t,\dots}\right) = E(p_{t+h} | p_t) = p_t$$

RW as an MA

- Note also that we may write

$$p_t = a_t + a_{t-1} + \cdots$$

- Hence the h-step forecast error is

$$e_t(h) = a_{t+h} + \cdots + a_{t+1}$$

RW with drift

- In stock price applications it is natural to assume that the log-prices are governed by

$$p_t = \mu + p_{t-1} + a_t$$

- And of course we would like to see that μ is significantly positive if we are to invest in the stock, assuming that the distribution of a_t is symmetric

RW with drift

- Note that

$$
\begin{aligned}
p_t &= \mu + p_{t-1} + a_t \\
&= \mu + \mu + p_{t-2} + a_{t-1} + a_t \\
&\vdots \\
&= \mu t + p_0 + \sum_{i=1}^{t} a_i
\end{aligned}
$$

RW with drift

- We get that

$$E(p_t) = \mu t + p_0$$

$$Var(p_t) = t\sigma^2{}_a$$

$$Cov(p_t, p_s) = \min\{s, t\}\sigma^2{}_a$$

Trend-stationary time series

- The model defined by

$$p_t = \beta_0 + \beta_1 t + r_t$$

where $\{r_t\}$ is stationary is called a trend-stationary model

- Note that $E(p_t) = \beta_0 + \beta_1 t$ and $Var(p_t) = Var(r_t)$ so the variance does not depend on time

General unit-root nonstationary models

- An ARMA model may be extended so that the AR-polynomial has 1 as a characteristic root

- Doing so gives us an ARIMA model

- We say that the time series $\{y_t\}$ follows an ARIMA($p, 1, q$) model if it holds that the change series or increment series $\{c_t\}$ given by $c_t = y_t - y_{t-1}$ follows a stationary and invertible ARMA(p, q) model

ARIMA

- Is common belief that stock prices are non-stationary but that log-returns are stationary

- Under these assumptions the log-price will be unit-root nonstationary

- It may also be the case that $\{y_t\}$ and $\{c_t\}$ are unit-root nonstationary but $\{s_t\}$ given by $s_t = c_t - c_{t-1}$ is ARMA(p, q). In this case we say that $\{y_t\}$ is ARIMA$(p, 2, q)$

Unit-root test

- If we want to test if the log-price of an asset follows an RW or an RW w drift we assume the models

$$p_t = \varphi_1 p_{t-1} + e_t$$
$$p_t = \varphi_0 + \varphi_1 p_{t-1} + e_t$$

where e_t is the error term, the null hypothesis is H_0: $\varphi_1 = 1$ and the alternative hypothesis is H_1: $\varphi_1 < 1$

Unit-root test (Dickey-Fuller)

- The least squares estimate of φ_1 is

$$\hat{\varphi}_1 = \frac{\sum_{t=1}^{T} p_t p_{t-1}}{\sum_{t=1}^{T} (p_{t-1})^2}$$

where $p_0 = 0$, also

$$\hat{\sigma}^2{}_e = \frac{\sum_{t=1}^{T} (p_t - \hat{\varphi}_1 p_{t-1})^2}{T-1}$$

Unit-root test

- The test statistic is

$$\frac{\hat{\varphi}_1 - 1}{\sqrt{Var(\hat{\varphi}_1)}} = \frac{\sum_{t=1}^{T} e_t p_{t-1}}{\hat{\sigma}_e \sqrt{\sum_{t=1}^{T} (p_{t-1})^2}}$$

- Critical values are found in the econometrics literature, e.g. Enders, Applied Econometric Time Series, Wiley

ARIMA(p, d, q)

- For many economic series appropriate models may be found within the ARIMA(p, d, q)-family

- To justify the use of such models we have to test $H_0: \beta = 1$ vs. $H_a: \beta < 1$ for the regression

$$x_t = c_t + \beta x_{t-1} + \sum_{i=1}^{p-1} \varphi_i \Delta x_{t-i} + e_t$$

where c_t is zero, constant or a linear function of t

Augmented Dickey-Fuller

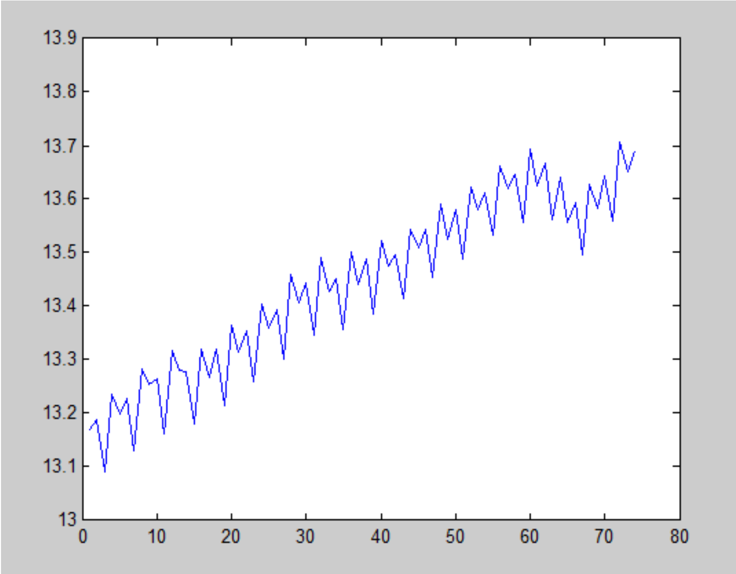
- The test function is

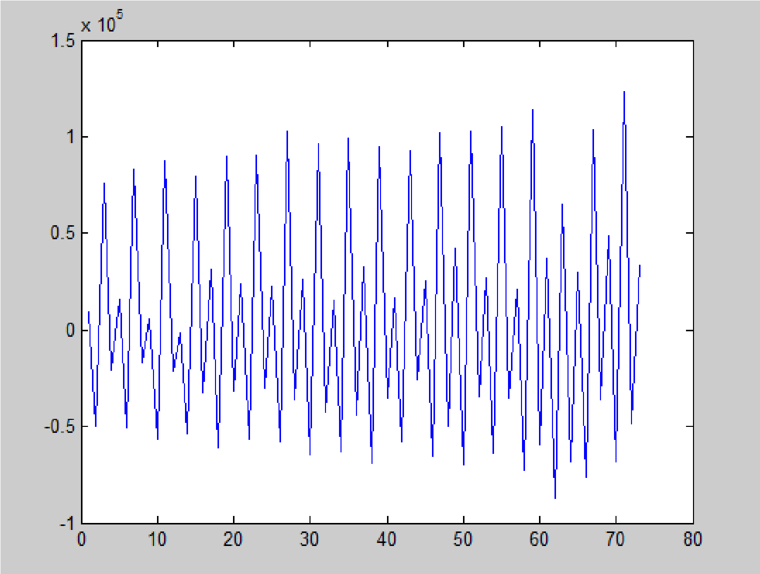
$$\frac{\hat{\beta} - 1}{SD(\hat{\beta})}$$

- Test are available in matlab as ”adftest”

Example Dickey-Fuller test

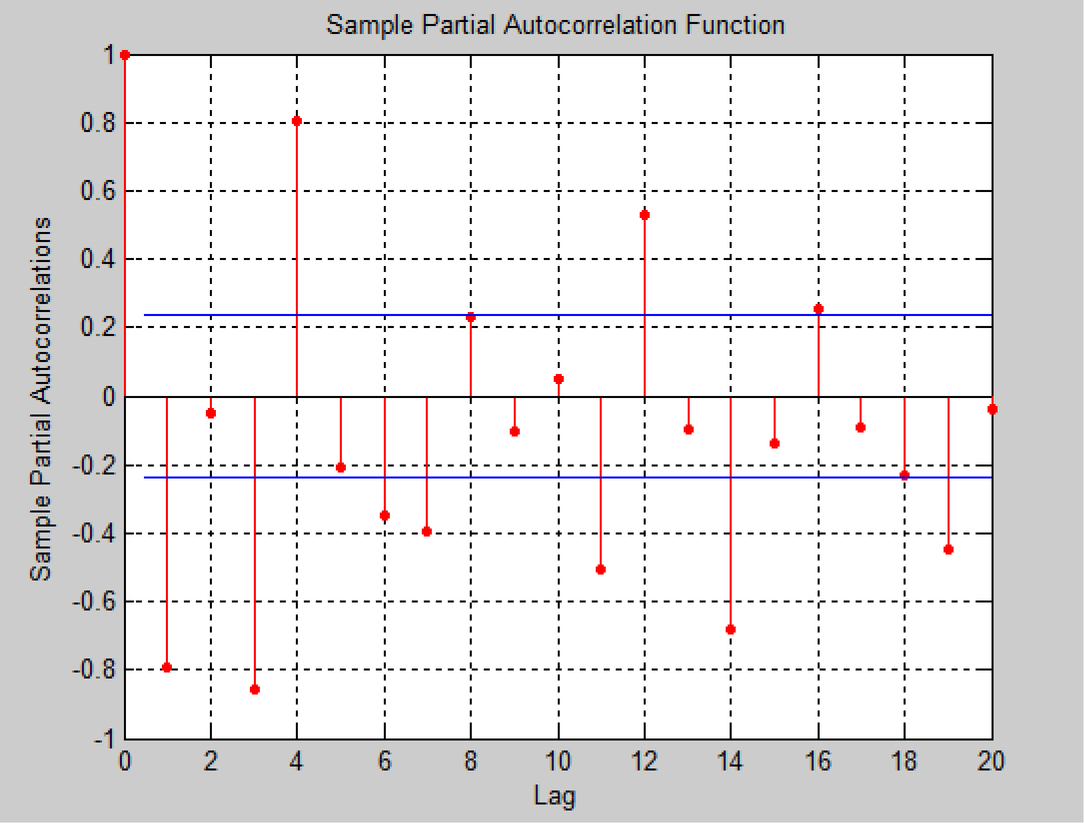
- Quarterly log Swedish GDP ’93 Q1 through ’11 Q2





Example Dickey-Fuller test

- Partial autocorrelation of differenced series



Example Dickey-Fuller test

- Clearly there seems to be a linear trend…
- Is it reasonable to assume an ARIMA model?
- To check for trend stationary models with zero to for lags, we type

```
>> [h,~,~,~,reg] = adftest(log(bnp),'model','TS','lags',0:4);
```

Example Dickey-Fuller test

- To find the best model we may use the Bayesian Information Criterion (BIC) given by

$$-2l + klnT$$

where l is the log-likelihood, k is the number of parameters in the model and T is the number of observations used to fit the model

- Lower values indicate better models

Example Dickey-Fuller test

- It turns out that the model with 4 lags has the lowest BIC

- This in turn means that our model is given by

$$x_t = \beta_0 + \beta_1 t + x_{t-1} + \sum_{i=1}^{4} \varphi_i \Delta x_{t-i} + e_t$$

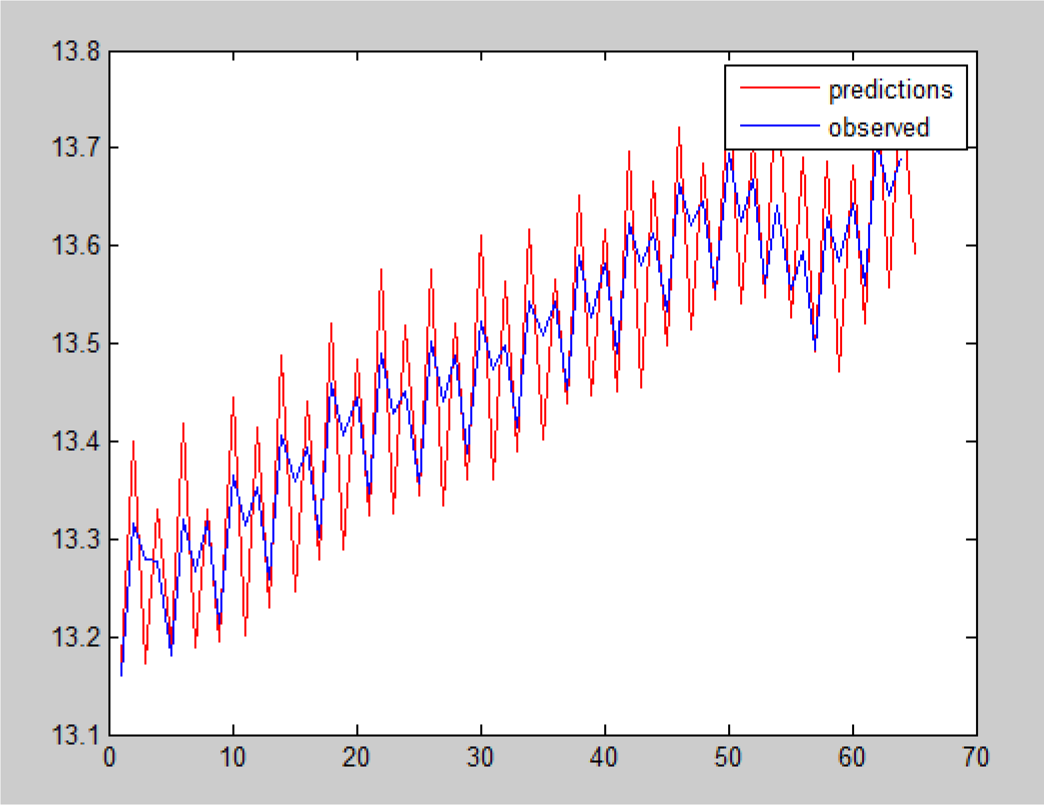
- I.e., we have an ARIMA(5,1,1)

Model fit

- In fact the prediction model is given by

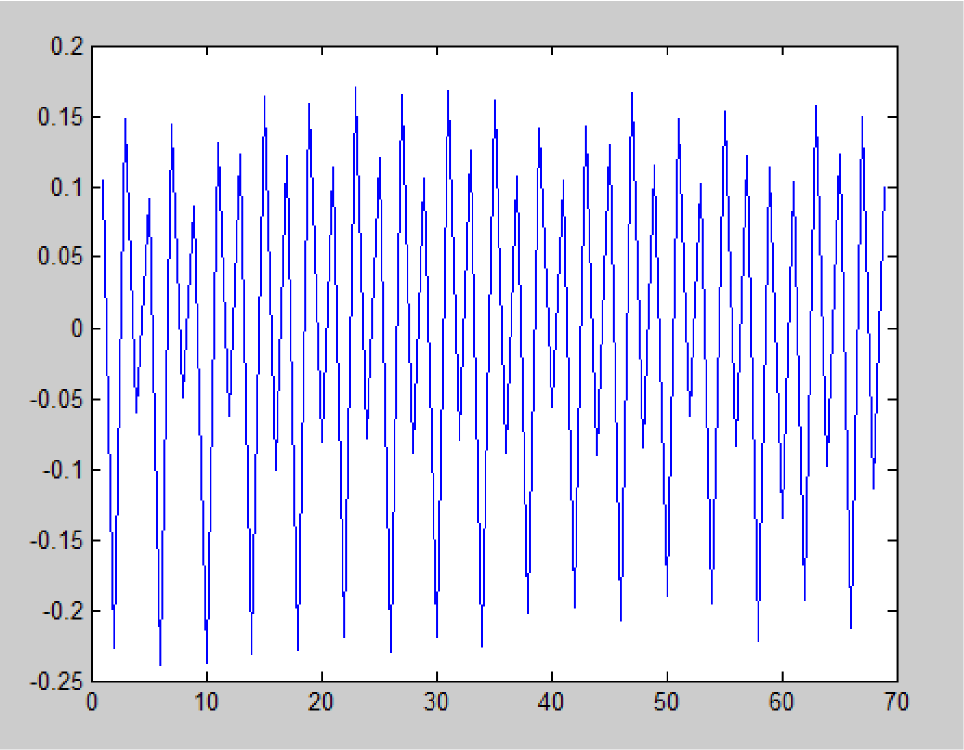
$$\begin{aligned}\hat{x}_t = 3.073 + 0.0016t + 0.7676x_{t-1} \\ + 0.0063\Delta x_{t-1} - 0.0175\Delta x_{t-2} \\ - 0.0868\Delta x_{t-3} + 0.8189\Delta x_{t-4}\end{aligned}$$

Model fit



13.8
13.7
13.6
13.5
13.4
13.3
13.2
13.1
0
10
20
30
40
50
60
70
predictions
observed

Residuals?



- There seems to be a seasonal component left in th resdual series? Next time we will talk about seasonal models