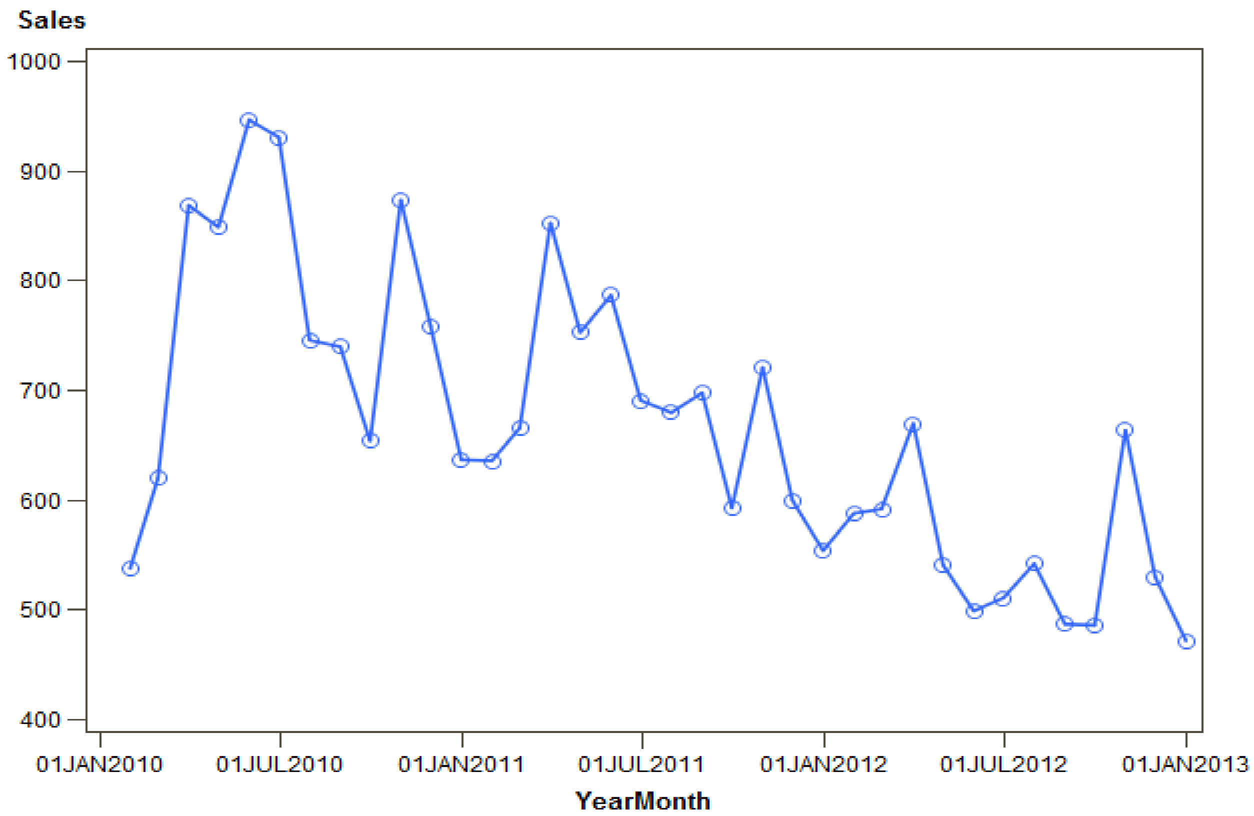
Financial Times Series

Lecture 4

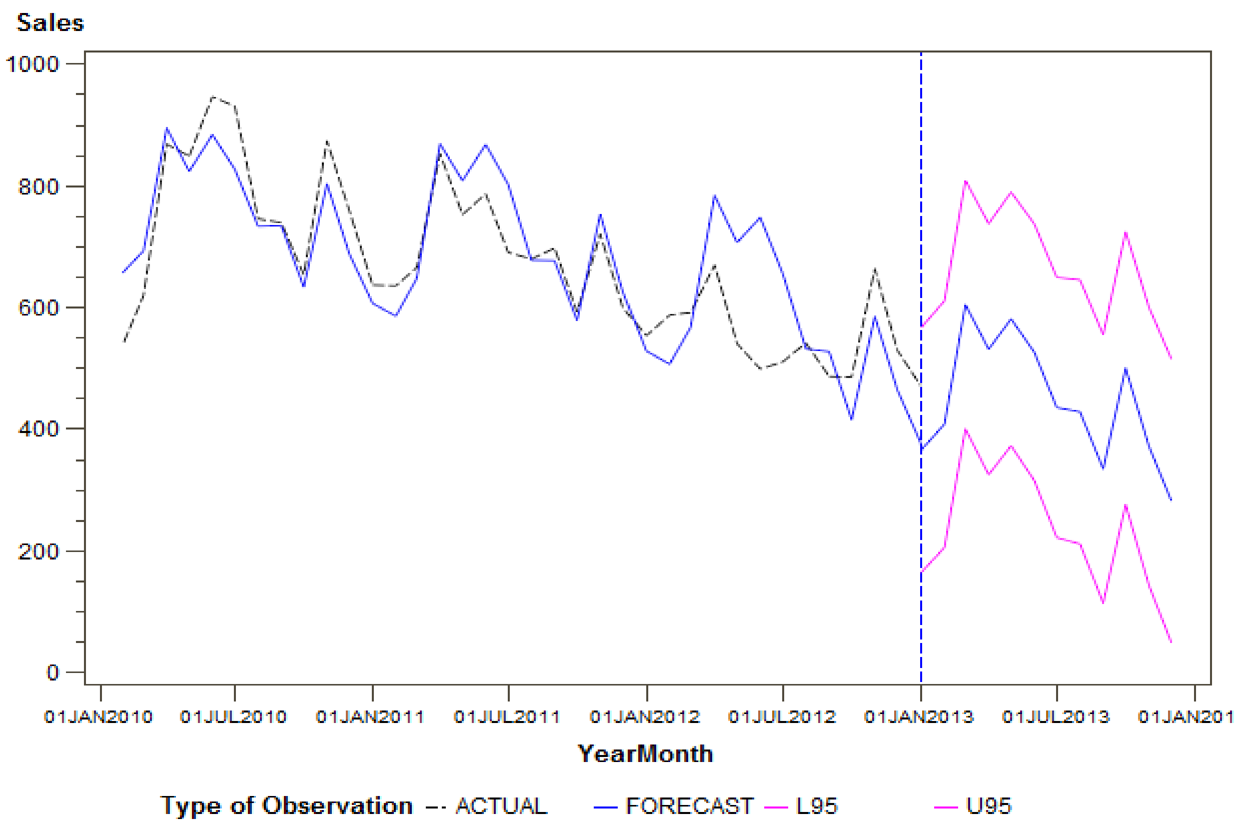
Trends and Seasonalities



To make predictions

- We need to estimate the trend and seasonality
- Fair to assume additive model (?)
- Linear trend?
- Period of the seasonality?

Possible prediction



Additive model

- $x_t = m_t + s_t + e_t$ (trend, seasonality, stationary mean zero noise)

- We first consider the case where no seasonality is present

- $x_t = m_t + e_t$

Estimation of trend (1)

- Let $x_1, \dots, x_T$ be the observed time series values

- By occular inspection try/find reasonable polynomial $m_t = \sum_{i=0}^{k} a_i t^i$ that minimizes

$$\sum_{t=1}^{T} (x_t - m_t)^2$$

Estimation of trend (1)

- The procedure yields a polynomial

$$\widehat{m}_t = \sum_{i=0}^{k} \hat{a}_i t^i$$

which in turn is a predictor for X_t under the assumption that the noise process has mean zero

Estimation of trend (1)

- It may turn out that the noise series exhibits correlation
- This may be used to improve the prediction

Estimation of trend (2)

- We may also use two-sided moving averages

$$\hat{m}_t = \frac{1}{2q+1}\sum_{j=-q}^{q} x_{t+j}, \quad q+1 \leq t \leq T-q$$

- Or exponential smoothing

$$\hat{m}_t = a x_t + (1-a)\hat{m}_{t-1}, 0 \leq a \leq 1, t = 2, \dots, T,$$

$$\hat{m}_1 = x_1$$

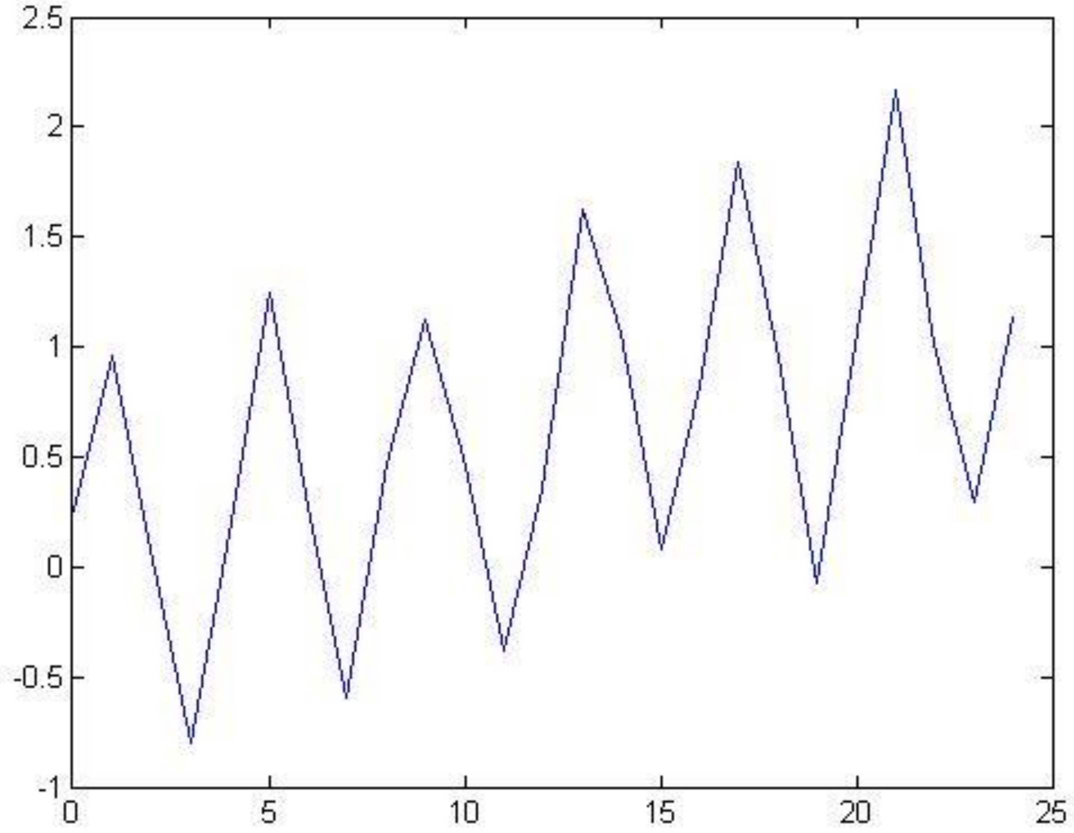
Estimation of seasonality

- If there is no obvious trend it may be resonable to assume that the seasonality component is given by the periodic function

$$s_t = a_0 + \sum_{j=1}^{k} \left(a_j cos(\lambda_j t) + b_j sin(\lambda_j t) \right),$$

where $s_t = s_{t+d}$ since λ_j is an integer multiple of $2\pi/d$

Seasonality and trend



Seasonality and trend, classical

- First, (pre-)estimate the trend for $q < t \leq T - q$ by

$$\widehat{m}_t = (0.5x_{t-q} + x_{t-q+1} + \cdots + x_{t+q-1} + 0.5x_{t+q})/d,$$

if the period d of the seasonality component is $2q$ or by the same two-sided MA as for trend only if the period is odd.

Seasonality and trend, classical

- Now define, for $1 \leq k \leq d$, w_k as the average of $x_{k+jd} - \widehat{m}_{k+jd}$

- The seasonality estimate is then given by

$$\hat{s}_k = w_k - \frac{1}{d}\sum_{i=1}^{d} w_i\,, \qquad k = 1, \dots, d$$

Seasonality and trend, classical

- In turn we get the deseasonalized data as

$$d_t = x_t - \hat{s}_t$$

- From this data we re-estimate the trend using one of the above methods

Seasonality and trend, classical

- Finally, we end upp with the estimated noise

$$\hat{e}_t = x_t - \widehat{m}_t - \hat{s}_t$$

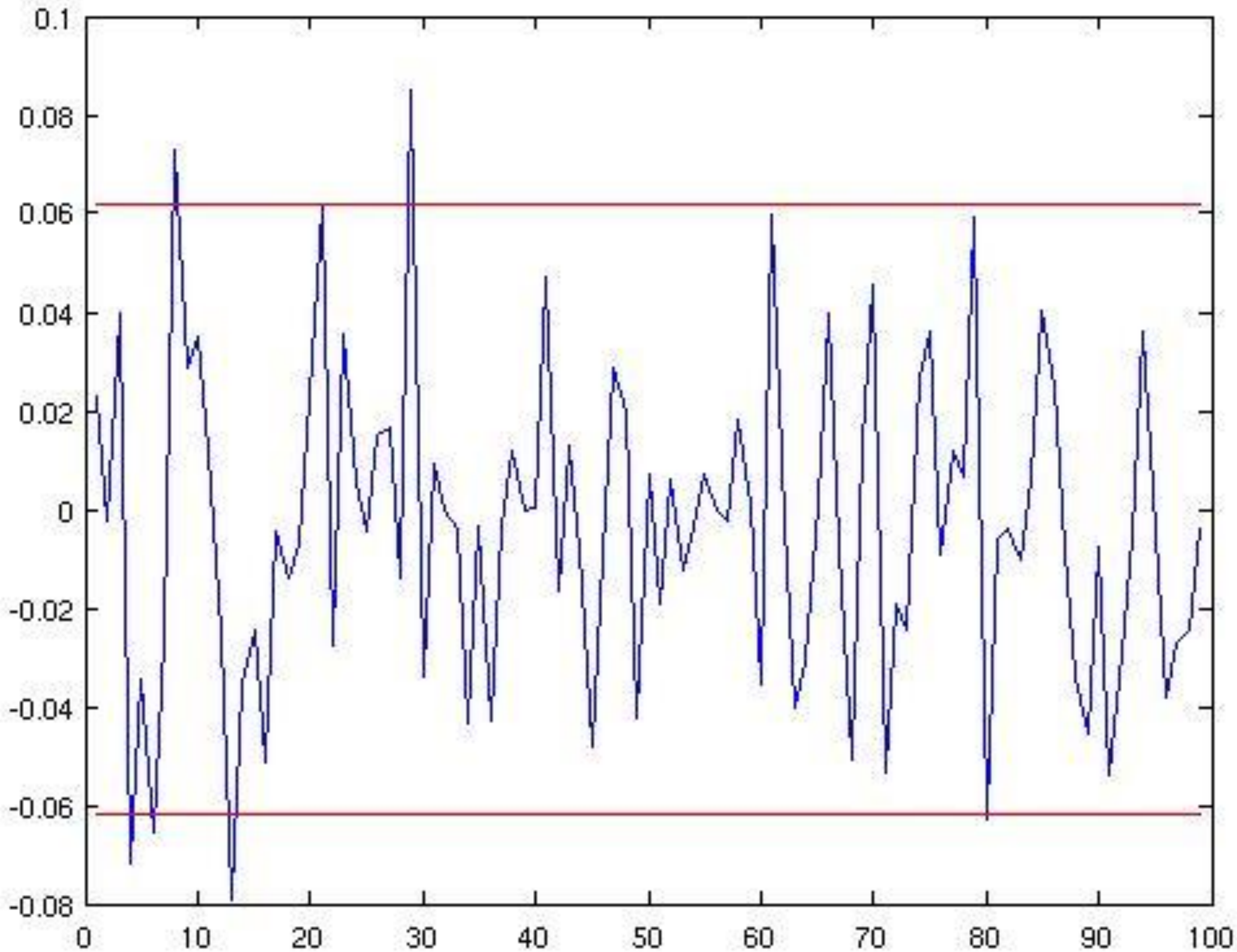
- Now we want to investigate the properties of the estimated noise
- If it turns out that there is no dependence in the noise, all further modeling that can be done is to estimate the mean and variance of the noise

Checking noise properties

- For an iid, zero mean and finite variance series it can be shown that the sample acf:s, $\hat{\rho}(h)$, asymptotically follow a $N(0, T^{-1})$ distribution

- So if we uncorrelated residuals 95% of the sample acf:s should be expected to fall within $\pm 1.96/\sqrt{T}$

ACF for 1000 iid, first 100 lags



Checking noise properties

- Portmanteau test

$$Q = T \sum_{j=1}^{h} \hat{\rho}^2(j) \sim {\chi^2}_{1-\alpha}(h)$$

- Ljung-Box (faster convergence)

$$Q_{LB} = T(T+2) \sum_{j=1}^{h} \frac{\hat{\rho}^2(j)}{T-j} \sim {\chi^2}_{1-\alpha}(h)$$

Portmanteau

- The series used for the plotted sample acf:s is based on 1000 observations
- The observed value of the Portmanteau statistic is 102.3
- The corresponding 95% chi2(100) quantile is 124.3 so we cannot reject the null hypothesis of the series being oncorrelated at the 5% level.

Exercise

- Play around in matlab with, e.g.

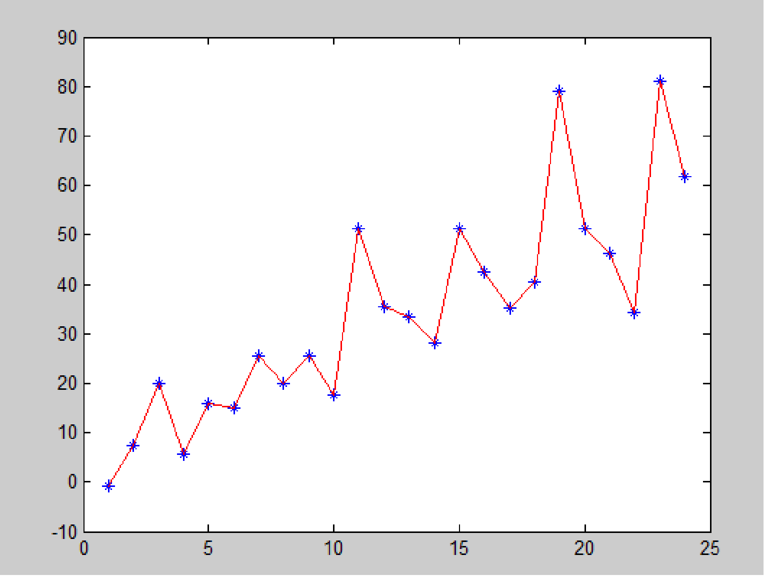
```
%Season and trend time series
time=0:24;
season=sin(pi/2*time);
trend=0.05*time;
WN=normrnd(0,0.2,1,length(time));
Y=season+trend+WN;
plot(time,Y)
```

Exercise

- Try the classical decomposition scheme and check for iid residuals using the suggested tests.

Mutiplicative seasonal models

- $x_t = m_t s_t e_t$ (trend, seasonality, stationary mean zero noise)
- We use this model if the seasonal variation seems to increase with increased trend or decrease with decreasing trend



Mutiplicative seasonal models

- If the periodicity is denoted d it is often the case that a seasonal series $\{x_t\}$ can be defined by

$$(1 - B^d)(1 - B)x_t = (1 - \theta B)(1 - \Theta B^d)a_t$$

where $|\theta| < 1$, $|\Theta| < 1$ and a_t is WN

Mutiplicative seasonal models

- Note that the AR part (LHS) contains difference and seasonal differences, i.e.,

$$(1-B^d)(1-B)x_t = (1-B^d)(x_t - x_{t-1})$$
$$= x_t - x_{t-1} - (x_{t-d} - x_{t-d-1})$$

whereas for the MA part (RHS) it holds that its expectation is zero and its ACF is (show this)

$$\rho_1 = \frac{-\theta}{1+\theta^2}, \rho_d = \frac{-\Theta}{1+\Theta^2},$$

$\rho_{d-1} = \rho_{d+1} = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}$ and 0 otherwise

Mutiplicative seasonal models

- The model with the MA part as above is called a multiplicative model

- If the MA part instead is

$$x_t = (1 - \theta B - \Theta B^s)a_t$$

the model is called non-multiplicative

Regression models with correlated errors

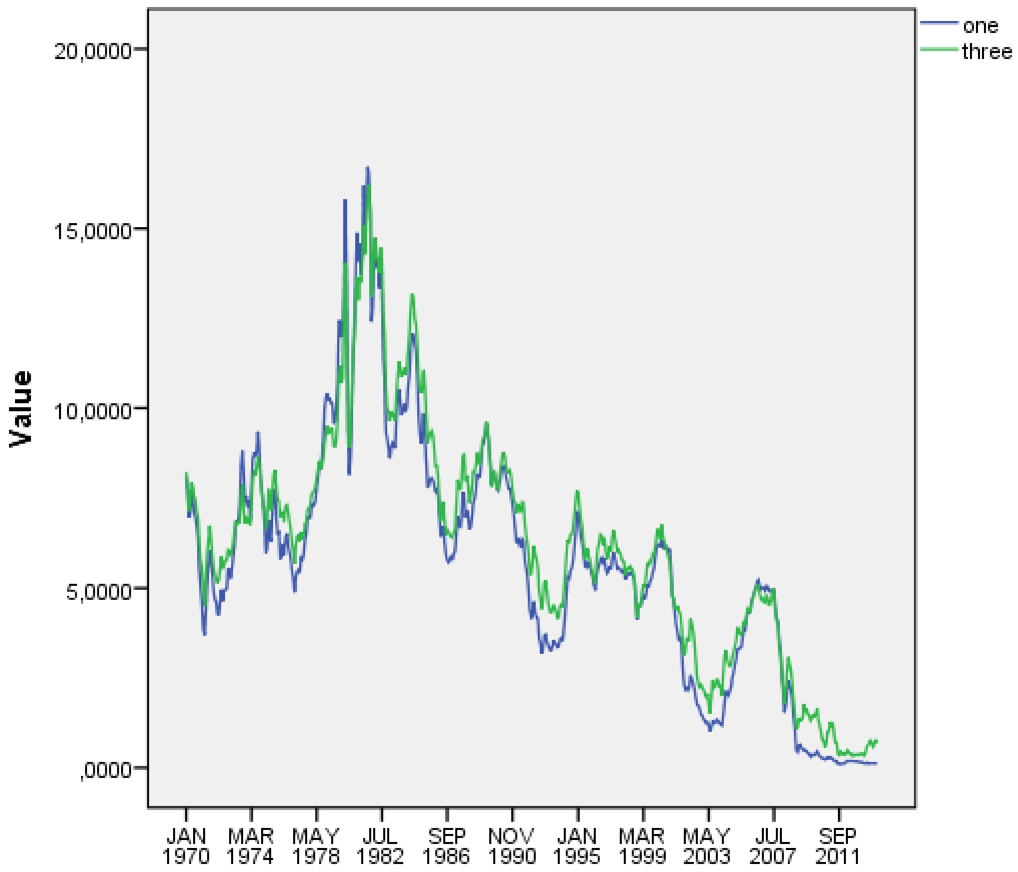
- In many situations we want to fit a model that predicts returns of a single stock from the returns of an index

$$r_{1t} = \alpha + \beta r_{2t} + e_t$$

- Using OLS to fit the model, i.e. to estimate α and β is OK if $\{e_t\}$ is WN

- However, in practice it is common that $\{e_t\}$ exhibits serial correlation, i.e. autocorrelation and if so OLS will not give consistent parameter estimates

Example, 3-year interest rates vs. 1-year interest rates



Example, 3-year interest rates vs. 1-year interest rates

- We want to predict the 3-year rate from the 1-year rate

$$r_{3t} = \alpha + \beta r_{1t} + e_t$$

- OLS gives

Model Summary[b]

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	,987[a]	,975	,975	,5267994	,092

a. Predictors: (Constant), one

b. Dependent Variable: three

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	5702,540	1	5702,540	20548,392	,000[b]
	Residual	146,529	528	,278		
	Total	5849,069	529			

a. Dependent Variable: three

b. Predictors: (Constant), one

OLS assumptions

- Remember that OLS assumes uncorrelated constant variance (homoskedastic) normal residuals
- This is many times overlooked by practitioners
- May lead to inconsistent models/tests…

Durbin-Watson test for correlated residuals

- The null hypothesis is that there is no (one lag) autocorrelation in the residuals $e_t, t = 1,2, \dots, n$

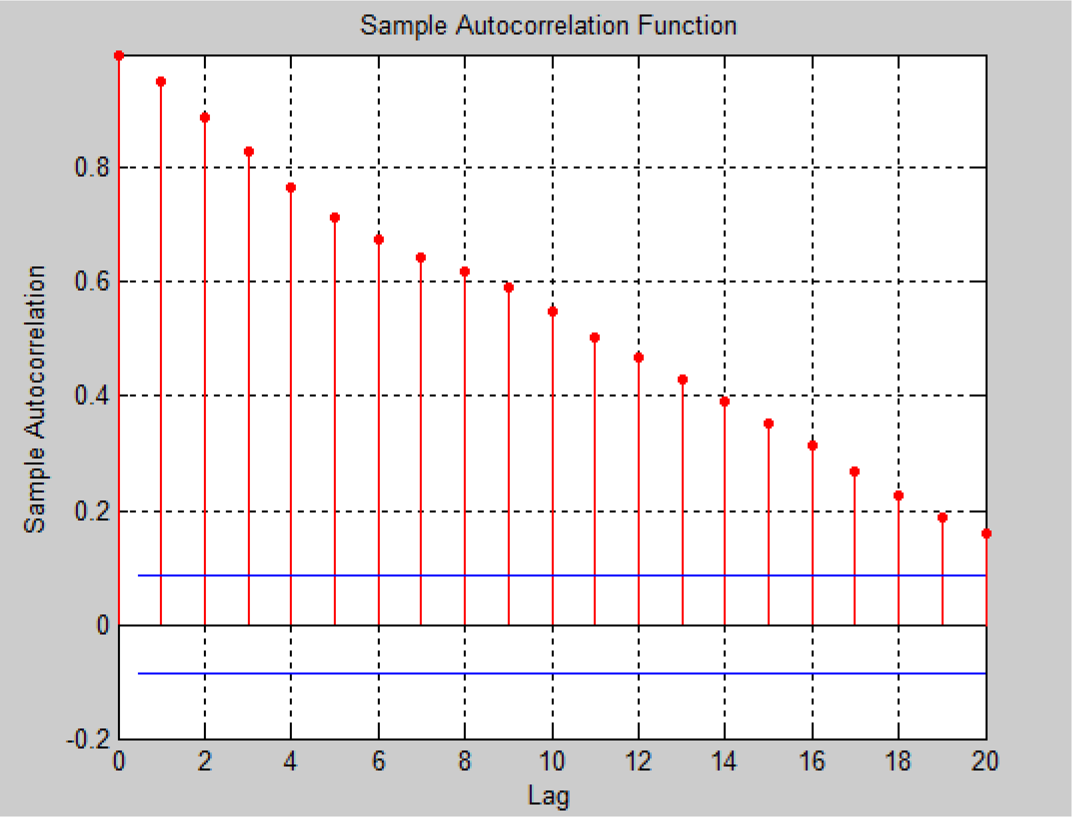
- The test statistic is

$$D = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n}(e_t)^2}$$

- If we observe $D > d_U$ we do not reject H_0, if we observe $D < d_L$ we reject H_0 but if $d_L < D < d_U$ the test is indecisive. Values of d_L and d_U can be found in the econometrics literature

- In our example $d_L \approx 1.85$ for 5% significance level

ACF of residuals



- So clearly we have autocorrelation in the residual series

Remedy? Differencing

- We could try to model

$$(1-B)r_{3t} = \alpha + \beta(1-B)r_{1t} + e_t$$

- OLS gives

Model Summary[b]

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	,933[a]	,871	,871	,14370	1,637

a. Predictors: (Constant), onediff

b. Dependent Variable: threediff

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	73,320	1	73,320	3550,886	,000[b]
	Residual	10,882	527	,021		
	Total	84,202	528			

a. Dependent Variable: threediff

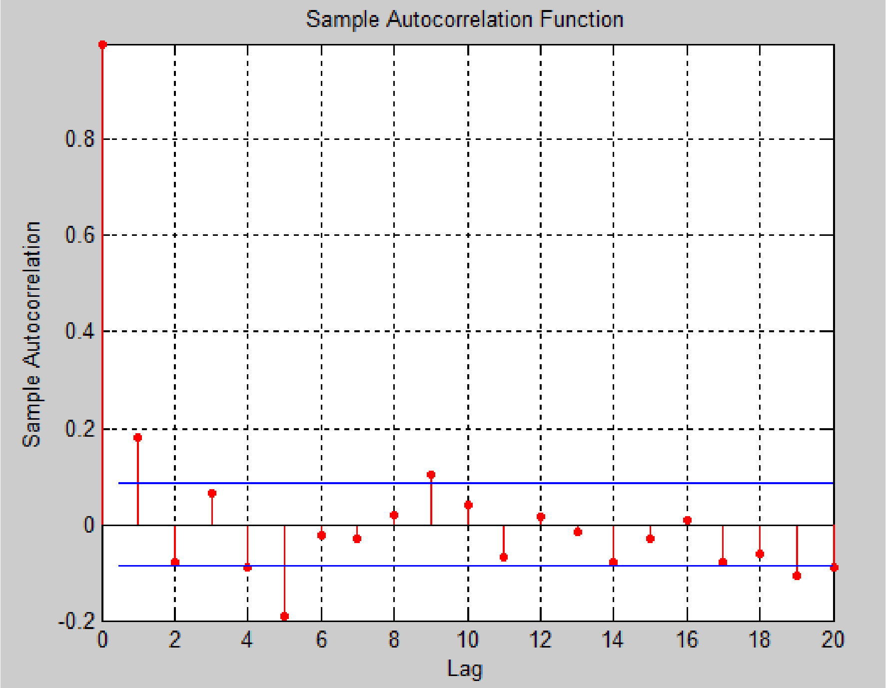
b. Predictors: (Constant), onediff

Remedy…

- The Durbin-Watson test still says that there is autocorrelation at the 5% level but

- Only one lag

- So if we model the residuals as an AR(1) we are good to go



Final model

- So ur final model is

$$(1-B)r_{3t} = \alpha + \beta(1-B)r_{1t} + e_t$$

$$e_t = a_t + \rho a_{t-1}$$

- Which is equivalent to

$$\begin{aligned}(1-B)&r_{3t}\\ &= \alpha + \beta(1-B)r_{1t} + \rho(1-B)r_{3t-1} - \beta\rho(1-B)r_{1t-1} + \tilde{a}_t\end{aligned}$$

where $\tilde{a}_t = e_t - \rho e_{t-1}$ constitutes a WN series

Model fit

Model Summary[b]

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	,936[a]	,876	,875	,14118	1,961

a. Predictors: (Constant), onedifflag, onediff, threedifflag

b. Dependent Variable: threediff

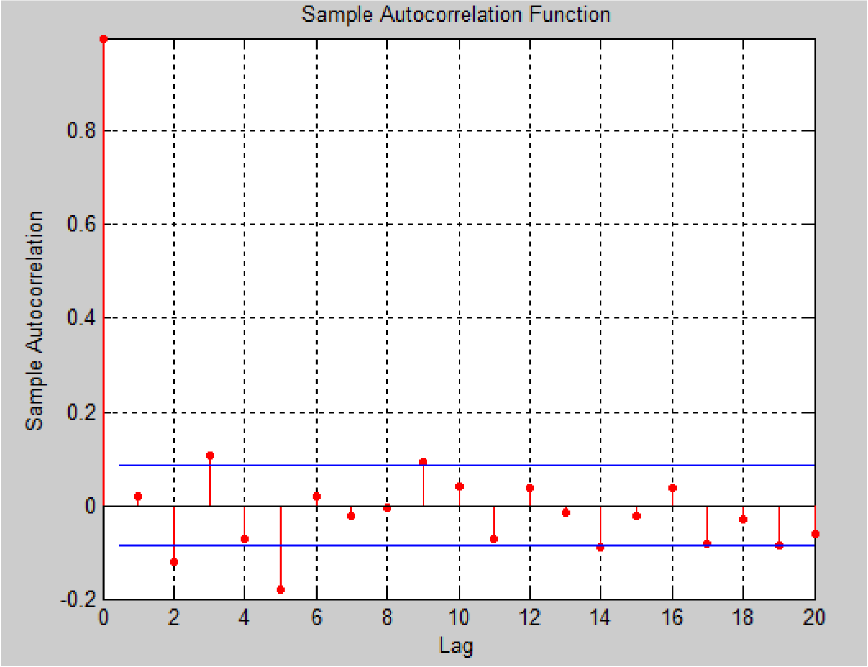
ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	73,567	3	24,522	1230,245	,000[b]
	Residual	10,445	524	,020		
	Total	84,012	527			

a. Dependent Variable: threediff

b. Predictors: (Constant), onedifflag, onediff, threedifflag

- Here the observed value of the DW-test is 1.96 which is above $d_U \approx 1.86$ for 5% significance level



Compensation of autocorrelation and/or heteroskedasticity

- What if our main goal is to make inference about α and β in the regression model but residuals exhibit autocorrelation and/or heteroskedasticity?

- The problems can be solved by using the approprioate covariance structures for the coefficient estimators

OLS as we (used to) know it

- So we want to fit the model

$$y_t = \boldsymbol{x}'_t \boldsymbol{\beta} + e_t$$

where $\boldsymbol{x}_t = (x_{1t}, \dots, x_{kt})'$ where $x_{1t} = 1$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$

- The OLS estimates are given by

$$\widehat{\boldsymbol{\beta}} = \left[\sum_{t=1}^{T} \boldsymbol{x}_t \, \boldsymbol{x}'_t\right]^{-1} \sum_{t=1}^{T} \boldsymbol{x}_t \, y_t, \qquad Cov\left(\widehat{\boldsymbol{\beta}}\right) = \sigma^2{}_e \left[\sum_{t=1}^{T} \boldsymbol{x}_t \, \boldsymbol{x}'_t\right]^{-1}$$

Autocorrelation and/or heteroskedacticity

- To compensate for heteroskedacticity use

$$Cov(\widehat{\boldsymbol{\beta}})_{HC} = \left[\sum_{t=1}^{T} \boldsymbol{x}_t \boldsymbol{x}'_t\right]^{-1} \left[\frac{T}{T-k}\sum_{t=1}^{T} (\hat{e}_t)^2 \boldsymbol{x}_t \boldsymbol{x}'_t\right] \left[\sum_{t=1}^{T} \boldsymbol{x}_t \boldsymbol{x}'_t\right]^{-1}$$

where $\hat{e}_t = y_t - \boldsymbol{x}'_t \widehat{\boldsymbol{\beta}}$

Autocorrelation and/or heteroskedacticity

- To compensate for autocorrelation and heteroskedacticity use

$$Cov(\widehat{\boldsymbol{\beta}})_{HAC} = \left[\sum_{t=1}^{T} \boldsymbol{x}_t \, \boldsymbol{x}'_t\right]^{-1} \hat{C}_{HAC} \left[\sum_{t=1}^{T} \boldsymbol{x}_t \, \boldsymbol{x}'_t\right]^{-1}$$

where

$$\hat{C}_{HAC} = \sum_{t=1}^{T} (\hat{e}_t)^2 \boldsymbol{x}_t \, \boldsymbol{x}'_t + \sum_{j=1}^{l} w_j \sum_{t=j+1}^{T} \left(\boldsymbol{x}_t \hat{e}_t \hat{e}_{t-j} \boldsymbol{x}'_{t-j} + \boldsymbol{x}_{t-j} \hat{e}_{t-j} \hat{e}_t \boldsymbol{x}'_t\right)$$

for $w_j = 1 - \frac{j}{l+1}$ and (suggested) $l = 4(T/100)^{2/9}$