Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]). August 19^{th}, 2015

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REMARK: No aids permitted

1. **Theorems.**

   a) Let $\Pi_Y^{(a,t_0)}(t)$ denote the Black-Scholes price at time $t \in [0,T]$ of a European derivative with pay-off $Y = g(S(T))$ assuming that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in [0,T]$, where $0 < a < 1$. Show that

   $$\Pi_Y^{(a,t_0)}(t) = \begin{cases} v(t,(1-a)S(t)), & \text{for } t < t_0, \\ v(t,S(t)), & \text{for } t \geq t_0, \end{cases}$$

   where $v(t,x)$ is the Black-Scholes price function in the absence of dividends (max. 3 points)

   b) Let $\Pi_X(t)$ denotes the Black-Scholes price of a derivative with pay-off X. Consider a standard European derivative with pay-off $Y = g(S(T))$ at maturity T and another derivative with pay-off $Z = \Pi_Y(t_*)$ at maturity $t_* < T$. Show that $\Pi_Z(t) = \Pi_Y(t)$, $t \in [0,t_*]$ (max. 2 points).

   Solution: See Theorems 4.1 Ref. [7] and 5.1.1 in Ref. [1].

2. Compute the Black-Scholes price $\Pi_Y(0)$ at time $t=0$ of a European derivative with pay-off $Y = \max(S(T), B(T))$, where $B(t)$ is the price of the bond, $S(t)$ is the price of the underlying stock and T is the time of maturity of the derivative (max. 3 points). Derive the low volatility limit ($\sigma \to 0^+$) and the high volatility limit ($\sigma \to +\infty$) of $\Pi_Y(0)$ (max. 2 points).

   Solution: The Black-Scholes price at time $t = 0$ is given by $\Pi_Y(0) = v(S_0)$, where $S_0 = S(0)$ and

   $$v(x) = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(xe^{(r-\frac{1}{2}\sigma^2)T} e^{\sigma\sqrt{T}y}) e^{-\frac{y^2}{2}}\, dy.$$

   Computing the integral with the given pay-off function $g(z) = \max(z, B(T))$ and $B(T) = B_0 e^{rT}$ we obtain

   $$\Pi_Y(0) = S_0 - S_0\Phi(d - \sigma\sqrt{T}) + B_0\Phi(d),$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}} dy$ is the standard normal distribution and

$$d = \frac{1}{2}\sigma\sqrt{T} + \frac{1}{\sigma\sqrt{T}} \log\left(\frac{B_0}{S_0}\right).$$

This solves the first part of the exercise (3 points). As to the second part, we used that

$$\lim_{\sigma\to+\infty} d = +\infty, \qquad \lim_{\sigma\to+\infty} d - \sigma\sqrt{T} = -\infty$$

and

$$\lim_{\sigma\to 0^+} d = \lim_{\sigma\to 0^+} d - \sigma\sqrt{T} = \begin{cases} -\infty & \text{for } B_0 < S_0 \\ +\infty & \text{for } B_0 > S_0 \\ 0 & \text{for } B_0 = S_0. \end{cases}$$

Hence

$$\lim_{\sigma\to+\infty} \Pi_Y(0) = S_0 + B_0, \qquad \lim_{\sigma\to 0^+} \Pi_Y(0) = \max(S_0, B_0).$$

This solves the second part of the exercise (2 points).

3. Consider a standard European derivative with pay-off $Y = g(S(2))$ at the time of maturity 2. Assume that the price of the underlying stock follows the 2-period arbitrage-free binomial model

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p_u, \\ S(t-1)e^d & \text{with probability } p_d. \end{cases} \qquad t = 1, 2$$

and that the interest rate of the bond is a constant $r > 0$. Let

$$\Delta = g(S_0 e^{2d}) - e^{d-u} g(S_0 e^{2d}) - g(S_0 e^{u+d}) + g(S_0 e^{2u}) e^{d-u}.$$

Show that a constant predictable hedging portfolio (h_S, h_B) exists if and only if $\Delta = 0$ and find such portfolio (max. 5 points).

Solution: The hedging condition reads

$$h_S S(2) + h_B B_0 e^{2r} = g(S(2)).$$

Since the portfolio is constant and is required to be predictable, then it can only depend on $S_0 = S(0)$ and not on $S(1), S(2)$. Hence we have to express $S(2)$ in terms of S_0 in the previous equation. Since $S(2) \in \{S(0)e^{2u}, S_0 e^{u+d}, S_0 e^{2d}\}$, we obtain the system

$$\begin{aligned} h_S S_0 e^{2u} + h_B B_0 e^{2r} &= g(S_0 e^{2u}) \\ h_S S_0 e^{u+d} + h_B B_0 e^{2r} &= g(S_0 e^{u+d}) \\ h_S S_0 e^{2d} + h_B B_0 e^{2r} &= g(S_0 e^{2d}). \end{aligned}$$

It is straightforward to show that the previous system has a (unique) solution (h_S, h_B) if and only if $\Delta = 0$ and in this case the solution is given by

$$h_B = \frac{e^u g(S_0 e^{u+d}) - g(S_0 e^{2u}) e^d}{B_0 e^{2r}(e^d - e^u)}, \qquad h_S = \frac{g(S_0 e^{2u})(2e^d - e^u) - e^u g(S_0 e^{u+d})}{S_0 e^{2u}(e^d - e^u)}.$$