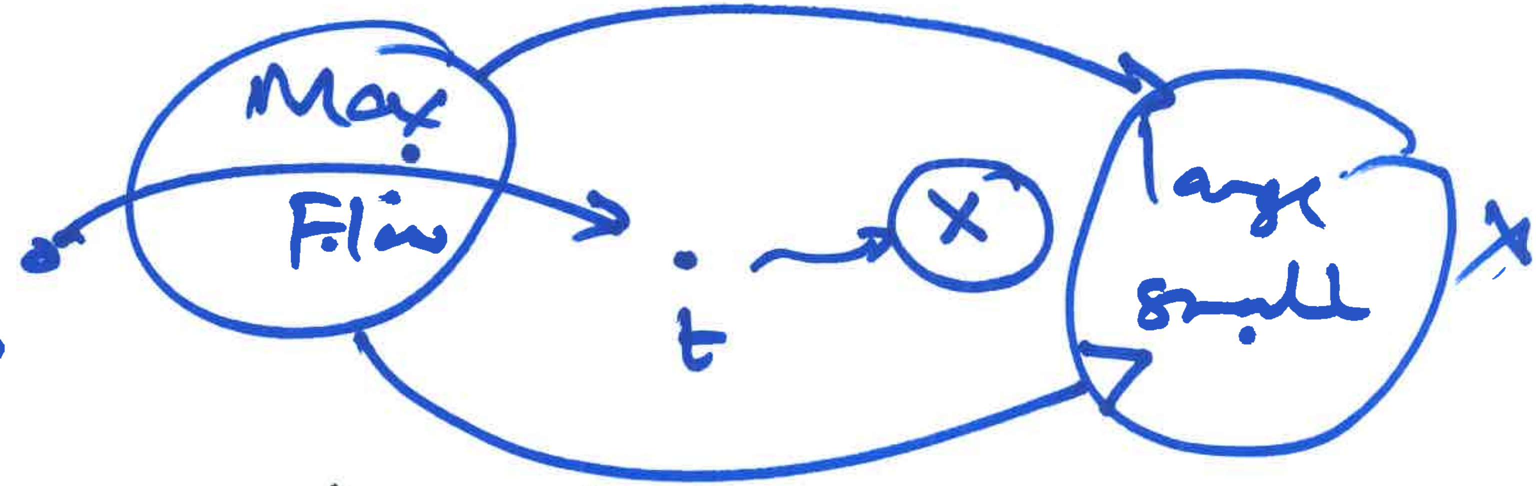
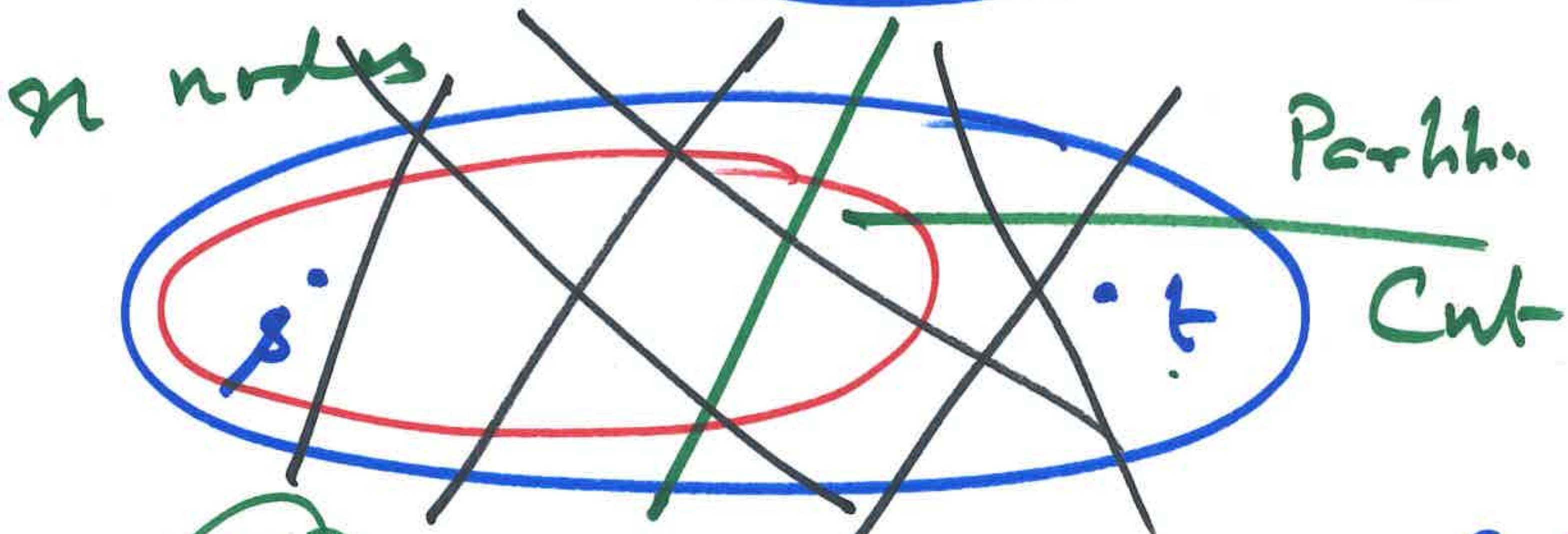
Class Notes on Network Flow





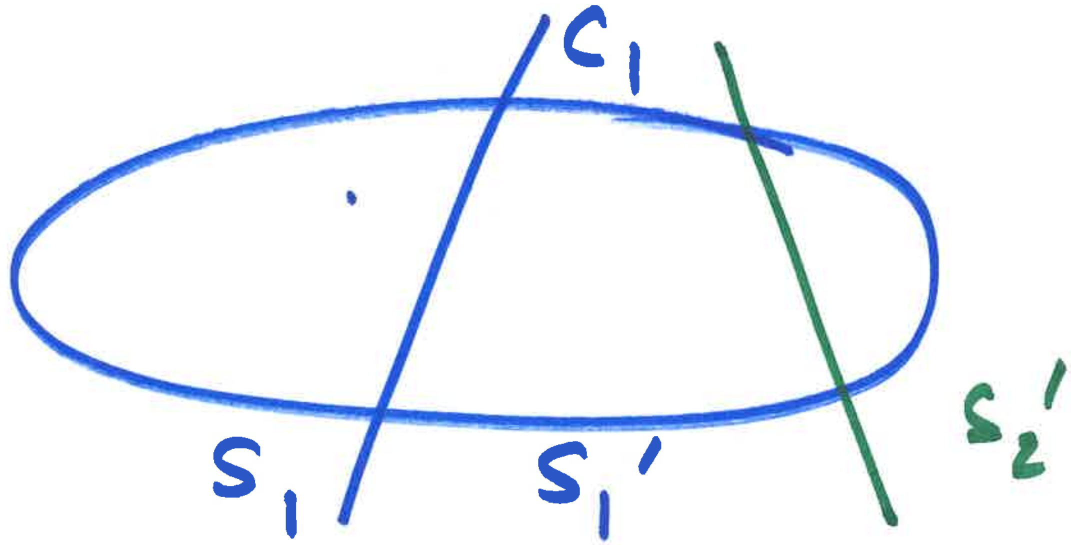
Source ~~Sink~~ S : $S \subseteq V \Rightarrow s \in S$, $t \notin S$

~~Source~~ Sink. S' : $\underline{V - S}$

$$u \xrightarrow{e} v$$

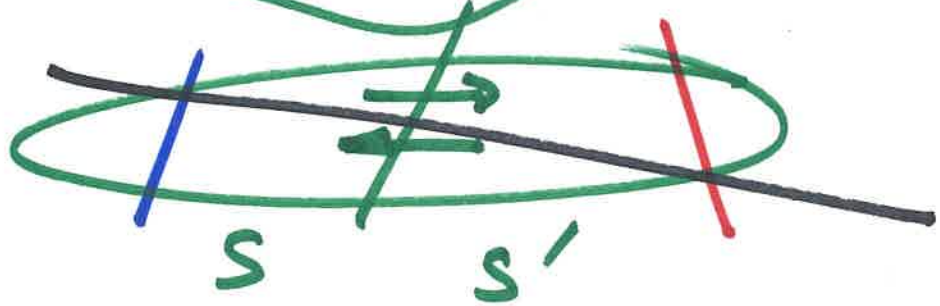
$$c(e) \,/\, f(e)$$

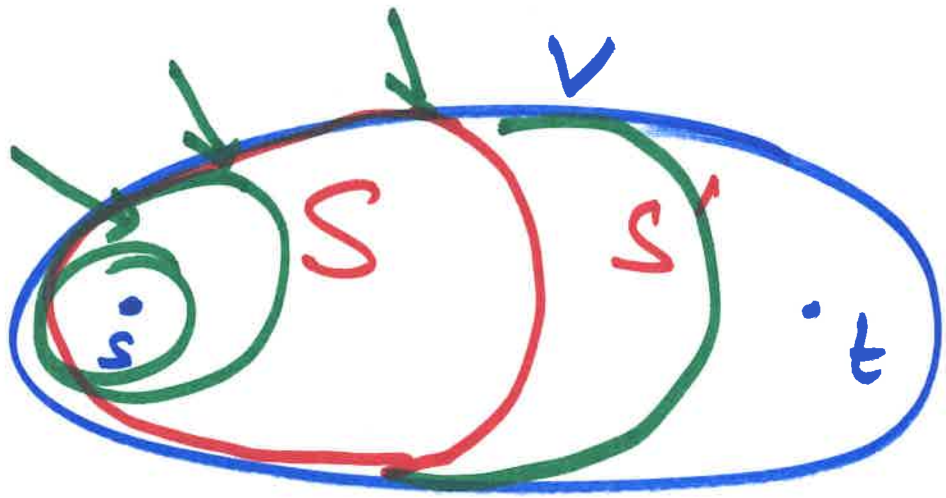
$$c(e) \geq f(e)$$



S_2

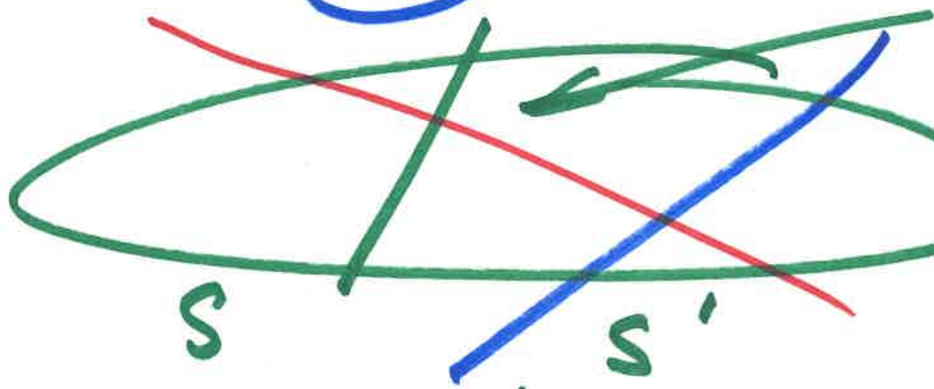
$$F = \sum_{e \in (s:s')} f(e) - \sum_{e \in (s':s)} f(e)$$

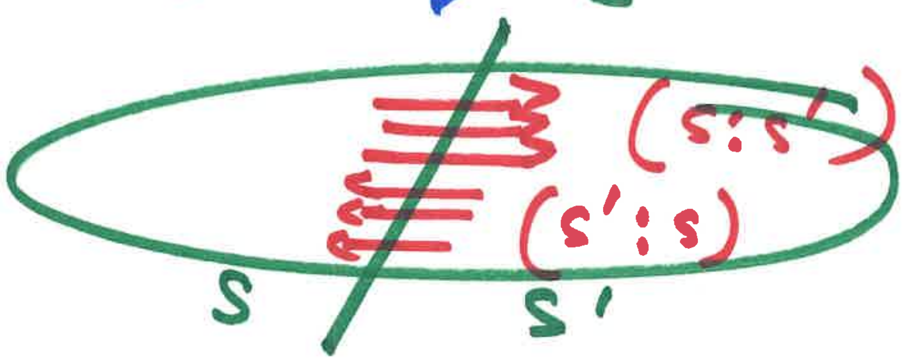


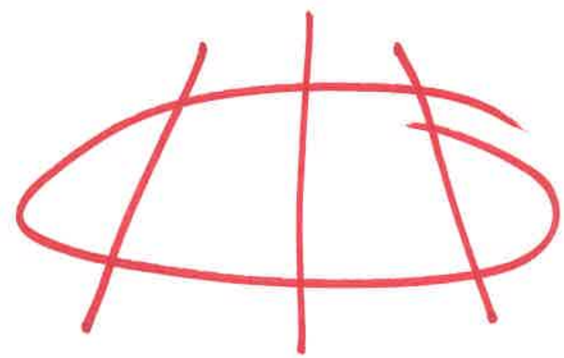


$\underline{S} : s'$

$\underline{\underline{S}} \subseteq V$



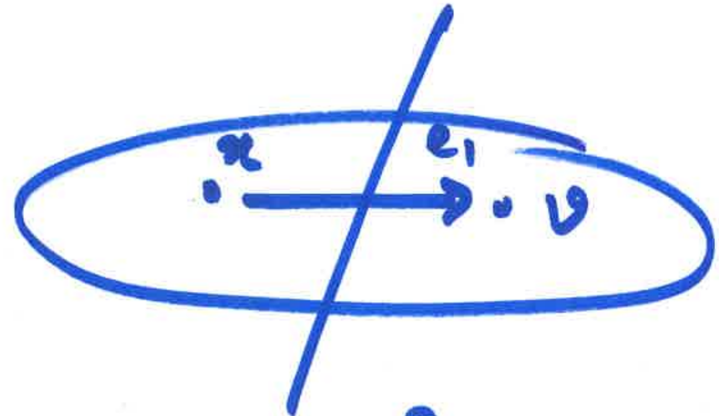




$x \xrightarrow{e_1} v_1$

$\underline{\underline{x \notin S'}}$ $\quad x \in S$

$f(e_1)$

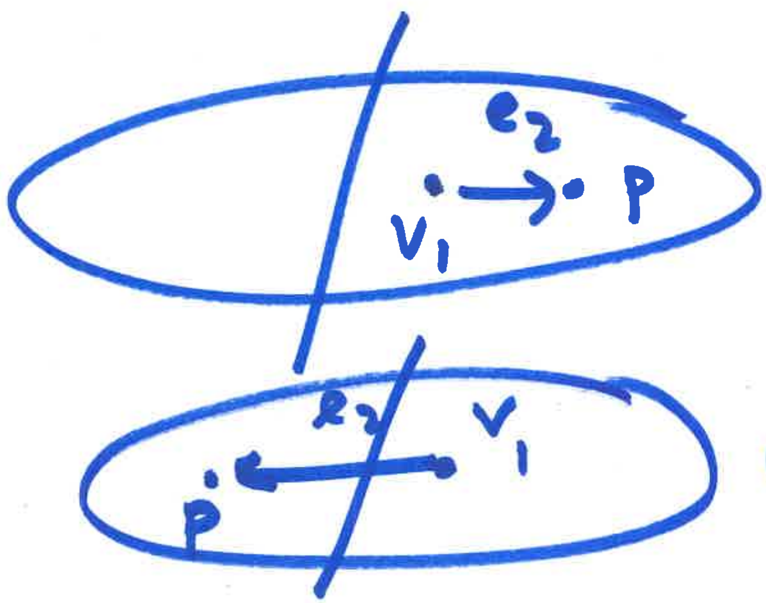


$+ f(e_1)$

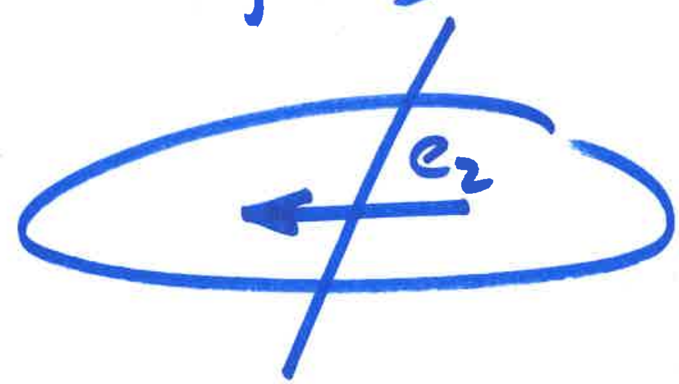
$v_1 \xrightarrow{e_2} p$

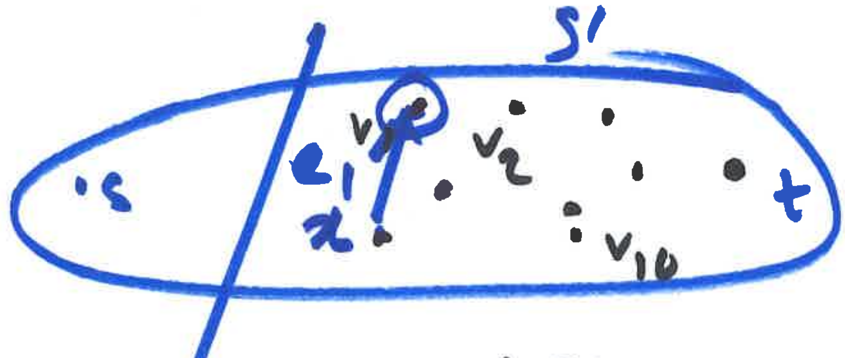
$f(e_2)$

$\underline{\cancel{p \in S'}}$ $\quad p \in S$



$f(e_3)$





$$\underbrace{\text{inflow} - \text{outflow}}_{\text{at sink}} = F \quad ($$

$$\text{inflow} - \text{outflow} = 0 \quad v \neq s, \ \neq t$$

$$(t): \quad F = \sum_{e \in \alpha(t)} f(e) - \sum_{e \in \beta(t)} f(e)$$

$$(v_1) \quad 0 = \underbrace{\sum_{e \in \alpha(v)} f(e)}_{f(e_1)} - \sum_{e \in \beta(v)} f(e)$$

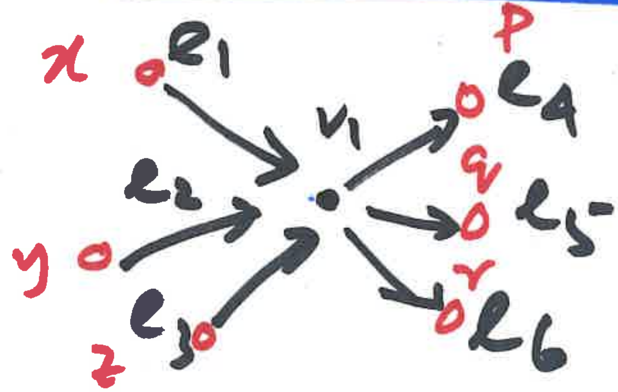
(v_2)

$\vdots$

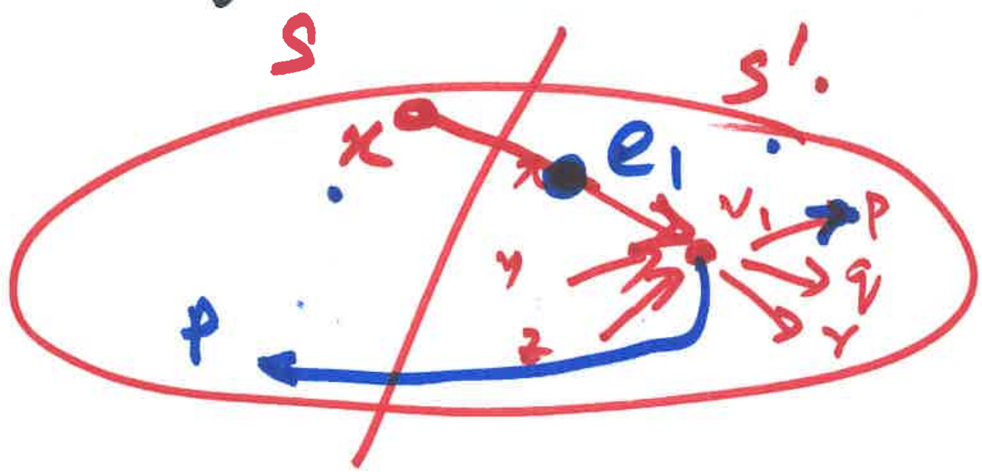
$v_{10} \quad 0 = - \quad -$

$$F = \underline{f(e_1) + f(e_2) + f(e_3)} + \cdots$$
$$- f(e_{10}) - f(e_{11}) - f(e_{12})$$

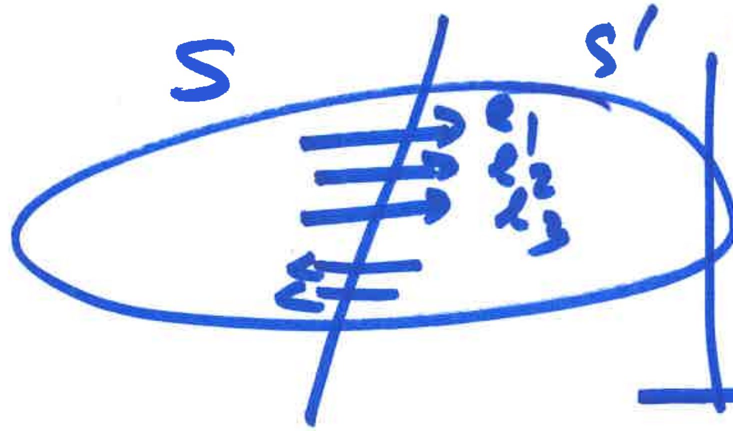
v_1



$$(v_1) : 0 = f(e_1) + f(e_2) + f(e_3) - f(e_4) - f(e_5) - f(e_6)$$

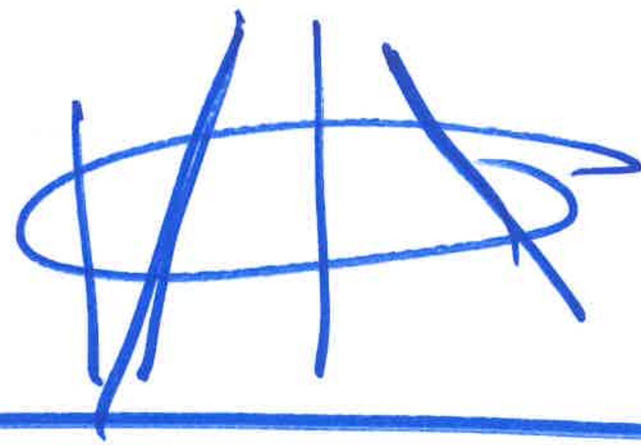


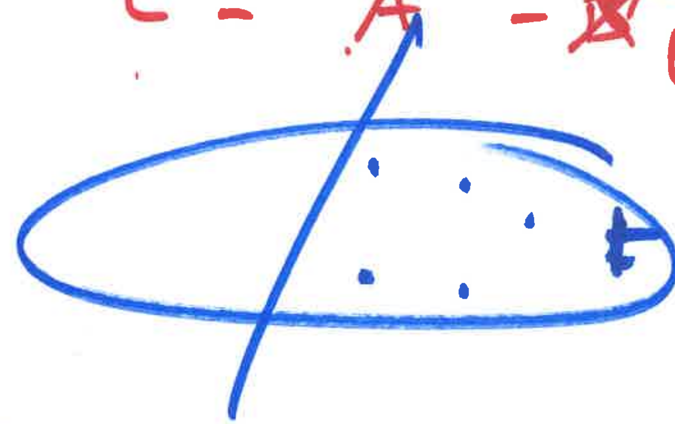
$x \in S$
$x \in S'$



$$\boxed{F = \sum_{e \in (s:s')} f(e) - \sum_{e \in (s':s)} f(e)}$$

$C = A - \cancel{B}\ D$

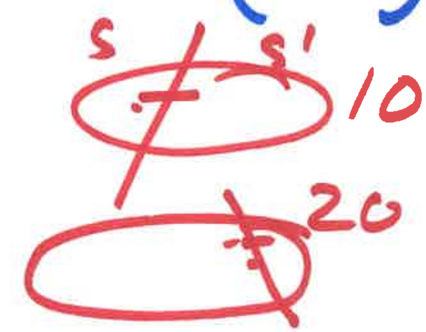


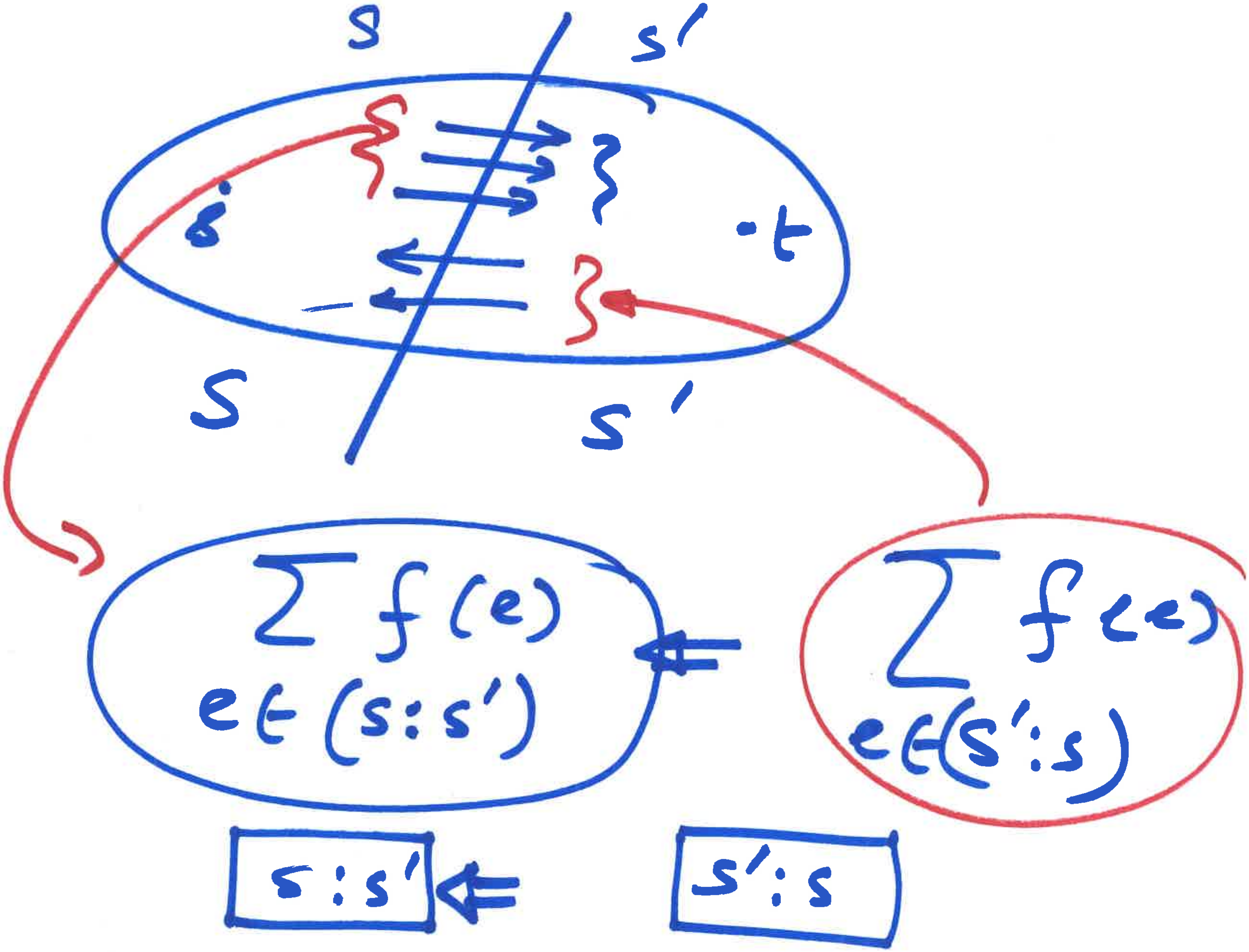


$\#$ $\quad c(s) = \sum_{e \in (s:s')} c(e) \leftarrow$ Capacity of the cut $(s:s')$

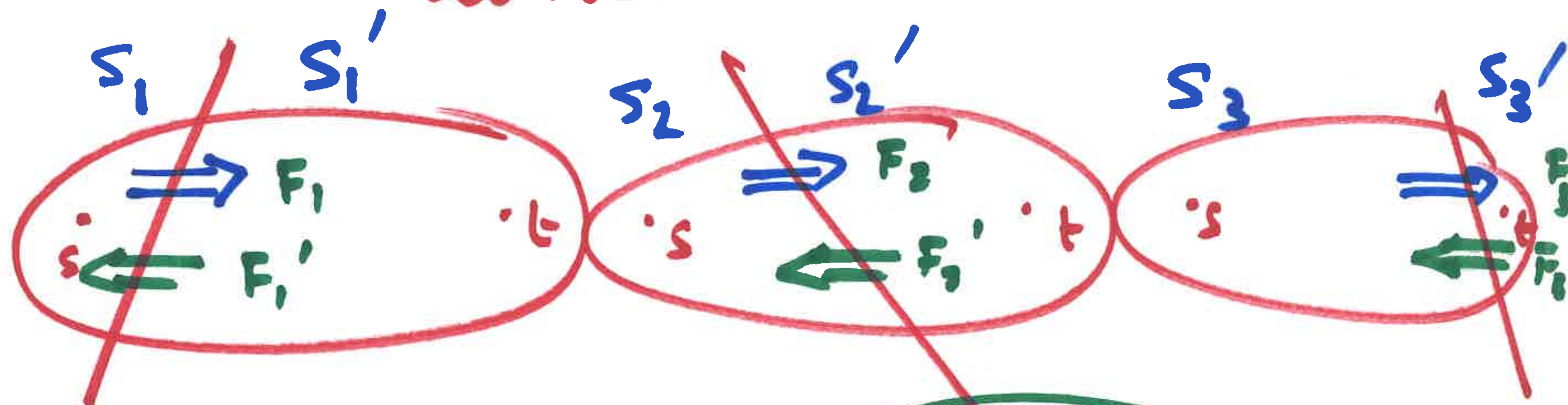
$$F_{max} = \sum_{e \in (s:s')} f(e) \leq \sum_{e \in (s:s')} c(e)$$

$$F_{max} \leq \sum_{e \in (s:s')} c(e)$$




s
s'
s
t
S
s'
$\sum_{e \in (s:s')} f(e)$
$\sum_{e \in (S':s)} f(e)$
s:s'
s':s

$$\underline{\underline{F}} = \underbrace{\sum_{e \in \alpha(t)} f(e)}_{\text{incoming flow at dest.}} - \underbrace{\sum_{e \in \beta(t)} f(e)}_{\text{outgoing flow at } t}$$

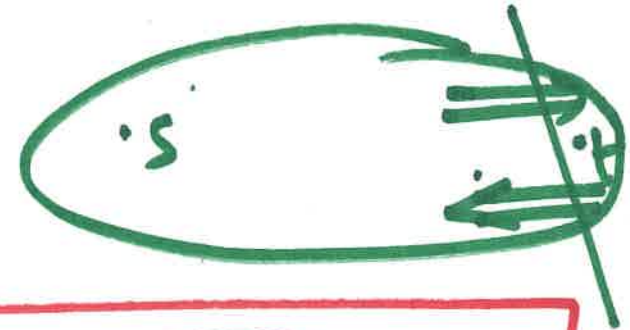


$$\underbrace{\sum_{e \in (S:S')} f(e)} - \sum_{e \in (S':S)} f(e)$$

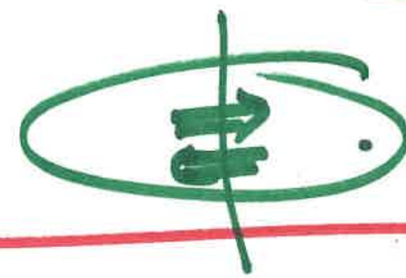
$F_1 - F_1'$ X

$F_2 - F_2'$ Y

$F_3 - F_3'$ Z



For every S, $F = \sum_{e \in (S:S')} f(e) - \sum_{e \in (S':S)} f(e)$

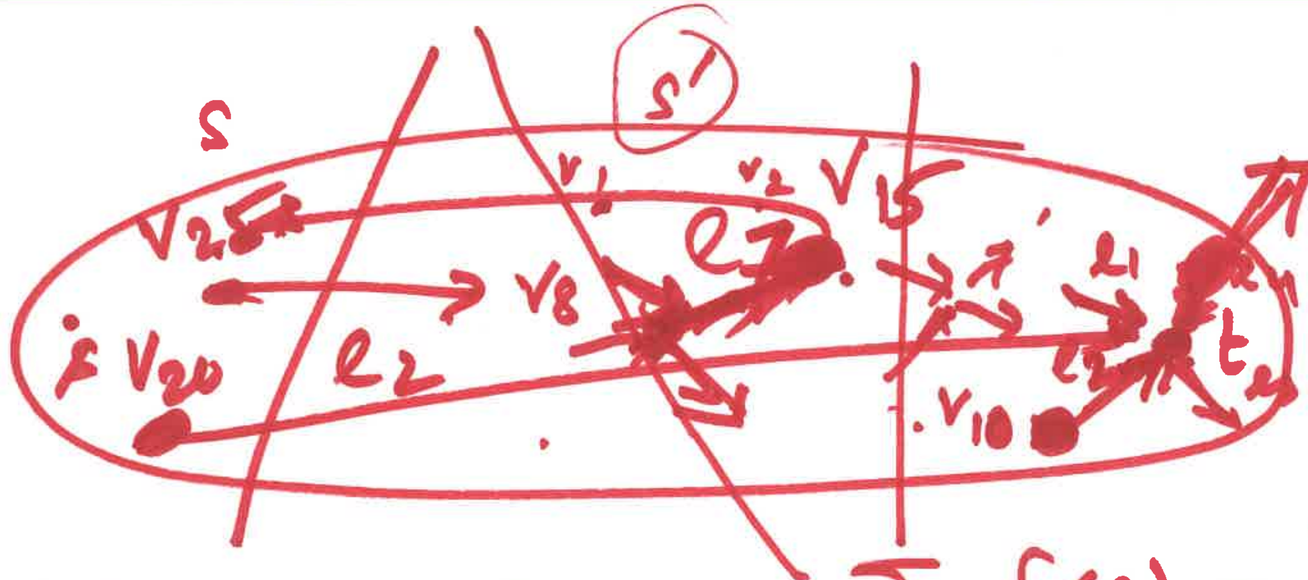


$$F = \sum_{e \in (s:s')} f(e) - \sum_{e \in (s':s)} f(e)$$

$$C(s) = \sum_{e \in (s:s')} c(e)$$

$$\boxed{F_{max}} = \boxed{\sum_{e \in (s:s')} f(e)} \leq C(s:s')$$

$$\boxed{F_{max} \leq \underbrace{C(s:s')}_{\text{capacity of the cut}}}$$



at t: $F = \sum_{e \in \alpha(t)} f(e) - \sum_{e \in \beta(t)} f(e)$

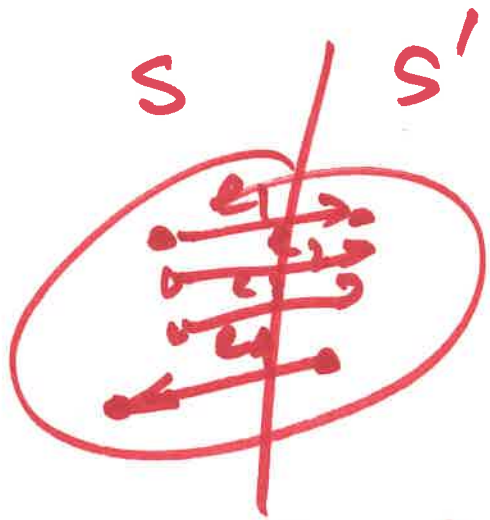
$\underline{\underline{v_8}} \rightarrow 0 = \underline{\sum_{e \in \alpha(v_1)} f(e) - \sum_{e \in \beta(v_1)} f(e)}$

$v_{10} \rightarrow 0 = \sum_{e \in \alpha(v_2)} f(e) - \sum_{e \in \beta(t)} f(e)$

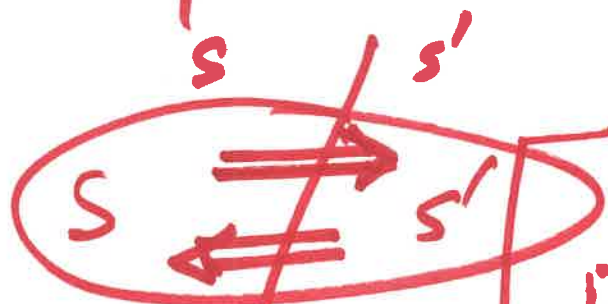
v_{15}

$F = f(e_1) + \underline{\underline{f(e_2)}} + f(e_3) - f(e_4) - f(e_5)$

$$= f(e_2) + f(e_5) + f(e_8) - f(e_7)$$

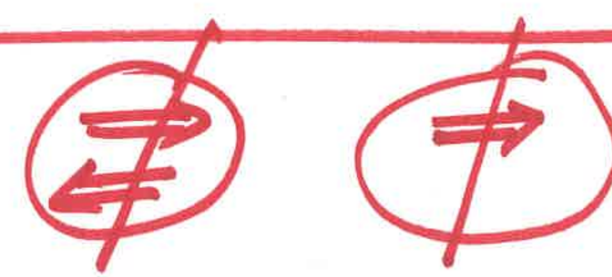


$$= \boxed{f(e_1) + f(e_2) + f(e_3)} \boxed{- f(e_4) - f(e_5) - f(e_6)}$$



$$C(S) = \sum_{e \in (S:S')} c(e)$$

$$\boxed{F = \sum_{e \in (S:S')} f(e) - \sum_{e \in (S':S)} f(e)}$$



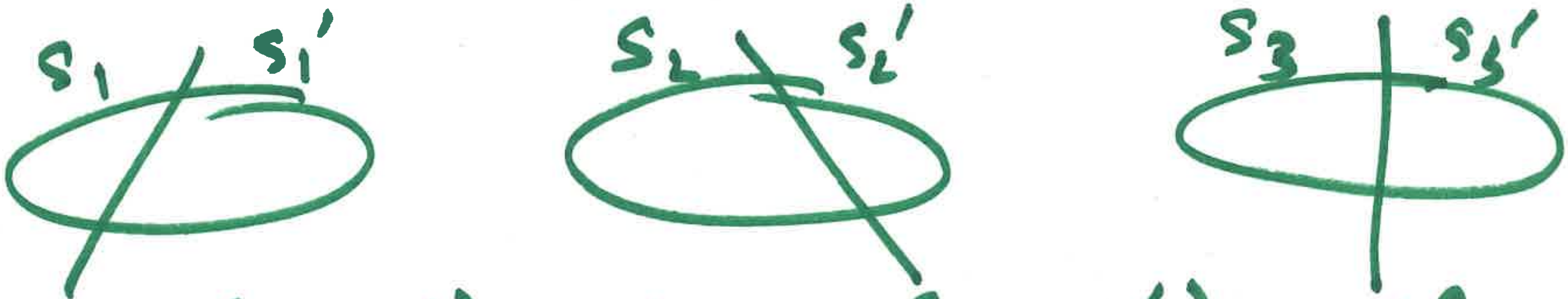
$$F_{max} = \sum_{e \in (s:s')} f(e)$$

$$\underbrace{\sum_{e \in (s:s')} f(e)}_{F_{max}} \leq \underbrace{\sum_{e \in (s:s')} c(e)}_{\text{Capacity of } (s:s')}$$

$$\boxed{F_{max} \leq \text{Capacity of a cut } (s:s')}$$

$F_{max} \leq$ Lowest Capacity of cut

$\leq$ <u>min</u>- cut.

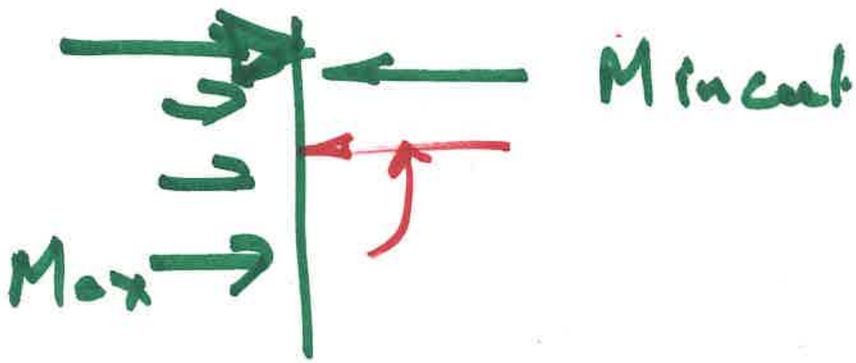


$$C(s_1 : s_1') \neq C(s_2 : s_2') \neq C(s_3 : s_3')$$

15   20   10

$$\boxed{\text{Max Flow} \leq \text{Min\_Cut}}$$

Upper Bound

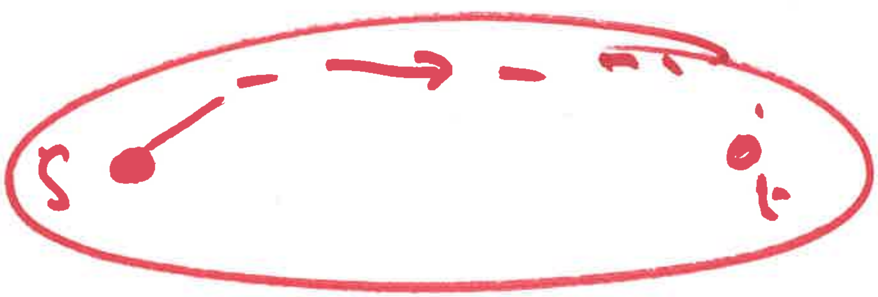


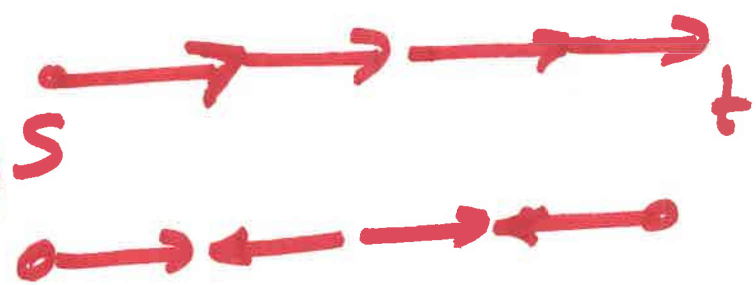
$\boxed{\text{max. flow} \le \text{min. cut}}$

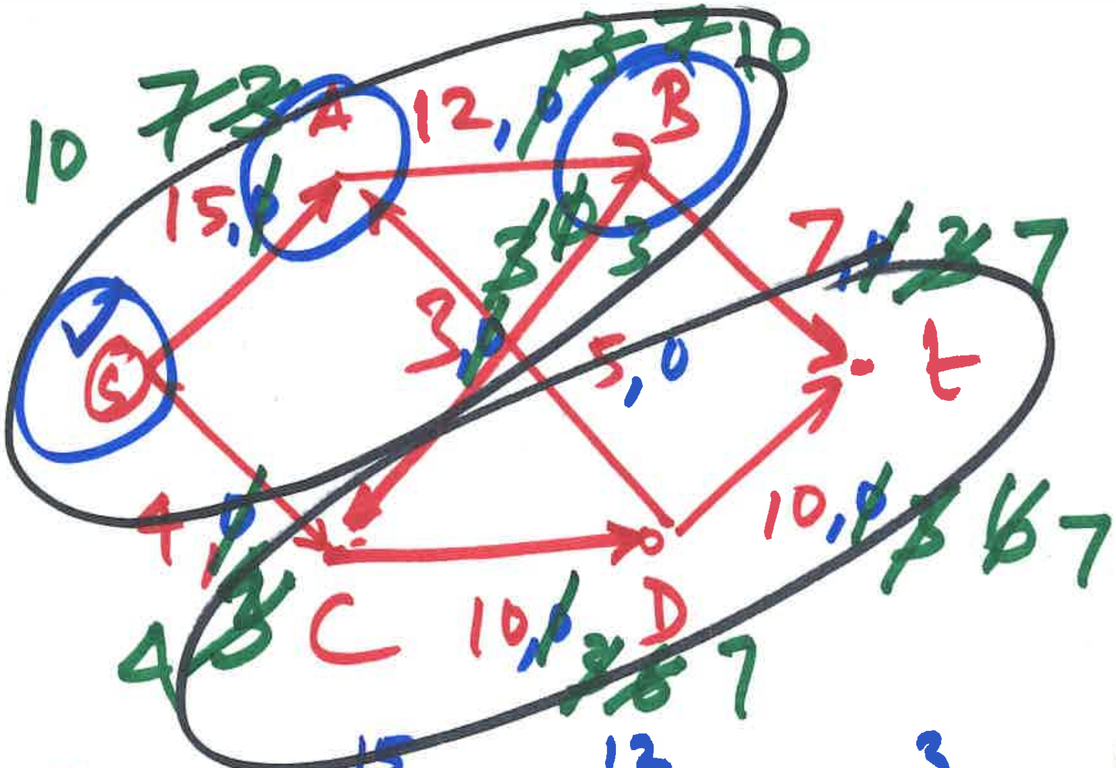
$\boxed{\text{max. flow} \le 20}$ ✓

$\boxed{\text{max. flow} = 20}$ (? Yes.

Ford-Fulkerson Alg $\Rightarrow$ $\boxed{\text{Max Flow} = \text{Min Cut}}$







$u \longrightarrow v$

(L) (NL) $c(e) > f(e)$

$u \longleftarrow v$

(L) (NL) $f(e) > 0$

P_1: $S \xrightarrow{15} A \xrightarrow{12} B \xrightarrow{3} C \xrightarrow{10} D \xrightarrow{10} t$ $\Delta = 3$

L L L L L L

P_2: $S \xrightarrow{4} C \xleftarrow{3} B \xrightarrow{7} t$ $\Delta = 3$

P_3: $S \xrightarrow[12]{} A \xrightarrow[9]{} B \xrightarrow[4]{} t$ $\Delta = 4$

P_4: $S \xrightarrow[8]{} A \xrightarrow[5]{} B \xrightarrow[3]{} C \xrightarrow[7]{} D \xrightarrow[7]{} t$ $\Delta = 3$

P_5: $S \rightarrow C \rightarrow D \rightarrow t$ $\Delta = 1$

$$
\begin{tikzcd}
S \arrow[r] \arrow[d, rightsquigarrow] & A \arrow[r] \arrow[d, rightsquigarrow] \arrow[l, dashed, bend left] & B \arrow[l, bend left] \arrow[r, dashed] & X \\
X & X & &
\end{tikzcd}
$$