SUPPLEMENTARY TEXTUAL MATERIAL

IN

MATHEMATICS

(CLASS XII)

CONTENT

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CHAPTER - 5

CONTINUITY AND DIFFERENTIATION

I. Derivative of an exponential function

Let $f(x) = e^x$

Then by definition

$$f^1(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right)$$

$$= e^x \lim_{h \to 0} \frac{\left(e^h - 1\right)}{h} = e^x . 1 = e^x$$

Hence, $\frac{d}{dx}\left(e^x\right) = e^x$

Similarly, we can write that $\frac{d}{dx}(a^x) = a^x \log a$.

II. Derivative of a logarithmic function

Let $f(x) = \log_e x$, where $x > 0$

Then, $f^1(x) = \lim_{h \to 0} \frac{f\left(x+h\right) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\log_e\left(x+h\right) - \log_e x}{h}$$

$$= \lim_{h \to 0} \frac{\log_e\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\log_e\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h \to O} \frac{\log_e\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \frac{1}{x} = 1.\frac{1}{x} = \frac{1}{x}$$

Hence, $\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$

CHAPTER – 7

INTEGRALS

I. **Integrals of Type :** $\int \sqrt{ax^2 + bx + c}\ dx$

In order to evaluate the above type of integral, we put $ax^2 + bx + c$ in the form :

$$\begin{cases} a\left[\left(x + \dfrac{b}{2a}\right)^2 + \left(\dfrac{\sqrt{4ac - b^2}}{2a}\right)^2\right] \text{ when } b^2 < 4ac. \\ a\left[\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right] \text{ when } b^2 > 4ac. \end{cases}$$

and the integral reduces to one of the following three forms:

$\int \sqrt{a^2 + x^2}\,dx$, $\int \sqrt{a^2 - x^2}\,dx$, $\int \sqrt{x^2 - a^2}\,dx$, which can be evaluated by using standard formulae.

Example 1: Evaluate

(i) $\int \sqrt{x^2 + 4x + 8}\,dx$, (ii) $\int \sqrt{(x-5)(7-x)}\ dx$

(ii) $\int \sqrt{14x - 20 - 2x^2}\ dx$ (iv) $\int \sqrt{4a - x^2}\,dx$

Solution : (i) $\int \sqrt{x^2 + 4x + 8}\ dx = \int \sqrt{x^2 + 4x + 4 + 4}\ dx$

$$= \int \sqrt{(x+2)^2 + (2)^2}\ dx$$

$$= \frac{x+2}{2}\sqrt{x^2 + 4x + 8} + \frac{4}{2}\log\left|(x+2) + \sqrt{x^2 + 4x + 8}\right| + c$$

$$= \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 8} + 2\log\left|(x+2) + \sqrt{x^2 + 4x + 8}\right| + c$$

(ii) $\int \sqrt{(x-5)(7-x)}\ dx = \int \sqrt{12x - 35 - x^2}\ dx$

$$= \int \sqrt{-35 - (x^2 - 12x)}\,dx = \int \sqrt{-35 - (x^2 - 12x + 36 - 36)}\ dx$$

$$= \int \sqrt{(36-35) - (x-6)^2}\,dx = \int \sqrt{1^2 - (x-6)^2}\ dx$$

$$= \frac{x-6}{2}\sqrt{(x-5)(7-x)} + \frac{1}{2}\sin^{-1}\left(\frac{x-6}{1}\right) + c$$

(iii) $\int \sqrt{14x - 20 - 2x^2}\, dx = \sqrt{2}\int \sqrt{-10 + 7x - x^2}\, dx$

$$= \sqrt{2}\int \sqrt{-10 - (x^2 - 7x)}\, dx = \sqrt{2}\int \sqrt{-10 - \left(x^2 - 7x + \frac{49}{4} - \frac{49}{4}\right)}\, dx$$

$$= \sqrt{2}\int \sqrt{-10 + \frac{49}{4} - \left(x - \frac{7}{2}\right)^2}\, dx = \sqrt{2}\int \sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}\, dx$$

$$= \sqrt{2}\left[\frac{x - 7/2}{2}\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - 7/2\right)^2} + \frac{9}{8}\sin^{-1}\left(\frac{x - \frac{7}{2}}{\frac{3}{2}}\right)\right] + c$$

$$= \sqrt{2}\left[\frac{2x-7}{4}\sqrt{7x - 10 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x-7}{3}\right)\right] + c$$

(iv) $\int \sqrt{4ax - x^2}\, dx = \int \sqrt{-\left(x^2 - 4ax + 4a^2 - 4a^2\right)}\, dx$

$$= \int \sqrt{4a^2 - \left(x^2 - 4ax + 4a^2\right)}\, dx = \int \sqrt{(2a)^2 - (x - 2a)^2}\, dx$$

$$= \frac{x-2a}{2}\sqrt{4ax - x^2} + \frac{4a^2}{2}\sin^{-1}\left(\frac{x-2a}{2a}\right) + c$$

$$= \frac{1}{2}(x-2a)\sqrt{4ax - x^2} + 2a^2\sin^{-1}\left(\frac{x-2a}{2a}\right) + c$$

II. Integrals of Type : $\int (px+q)\sqrt{ax^2+bx+c}\, dx$

Here, $px + q$ is written as

$px + q = A\left[\frac{d}{dx}\left(ax^2+bx+c\right)\right] + B$ and the values of A and B are determined by equating the coefficients of x and constant terms on both sides.

Then, writing

$$\int (px+q)\sqrt{ax^2+bx+c}\, dx = A\int (2ax+b)\sqrt{ax^2+bx+c}\, dx + B\int \sqrt{ax^2+bx+c}\, dx$$

$$= \frac{2A}{3}(ax^2 + bx + c)^{3/2} + B\int \sqrt{ax^2 + bx + c}\, dx$$

The second part is evaluated as explained in above example.

Example 2: Evaluate

(i) $\int (x-3)\sqrt{x^2+4x+3}\, dx$ (ii) $\int (3x+5)\sqrt{2x^2+3x+7}\, dx$

(ii) $\int (x-4)\sqrt{4+3x-x^2}\, dx$ (iii) $\int (5x-1)\sqrt{6+5x-2x^2}\, dx$

Solution :

(i) Let $I = \int (x-3)\sqrt{x^2+4x+3}\, dx$

$$x - 3 = A(2x+4) + B$$

$$\Rightarrow A = \frac{1}{2},\ B = -5$$

$$\therefore I = \int \frac{1}{2}(2x+4)\sqrt{x^2+4x+3}\, dx - 5\int \sqrt{x^2+4x+3}\, dx$$

$$= \frac{1}{3}\left(x^2+4x+3\right)^{3/2} - 5\int \sqrt{(x+2)^2 - (1)^2}\, dx$$

$$= \frac{1}{3}\left(x^2+4x+3\right)^{3/2} - 5\left[\frac{x+2}{2}\sqrt{x^2+4x+3}\ \frac{-1}{2}\log\left|(x+2) + \sqrt{x^2+4x+3}\right|\right] + c$$

(ii) $I = \int (3x+5)\sqrt{2x^2+3x+7}\, dx$

$$3x + 5 = A(4x+3) + B \Rightarrow 4A = 3 \text{ and } 3A + B = 5$$

$$\Rightarrow A = 3/4 \text{ and } B = \frac{11}{4}$$

$$I = \int \frac{3}{4}(4x+3)\sqrt{2x^2+3x+7}\, dx + \frac{11}{4}\int \sqrt{2}\sqrt{x^2 + \frac{3}{2}x + \frac{7}{2}}\, dx$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\int \sqrt{\left(x + 3/4\right)^2 + \frac{7}{2} - \frac{9}{16}}\, dx$$

$$= \quad \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\int\sqrt{\left(x+\tfrac{3}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2}\,dx$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\left[\frac{x+\tfrac{3}{4}}{2}\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}+\frac{47}{8}\log\left|\left(x+\tfrac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}\right|\right]+C$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\left[\frac{4x+3}{8}+\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}+\frac{47}{8}\log\left|\frac{4x+3}{4}+\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}\right|\right]+C$$

(iii) $I = \int (x-4)\sqrt{4+3x-x^2}\,dx$

$$x-4 = A\,(3-2x)+B \Rightarrow A = -\frac{1}{2},\ B = -\frac{5}{2}$$

$$\therefore I = \int -\frac{1}{2}(3-2x)\sqrt{4+3x-x^2}\,dx + \left(\frac{-5}{2}\right)\int\sqrt{4+3x-x^2}\,dx$$

$$= \quad -\frac{1}{3}\left(4-3x-x^2\right)^{3/2} - \frac{5}{2}\left[\frac{2x-3}{4}\sqrt{4+3x-x^2}+\frac{25}{8}\sin^{-1}\frac{2x-3}{5}\right]+C$$

(iv) $\therefore I = \int (5x-1)\sqrt{6+5x-2x^2}\,dx$

$$5x-1 = A\left[5-4x\right]B,\ A = -\tfrac{5}{4},\ B = \frac{21}{4}$$

$$\therefore I = \int \frac{-5}{4}(5-4x)\sqrt{6+5x-2x^2}\,dx + \left(\frac{21}{4}\right)\int\sqrt{2}\sqrt{3+\frac{5}{2}x-x^2}\,dx$$

$$= \quad \frac{-5}{6}\left(6+5x-2x^2\right)^{3/2} + \frac{21\sqrt{2}}{4}\int\sqrt{\left(\frac{73}{4}\right)^2-\left(x-\tfrac{5}{4}\right)^2}\,dx$$

$$= \quad \frac{-5}{6}\left(6+5x-2x^2\right)^{3/2} + \frac{21\sqrt{2}}{4}\left[\frac{x-\tfrac{5}{4}}{2}\sqrt{3+\frac{5}{2}x-x^2}+\frac{73}{32}\sin^{-1}\left(\frac{4x-5}{\sqrt{73}}\right)\right]+C$$

$$= \quad \frac{-5}{6}\left(6+5x-2x^2\right)^{3/2} + \frac{21\sqrt{2}}{4}\left[\frac{4x-5}{8}\sqrt{3+\frac{5}{2}x-x^2}+\frac{73}{32}\sin^{-1}\left(\frac{4x-5}{\sqrt{73}}\right)\right]+C$$

Note: The above integral can also be evalnated by substituting $\tan \frac{x}{2} = t$.

Exercise 1.

1. Evaluate:

(i) $\int \sqrt{3x^2 + 4x + 1}\, dx$ (ii) $\int \sqrt{1 + 2x - 3x^2}\, dx$

(iii) $\int \sqrt{x^2 + 4x + 1}\, dx$ (iv) $\int \sqrt{3 - 2x - 2x^2}\, dx$

(v) $\int \sqrt{(x-3)(5-x)}\, dx$ (vi) $\int \sqrt{(2ax - x^2)}\, dx$

2. Evaluate:

(i) $\int (2x+3)\sqrt{x^2 + 4x + 3}\, dx$ (ii) $\int (2x-5)\sqrt{2 + 3x - x^2}\, dx$

(iii) $\int (2x+3)\sqrt{4x^2 + 5x + 6}\, dx$ (iv) $\int (2x-5)\sqrt{x^2 + 4x + 3}\, dx$

(v) $\int (x+1)\sqrt{1 - x - x^2}\, dx$ (vi) $\int (6x+5)\sqrt{x + 6 - 2x^2}\, dx$

ANSWERS:

EXERCISE : 1

1.(i) $\frac{1}{6}(3x+2)\sqrt{3x^2 + 4x + 1} - \frac{\sqrt{3}}{18} \log \left| \left(x + \frac{2}{3}\right) + \sqrt{x^2 + \frac{4}{3}x + \frac{1}{3}} \right| + C$

(ii) $\frac{1}{6}(3x-1)\sqrt{1 + 2x - 3x^2} + \frac{2\sqrt{3}}{9} \sin^{-1}\left(\frac{3x-1}{2}\right) + C$

(iii) $\frac{1}{2}(x+2)\sqrt{x^2 + 4x + 1} - \frac{3}{2} \log \left| (x+2) + \sqrt{x^2 + 4x + 1} \right| + C$

(iv) $\sqrt{2}\left(\frac{2x+1}{4}\right)\sqrt{\frac{3}{2} - x - x^2} + \frac{7}{8} \sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$

(v) $\frac{1}{2}(x-4)\sqrt{(x-3)(5-x)} + \frac{1}{2} \sin^{-1}(x-4) + C$

(vi) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + C$

Q.2(i) $\frac{2}{3}\left(x^2+4x+3\right)^{3/2}$ - $\frac{x+2}{2}\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|\left(x+2\right)+\sqrt{x^2+4x+3}\right|+C$

(ii) $\frac{-2}{3}\left(2+3x-x^2\right)^{3/2}$ - $\left(\frac{2x-3}{2}\right)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\frac{(2x-3)}{\sqrt{17}}+C$

(iii) $\frac{1}{6}\left(4x^2+5x+6\right)^{3/2}+\frac{7}{16}(8x+5)\sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}+\frac{497}{256}\log\left|\frac{(8x+5)}{8}+\sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}\right|+C$

(iv) $\frac{2}{3}\left(x^2-4x+3\right)^{3/2}$ - $\frac{1}{2}(x-2)\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right|+C$

(v) $\frac{-1}{3}\left(1-x-x^2\right)^{3/2}+\frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{1-x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$

(vi) $-\left(6+x-2x^2\right)^{3/2}+\frac{13}{16}(4x-1)\sqrt{6+x-2x^2}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$

CHAPTER - 10

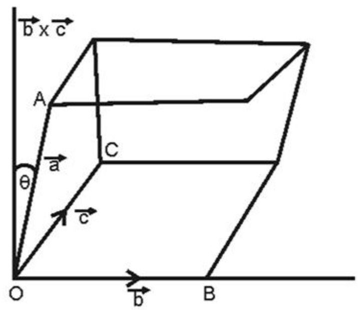
VECTOR ALGEBRA

I. SCALAR TRIPLE PRODUCT

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors. Then the scalar $(\vec{a} \times \vec{b}).\vec{c}$ is called the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$

$$\therefore\ [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}).\vec{c}$$

II. GEOMETRICAL INTERPRETATION OF A SCALAR TRIPLE PRODUCT



If three co-terminus edges OA, OB and OC of a parallelopiped are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, then $\vec{b} \times \vec{c}$ represents the vector area of the base of the paralleopiped and the height of the parallelopiped is the projection of $\vec{a}$ along the normal to the plane containing Vectors $\vec{b}$ and $\vec{c}$, i.e., along $\vec{b} \times \vec{c}$

$$\text{Magnitude of this projection} = \frac{\vec{a}.(\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}$$

$\therefore$ Volume of the parallelopiped

= (Area of base) x (Height)

$$= \frac{|\vec{b} \times \vec{c}|\ |\vec{a}.(\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|} = |\vec{a}.(\vec{b} \times \vec{c})|$$

{ Modulus has been taken as area is always positive}

Thus, if $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the three co-terminus edges of a parallelgram then its volume = $\vec{a}.(\vec{b} \times \vec{c})$

or $[\vec{a}\ \vec{b}\ \vec{c}]$

III. SCALAR TRIPLE PRODUCT IN TERMS OF RECTANGULAR COMPONENTS

Let $\vec{a} = a_1\ \hat{i} + a_2\ \hat{j} + a_3\ \hat{k}$, $\vec{b} = b_1\ \hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\ \hat{k}$

then $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left(b_2\ c_3 - b_3\ c_2\right)\ \hat{i} - \left(b_1\ c_3 - b_3 c_1\right)\ \hat{j} + \left(b_1 c_2 - b_2\ c_1\right)\ \hat{k}$

$$\vec{a}.\ \left(\vec{b} \times \vec{c}\right) = a_1\ (b_2\ c_3 - b_3\ c_2) - a_2\ (b_1 c_3 - b_3\ c_1) + a_3\ (b_1\ c_2 - b_2\ c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Remarks : If for any three vectors $\vec{a},\ \vec{b}$ and $\vec{c}$, $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$, then the volume of the parallelopipped with the three co-terminus edges $\vec{a},\ \vec{b}$ and $\vec{c}$, is zero, which is possible only if $\vec{a},\ \vec{b}$ and $\vec{c}$, are co-planar vectors.

Thus, $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0 \Leftrightarrow \vec{a},\ \vec{b}$ and $\vec{c}$ are co-planar

IV. PROPERTIES OF SCALAR TRIPLE PRODUCT

1. If $\vec{a},\ \vec{b}$ and $\vec{c}$ are any three vectors, then

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{c}\ \vec{a}\ \vec{b}\right]$$

Proof : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = \vec{b_1}\hat{i} + \vec{b_2}\hat{j} + b_3\hat{k}$ and $\vec{c} = \vec{c_1}\ \hat{i} + \vec{c_2}\hat{j} + \vec{c_3}\hat{k}$, then

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \left[\vec{b} \ \ \vec{c} \ \ \vec{a}\right]$$

$$\therefore \left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = \left[\vec{b} \ \ \vec{c} \ \ \vec{a}\right] \_\_\_(i)$$

Similarly, it can be vertified that $\left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = \left[\vec{c} \ \ \vec{a} \ \ \vec{b}\right] \_\_\_(ii)$

from (i) and (ii), we see that

$$\left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = \left[\vec{b} \ \ \vec{c} \ \ \vec{a}\right] = \left[\vec{c} \ \ \vec{a} \ \ \vec{b}\right]$$

$\Rightarrow$ If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are cyclically permuted, the value of the scalar Triple Product remains unaltered.

2. In scalar triple product, the position of dot and cross can be interchanged, provided the cyclic order of vectors remains the same.

Proof: Since $\left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = \left[\vec{c} \ \ \vec{a} \ \ \vec{b}\right]$

$$\Rightarrow \vec{a}.\left(\vec{b} \times \vec{c}\right) = \vec{c}.\left(\vec{a} \times \vec{b}\right)$$
$$\text{or } \vec{a}.\left(\vec{b} \times \vec{c}\right) = \vec{c}.\left(\vec{a} \times \vec{b}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$$

3. The value of the scalar triple product remains the same in magnitude, but changes the sign, if the cyclic order of $\vec{a}$, $\vec{b}$ and $\vec{c}$ is changed.

Proof: $\left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = \vec{a}.\left(\vec{b} \times \vec{c}\right) = -\vec{a}.\left(-\vec{c} \times \vec{b}\right) = \vec{a}\left(\vec{c} \times \vec{b}\right) = -\left[\vec{a} \ \ \vec{c} \ \ \vec{b}\right]$

4. The scalar triple product of three vectors is zero if any two of the given vectors are equal.

Proof: Let $\vec{a} = \vec{b}$

$$\therefore \left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = \left[\vec{a} \ \ \vec{a} \ \ \vec{c}\right] = \left(\vec{a} \times \vec{a}\right).\vec{c} = 0$$

Similarly, if $\vec{b} = \vec{c}$ or $\vec{c} = \vec{a}$, $\left[\vec{a} \ \ \vec{b} \ \ \vec{c}\right] = 0$

5. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and scalar λ, we have

$$\therefore \left[\lambda\vec{a} \quad \vec{b} \quad \vec{c}\right] = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$$

Proof: $\left[\lambda\vec{a} \quad \vec{b} \quad \vec{c}\right] = \left(\lambda\vec{a} \times \vec{b}\right).\vec{c}$

$$= \lambda\left(\vec{a} \times \vec{b}\right).\vec{c} = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$$

6. The scalar triple product of three vector is zero if any two of them are parallel or collinear

Proof: let $\vec{a}$ be parallel (or collinear) to $\vec{b}$

$\therefore \vec{a} = \lambda\vec{b}$ for some scalar λ

$$\therefore \left[\vec{a} \quad \vec{b} \quad \vec{c}\right] = \left[\lambda\vec{b} \quad \vec{b} \quad \vec{c}\right] = \lambda\left[\vec{b} \quad \vec{b} \quad \vec{c}\right] = \lambda.0 = 0$$

Let us now take some examples:

Example1: If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} + 4\hat{j} - \hat{k}$`

then find $\vec{a}.\left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right).\vec{c}$. Is $\vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$?

Solution :

$$\vec{a}.\left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ 3 & 4 & -1 \end{vmatrix} = 2\ (-2+12) + 3\ (-1+9) + 4\ (4-6) = 20 + 24 - 8 = 36$$

$$\left(\vec{a} \times \vec{b}\right).\vec{c} = \vec{c}.\left(\vec{a} \times \vec{b}\right) = \begin{vmatrix} 3 & 4 & -1 \\ 2 & -3 & 4 \\ 1 & 2 & -3 \end{vmatrix} = 3\ (9-8) - 4\ (-6-4) - 1\ (4+8(= 3 + 40 - 7 = 36$$

$$\therefore \vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$$

Example2: Find the volume of the parallelopipped whose edges are

$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}, \quad \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$`

Solution: Volume of the parallelopipped =

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = 2(4-1) + 3(2+2) + 4(-1-4) = 6+12-20 = -2$$

As volume is always positive, required volume is 2 cubic units

Example3: Show that the vectors $\vec{a} = -4\hat{i} - 6\hat{j} - 2\hat{k}, \quad \vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$ are co-planar

Solution: Three vector $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3) + 6(-3+24) - 2(1+32) = -60 + 126 - 66 = 0$$

$\therefore \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar

Example4: Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \quad \vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + \lambda\hat{k}$ are co-planar if $\lambda = 5$

Solution: Three vector $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = O$

$$\therefore \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & \lambda \end{vmatrix} = 0 \text{ or } 1(3\lambda-12) + 2(-2\lambda+4) + 3(6-3) = 0$$

or $3\lambda - 12 - 4\lambda + 8 + 9 = 0$

$-\lambda + 5 = 0 \Rightarrow \lambda = 5$

Example 5: Show that four points with position vectors

$6\hat{i} - 7\hat{j}, \quad 16\hat{i} - 19\hat{j} - 4\hat{k}, \quad 3\hat{j} - 6\hat{k}$ and $2\hat{i} + 5\hat{j} + 10\hat{k}$ are not co-planar

Solution: Let A = $6\hat{i} - 7\hat{j}$, $B = 16\hat{i} - 19\hat{j} - 4\hat{k}$, $C = 3\hat{j} - 6\hat{k}$ and $D = 2\hat{i} + 5\hat{j} + 10\hat{k}$

$$\therefore\ \vec{a} = \overline{AB} = \left(16\hat{i} - 19\hat{j} - 4\hat{k}\right) - \left(6\hat{i} - 7\hat{j}\right) = 10\hat{i} - 12\hat{j} - 4\hat{k}$$

$$\vec{b} = \overline{AC} = \left(3\hat{j} - 6\hat{k}\right) - \left(6\hat{i} - 7\hat{j}\right) = -6\hat{i} + 10\hat{j} - 6\hat{k}$$

$$\vec{c} = \overline{AD} = \left(2\hat{i} + 5\hat{j} + 10\hat{k}\right) - \left(6\hat{i} - 7\hat{j}\right) = -4\hat{i} + 12\hat{j} + 10\hat{k}$$

$\vec{a},\ \vec{b},\ \vec{c}$ are coplaner if $\left[\vec{a},\ \vec{b},\ \vec{c}\right] = 0$

Let us evaluate $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$$\text{i.e.} \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 10\,(100 + 72) + 12\,(-60 - 24) - 4\,(-72 + 40) = 1720 - 1008 + 128 = 840 \neq 0$$

$\therefore \vec{a},\ \vec{b},\ \vec{c}$ are not coplanar

i.e. points A, B, C and D are not co-planar

Example 6 : For any three vectors $\therefore \vec{a},\ \vec{b},$ and $\vec{c}$, prove that

$$\left[\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}\right] = 2\ \left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

Solution:
$$\left[\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}\right] = \left[\left(\vec{a} + \vec{b}\right) \times \left(\vec{b} + \vec{c}\right)\right].\left(\vec{c} + \vec{a}\right)$$

$$= \left[\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c}\right].\left(\vec{c} + \vec{a}\right)$$

$$= \left(\vec{a} \times \vec{b}\right).\vec{c} + \left(\vec{a} \times \vec{b}\right).\vec{a} + \left(\vec{a} \times \vec{c}\right).\vec{c} + \left(\vec{a} \times \vec{c}\right).\vec{a} + \left(\vec{b} \times \vec{c}\right).\vec{c} + \left(\vec{b} \times \vec{c}\right).\vec{a}$$

$$= \left(\vec{a} \times \vec{b}\right).\vec{c} + \left(\vec{b} \times \vec{c}\right).\vec{a} = \left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 2\ \left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

EXERCISE1.

1. If $\vec{a} = 7\hat{i} - 2\hat{j} + 3\hat{k}$, $b = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + 8\hat{j}$, then find $\vec{a}.\left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right).\vec{c}$ Also Find whether if $\vec{a}.\left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right).\vec{c}$ are equal

2. If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$, then

(i) find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

(ii) find $\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right]$

3. Find the volumes of the following parallelopipeds whose three co-terminus edges are:

(i) $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$

(ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, and $\vec{c} = 2\hat{i} + \hat{j} - \hat{k}$

4. Show that

(i) the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$, and $\vec{c} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ are coplanar

(ii) the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$, and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar

5. Find the value of λ if the following vectors are co-planar

(i) $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$, and $\vec{c} = 3\hat{i} - \lambda\hat{j} + 5\hat{k}$

(ii) $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \lambda\hat{j} + \hat{k}$, and $\vec{c} = 5\hat{i} + \hat{j} - 3\hat{k}$

6. If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are three mutually perpendicular vectors, prove that $\hat{i}.\left(\hat{k} \times \hat{j}\right) = \hat{j}.\left(\hat{i} \times \hat{k}\right) = \hat{k}.\left(\hat{j} \times \hat{i}\right) = -1$

7. Show that the four points A, B, C and D with position vectors

$4\hat{i} + 5\hat{j} + \hat{k}, -\left(\hat{j} + \hat{k}\right), 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$, respectively are co-planar.

8. Find the value of λ if the points A (-1, 4, -3), B = (3, λ, -5), C (-3, 8 -5) and D (-3, 2,1) are coplanar

9. Show that if the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are co-planar, then $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also co-planar.

10. If the vectors $\vec{A} = a\hat{i} + \hat{j} + \hat{k}$, $\vec{B} = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{C} = \hat{i} + \hat{j} + c\hat{k}$ are co-planar, then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$, where a, b, c $\neq$ 1

Answers:

(1) Equal (2) (i) -7 (ii) -14 (3) (i) 7 (ii) 0

(5) (i) $\lambda = 4$ (ii) $\lambda = -1$ (8) $\lambda = 2$