Exercise – 6.1

1. Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer:

(i) $3x^2 - 4x + 15$

(ii) $y^2 + 2\sqrt{3}$

(iii) $3\sqrt{x} + \sqrt{2}x$

(iv) $x - \frac{4}{x}$

(v) $x^{12} + y^3 + t^{50}$

Sol:

(i) $3x^2 - 4x + 15$ is a polynomial of one variable x.

(ii) $y^2 + 2\sqrt{3}$ is a polynomial of one variable y.

(iii) $3\sqrt{x} + \sqrt{2}x$ is not a polynomial as the exponents of $3\sqrt{x}$ is not a positive integer.

(iv) $x - \frac{4}{x}$ is not a polynomial as the exponent of $\frac{-4}{x}$ is not a positive integer.

(v) $x^{12} + y^3 + t^{50}$ is a polynomial of three variables x, y, t.

2. Write the coefficient of x^2 in each of the following:

(i) $17 - 2x + 7x^2$

(ii) $9 - 12x + x^3$

(iii) $\frac{\pi}{6}x^2 - 3x + 4$

(iv) $\sqrt{3}x - 7$

Sol:

Coefficient of x^2 in

(i) $17 - 2x + 7x^2$ is 7

(ii) $9 - 12x + x^3$ is 0

(iii) $\frac{\pi}{6}x^2 - 3x + 4$ is $\frac{\pi}{6}$

(iv) $\sqrt{3}x - 7$ is 0

3. Write the degrees of each of the following polynomials:

(i) $7x^3 + 4x^2 - 3x + 12$

(ii) $12 - x + 2x^3$

(iii) $5y - \sqrt{2}$

(iv) $7 = 7 \times x^{\circ}$

(v) 0

Sol:

Degree of polynomial

(i) $7x^2 + 4x^2 - 3x + 12$ is 3

(ii) $12 - x + 2x^3$ is 3

(iii) $5y - \sqrt{2}$ is 1

(iv) $7 = 7 \times x^{\circ}$ is 0

(v) 0 is un defined.

4. Classify the following polynomials as linear, quadratic, cubic and biquadratic polynomials:

(i) $x + x^2 + 4$

(ii) $3x - 2$

(iii) $2x + x^2$

(iv) 3y

(v) $t^2 + 1$

(vi) $7t^4 + 4t^3 + 3t - 2$

Sol:

Given polynomial

(i) $x + x^2 + 4$ is quadratic as degree of polynomial is 2.

(ii) $3x - 2$ is linear as degree of polynomial is 1.

(iii) $2x + x^2$ is quadratic as degree of polynomial is 2.

(iv) 3y is linear as degree of polynomial is 2.

(v) $t^2 + 1$ is quadratic as degree of polynomial is 2.

(vi) $7t^4 + 4t^3 + 3t - 2$ is bi-quadratic as degree of polynomial is 4.

5. Classify the following polynomials as polynomials in one-variable, two variables etc:

(i) $x^2 - xy + 7y^2$

(ii) $x^2 - 2tx + 7t^2 - x + t$

(iii) $t^3 - 3t^2 + 4t - 5$

(iv) $xy + yz + zx$

Sol:

(i) $x^2 - xy + 7y^2$ is a polynomial in two variables x, y.

(ii) $x^2 - 2tx + 7t^2 - x + t$ is a polynomial in 2 variables x, t.

(iii) $t^3 - 3t^2 + 4t - 5$ is a polynomial in 1 variables t.

(iv) $xy + yz + zx$ is a polynomial in 3 variables x, y, z.

6. Identify polynomials in the following:

(i) $f(x) = 4x^3 - x^2 - 3x + 7$

(ii) $g(x) = 2x^3 - 3x^2 + \sqrt{x} - 1$

(iii) $p(x) = \frac{2}{3}x^2 - \frac{7}{4}x + 9.$

(iv) $q(x) = 2x^2 - 3x + \frac{4}{x} + 2$

(v) $h(x) = x^4 - x^{\frac{3}{2}} + x - 1$

(vi) $f(x) = 2 + \frac{3}{x} + 4x$

Sol:

(i) $f(x) = 4x^3 - x^2 - 3x + 7$ is a polynomial

(ii) $g(x) = 2x^3 - 3x^2 + \sqrt{x} - 1$ is not a polynomial as exponent of x in $\sqrt{x}$ is not a positive integer.

(iii) $p(x) = \frac{2}{3}x^2 - \frac{7}{4}x + 9.$ is a polynomial as all the exponents are positive integers.

(iv) $q(x) = 2x^2 - 3x + \frac{4}{x} + 2$ is not a polynomial as exponent of x in $\frac{4}{x}$ is not a positive integer.

(v) $h(x) = x^4 - x^{\frac{3}{2}} + x - 1$ is not a polynomial as exponent of x in $-x^{\frac{3}{2}}$ is not a positive integer.

(vi) $f(x) = 2 + \frac{3}{x} + 4x$ is not a polynomial as exponent of x in $\frac{3}{x}$ is not a positive integer.

7. Identify constant, linear, quadratic and cubic polynomials from the following polynomials:

(i) $f(x)=0$

(ii) $g(x)=2x^3-7x+4$

(iii) $h(x)=-3x+\frac{1}{2}$

(iv) $p(x)=2x^2-x+4$

(v) $q(x)=4x+3$

(vi) $r(x)=3x^2+4x^2+5x-7$

Sol:

Given polynomial

(i) $f(x)=0$ is a constant polynomial as 0 is a constant

(ii) $g(x)=2x^3-7x+4$ is a cubic polynomial as degree of the polynomial is 3.

(iii) $h(x)=-3x+\frac{1}{2}$ is a linear polynomial as degree of the polynomial is 1.

(iv) $p(x)=2x^2-x+4$ is a quadratic as the degree of the polynomial is 2.

(v) $q(x)=4x+3$ is a linear polynomial as the degree of the polynomial is 1.

(vi) $r(x)=3x^2+4x^2+5x-7$ is a cubic polynomial as the degree is 3.

8. Give one example each of a binomial of degree 35, and of a monomial of degree 100.

Sol:

Example of a binomial with degree 35 is $7x^{35}-5$

Example of a monomial with degree 100 is $2t^{100}$

Exercise – 6.2

1. If f(x) = $2x^3 - 13x^2 + 17x + 12$, find (i) f(2) (ii) f (-3) (iii) f(0)

Sol:

We have

$f(x) = 2x^3 - 13x^2 + 17x + 12$

(i) $f(2) = 2\times(2)^3 - 13\times(2)^2 + 17\times(2) + 12$

$= (2\times8) - (13\times4) + (17\times2) + 12$

$= 16 - 52 + 34 + 12 = 10$

(ii) $f(-3) = 2\times(-3)^3 - 13\times(-3)^2 + 17\times(-3) + 12$

$= 2\times(-27) - 13\times(9) + 17\times(-3) + 12$

$= -54 - 117 - 51 + 12 = -210$

(iii) $f(0) = 2\times(0)^3 - 13\times(0)^2 + 17\times(0) + 12$

$= 0 - 0 + 0 + 12 = 12$

2. Verify whether the indicated numbers are zeroes of the polynomials corresponding to them in the following cases:

(i) $f(x) = 3x + 1, x = -\frac{1}{3}$

(ii) $f(x) = x^2 - 1, x = 1, -1$

(iii) $g(x) = 3x^2 - 2, x = \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$

(iv) $p(x) = x^3 - 6x^2 + 11x - 6, x = 1, 2, 3$

(v) $f(x) = 5x - \pi, x = \frac{4}{5}$

(vi) $f(x) = x^2$ *and* $x = 0$

(vii) $f(x) = lx + m,\ x = -\frac{m}{l}$

(viii) $f(x) = 2x + 1, x = \frac{1}{2}$

Sol:

(i) $f(x) = 3x + 1, x = -\frac{1}{3}$

We have

$f(x) = 3x + 1$

Put $x=-\frac{1}{3} \Rightarrow f\left(-\frac{1}{3}\right)=\cancel{3}\times\left(-\frac{1}{\cancel{3}}\right)+1=-1+1=0$

$\therefore x=-\frac{1}{3}$ is a root of $f(x)=3x+1$

(ii) $f(x)=x^2-1, x=1,-1$

We have $f(x), x^2-1$

Put $x=1$ and $x=-1$

$\Rightarrow f(1)=(1)^2-1$ and $f(-1)=(-1)^2-1$

$=1-1=0$ $\quad$ $=1-1=0$

$\therefore x=1,-1$ are the roots of $f(x)=x^2-1$

(iii) $g(x)=3x^2-2, x=\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}}$

We have $g(x)=3x^2-2$

Put $x=\frac{2}{\sqrt{3}}$ and $x=-\frac{2}{\sqrt{3}}$

$\Rightarrow g\left(\frac{2}{\sqrt{3}}\right)=3\left(\frac{2}{\sqrt{3}}\right)^2-2$ and $g\left(\frac{-2}{\sqrt{3}}\right)=3\left(\frac{-2}{\sqrt{3}}\right)^2-2$

$=\cancel{3}\times\frac{4}{\cancel{3}}-2$ $\quad$ $=\cancel{3}\left(\frac{4}{\cancel{3}}\right)-2$

$=4-2=2\neq 0$ $\quad$ $=4-2=2\neq 0$

$\therefore x=\frac{2}{\sqrt{3}},\frac{-2}{\sqrt{3}}$ are not roots of $g(x)=3x^2-2$

(iv) $p(x)=x^3-6x^2+11x-6, x=1,2,3$

Put $x=1 \Rightarrow p(1)=(1)^3-6(1)^2+n(1)-6=1-6+11-6=0$

$x=2 \Rightarrow p(2)=(2)^3-6(2)^2+11(2)-6=8-24+22-6=0$

$x=3 \Rightarrow p(3)=3^3-6(3^2)+11(3)-6=27-54+33-6=0$

$\therefore x=1,2,3$ are roots of $p(x)=x^3-6x^2+11x-6$

(v) We know $f(x)=5x-\pi, x=\frac{4}{5}$

Put $x=\frac{4}{5} \Rightarrow f\left(\frac{4}{5}\right)=\cancel{5}\times\frac{4}{\cancel{5}}-\pi=4-\pi\neq 0$

$\therefore x=\frac{4}{5}$ is not a root of $f(x)=5x-\pi$

(vi) We have $f(x) = x^2$ *and* $x = 0$

Put $x = 0 \Rightarrow f(0) = (0)^2 = 0$

$\therefore x = 0$ is a root of $f(x) = x^2$

(vii) $f(x) = lx + m$ and $x = -\frac{m}{l}$

Put $x = \frac{-m}{l} \Rightarrow f\left(\frac{-m}{l}\right) = l \times \left(\frac{-m}{l}\right) + m = -m + m = 0$

$\therefore x = -\frac{m}{l}$ is a root of $f(x) = lx + m$

(viii) $f(x) = 2x + 1, x = \frac{1}{2}$

Put $x = \frac{1}{2} \Rightarrow f\left(\frac{1}{x}\right) = \cancel{2} \times \left(\frac{1}{\cancel{2}}\right) + 1 = 1 + 1 = 2 \neq 0$

$\therefore x = \frac{1}{2}$ is not a root of $f(x) = 2x + 1$

3. If x = 2 is a root of the polynomial f(x) = $2x^2 - 3x + 7a$, find the value of a.

Sol:

We have $f(x) = 2x^2 - 3x + 7a$

Put $x = 2 \Rightarrow f(2) = 2(2)^2 - 3(2) + 7a$

$= 2 \times 4 - 3 \times 2 + 7a = 8 - 6 + 7a$

$= 2 + 7a$

Given $x = 2$ is a root of $f(x) = 2x^2 - 3x + 7a$

$\Rightarrow f(2) = 0$

$\therefore 2 + 7a = 0$

$\Rightarrow 7a = -2 \Rightarrow \boxed{a = -\frac{2}{7}}$

4. If x = $-\frac{1}{2}$ is a zero of the polynomial p(x) = $8x^3 - ax^2 - x + 2$, find the value of a.

Sol:

We have $p(x) = 8x^3 - ax^2 - x + 2$

Put $x = -\frac{1}{2}$

$\Rightarrow P\left(-\frac{1}{2}\right) = 8 \times \left(-\frac{1}{2}\right)^3 - ax\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 2$

$= \cancel{8} \times \frac{-1}{\cancel{8}} - a \times \frac{1}{4} + \frac{1}{2} + 2$

$= -1 - \frac{a}{4} + \frac{1}{2} + 2$

$= \frac{3}{2} - \frac{a}{4}$

Given that $x = \frac{-1}{2}$ is a root of $p(x)$

$\Rightarrow P\left(\frac{-1}{2}\right) = 0$

$\therefore \frac{3}{2} - \frac{a}{4} = 0 \Rightarrow \frac{a}{4} = \frac{3}{2} \Rightarrow a = \frac{3}{\cancel{2}} \times \cancel{4}^{2}$

$\Rightarrow \boxed{a = 6}$

5. If x = 0 and x = -1 are the roots of the polynomial f(x) = $2x^3 - 3x^2 + ax + b$, find the value of a and b.

Sol:

We have $f(x) = 2x^3 - 3x^2 + ax + b$

Put

$x = 0 \Rightarrow f(0) = 2 \times (0)^3 - 3(0)^2 + a(0) + b = 0 - 0 + 0 + b = b$

$x = -1 \Rightarrow f(-1) = 2 \times (-1)^3 - 3 \times (-1)^2 + a(-1) + b = 2 \times (-1) - 3 \times (1) - a + b$

$= -2 - 3 - a + b$

$= -5 - a + b$

Since $x = 0$ and $x = -1$ are roots of $f(x)$

$\Rightarrow f(0) = 0$ *and* $f(-1) = 0$

$\Rightarrow b = 0 \qquad \Rightarrow -5 - a + b = 0$

$\boxed{b = 0}$ and $a - b = -5$

$\Rightarrow a - 0 = -5$

$\Rightarrow \boxed{a = -5}$

$\therefore a = -5$ and $b = 0$

6. Find the integral roots of the polynomial f(x) = $x^3 + 6x^2 + 11x + 6$.

Sol:

We have

$f(x) = x^3 + 6x^2 + 11x + 6$

Clearly, $f(x)$ is a polynomial with integer coefficient and the coefficient of the highest degree term i.e., the leading coefficients is 1.

Therefore, integer roots of $f(x)$ are limited to the integer factors of 6, which are $\pm 1, \pm 2, \pm 3, \pm 6$

We observe that

$f(-1) = (-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0$

$f(-2) = (-2)^3 + 6(-2)^2 + 11(-2) + 6 = -8 + 24 - 22 + 6 = 0$

$f(-3) = (-3)^2 + 6(-3)^2 + 11(-3) + 6 = -27 + 54 - 33 + 6 = 0$

$\therefore$ Hence, integral roots of $f(x)$ are $-1, -2, -3$.

7. Find rational roots of the polynomial f(x) = $2x^3 + x^2 - 7x - 6$

Sol:

We have

$f(x) = 2x^3 + x^2 - 7x - 6$

Clearly, $f(x)$ is a cubic polynomial with integer coefficient. If $\frac{b}{c}$ is a rational roots in lowest terms, then the value of b are limited to the factors of 6 which $\pm 1, \pm 2, \pm 3, \pm 6$ and values of c are limited to the factors of 2 which are $\pm 1, \pm 2$.

Hence, the possible rational roots of $f(x)$ are

$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

We observe that

$f(-1) = 2(-1)^3 + (-1)^2 - 7(-1) - 6 = -2 + 1 + 7 - 6 = 0$

$f(2) = 2(2)^3 + (2)^2 - 7(2) - 6 = 16 + 4 - 14 - 6 = 0$

$f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) - 6 = -\frac{27}{4} + \frac{9}{4} + \frac{21}{2} - 6 = 0$

$\therefore$ Hence, $-1, 2, \frac{-3}{2}$ are the rational roots of $f(x)$

Exercise– 6.3

In each of the following, using the remainder theorem, find the remainder when f(x) is divided by g(x) and verify the result by actual division: (1 – 8)

1. $f(x)=x^3+4x^2-3x+10$, $g(x)=x+4$

Sol:

We have $f(x)=x^3+4x^2-3x+10$ and $g(x)=x+4$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x)=x-(-4)$, the remainder is equal to $f(-4)$

Now, $f(x)=x^3+4x^2-3x+10$

$\Rightarrow f(-4)=(-4)^3+(-4)^2-3(-4)+10$

$=-64+4\times16+12+10$

$=-64+64+12+10=22$

Hence, required remainder is 22.

2. $f(x)=4x^4-3x^3-2x^2+x-7$, $g(x)=x-1$

Sol:

We have

$f(x)=4x^4-3x^3-2x^2+x-7$ and $g(x)=x-1$

Therefore by remainder theorem when $f(x)$ is divide by $g(x)=x-1$, the remainder is equal to $f(+1)$

Now, $f(x)=4x^4-3x^3-2x^2+x-7$

$\Rightarrow f(1)=4(+1)^4-3(+1)^3-2(+1)^2+(+1)-7$

$=4\times1-3(+1)-2(1)+1-7$

$=4-3-2+1-7=-7$

Hence, required remainder is -7

3. $f(x)=2x^4-6x^3+2x^2-x+2$, $g(x)=x+2$

Sol:

We have

$f(x)=2x^4-6x^3+2x^2-x+2$ and $g(x)=x+2$

Therefore, by remainder theorem when $f(x)$ is divide by $g(x) = x-(-2)$, the remainder is equal to $f(-2)$

Now, $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$

$\Rightarrow f(-2) = 2(-2)^4 - 6(-2)^3 + 2(-2)^2 - (-2) + 2$

$= 2\times 16 - 6\times(-8) + 2\times 4 + 2 + 2$

$= 32 + 48 + 8 + 4 = 92$

Hence, required remainder is 92.

4. $f(x) = 4x^3 - 12x^2 + 14x - 3$, $g(x) = 2x - 1$

Sol:

We have

$f(x) = 4x^3 - 12x^2 + 14x - 3$ and $g(x) = 2x - 1$

Therefore, by remainder theorem when $f(x)$ is divide by $g(x) = 2\left(x - \frac{1}{2}\right)$, the remainder is equal to $f\left(\frac{1}{2}\right)$

Now, $f(x) = 4x^3 - 12x^2 + 14x - 3$

$\Rightarrow f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 3$

$= \left(\cancel{4}\times\frac{1}{\cancel{8}_2}\right) - \left(\cancel{12}^3\times\frac{1}{\cancel{4}}\right) + \left(\cancel{14}^7\times\frac{1}{\cancel{2}}\right) - 3$

$= \frac{1}{2} - 3 + 7 - 3 = \frac{1}{2} + 1 = \frac{3}{2}$

Hence, required remainder is $\frac{3}{2}$.

5. $f(x) = x^3 - 6x^2 + 2x - 4$, $g(x) = 1 - 2x$

Sol:

We have

$f(x) = x^3 - 6x^2 + 2x - 4$ and $g(x) = 1 - 2x$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = -2\left(x - \frac{1}{2}\right)$, the remainder is equal to $f\left(\frac{1}{2}\right)$

Now, $f(x) = x^3 - 6x^2 + 2x - 4$

$\Rightarrow f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4$

$= \frac{1}{8} - \left(\overset{3}{\cancel{6}} + \frac{1}{\cancel{4}_2}\right) + \cancel{2} \times \frac{1}{\cancel{2}} - 4$

$= \frac{1}{8} - \frac{3}{2} + 1 - 4 = -\frac{35}{8}$

Hence, the required remainder is $-\frac{35}{8}$

6. $f(x) = x^4 - 3x^2 + 4$, $g(x) = x - 2$

Sol:

We have

$f(x) = x^4 - 3x^2 + 4$ and $g(x) = x - 2$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - 2$, the remainder is equal to $f(2)$

Now, $f(x) = x^4 - 3x^2 + 4$

$\Rightarrow f(2) = 2^4 - 3(2)^2 + 4$

$= 16 - (3 \times 4) + 4 = 16 - 12 + 4 = 20 - 12 = 8$

Hence, required remainder is 8.

7. $f(x) = 9x^3 - 3x^2 + x - 5$, $g(x) = x - \frac{2}{3}$

Sol:

We have $f(x) = 9x^3 - 3x^2 + x - 5$ and $g(x) = x - \frac{2}{3}$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - \frac{2}{3}$, the remainder is equal to $f\left(\frac{2}{3}\right)$

Now, $f(x) = 9x^3 - 3x^2 + x - 5$

$\Rightarrow f\left(\frac{2}{3}\right) = 9\left(\frac{2}{3}\right)^3 - 3\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right) - 5$

$$= \left(\cancel{9} \times \frac{8}{\underset{3}{\cancel{27}}} \right) - \left(\cancel{3} \times \frac{4}{\underset{3}{\cancel{9}}} \right) + \frac{2}{3} - 5$$

$$= \frac{8}{3} - \frac{4}{3} + \frac{2}{3} - 5 = \frac{6}{3} - 5 = 2 \cdot 5 = -3$$

Hence, the required remainder is -3.

8. $f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27}$, $g(x) = x + \frac{2}{3}$

Sol:

We have

$f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27}$ and $g(x) = x + \frac{2}{3}$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - \left(-\frac{2}{3}\right)$, the remainder is equal to $f\left(-\frac{2}{3}\right)$

Now, $f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27}$

$$\Rightarrow f\left(-\frac{2}{3}\right) = 3 \times \left(\frac{-2}{3}\right)^4 + 2\left(\frac{-2}{3}\right)^3 - \frac{\left(\frac{-2}{3}\right)^2}{3} - \frac{\left(\frac{-2}{3}\right)}{9} + \frac{2}{27}$$

$$= \cancel{3} \times \frac{16}{\underset{27}{\cancel{81}}} + 2 \times \frac{-8}{27} - \frac{4}{9 \times 3} - \left(\frac{-2}{3 \times 9}\right) + \frac{2}{27}$$

$$= \frac{16}{27} - \frac{16}{27} - \frac{4}{27} + \frac{2}{27} + \frac{2}{27}$$

$$= \frac{16 - 16 - 4 + 2 + 2}{27} = \frac{0}{27} = 0$$

Hence, required remainder is 0.

9. If the polynomials $2x^3 + ax^2 + 3x - 5$ and $x^3 + x^2 - 4x + a$ leave the same remainder when divided by x – 2, find the value of a.

Sol:

Let $p(x) = 2x^3 + ax^2 + 3x - 5$ and $q(x) = x^3 + x^2 - 4x + a$ be the given polynomials

The remainders when $p(x)$ and $q(x)$ are divided by $(x-2)$ are $p(2)$ and $q(2)$ respectively.

By the given condition we have

$p(2) = q(2)$

$\Rightarrow 2(2)^3 + a(2)^2 + 3(2) - 5 = 2^3 + 2^2 - 4(2) + a$

$\Rightarrow 16 + 4a + 6 - 5 = 8 + 4 - 8 + a$

$\Rightarrow 3a + 13 = 0 \Rightarrow 3a = -13 \Rightarrow \boxed{a = \frac{-13}{3}}$

10. The polynomials $ax^3 + 3x^2 - 3$ and $2x^3 - 5x + a$ when divided by (x – 4) leave the remainders R_1 and R_2 respectively. Find the values of a in each of the following cases, if (i) $R_1 = R_2$ (ii) $R_1 + R_2 = 0$ (iii) $2R_1 - R_2 = 0$.

Sol:

Let $p(x) = ax^3 + 3x^2 - 3$ and $q(x) = 2x^3 - 5x + a$ be the given polynomials.

Now,

$R_1 =$ Remainder when $p(x)$ is divided by $x - 4$

$\Rightarrow R_1 = p(4)$

$\Rightarrow R_1 = a(4)^3 + 3(4)^2 - 3 \qquad [\because p(x) = ax^3 + 3x^2 - 3]$

$\Rightarrow R_1 = 64a + 48 - 3$

$\Rightarrow \boxed{R_1 = 64a + 45}$

And,

$R_2 =$ Remainder when $q(x)$ is divided by $x - 4$

$\Rightarrow R_2 = q(4)$

$\Rightarrow R_2 = q(4)^3 - 5(4) + a \qquad [\because q(x) = 2x^3 - 5x + a]$

$\Rightarrow R_2 = 128 - 20 + a$

$\Rightarrow \boxed{R_2 = 108 + a}$

(i) Given condition is $R_1 = R_2$

$\Rightarrow 64a + 45 = 108 + a$

$\Rightarrow 63a - 63 = 0 \Rightarrow 63a = 63 \Rightarrow \boxed{a = 1}$

(ii) Given condition is $R_1 + R_2 = 0$

$\Rightarrow 64a + 45 + 108 + a = 0$

$\Rightarrow 65a + 153 = 0 \Rightarrow 65a = -153 \Rightarrow \boxed{a = \frac{-153}{65}}$

(iii) Given condition is $2R_1 - R_2 = 0$

$\Rightarrow 2(64a+45)-(108+a)=0$

$\Rightarrow 128a+90-108-a=0$

$\Rightarrow 127a-18=0 \Rightarrow 127a=18 \Rightarrow \boxed{a=\frac{18}{127}}$

11. If the polynomials ax^3+3x^2-13 and $2x^3-5x+a$, when divided by (x – 2) leave the same remainder, find the value of a.

Sol:

Let $p(x)=ax^3+3x^2-13$ and $q(x)=2x^3-5x+a$ be the given polynomials

The remainders when $p(x)$ and $q(x)$ are divided by $(x-2)$ are $p(2)$ and $q(2)$.

By the given condition we have

$p(2)=q(2)$

$\Rightarrow a(2)^3+3(2)^2-13=2(2)^3-5(2)+a$

$\Rightarrow 8a+12-13=16-10+a$

$\Rightarrow 7a-7=0 \Rightarrow 7a=7 \Rightarrow a=\frac{7}{7} \Rightarrow \boxed{a=1}$

(i) $x \Rightarrow x-0$

By remainder theorem, required remainder is equal to $f(0)$

Now, $f(x)=x^3+3x^2+3x+1$

$\Rightarrow f(0)=0^3+3\times 0^2+3\times 0+1=0+0+0+1=1$

Hence, required remainder is 1.

(ii) $x+\pi \Rightarrow x-(-\pi)$

By remainder theorem, required remainder is equal to $f(-\pi)$

Now, $f(x)=x^3+3x^2+3x+1$

$\Rightarrow f(-\pi)=(-\pi)^3+3(-\pi)^2+3(-\pi)+1$

$=-\pi^3+3\pi^2-3\pi+1$

Hence, required remainder is $-\pi^3+3\pi^2-3\pi+1$.

(iii) $5+2x \Rightarrow 2\left(x-\left(-\frac{5}{2}\right)\right)$

By remainder theorem, required remainder is equal to $f\left(-\frac{5}{2}\right)$.

Now, $f(x)=x^3+3x^2+3x+1$

$$\Rightarrow f\left(-\frac{5}{2}\right)=\left(-\frac{5}{2}\right)^3+3\left(-\frac{5}{2}\right)^2+3\left(-\frac{5}{2}\right)+1$$
$$=\frac{-125}{8}+\frac{3\times 25}{4}+\frac{3\times -5}{2}+1$$
$$=\frac{-128}{8}+\frac{75}{4}-\frac{15}{2}+1$$
$$=\frac{-27}{8}$$

Hence, required remainder is $\frac{-27}{8}$.

Exercise – 6.4

In each of the following, use factor theorem to find whether polynomial g(x) is a factor of polynomial f(x) or, not: (1 – 7)

1. $f(x)=x^3-6x^2+11x-6$; $g(x)=x-3$

Sol:

We have $f(x)=x^3-6x^2+11x-6$ and $g(x)=x-3$

In order to find whether polynomial $g(x)=x-3$ is a factor of $f(x)$, it is sufficient to show that $f(3)=0$

Now, $f(x)=x^3-6x^2+11x-6$

$$\Rightarrow f(3)=3^3-6(3)^2+11(3)-6$$
$$=27-54+33-6=60-60=0$$

Hence, $g(x)$ is a factor of $f(x)$

2. $f(x)=3x^4+17x^3+9x^2-7x-10$; $g(x)=x+5$

Sol:

We have $f(x)=3x^4+17x^3+9x^2-7x-10$ and $g(x)=x+5$

In order to find whether $g(x)=x-(-5)$ is a factor of $f(x)$ or not, it is sufficient to show that $f(-5)=0$

Now, $f(x)=3x^4+17x^3+9x^2-7x-10$

$$\Rightarrow f(-5)=3(-5)^4+17(-5)^3+9(-5)^2-7(-5)-10$$

$= 3 \times 625 + 17 \times (-125) + 9 \times 25 + 35 - 10$

$= 1875 - 2125 + 225 + 35 - 10$

$= 0$

Hence, $g(x)$ is a factor of $f(x)$

3. $f(x) = x^5 + 3x^4 - x^3 - 3x^2 + 5x + 15$, $g(x) = x + 3$

Sol:

We have $f(x) = x^5 + 3x^4 - x^3 - 3x^2 + 5x + 15$ and $g(x) = x + 3$

In order to find whether $g(x) = x - (-3)$ is a factor of $f(x)$ or not, it is sufficient to prove that $f(-3) = 0$

Now, $f(x) = x^5 + 3x^4 - x^3 - 3x^2 + 5x + 15$

$\Rightarrow f(-3) = (-3)^5 + 3(-3)^4 - 3(-3)^2 + 5(-3) + 15$

$= -243 + 243 - (-27) - 3(9) + 5(-3) + 15$

$= -243 + 243 + 27 - 27 - 15 + 15$

$= 0$

Hence, $g(x)$ is a factor of $f(x)$

4. $f(x) = x^3 - 6x^2 - 19x + 84$, $g(x) = x - 7$

Sol:

We have $f(x) = x^3 - 6x^2 - 19x + 84$ and $g(x) = x - 7$

In order to find whether $g(x) = x - 7$ is a factor of $f(x)$ or not, it is sufficient to show that $f(7) = 0$

Now, $f(x) = x^3 - 6x^2 - 19x + 84$

$\Rightarrow f(7) = 7^3 - 6(7)^2 - 19(7) + 84$

$= 343 - 294 - 133 + 84 = 427 - 427$

$= 0$

Hence $g(x)$ is a factor $f(x)$

5. $f(x) = 3x^3 + x^2 - 20x + 12$ and $g(x) = 3x - 2$

Sol:

We have

$f(x) = 3x^3 + x^2 - 20x + 12$ and $g(x) = 3x - 2$

In order to find whether $g(x)=3\left(x-\frac{2}{3}\right)$ is a factor of $f(x)$ or not, it is sufficient to prove that $f\left(\frac{2}{3}\right)=0$

Now, $f(x)=3x^3+x^2-20x+12$

$\Rightarrow f\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^2-20\left(\frac{2}{3}\right)+12$

$=3\times\frac{8}{\cancel{27}_{9}}+\frac{4}{9}=\frac{40}{3}+12$

$=\frac{12}{9}-\frac{40}{3}+12=\frac{12-20+108}{9}=\frac{120-120}{9}=0$

Hence $g(x)=3x-2$ is a factor of $f(x)$

6. $f(x)=2x^3-9x^2+x+12$, $g(x)=3-2x$

Sol:

We have $f(x)=2x^3-9x^2+x+12$ and $g(x)=3-2x$

In order to find whether $g(x)=3-2x=-2\left(x-\frac{3}{2}\right)$ is a factor of $f(x)$ or not, it is sufficient to prove that $f\left(\frac{3}{2}\right)=0$

Now, $f(x)=2x^3-9x^2+x+12$

$\Rightarrow f\left(\frac{3}{2}\right)=2\left(\frac{3}{2}\right)^3-9\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)+12$

$=\cancel{2}\times\frac{27}{\cancel{8}_{4}}-9\times\frac{9}{4}+\frac{3}{2}+12$

$=\frac{27}{4}-\frac{81}{4}+\frac{3}{2}+12=\frac{27-81+6+48}{4}=\frac{81-81}{4}$

$=0$

Hence $g(x)=3-2x$ is a factor of $f(x)$

7. $f(x)=x^3-6x^2+11x-6$, $g(x)=x^2-3x+2$

Sol:

We have

$f(x)=x^3-6x^2+11x-6$ and $g(x)=x^2-3x+2$

$\Rightarrow g(x) = x^2 - 3x + 2 = (x-1)(x-2)$

Clearly, $(x-1)$ and $(x-2)$ are factors of $g(x)$

In order to find whether $g(x) = (x-1)(x-2)$ is a factor of $f(x)$ or not, it is sufficient to prove that $(x-1)$ and $(x-2)$ are factors of $f(x)$.

i.e., we should prove that $f(1) = 0$ and $f(2) = 0$

Now, $f(x) = x^3 - 6x^2 + 11x - 6$

$\Rightarrow f(1) = 1^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 12 - 12 = 0$

$\Rightarrow f(2) = 2^3 - 6(2)^2 + 11(2) - 6 = 8 - 24 + 22 - 6 = 30 - 30 = 0$

$\therefore (x-1)$ and $(x-2)$ are factors of $f(x)$

$\Rightarrow g(x) = (x-1)(x-2)$ is factor of $f(x)$

8. Show that (x – 2), (x + 3) and (x – 4) are factors of $x^3 - 3x^2 - 10x + 24$.

Sol:

Let $f(x) = x^3 - 3x^2 - 10x + 24$ be the given polynomial.

In order to prove that $(x-2), (x+3), (x-4)$ are factors of $f(x)$, it is sufficient to prove that $f(2) = 0$, $f(-3) = 0$ and $f(4) = 0$ respectively.

Now $f(x) = x^3 - 3x^2 - 10x + 24$

$\Rightarrow f(2) = 2^3 - 3(2)^2 - 10(2) + 24 = 8 - 12 - 20 + 24 = 0$

$\Rightarrow f(-3) = (-3)^3 - 3(-3)^2 - 10(-3) + 24 = -27 - 27 + 30 + 24 = 0$

$\Rightarrow f(4) = 4^3 - 3(4)^2 - 10(4) + 24 = 64 - 48 - 40 + 24 = 0$

Hence, $(x-2), (x+3)$ and $(x-4)$ are factors of the given polynomial.

9. Show that (x + 4), (x – 3) and (x – 7) are factors $x^3 - 6x^2 - 19x + 84$

Sol:

Let $f(x) = x^3 - 6x^2 - 19x + 84$ be the given polynomial

In order to prove that $(x+4), (x-3)$ and $(x-7)$ are factors of $f(x)$, it is sufficient to prove that $f(-4) = 0$, $f(3) = 0$ and $f(7) = 0$ respectively

Now, $f(x) = x^3 - 6x^2 - 19x + 84$

$\Rightarrow f(-4) = (-4)^3 - 6(-4)^2 - 19(-4) + 84 = -64 - 96 + 76 + 84 = 0$

$\Rightarrow f(3) = (3)^3 - 6(3)^2 - 19(3) + 84 = 27 - 54 - 57 + 84 = 0$

$\Rightarrow f(7) = 7^3 - 6(7)^2 - 19(7) + 84 = 343 - 294 - 133 + 84 = 0$

Hence, $(x+4),(x-3)$ and $(x-7)$ are factors of the given polynomial $x^3 - 6x^2 - 19x + 84$.

10. For what value of a is (x – 5) a factor of $x^3 - 3x^2 + ax - 10$?

Sol:

Let $f(x) = x^3 - 3x^2 + ax - 10$ be the given polynomial

From factor theorem,

If $(x-5)$ is a factor of $f(x)$ then $f(5) = 0$

Now, $f(x) = x^3 - 3x^2 + ax - 10$

$\Rightarrow f(5) = 5^3 - 3(5)^2 + a(5) - 10 = 0$

$\Rightarrow 125 - 3(25) + 5a - 10 = 0$

$\Rightarrow 5a + 40 = 0$

$\Rightarrow 5a = -40$

$\Rightarrow \boxed{a = -8}$

Hence $(x-5)$ is a factor of $f(x)$ if $a = -8$

11. Find the value of a such that (x – 4) is a factor of $5x^3 - 7x^2 - ax - 28$.

Sol:

Let $f(x) = 5x^3 - 7x^2 - ax - 28$ be the given polynomial from factor theorem, if $(x-4)$ is a factor of $f(x)$ then $f(4) = 0$

$\Rightarrow f(4) = 0$

$\Rightarrow 5(4)^3 - 7(4)^2 - a(4) - 28 = 0$

$\Rightarrow 5 \times 64 - 7 \times 16 - 4a - 28 = 0$

$\Rightarrow 320 - 112 - 4a - 28 = 0$

$\Rightarrow 180 - 4a = 0$

$\Rightarrow 4a = 180$

$\Rightarrow a = \frac{180}{4} = 45$

Hence $(x-4)$ is a factor of $f(x)$ when $a = 45$

12. For what value of a, if x + 2 is a factor of factor of $4x^4 + 2x^3 - 3x^2 + 8x + 5a$.

Sol:

Let $f(x) = 4x^4 + 2x^3 - 3x^2 + 8x + 5a$ be the given polynomial

From factor theorem if $(x+2)$ is a factor of $f(x)$ then $f(-2) = 0$

Now, $f(x) = 4x^4 + 2x^3 - 3x^2 + 8x + 5a$

$\Rightarrow f(-2) = 0$

$\Rightarrow 4(-2)^4 + 2(-2)^3 - 3(-2)^2 + 8(-2) + 5a = 0$

$\Rightarrow 64 - 16 - 12 - 16 + 5a = 0 \Rightarrow 5a + 20 = 0$

$\Rightarrow 5a = 20$

$\Rightarrow \boxed{a = -4}$

Hence $(x+2)$ is a factor of $f(x)$ when $a = -4$

13. Find the value of k if x – 3 is a factor of $k^2x^3 - kx^2 + 3kx - k$.

Sol:

Let $f(x) = k^2x^3 - kx^2 + 3kx - k$ be the given polynomial from factor theorem if $(x-3)$ is a factor of $f(x)$ then $f(3) = 0$

$\Rightarrow k^2(3)^3 - k(3)^2 + 3k(3) - k = 0$

$\Rightarrow 27k^2 - 9k + 9k - k = 0$

$\Rightarrow 27k^2 - k = 0 \Rightarrow k(27k - 1) = 0$

$\Rightarrow k = 0$ and $27k - 1 = 0 \Rightarrow k = \frac{1}{27}$

Hence, $(x-3)$ is a factor of $f(x)$ when $k = 0$ or $k = \frac{1}{27}$

14. Find the values of a and b, if $x^2 - 4$ is a factor of $ax^4 + 2x^3 - 3x^2 + bx - 4$.

Sol:

Let $f(x) = ax^4 + 2x^3 - 3x^2 + bx - 4$ and $g(x) = x^2 - 4$

We have $g(x) = x^2 - 4 = (x-2)(x+2)$

Given $g(x)$ is a factor of $f(x)$.

$\Rightarrow (x-2)$ and $(x+2)$ are factors of $f(x)$

From factor theorem,

If $(x-2)$ and $(x+2)$ are factors of $f(x)$ then $f(2) = 0$ and $f(-2) = 0$ respectively

$\Rightarrow f(2) = 0 \Rightarrow a(2)^4 + 2(2)^3 - 3(2)^2 + b(2) - 4 = 0$

$\Rightarrow 16a+16-12+2b-4=0$

$\Rightarrow 16a+2b=0 \Rightarrow 2(8a+b)=0$

$\Rightarrow \boxed{8a+b=0}$ ....(1)

Similarly $f(-2)=0 \Rightarrow a(-2)^4+2(-2)^3-3(-2)^2+b(-2)-4=0$

$\Rightarrow 16a-16-12-2b-4=0$

$\Rightarrow 16a-2b-32=0 \Rightarrow 2(8a+b)=32$

$\Rightarrow \boxed{8a-b=16}$ ....(2)

Adding equation (1) and (2)

$8a+b+8a-b=16 \Rightarrow 16a=16 \Rightarrow \boxed{a=1}$

Put $a=1$ in equation (1)

$\Rightarrow 8\times1+b=0 \Rightarrow \boxed{b=-8}$

Hence, $a=1$ and $b=-8$

15. Find α and β, if x + 1 and x + 2 are factors of $x^3+3x^2-2\alpha x+\beta$.

Sol:

Let $f(x)=x^3+3x^2-2\alpha x+\beta$ be the given polynomial from factor theorem, if $(x+1)$ and $(x+2)$ are factors of $f(x)$ then $f(-1)=0$ and $f(-2)=0$

$\Rightarrow f(-1)=0 \Rightarrow (-1)^3+3(-1)^2-2\alpha(-1)+\beta=0$

$\Rightarrow -1+3+2\alpha+\beta=0 \Rightarrow 2\alpha+\beta+2=0$ ....(1)

Similarly,

$f(-2)=0 \Rightarrow (-2)^3+3(-2)^2-2\alpha(-2)+\beta=0$

$\Rightarrow -8+12+4\alpha+\beta=0 \Rightarrow 4\alpha+\beta+4=0$ ....(2)

Subtract equation (1) from (2)

$\Rightarrow 4\alpha+\beta+4-(2\alpha+\beta+2)=0-0$

$\Rightarrow 4\alpha+\beta+4-2\alpha-\beta-2=0$

$\Rightarrow 2\alpha+2=0 \Rightarrow 2\alpha=-2 \Rightarrow \boxed{\alpha=-1}$

Put $\alpha=-1$ in equation (1)

$\Rightarrow 2(-1)+\beta+2=0 \Rightarrow -2+\beta+2=0 \Rightarrow \boxed{\beta=0}$

Hence, $\alpha=-1$ and $\beta=0$

16. Find the values of p and q so that $x^4 + px^3 + 2x^2 - 3x + q$ is divisible by $(x^2 - 1)$

Sol:

Let $f(x) = x^4 + px^3 + 2x^2 - 3x + q$ be the given polynomial.

and let $g(x) = x^2 - 1 = (x-1)(x+1)$

Clearly, $(x-1)$ and $(x+1)$ are factors of $g(x)$

Given $g(x)$ is a factor of $f(x)$

$\Rightarrow (x-1)$ and $(x+1)$ are factors of $f(x)$

From factor theorem,

If $(x-1)$ and $(x+1)$ are factors of $f(x)$ then $f(1) = 0$ and $f(-1) = 0$ respectively

$\Rightarrow f(1) = 0 \Rightarrow 1^4 + p(1)^3 + 2(1)^2 - 3(1) + q = 0$

$\Rightarrow 1 + p + 2 - 3 + q = 0 \Rightarrow p + q = 0$ ....(1)

$\Rightarrow f(-1) = 0 \Rightarrow (-1)^4 + p(-1)^3 + 2(-1)^2 - 3(-1) + q = 0$

$\Rightarrow 1 + (-p) + 2 + 3 + q = 0 \Rightarrow q - p + 6 = 0$ ....(2)

Adding equation (1) and (2)

$\Rightarrow p + q + q - p + 6 = 0 \Rightarrow 2q + 6 = 0 \Rightarrow 2q = -6 \Rightarrow \boxed{q = -3}$

Put $q = -3$ in equation (1)

$\Rightarrow p - 3 = 0 \Rightarrow \boxed{p = 3}$

Hence $x^2 - 1$ is divisible by $f(x)$ when $p = 3, q = -3$

17. Find the values of a and b so that (x + 1) and (x – 1) are factors of $x^4 + ax^3 - 3x^2 + 2x + b$.

Sol:

Let $f(x) = x^4 + ax^3 - 3x^2 + 2x + b$ be the given polynomial.

From factor theorem; if $(x+1)$ and $(x-1)$ are factors of $f(x)$ then $f(-1) = 0$ and $f(1) = 0$ respectively.

$\Rightarrow f(-1) = 0 \Rightarrow (-1)^4 + a(-1)^3 - 3(-1)^2 + 2(-1) + b = 0$

$\Rightarrow 1 - a - 3 - 2 + b = 0 \Rightarrow b - a - 4 = 0$ ....(1)

$\Rightarrow f(1) = 0 \Rightarrow (1)^4 + a(1)^3 - 3(1)^2 + 2(1) + b = 0$

$\Rightarrow 1 + a - 3 + 2 + b = 0 \Rightarrow a + b = 0$ ....(2)

Adding equation (1) and (2)

$\Rightarrow b-a-4+a+b=0+0$

$\Rightarrow 2b-4=0 \Rightarrow 2b=4 \Rightarrow b=\frac{\cancel{4}^{2}}{\cancel{2}} \Rightarrow \boxed{b=2}$

Substitute $b=2$ in equation (2)

$\Rightarrow a+2=0 \Rightarrow \boxed{a=-2}$

Hence, $a=-2$ and $b=2$

18. If $x^3+ax^2-bx+10$ *is divisible by* x^2-3x+2, find the values of a and b.

Sol:

Let $f(x)=x^3+ax^2-bx+10$ and $g(x)=x^2-3x+2$ be the given polynomials.

We have $g(x)=x^2-3x+2=(x-2)(x-1)$

$\Rightarrow$ Clearly, $(x-1)$ and $(x-2)$ are factors of $g(x)$

Given that $f(x)$, is divisible by $g(x)$

$\Rightarrow g(x)$ is a factor of $f(x)$

$\Rightarrow (x-2)$ and $(x-1)$ are factors of $f(x)$

From factor theorem,

If $(x-1)$ and $(x-2)$ are factors of $f(x)$ then $f(1)=0$ and $f(2)=0$ respectively.

$\Rightarrow f(1)=0 \Rightarrow (1)^3+a(1)^2-b(1)+10=0$

$\Rightarrow 1+a-b+10=0 \Rightarrow a-b+11=0$ ....(1)

$\Rightarrow f(2)=0 \Rightarrow (2)^3+a(2)^2-b(2)+10=0$

$\Rightarrow 8+4a-2b+10=0$

$\Rightarrow 4a-2b+18=0$

$\Rightarrow 2(2a-b+9)=0$

$\Rightarrow 2a-b+9=0$ ....(2)

Subtract equation (1) from (2)

$\Rightarrow 2a-b+9-(a-b+11)=0-0$

$\Rightarrow 2a-b+9-a+b-11=0 \Rightarrow a-2=0 \Rightarrow \boxed{a=2}$

Put $a=2$ in equation (1)

$\Rightarrow a-b+11=0 \Rightarrow 2-b+11=0 \Rightarrow 13-b=0 \Rightarrow \boxed{b=13}$

Hence, $a=2$ and $b=13$

$\therefore x^3+ax^2-bx+10$ is divisible by x^2-3x+2 when $a=2$ and $b=13$

19. If both x + 1 and x – 1 are factors of $ax^3 + x^2 - 2x + b$, find the values of a and b.

Sol:

Let $f(x) = ax^3 + x^2 - 2x + b$ be the given polynomial.

Given $(x+1)$ and $(x-1)$ are factor of $f(x)$.

From factor theorem,

If $(x+1)$ and $(x-1)$ are factors of $f(x)$ then $f(-1) = 0$ and $f(1) = 0$ respectively.

$\Rightarrow f(-1) = 0 \Rightarrow a(-1)^3 + (-1)^2 - 2(-1) + b = 0$

$\Rightarrow -a + 1 + 2 + b = 0 \Rightarrow b - a + 3 = 0$ ....(1)

$\Rightarrow f(1) = 0 \Rightarrow a(1)^3 + (1)^2 - 2(1) + b = 0$

$\Rightarrow a + 1 - 2 + b = 0 \Rightarrow b + a - 1 = 0$ ....(2)

Adding equation (1) and (2)

$\Rightarrow b - a + 3 + b + a - 1 = 0 + 0$

$\Rightarrow 2b + 2 = 0 \Rightarrow 2b = -2 \Rightarrow \boxed{b = -1}$

Put $b = -1$ in equation (1)

$\Rightarrow -1 - a + 3 = 0 \Rightarrow 2 - a = 0 \Rightarrow \boxed{a = 2}$

Hence the values of a, b are 2, -1 respectively.

20. What must be added to $x^3 - 3x^2 - 12x + 19$ so that the result is exactly divisibly by $x^2 + x - 6$?

Sol:

Let $p(x) = x^3 - 3x^2 - 12x + 19$ and $q(x) = x^2 + x - 6$.

By division algorithm, when $p(x)$ is divided by $q(x)$, the remainder is a linear expression in x.

So, let $r(x) = ax + b$ is added to $p(x)$ so that $p(x) + r(x)$ is divisible by $q(x)$.

Let $f(x) = p(x) + r(x)$

$\Rightarrow f(x) = x^3 - 3x^2 - 12x + 19 + ax + b$

$\Rightarrow \boxed{f(x) = x^3 - 3x^2 + x(a - 12) + b + 19}$

We have,

$q(x) = x^2 + x - 6 = (x+3)(x-3)$

Clearly, $q(x)$ is divisible by $(x-2)$ and $(x+3)$

i.e., $(x-2)$ and $(x+3)$ are factors of $q(x)$

We have,

$f(x)$ is divisible by $q(x)$

$\Rightarrow (x-2)$ and $(x+3)$ are factors of $f(x)$

From factors theorem,

If $(x-2)$ and $(x+3)$ are factors of $f(x)$ then $f(2)=0$ and $f(-3)=0$ respectively.

$\Rightarrow f(2)=0 \Rightarrow 2^3-3(2)^2+2(a-12)+b+19=0$

$\Rightarrow 8-12+2a-24+b+19=0$

$\Rightarrow 2a+b-9=0$ ....(1)

Similarly

$f(-3)=0 \Rightarrow (-3)^3-3(-3)^2+(-3)(a-12)+b+19=0$

$\Rightarrow -27-27-3a+36+b+19=0$

$\Rightarrow b-3a+1=0$ ....(2)

Subtract equation (1) from (2)

$b-3a+1-(2a+b-9)=0-0$

$\Rightarrow b-3a+1-2a-6+9=0$

$\Rightarrow -5a+10=0 \Rightarrow 5a=10 \Rightarrow \boxed{a=2}$

Put $a=2$ in equation (2)

$\Rightarrow b-3\times 2+1=0 \Rightarrow b-6+1=0 \Rightarrow b-5=0 \Rightarrow \boxed{b=5}$

$\therefore r(x)=ax+b \Rightarrow \boxed{r(x)=2x+5}$

Hence, $x^3-3x^2-12x+19$ is divisible by x^2+x-6 when $2x+5$ is added to it.

21. What must be subtracted from $x^3-6x^2-15x+80$ so that the result is exactly divisible by x^2+x-12 ?

Sol:

Let $p(x)=x^3-6x^2-15x+80$ and $q(x)=x^2+x-12$

By division algorithm, when $p(x)$ is divided by $q(x)$ the remainder is a linear expression in *x*.

So, let $r(x)=ax+b$ is subtracted from $p(x)$, So that $p(x)-r(x)$ is divisible by $q(x)$

Let $f(x)=p(x)-r(x)$

Clearly, $(3x-2)$ and $(x+3)$ are factors of $q(x)$

Therefore, $f(x)$ will be divisible by $q(x)$ if $(3x-2)$ and $(x+3)$ are factors of $f(x)$

i.e., from factor theorem,

$f\left(\frac{2}{3}\right)=0$ and $f(-3)=0$ $\left[\because 3x-2=0 \Rightarrow x=\frac{2}{3} \text{ and } x+3=0 \Rightarrow x=-3\right]$

$\Rightarrow f\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^2+\frac{2}{3}(a-22)+b+9=0$

$\Rightarrow 3\times\frac{8}{27}+\frac{4}{9}+\frac{2}{3}a-\frac{44}{3}+b+9=0$

$\Rightarrow \frac{12}{9}+\frac{2}{3}a-\frac{44}{3}+b+9=0$

$\Rightarrow \frac{12+6a-132+9b+81}{9}=0$

$\Rightarrow 6a+9b-39=0$

$\Rightarrow 3(2a+3b-13)=0 \Rightarrow 2a+3b-13=0$ ....(1)

Similarly,

$f(-3)=0 \Rightarrow 3(-3)^3+(-3)^2+(-3)(a-22)+b+9=0$

$\Rightarrow -81+9-3a+66+b+9=0$

$\Rightarrow b-3a+3=0$

$\Rightarrow 3(b-3a+3)=0 \Rightarrow 3b-9a+9=0$ ....(2)

Subtract equation (1) from (2)

$\Rightarrow 3b-9a+9-(2a+3b-13)=0-0$

$\Rightarrow 3b-9a+9-2a-3b+13=0$

Put $a=4$ in equation (2)

$\Rightarrow 4\times 4-b-20=0$

$\Rightarrow 16-b-20=0 \Rightarrow -b-4=0 \Rightarrow \boxed{b=-4}$

Putting the value of a and b in $r(x)=ax+b,$

We get $\boxed{r(x)=4x-4}$

Hence, $p(x)$ is divisible by $q(x)$, if $r(x)=4x-4$ is subtracted from it.

22. What must be added to $3x^3+x^2-22x+9$ so that the result is exactly divisible by $3x^2+7x-6$?

Sol:

Let $p(x)=3x^3+x^2-22x+9$ and $q(x)=3x^2+7x-6$

By division algorithm,

When $p(x)$ is divided by $q(x)$, the remainder is a linear equation in x.

So, let $r(x)=ax+b$ is added to $p(x)$, so that $p(x)+r(x)$ is divisible by $q(x)$

Let $f(x)=p(x)+r(x)$

$\Rightarrow f(x) = 3x^3 + x^2 - 22x + 9 + (ax+b)$

$\Rightarrow \boxed{f(x) = 3x^3 + x^2 + x(a-22) + b + 9}$

We have,

$q(x) = 3x^2 + 7x - 6 = 3x^2 + 9x - 2x - 6 = 3x(x+3) - 2(x+3)$

$= (3x-2)(x+3)$

$= f(x) = x^3 - 6x^2 - 15x + 80 - (ax+b)$

$\Rightarrow \boxed{f(x) = x^3 - 6x^2 - x(a+15) + 80 - b}$

We have,

$q(x) = x^2 + x - 12 = x^2 + 4x - 3x - 18 = x(x+4) - 3(x+4)$

$= (x-3)(x+4)$

Clearly, $(x-3)$ and $(x+4)$ are factors of $q(x)$.

Therefore, $f(x)$ will be divisible by $q(x)$ if $(x-3)$ and $(x+4)$ are factor of $f(x)$.

i.e., from factors theorem,

$f(3) = 0$ and $f(-4) = 0$ $\quad [\because x-3=0 \Rightarrow x=3$ and $x+4=0 \Rightarrow x=-4]$

$\Rightarrow f(3) = 0 \Rightarrow (3)^3 - 6(3)^2 - 3(a+15) + 80 - b = 0$

$\Rightarrow 27 - 54 - 3a - 45 + 80 - b = 0$

$\Rightarrow 8 - 3a - b = 0 \quad ....(1)$

$\Rightarrow f(-4) = 0 \Rightarrow (-4)^3 - 6(-4)^2 - (4)(a+15) + 80 - b = 0$

$\Rightarrow -64 - 96 - 4a + 60 + 80 - b = 0$

$\Rightarrow 4a - b - 20 = 0 \quad ....(2)$

Subtract equation (1) from (2)

$\Rightarrow 4a - b - 20 - (8 - 3a - b) = 0 - 0$

$\Rightarrow 4a - b - 20 - 8 + 3a + b = 0$

$\Rightarrow 7a - 28 = 0 \Rightarrow 7a = 28 \Rightarrow \boxed{a = 4}$

$\Rightarrow -11a + 22 = 0 \Rightarrow 11a = 22 \Rightarrow \boxed{a = 2}$

Put $a = 2$ in equation (1)

$\Rightarrow 2 \times 2 + 3b - 13 = 0$

$\Rightarrow 4 + 3b - 13 = 0 \Rightarrow 3b - 9 = 0 \Rightarrow 3b = 9 \Rightarrow \boxed{b = 3}$

Putting the value of a and b in $r(x) = ax + b$,

We get, $\boxed{r(x) = 2x + 3}$

Hence, $3x^3 + x^2 - 22x + 9$ will be divisible by $3x^2 + 7x - 6$, if $2x + 3$ is added to it.

23. If x – 2 is a factor of each of the following two polynomials, find the values of a in each case:

(i) x^3-2ax^2+ax-1

(ii) $x^5-3x^4-ax^3+3ax^2+2ax+4$

Sol:

(i) Let $f(x)=x^3-2ax^2+ax-1$ be the given polynomial.

From factor theorem,

If $(x-2)$ is a factor of $f(x)$ then $f(2)=0$ $[\because x-2=0\Rightarrow x=2]$

$\Rightarrow f(2)=0\Rightarrow 2^3-2a(2)^2+a(2)-1=0$

$\Rightarrow 8-8a+2a-1=0$

$\Rightarrow 7-6a=0\Rightarrow 6a=7\Rightarrow \boxed{a=\frac{7}{6}}$

Hence, $(x-2)$ is a factor of $f(x)$ when $a=\frac{7}{6}$

(ii) Let $f(x)=x^5-3x^4-ax^3+3ax^2+2ax+4$ be the given polynomial.

From factor theorem

If $x-2$ is a factor of $f(x)$ then $f(2)=0$ $[\because x-2=0\Rightarrow x=2]$

$\Rightarrow f(2)=0\Rightarrow 2^5-3(2)^4-a(2)^3+3a(2)^2+2a(2)+4=0$

$\Rightarrow 32-48-8a+12a+4a+4=0$

$\Rightarrow 8a-12=0\Rightarrow 8a=12\Rightarrow \boxed{a=\frac{3}{2}}$

Hence, $(x-2)$ is a factor of $f(x)$ when $a=\frac{3}{2}$

24. In each of the following two polynomials, find the value of a, if x – a is a factor:

(i) $x^6-ax^5+x^4-ax^3+3x-a+2$

(ii) $x^5-a^2x^3+2x+a+1$

Sol:

(i) Let $f(x)=x^6-ax^5+x^4-ax^3+3x-a+2$ be the given polynomial.

From factor theorem,

If $(x-a)$ is a factor of $f(x)$ then $f(a)=0$ $[\because x-a=0\Rightarrow x=a]$

$\Rightarrow f(a)=0\Rightarrow a^6-a(a)^5+a^4-a(a)^3+3(a)-a+2=0$

$\Rightarrow a^6-a^6+a^4-a^4+3a-a+2=0$

$\Rightarrow 2a+2=0\Rightarrow 2a=-2\Rightarrow \boxed{a=-1}$

Hence, $(x-a)$ is a factor of $f(x)$, if $\boxed{a=-1}$

(ii) Let $f(x) = x^5 - a^2x^3 + 2x + a + 1$ be the given polynomial.

From factor theorem,

If $(x-a)$ is a factor of $f(x)$ then $f(a) = 0$ $[\because x - a = 0 \implies x = a]$

$\Rightarrow f(a) = 0 \Rightarrow a^5 - a^2(a)^3 + 2(a) + a + 1 = 0$

$\Rightarrow a^5 - a^5 + 2a + a + 1 = 0$

$\Rightarrow 3a + 1 = 0 \Rightarrow 3a = -1 \Rightarrow \boxed{a = -\frac{1}{3}}$

Hence, $(x-a)$ is a factor of $f(x)$, if $a = -\frac{1}{3}$

25. In each of the following two polynomials, find the value of a, if x + a is a factor:

(i) $x^3 + ax^2 - 2x + a + 4$

(ii) $x^4 - a^2x^2 + 3x - a$

Sol:

(i) Let $f(x) = x^3 + ax^2 - 2x + a + 4$ be the given polynomial.

From factor theorem,

If $(x+a)$ is a factor of $f(x)$ then $f(-a) = 0$ $[\because x + a = 0 \Rightarrow x = -a]$

$\Rightarrow f(-a) = 0 \Rightarrow (-a)^3 + a(-a)^2 - 2(-a) + a + 4 = 0$

$\Rightarrow -a^3 + a^3 + 2a + a + 4 = 0$

$\Rightarrow 3a + 4 = 0 \Rightarrow 3a = -4 \Rightarrow \boxed{a = -\frac{4}{3}}$

Hence, $(x+a)$ is a factor of $f(x)$, if $a = -\frac{4}{3}$

(ii) Let $f(x) = x^4 - a^2x^2 + 3x - a$ be the given polynomial

From factor theorem,

If $(x+a)$ is a factor of $f(x)$ then $f(-a) = 0$ $[\because x + a = 0 \Rightarrow x = -a]$

$\Rightarrow f(-a) = 0 \Rightarrow (-a)^4 - a^2(-a)^2 + 3(-a) - a = 0$

$\Rightarrow a^4 - a^4 - 3a - a = 0$

$\Rightarrow -3a - a = 0 \Rightarrow -4a = 0 \Rightarrow \boxed{a = 0}$

Hence, $(x+a)$ is a factor of $f(x)$, if $a = 0$

Exercise – 6.5

Using factor theorem, factorize each of the following polynomials:

1. $x^3+6x^2+11x+6$

Sol:

Let $f(x)=x^3+6x^2+11x+6$ be the given polynomial.

The constant term in $f(x)$ is 6 and factors of 6 are $\pm1, \pm2, \pm3$ and ±6

Putting $x=-1$ in $f(x)$, we have

$f(-1)=(-1)^3+6(-1)^2+11(-1)+6=-1+6-11+6=0$

$\therefore (x+1)$ is a factor of $f(x)$

Similarly, $(x+2)$ and $(x+3)$ are factors of $f(x)$.

Since $f(x)$ is a polynomial of degree 3. So, it cannot have more than three linear factors.

$\therefore f(x)=k(x+1)(x+2)(x+3)$

$\Rightarrow x^3+6x^2+11x+6=k(x+1)(x+2)(x+3)$

Putting $x=0$ on both sides, we get

$0+0+0+6=k(0+1)(0+2)(0+3)$

$6=k(1)(2)(3)$

$\Rightarrow 6=6k \Rightarrow \boxed{k=1}$

Putting $k=1$ in $f(x)=k(x+1)(x+2)(x+3)$, we get

$\boxed{f(x)=(x+1)(x+2)(x+3)}$

Hence, $x^3+6x^2+11x+6=(x+1)(x+2)(x+3)$

2. x^3+2x^2-x-2

Sol:

Let $f(x)=x^3+2x^2-x-2$

The constant term in $f(x)$ is equal to -2 and factors of -2 and $\pm1, \pm2$.

Putting $x=1$ in $f(x)$, we have

$f(1)=1^3+2(1)^2-1-2=0$

$\therefore (x-1)$ is a factor of $f(x)$

Similarly, $(x+1), (x+2)$ are factors of $f(x)$.

Since $f(x)$ is a polynomial of degree 3. So, it cannot have more than three linear factors.

$\therefore f(x)=k(x-1)(x+1)(x+2)$

$\Rightarrow x^3+2x^2-x-2=k(x-1)(x+1)(x+2)$

Putting $x=0$ on both sides, we get $0+0-0-2=k(-1)(+1)(+2)$

$-2=-2k \Rightarrow \boxed{k=1}$

Putting $k=1$ in $f(x)=k(x-1)(x+1)(x+2)$, we get

$\boxed{f(x)=(x-1)(x+1)(x+2)}$

Hence, $x^3+2x^2-x-2=(x-1)(x+1)(x+2)$

3. $x^3-6x^2+3x+10$

Sol:

Let $f(x)=x^3-6x^2+3x+10$

The constant term in $f(x)$ is equal to 10 and factors of 10 are $\pm1, \pm2, \pm5$ and ±10

Putting $x=-1$ in $f(x)$, we have

$f(-1)=(-1)^3-6(-1)^2+3(-1)+10=-1-6-3+10=0$

$\therefore (x+1)$ is a factor of $f(x)$

Similarly, $(x-2)$ and $(x-5)$ are factors of $f(x)$

Since $f(x)$ is a polynomial of degree 3. So, it cannot have more than three linear factors.

$\therefore f(x)=k(x+1)(x-2)(x-5)$

Putting $x=0$ on both sides, we get

$\Rightarrow x^3-6x^2+3x+10=k(x+1)(x-2)(x-5)$

$0-0+0+10=k(1)(-2)(-5)$

$\Rightarrow 10=10k \Rightarrow \boxed{k=1}$

Putting $k=1$ in $f(x)=k(x+1)(x-2)(x-5)$, we get

$\boxed{f(x)=(x+1)(x-2)(x-5)}$

Hence, $x^3-6x^2+3x+10=(x+1)(x+2)(x-5)$

4. $x^4 - 7x^3 + 9x^2 + 7x - 10$

Sol:

Let $f(x) = x^4 - 7x^3 + 9x^2 + 7x - 10$

The constant term in $f(x)$ is equal to -10 and factors of -10 are $\pm 1, \pm 2, \pm 5$ and ± 10

Putting $x = 1$ in $f(x)$, we have

$f(1) = 1^4 - 7(1)^3 + 9(1)^2 + 7(1) - 10 = 1 - 7 + 9 + 7 - 10 = 0$

$\therefore (x-1)$ is a factor of $f(x)$

Similarly $(x+1), (x-2), (x-5)$ are also factors of $f(x)$

Since $f(x)$ is a polynomial of degree 4. So, it cannot have more than four linear factors

$\therefore f(x) = k(x-1)(x+1)(x-2)(x-5)$

$\Rightarrow x^4 - 7x^3 + 9x^2 + 7x - 10 = k(x-1)(x+1)(x-2)(x-5)$

Putting $x = 0$ on both sides, we get

$\Rightarrow 0 - 0 + 0 + 0 - 10 = k(-1)(1)(-2)(-5)$

$\Rightarrow -10 = k(-10)$

$\Rightarrow \boxed{k = 1}$

Putting $k = 1$ in $f(x) = k(x-1)(x+1)(x-2)(x-5)$, we get

$\boxed{f(x) = (x+1)(x-1)(x-2)(x-5)}$

Hence, $x^4 - 7x^3 + 9x^2 + 7x - 10 = (x+1)(x-1)(x-2)(x-5)$

5. $x^4 - 2x^3 - 7x^2 + 8x + 12$

Sol:

Let $f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

The constant term in $f(x)$ is equal to $+12$ and factors of $+12$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$ and ± 12

Putting $x = -1$ in $f(x)$, we have

$f(-1) = (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12$

$= 1 + 2 - 7 - 8 + 12 = 0$

$\therefore (x+1)$ is a factor of $f(x)$

Similarly $(x+2), (x-2), (x-3)$ are also factors of $f(x)$

Since $f(x)$ is a polynomial of degree 4. So, it cannot have more than four linear factors.

$\therefore f(x) = k(x+1)(x+2)(x-2)(x-3)$

$\Rightarrow x^4 - 2x^3 - 7x^2 + 8x + 12 = k(x+1)(x+2)(x-2)(x-3)$

Putting $x = 0$ on both sides, we get

$\Rightarrow 0 - 0 - 0 + 0 + 12 = k(1)(2)(-2)(-3)$

$\Rightarrow \cancel{12} = k(\cancel{12})$

$= \boxed{k = 1}$

Putting $k = 1$ in $f(x) = k(x+1)(x+2)(x-2)(x-3)$, we get

$\boxed{f(x) = (x+1)(x+2)(x-2)(x-3)}$

Hence, $x^4 - 2x^3 - 7x^2 + 8x + 12 = (x+1)(x+2)(x-2)(x-3)$

6. $x^4 + 10x^3 + 35x^2 + 50x + 24$

Sol:

Let $f(x) = x^4 + 10x^3 + 35x^2 + 50x + 24$

The constant term in $f(x)$ is equal to $+24$ and factors of $+24$ are

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

Putting $x = -1$ in $f(x)$, we have

$f(-1) = (-1)^4 + 10(-1)^3 + 35(-1)^2 + 50(-1) + 24$

$= 1 - 10 + 35 - 50 + 24 = 0$

$\therefore (x+1)$ is a factor of $f(x)$

Similarly, $(x+2), (x+3)$ and $(x+4)$ are also factors of $f(x)$.

Since $f(x)$ is polynomial of degree 4. So, it cannot have more than four linear factors.

$\therefore f(x) = k(x+1)(x+2)(x+3)(x+4)$

$\Rightarrow x^4 + 10x^3 + 35x^2 + 50x + 24 = k(x+1)(x+2)(x+3)(x+4)$

Putting $x = 0$ on both sides, we get

$\Rightarrow 0 + 0 + 0 + 0 + 24 = k(1)(2)(3)(4)$

$\Rightarrow 24 = 24k \Rightarrow \boxed{k = 1}$

Putting $k = 1$ in $f(x) = k(x+1)(x+2)(x+3)(x+4)$, we get

$\boxed{f(x) = (x+1)(x+2)(x+3)(x+4)}$

Hence, $x^4 + 10x^3 + 35x^2 + 50x + 24 = (x+1)(x+2)(x+3)(x+4)$

7. $2x^4-7x^3-13x^2+63x-45$

Sol:

Let $f(x)=2x^4-7x^3-13x^2+63x-45$

The factors of the constant term -45 are $\pm1,\pm3,\pm5,\pm9,\pm15$ and ±45

The factor of the coefficient of x^4 is 2. Hence possible rational roots of $f(x)$ are

$\pm1,\pm3,\pm5,\pm9,\pm15,\pm45,\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\pm\frac{9}{2},\pm\frac{15}{2},\pm\frac{45}{2}$

We have,

$f(1)=2(1)^4-7(1)^3-13(1)^2+63(1)-45$

$=2-7-13+63-45=0$

And $f(3)=2(3)^4-7(3)^3-13(3)^2+63(3)-45$

$=162-189-117+189-45=0$

And $f(3)=2(3)^4-7(3)^3-13(3)^2+63(3)-45$

$=162-189-117+189-45=0$

So, $(x-1)$ and $(x-3)$ are factors of $f(x)$

$\Rightarrow(x-1)(x-3)$ is also a factor of $f(x)$

$\Rightarrow(x^2-4x+3)$ is a factor of $f(x)$

Let us now divide

$f(x)=2x^4-7x^3-13x^2+63x-45$ by (x^2-4x+3) to get thee other factors of $f(x)$.

By long division, we have

$$
\begin{array}{r|l}
x^2-4x+3 & 2x^4-7x^3-13x^2+63x-45 \;(2x^2+x-15 \\
 & 2x^4-8x^3+6x \\
 & -\quad+\quad- \\
\hline
 & x^3-19x^2+63x \\
 & x^3-4x^2+3x \\
 & -\quad+\quad- \\
\hline
 & -15x^2+60x-45 \\
 & -15x^2+60x-45 \\
 & +\quad-\quad+ \\
\hline
 & 0
\end{array}
$$

$\therefore 2x^4-7x^3-13x^2+63x-45=(x^2-4x+3)(2x^2+x-15)$

$\Rightarrow 2x^4-7x^3-13x^2+63x-45=(x-1)(x-3)(2x^2+x-15)$

Now,

$2x^2+x-15=2x^2+6x-5x-15=2x(x+3)-5(x+3)$

$=(2x-5)(x+3)$

Hence $2x^4-7x^3-13x^2+63x-45=(x-1)(x-3)(x+3)(2x-5)$

8. $3x^3-x^2-3x+1$

Sol:

Let $f(x)=3x^3-x^2-3x+1$

The factors of the constant term $+1$ is ± 1.

The factors of the coefficient of x^3 is 3.

Hence possible rational roots of $f(x)$ are $\pm 1, \pm\frac{1}{3}$

We have,

$f(1)+3(1)^3-(1)^2-3(1)+1=3-1-3+1=0$

So, $(x-1)$ is a factor of $f(x)$.

Let us now divide $f(x)=3x^3-x^2-3x+1$ by $(x-1)$ to get the other factors.

By long division method, we have

$$
\begin{array}{l}
x-1\overline{)3x^3-x^2-3x+1}(3x^2+2x-1 \\
\quad 3x^3-3x^2 \\
\quad \underline{-\quad +} \\
\qquad 2x^2-3x \\
\quad 2x^2-2x \\
\quad \underline{-\quad +} \\
\quad -x+1 \\
\quad -x+1 \\
\quad \underline{+\quad -} \\
\qquad 0
\end{array}
$$

$\therefore 3x^4-x^2-3x+1=(x-1)(3x^2+2x-1)$

Now,

$3x^2+2x-1=3x^2+3x-x-1=3x(x+1)-1(x+1)=(3x-1)(x+1)$

Hence, $3x^3-x^2-3x+1=(x-1)(x+1)(3x-1)$

9. $x^3-23x^2+142x-120$

Sol:

Let $f(x)=x^3-23x^2+142x-120$

The constant term in $f(x)$ is equal to –120 and factors of –120 are

$\pm1,\pm2,\pm3,\pm4,\pm5,\pm6,\pm8,\pm10,\pm12,\pm15,\pm20,\pm24,\pm30,\pm40,\pm60,\pm120$.

Putting $x=1$ we have

$f(1)=1^3-23(1)^2+142(1)-120=1-23+142-120=0$

So, $(x-1)$ is a factor of $f(x)$.

Let us now divide $f(x)=x^3-23x^2+142x-120$ by $(x-1)$ to get the other factors.

By long division, we have

$$
\begin{array}{l}
x-1\overline{)x^3-23x^2+142x-120}(x^2-22x+120 \\
\quad x^3-x^2 \\
\quad - \quad + \\
\hline
\quad -22x^2+142x \\
\quad -22x^2+22x \\
\quad + \qquad - \\
\hline
\quad 120x-120 \\
\quad 120x-120 \\
\quad - \qquad + \\
\hline
\qquad 0
\end{array}
$$

$\therefore x^3-23x^2+142x-120=(x-1)(x^2-22x+120)$

Now,

$x^2-22x+120=x^2-10x-12x+120=x(x-10)-12(x-10)$

$=(x-12)(x-10)$

Hence, $x^3-23x^2+142x-120=(x-1)(x-10)(x-12)$

10. y^3-7y+6

Sol:

Let $f(y)=y^3-7y+6$

The constant term in $f(y)$ is $+6$ and factors of $+6$ are $\pm1,\pm2,\pm3$ and ±6

Putting $y=1$ we have

$f(1)=1^3-7(1)+6=1-7+6=0$

$\therefore \left(y-1\right)$ is a factor of $f\left(y\right)$

Similarly it can be verified that $\left(y-2\right)$ and $\left(y+3\right)$ are also factors of $f\left(y\right)$

Since $f\left(y\right)$ is a polynomial of degree 3. So, it cannot have more than 3 linear factors.

$\therefore f\left(y\right)=k\left(y-1\right)\left(y-2\right)\left(y+3\right)$

$\Rightarrow y^3+7y+6=k\left(y-1\right)\left(y-2\right)\left(y+3\right)$

Putting $y=0$ on both sides, we get

$\Rightarrow 0-0+6=k\left(-1\right)\left(-2\right)\left(3\right)$

$\Rightarrow 6=6k \Rightarrow \boxed{k=1}$

Putting $k=1$ in $f\left(y\right)=k\left(y-1\right)\left(y-2\right)\left(y+3\right)$, we get

$\boxed{f\left(y\right)=\left(y-1\right)\left(y-2\right)\left(y+3\right)}$

Hence, $y^3-7y+6=\left(y-1\right)\left(y-2\right)\left(y+3\right)$

11. $x^3-10x^2-53x-42$

Sol:

Let $f\left(x\right)=x^3-10x^2-53x-42$

The constant term in $f\left(x\right)$ is -42 and factors of -42 are $\pm1, \pm2, \pm3, \pm6, \pm7, \pm14, \pm21, \pm42$

Putting $x=-1$, we get

$f\left(-1\right)=\left(-1\right)^3-10\left(1\right)^2-53\left(-1\right)-42=-1-10+53-42=0$

So, $\left(x+1\right)$ is a factor of $f\left(x\right)$

Let us now divide $f\left(x\right)=x^3-10x^2-53x-42$ by $\left(x+1\right)$ to get the other factors.

By long division, we have

$$
\begin{array}{l}
x+1\overline{)x^3-10x^2-53x-42}\left(x^2-11x-42\right. \\
\quad x^3+x^2 \\
\quad - \;\; - \\
\hline
\quad -11x^2-53x \\
\quad -11x^2-11x \\
\quad + \qquad + \\
\hline
\quad -42x-42 \\
\quad -42x-42 \\
\quad + \qquad + \\
\hline
\qquad 0
\end{array}
$$

$\therefore x^3-10x^2-53x-42=\left(x+1\right)\left(x^2-11x-42\right)$

Now, $x^2-11x-42=x^2-14x+3x-42=x(x-14)+3(x-14)$

$=(x+3)(x-14)$

Hence, $x^3-10x^2-53x-42=(x+1)(x+3)(x-14)$

12. $y^3-2y^2-29y-42$

Sol:

Let $f(y)=y^3-2y^2-29y-42$

The constant term in $f(y)$ is -42 and factors of -42 are $\pm1,\pm2,\pm3,\pm6,\pm7,\pm14,\pm21,\pm42$

Putting $y=-2$ we get

$f(-2)=(-2)^3-2(-2)^2-29(-2)-42$

$=-8-8+58-42=0$

So, $(y+2)$ is a factor of $f(y)$

Let us now divide $f(y)=y^3-2y^2-29y-42$ by $(y+2)$ to get the other factors

By long division, we get

$$
\begin{array}{l}
y+2\overline{)\,y^3-2y^2-29y-42\,}\left(y^2-4y-21\right. \\
\qquad y^3+2y^2 \\
\qquad -\quad + \\
\hline
\qquad\qquad -4y^2-29y \\
\qquad\qquad -4y^2-8y \\
\qquad\qquad +\qquad + \\
\hline
\qquad\qquad -21y-42 \\
\qquad\qquad -21y-42 \\
\qquad\qquad +\qquad + \\
\hline
\qquad\qquad 0
\end{array}
$$

$\therefore y^3-2y^2-29y-42=(y+2)(y^2-4y-21)$

Now,

$y^2-4y-21=y^2-7y+3y-21=y(y-7)+3(y-7)$

$=(y+3)(y-7)$

Hence, $y^3-2y^2-29y-42=(y+2)(y+3)(y-7)$

$(y-2)$ to get the other factors.

By long division, we have

$$y-2\overline{)2y^3-5y^2-19y+42}\left(2y^2-y-21\right.$$
$$2y^3-4y^2$$
$$-\quad+$$
$$\overline{\quad -y^2-19y}$$
$$-y^2-2y$$
$$+\quad-$$
$$\overline{\quad -21y+42}$$
$$-21y+42$$
$$+\quad-$$
$$\overline{\quad 0 \quad}$$

$$\therefore 2y^3-5y^2-19y+42=(y-2)(2y^2-y-21)$$
$$=(y-2)(y+3)(2y-7)$$

13. $2y^3 - 5y^2 - 19y + 42$
Sol:
(y – 2) (y + 3) (2y – 7)

14. $x^3+13x^2+32x+20$
Sol:
Let $f(x)=x^3+13x^2+32x+20$

The constant term in $f(x)$ is 20 and factors of $+20$ are $\pm1,\pm2,\pm4,\pm5,\pm10,\pm20.$

Putting $x=-1,$ we get

$$f(-1)=(-1)^3+13(-1)^2+32(-1)+20$$
$$=-1+13-32+20=0$$

So, $(x+1)$ is a factor of $f(x)$.

Let us now divide $f(x)=x^3+13x^2+32x+20$ by $(x+1)$ to get the remaining factors.

By long division, we have

$$x+1\overline{)x^3+13x^2+32x+20}\left(x^2+12x+20\right.$$
$$x^3+x^2$$
$$-\quad-$$
$$\overline{\quad 12x^2+32x}$$
$$12x^2+32x$$
$$-\quad-$$
$$\overline{\quad 20x+20}$$
$$20x+20$$
$$-\quad-$$
$$\overline{\quad 0 \quad}$$

$\therefore x^3+13x^2+32x+20=(x+1)(x^2+12x+20)$

Now,

$x^2+12x+20=x^2+10x+2x+20=x(x+10)+2(x+10)$

$=(x+2)(x+10)$

Hence, $x^3+13x^2+32x+20=(x+1)(x+2)(x+10)$

15. x^3-3x^2-9x-5

Sol:

Let $f(x)=x^3-3x^2-9x-5$

The constant term in $f(x)$ is -5 and factors of -5 are $\pm 1, \pm 5.$

Putting $x=-1,$ we get

$f(-1)=(-1)^3-3(-1)^2-9(-1)-5=-1-3+9-5=0$

So, $(x+1)$ is a factor of $f(x)$.

Let us now divide $f(x)=x^3-3x^2-9x-5$ by $(x+1)$ to get the other factors.

By long division, we have

$$
\begin{array}{r|l}
x+1 & x^3-3x^2-9x-5 \;(x^2-4x-5 \\
 & x^3+x^2 \\
 & -\quad - \\
\hline
 & -4x^2-9x \\
 & -4x^2-4x \\
 & +\quad + \\
\hline
 & -5x-5 \\
 & -5x-5 \\
 & +\quad + \\
\hline
 & 0
\end{array}
$$

$\therefore x^3-3x^2-9x-5=(x+1)(x^2-4x-5)$

Now,

$(x^2-4x-5)=x^2-5x+x-5=x(x-5)+1(x-5)$

$=(x+1)(x-5)$

Hence, $x^3-3x^2-9x-5=(x+1)(x+1)(x-5)$

$=(x+1)^2(x-5)$

16. $2y^3+y^2-2y-1$

Sol:

Let $f(y)=2y^3+y^2-2y-1$

The factors of the constant term of y^3 is 2. Hence possible rational roots are $\pm1,\pm\frac{1}{2}$.

We have,

$f(1)=2(1)^3+(1)^2-2(1)-1=2+1-2-1=0$

So, $(y-1)$ is a factor of $f(y)$

Let us now divide $f(y)=2y^3+y^2-2y-1$ by $(y-1)$ to get the other factors.

By long division, we have

$$
\begin{array}{r|l}
y-1 & 2y^3+y^2-2y-1\,(2y^2+3y+1 \\
 & 2y^3-2y^2 \\
 & -\quad + \\
\hline
 & \quad 3y^2-2y \\
 & \quad 3y^2-2y \\
 & \quad -\quad + \\
\hline
 & \qquad y-1 \\
 & \qquad y-1 \\
 & \qquad -\ + \\
\hline
 & \qquad 0
\end{array}
$$

$\therefore 2y^3-y^2-2y-1=(y-1)(2y^2+3y+1)$

Now,

$2y^2+3y+1=2y^2+2y+y+1=2y(y+1)+1(y+1)$

$=(2y+1)(y+1)$

Hence, $2y^3+y^2-2y-1=(y-1)(y+1)(2y+1)$

17. x^3-2x^2-x+2

Sol:

Let $f(x)=x^3-2x^2-x+2$

The constant term in $f(x)$ is 2 and factors of 2 are $\pm1,\pm2$.

Putting $x=1$, we have

$f(1)=1^3-2(1)^2-1+2=1-2-1+2=0$

So, $(x-1)$ is a factor of $f(x)$

Let us now divide $f(x)=x^3-2x^2-x+2$ by $(x-1)$ to get the remaining factors.

By long division, we have

$$x-1\overline{)x^3-2x^2-x+2}\left(x^2-x-2\right.$$

$$\begin{array}{l}
x^3-x^2 \\
-\quad + \\
\hline
\quad -x^2-x \\
\quad -x^2+x \\
\quad +\quad - \\
\hline
\quad -2x+2 \\
\quad -2x+2 \\
\quad +\quad - \\
\hline
\quad 0
\end{array}$$

$\therefore x^3-2x^2-x+2=(x-1)\left(x^2-x-2\right)$

Now,

$x^2-x-2=x^2-2x+x-2=x(x-2)+1(x-2)$

$=(x+1)(x-2)$

Hence $x^3-2x^2-x+2=(x-1)(x+2)(x-2)$

18. Factorize each of the following polynomials:

(i) $x^3+13x^2+31x-45$ given that $x+9$ is a factor

(ii) $4x^3+20x^2+33x+18$ given that $2x+3$ is a factor

Sol:

(i) Let $f(x)=x^3+13x^2+31x-45$

Given that $(x+9)$ is a factor of $f(x)$

Let us divide $f(x)$ by $(x+9)$ to get the other factors. By long division, we have

$$x+9\overline{)x^3+12x^2+31x-45}\left(x^2+4x-5\right.$$

$$\begin{array}{l}
x^3+9x^2 \\
-\quad - \\
\hline
\quad 4x^2+31x \\
\quad 4x^2+36x \\
\quad -\quad - \\
\hline
\quad -5x-45 \\
\quad -5x-45 \\
\quad +\quad + \\
\hline
\quad 0
\end{array}$$

$\therefore f(x)=x^3+13x^2+31x-45$

$\Rightarrow f(x)=(x+9)\left(x^2+4x-5\right)$

Now,

$x^2+4x-5=x^2+5x-x-5=x(x+5)-1(x+5)$

$=(x-1)(x+5)$

$\Rightarrow f(x)=(x+9)(x+5)(x-1)$

$\therefore x^3+13x^2+31x-45=(x-1)(x+5)(x+9)$

(ii) Let $f(x)=4x^3+20x^2+33x+18$

Given that $2x+3$ is a factor of $f(x)$

Let us divide $f(x)$ by $(2x+3)$ to get the other factors. By long division, we have

$$
\begin{array}{r|l}
2x+3 & 4x^3+20x^2+33x+18 \;(\; 2x^2+7x+6 \\
 & 4x^3+6x^2 \\
 & -\quad\; - \\
\hline
 & \qquad 14x^2+33x \\
 & \qquad 14x^2+21x \\
 & \qquad +\qquad - \\
\hline
 & \qquad\quad 12x+18 \\
 & \qquad\quad 12x+18 \\
 & \qquad\quad -\quad\; - \\
\hline
 & \qquad\quad 0
\end{array}
$$

Now,

$4x^3+20x^2+33x+18=(2x+3)(2x^2+7x+6)$

We have,

$2x^2+7x+6=2x^2+4x^2+3x+6=2x(x+2)+3(x+2)$

$=(2x+3)(x+2)$

$\Rightarrow 4x^2+20x^2+33x+18=(2x+3)(2x+3)(x+2)$

$=(2x+3)^2(x+2)$

Hence, $4x^3+20x^2+33x+18=(x+2)(2x+3)^2$

We have,

$2x^2+7x+6=2x^2+4x+3x+6=2x(x+2)+3(x+2)$

$=(2x+3)(x+2)$

$\Rightarrow 4x^3+20x^2+33x+18=(2x+3)(2x+3)(x+2)$

$=(2x+3)^2(x+2)$

Hence, $4x^3+20x^2+33x+18=(x+2)(2x+3)^2$