Ex 20.1

Q1

(i) $4, -2, 1, -\frac{1}{2}, \ldots$

$$\frac{t_n}{t_{n-1}} = r = \text{common ratio} \qquad \text{---(i)}$$

$$\frac{t_2}{t_1} = \frac{-2}{4} = \frac{-1}{2}$$

$$\frac{t_3}{t_2} = \frac{1}{-2} = \frac{-1}{2}$$

(ii) $\frac{-2}{3}, -6, -54, \ldots$

Using (i)

$$\frac{t_2}{t_1} = \frac{-6}{\frac{-2}{3}} = \frac{18}{2} = 9$$

$$\frac{t_3}{t_2} = \frac{-54}{-6} = 9$$

$\therefore \quad r = 9$

(iii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \ldots$

Using (i)

$$\frac{t_3}{t_2} = \frac{\frac{9a^3}{16}}{\frac{3a^2}{4}} = \frac{9a^3}{16} \times \frac{4}{3a^2} = \frac{3a}{4}$$

$$\frac{t_2}{t_1} = \frac{\frac{3a^2}{4}}{a} = \frac{3a^2}{a}$$

$\therefore \quad r = \frac{3}{4}a$

(iii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \ldots$

Using (i)

$$\frac{t_3}{t_2} = \frac{\frac{9a^3}{16}}{\frac{3a^2}{4}} = \frac{9a^3}{16} \times \frac{4}{3a^2} = \frac{3a}{4}$$

$$\frac{t_2}{t_1} = \frac{\frac{3a^2}{4}}{a} = \frac{3a^2}{a}$$

$\therefore \quad r = \frac{3}{4}a$

(iv) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27} \ldots$

Using (i)

$$\frac{t_3}{t_2} = \frac{\frac{2}{9}}{\frac{1}{3}} = \frac{2}{3}$$

$$\frac{t_2}{t_1} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$\therefore \quad r = \frac{2}{3}$

Q2

$an = \frac{2}{3n}, n \in N$

Put $n = 1, 2, 3 \ldots$ because n is natural number

$$\frac{2}{3}, \frac{2}{3^2}, \frac{2}{3^3}, \ldots$$

$$\frac{t_3}{t_2} = \frac{\frac{2}{3^3}}{\frac{2}{3^2}} = \frac{1}{3}$$

$$\frac{t_2}{t_1} = \frac{\frac{2}{3^2}}{\frac{2}{3}} = \frac{1}{3}$$

Ratio of consecutive terms is solve

$\therefore \quad \frac{1}{3}$ is common ratio, Hence it is G.P $\forall n \in N$.

Q3

(i) 9^{th} term of G.P 1, 4, 16, 64, ...

$t_1 = 1 = a$

$t_2 = 4$

Because it is G.P

$\frac{t_2}{t_1} = \text{common ratio} = r$

$r = \frac{2}{1} = 4$

$t_n = ar^{n-1}$

$t_9 = ar^8 = 1(4)^8 = 4^8$

(ii) 10^{th} term of G.P $\frac{-3}{4}, \frac{1}{2}, \frac{-1}{3}, \frac{2}{9}, \ldots$

$a = \frac{-3}{4}$

Because it is G.P

$\therefore \quad r = \frac{t_2}{t_1} = \frac{\frac{1}{2}}{\frac{-3}{4}} = \frac{-2}{3}$

$t_n = ar^{n-1}$

$t_{10} = ar^9 = \left(\frac{-3}{4}\right)\left(\frac{-2}{3}\right)^9 = \frac{1}{2}\left(\frac{2}{3}\right)^8$

(iv) 12^{th} term of G.P $\frac{1}{a^3x^3}, ax, a^5x^5, \ldots$

$a = \frac{1}{a^3x^3}$

$r = \frac{t_n}{t_{n-1}} = \frac{t_2}{t_1} = \frac{ax}{\frac{1}{a^3x^3}} = a^4x^4$

$t_n = ar^{n-1}$

$t_{12} = ar^{11}$

$= \left(\frac{1}{a^3x^3}\right)\left(a^4x^4\right)^{11}$

$= (ax)^{41}$

(v) n^{th} term of G.P $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \ldots$

$r = \frac{t_n}{t_{n-1}} = \frac{t_2}{t_1} = \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}} = \frac{1}{3}$

$t_n = ar^{n-1}$

$t_n = \left(\sqrt{3}\right)\left(\frac{1}{3}\right)^{n-1}$

(vi) 10^{th} term of G.P $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \ldots$

$a = \sqrt{2}$

$r = \frac{t_n}{t_{n-1}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$

$t_n = ar^{n-1}$

$t_{10} = ar^9$

$= \left(\sqrt{2}\right)\left(\frac{1}{2}\right)^9$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)^8$

Q4

$$\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, \ldots .162$$

n^{th} term from the end

$$a_n = l\left(\frac{1}{r}\right)^{n-1}$$

$$l = 162, r = \text{common ratio} = \frac{t_2}{t_1}$$

$$= \frac{\frac{2}{9}}{\frac{2}{27}} = 3$$

$$n = 4$$

$$t_4 = (162)\left(\frac{1}{3}\right)^3$$

$$= \frac{162}{27}$$

$$= 6$$

Q5

$$0.004, 0.02, 0.1, \ldots \text{ is } 12.5$$

Here,

$$a = 0.004, \quad t_n = 12.5$$

$$r = \frac{t_2}{t_1} = \frac{0.02}{0.004} = 5$$

$$t_n = ar^{n-1}$$

$$12.5 = (0.004)(5)^{n-1}$$

$$\frac{12.5}{0.004} = (5)^{n-1}$$

$$\frac{125 \times 100}{4} = 5^{n-1}$$

$$5^5 = 5^{n-1}$$

$$= n - 1$$

$$n = 6$$

Q6

$\sqrt{2}, \frac{1}{\sqrt{2}}, \ldots$ is $\frac{1}{512\sqrt{2}}$

$$t_n = ar^{n-1}$$

$$a = \sqrt{2},\ r = \frac{t_n}{t_{n-1}} = \frac{t_2}{t_1} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}$$

$$t_n = \frac{1}{512\sqrt{2}},\ n = ?$$

$$t_n = ar^{n-1}$$

$$\frac{1}{512\sqrt{2}} = \left(\sqrt{2}\right)\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{512 \times \sqrt{2} \times \sqrt{2}} = \left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{1024} = \left(\frac{1}{2}\right)^{n-1}$$

$$\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{n-1}$$

$$10 = (n-1)$$

$$n = 11$$

$\therefore$ term is 11^{th}.

Q6(i)

$2, 2\sqrt{2}, 4, \ldots$ is 128

$$a = 2\quad r = \frac{t_n}{t_{n-1}} = \frac{2\sqrt{2}}{2} = \sqrt{2},\ n = ?$$

$$t_n = 128$$

Also,

$$t_n = ar^{n-1}$$

$$128 = (2)\left(\sqrt{2}\right)^{n-1}$$

$$\frac{128}{2} = \left(\sqrt{2}\right)^{n-1}$$

$$64 = \left(\sqrt{2}\right)^{n-1}$$

$$(2)^6 = \left(\sqrt{2}\right)^{n-1}$$

$\Rightarrow$ $$12 = n - 1$$

$$n = 13$$

$\therefore$ 13^{th} term is 128.

Q6(ii)

$\sqrt{3}, 3, 3\sqrt{3}, \ldots, 729$

$$a = \sqrt{3},\ r = \frac{t_n}{t_{n-1}},\ n = ?,\ t_n = 729$$

Now,

$$t_n = ar^{n-1}$$
$$729 = \left(\sqrt{3}\right)(r)^{n-1}$$

Now,

$$r = \frac{t_2}{t_1} = \frac{3}{\sqrt{3}} = \sqrt{3}$$
$$729 = \left(\sqrt{3}\right)\left(\sqrt{3}\right)^{n-1}$$
$$729 = \left(\sqrt{3}\right)^{n}$$
$$(3)^6 = \left(\sqrt{3}\right)^{n}$$
$$\left(\sqrt{3}\right)^{12} = \left(\sqrt{3}\right)^{n}$$

$\Rightarrow \quad n = 12$

$\therefore$ 12^{th} term is 729.

Q6(iii)

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots, \frac{1}{19683}$

$$a = \frac{1}{3},\ r = \frac{t_n}{t_{n-1}} = \frac{t_2}{t_1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3},\ t_n = \frac{1}{19683},\ n = ?$$

Now,

$$t_n = ar^{n-1}$$
$$\frac{1}{19683} = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n}$$
$$\left(\frac{1}{3}\right)^{9} = \left(\frac{1}{3}\right)^{n}$$

$\Rightarrow \quad n = 9$

$\therefore$ 9^{th} term of G.P is $\frac{1}{19683}$.

Q7

$18, -12, 8, \ldots$ is $\frac{512}{729}$

$$a = 18,\ n = ?,\ t_n = \frac{512}{729},\ r = \frac{t_{n-1}}{t_n}$$

$$r = \frac{t_2}{t_1} = \frac{-12}{18} = \frac{-2}{3}$$

Also,

$$t_n = ar^{n-1}$$

$$\frac{512}{729} = (18)\left(\frac{-2}{3}\right)^{n-1}$$

$$\frac{2^9}{36} \times \frac{1}{2 \times 3^2} = \left(\frac{-2}{3}\right)^{n-1}$$

$$\left(\frac{2}{3}\right)^8 = (-1)^{n-1}\left(\frac{2}{3}\right)^{n-1}$$

$$n = 9$$

Q8

$\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots, \frac{1}{4374}$

$$a = \frac{1}{2},\ l = \frac{1}{4374},\ r = \frac{t_n}{t_{n-1}} = \frac{t_2}{t_1} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Term from the end is

$$a_n = l\left(\frac{1}{r}\right)^{n-1}$$

$$t_4 = \left(\frac{1}{4374}\right)(3)^{n-1}$$

$$= \frac{1}{4374} \times 3^3$$

$$= \frac{1}{162}$$

$\therefore$ 4^{th} term from the end is $\frac{1}{162}$.

Q9

$t_4 = 27$
$t_7 = 729$

We know that $t_n = ar^{n-1}$

$t_4 = ar^3 = 27$

$t_7 = ar^6 = 729$

Now,

$$\frac{t_7}{t_4} = \frac{ar^6}{ar^3} = r^3 = \frac{729}{27}$$

$$r^3 = \left(\frac{9}{3}\right)^3$$

$r^3 = 3^3$

$r = 3$

$t_4 = ar^3 = 27$

$a(3)^3 = 27$

$a(27) = 27$

$a = 1$

Now G.P is $a, ar, ar^2, \ldots$

$1, 3, 9, \ldots$

Q10

$t_7 = 8t_4$
$t_5 = 48$

We know that $t_n = ar^{n-1}$

a = first term

r = common ratio

n = number of terms

$t_7 = ar^6 = 8(ar^3)$

$r^3 = 8$

$r = 2$

Also,

$t_5 = 48$

$ar^4 = 48$

$a(2)^4 = 48$

$$a = \frac{48}{16} = 3$$

$\therefore$ G.P is $a, ar, ar^2, \ldots$

$3, 6, 12, \ldots$

Q11

5, 10, 20, ... n term
1280, 640, 320, ..., n terms.

Let t_n be the general term of first G.P and t_n' be general term of second G.P whose nth terms are equal

a for first G.P = 5

a for second G.P = 1280

r for first G.P = $\frac{10}{5} = 2$

r for second G.P = $\frac{640}{1280} = \frac{1}{2}$

$t_n = ar^{n-1}$

Applying and equating for both G.P's

$$(5)(2)^{n-1} = 1280\left(\frac{1}{2}\right)^{n-1}$$

$$(2)^{n-1} = \frac{1280}{5}\left(\frac{1}{2}\right)^{n-1}$$

$$= 256\left(\frac{1}{2}\right)^{n-1}$$

$$= 2^8\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{(2)^{n-1}}{2^8} = \left(\frac{1}{2}\right)^{n-1} = 2^{n-9} = 2^{-n+1}$$

$\Rightarrow$ $2n = 10$

$n = 5$

Q12

We have

$$\left(a^2+b^2+c^2\right)p^2 - 2\left(ab+bc+cd\right)p + \left(b^2+c^2+d^2\right) \le 0$$

$$\left(a^2p^2 - 2abp + b^2\right) + \left(b^2p^2 - 2bcp + c^2\right) + \left(c^2p^2 - 2cdp + d^2\right) \le 0$$

$$(ap-b)^2 + (bp-c)^2 + (cp-d)^2 \le 0$$

This is only possible when

$ap - b = 0 \Rightarrow p = \frac{b}{a}$

$bp - c = 0 \Rightarrow p = \frac{c}{b}$

$cp - d = 0 \Rightarrow p = \frac{d}{c}$

Thus

$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

Hence a, b, c and d are in G.P

Q13

$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$, two show that a,b,c,d are in G.P

$\Rightarrow$ to show $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$ ---(i)

Now,

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx} \text{ and } \frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$$

Cross multiplying

$$(a+bx)(b-cx)=(b+cx)(a-bx)$$

$$ab-acx+b^2x-bcx^2=ab-b^2x+acx-bcx^2$$

Cancelling ab and $-bcx^2$ on both sides

$$-acx+b^2x=-b^2x+acx$$

$$x\left(b^2-ac\right)=-x\left(b^2-ac\right)$$

$$2b^2x=2acx$$

$$2b^2=2ac=b^2=ac$$

From (i) $b^2=ac$

Also,

$$\frac{cx+b}{b-cx}=\frac{c+dx}{c-dx}, \text{ cross multiplying}$$

$$c^2x-cdx^2+bc-bdx=bc+bdx-c^2x-cdx^2$$

$$2c^2x=2bdx$$

From (i) $c^2=bd$

Hence, a,b,c,d are in G.P.

Q14

We have

$a_5 = p$
$a_8 = q$
$a_{11} = s$

We have to show that

$$q^2 = ps$$
$$\Rightarrow \quad \frac{q}{p} = \frac{s}{q}$$

Now, $q = ar^7$
$p = ar^4$
$s = ar^{10}$

$$\therefore \quad \frac{q}{p} = \frac{s}{q}$$

$$\Rightarrow \quad \frac{ar^7}{ar^4} = \frac{ar^{10}}{ar^7}$$
$$\Rightarrow \quad r^3 = r^3$$

Hence proved.

Q15

Let a be the first term

then $a = -3$

Now we have

$$a_4 = (a_2)^2$$
$$\Rightarrow \quad ar^3 = (ar)^2$$
$$\Rightarrow \quad ar^3 = a^2r^2$$
$$\Rightarrow \quad r = a = -3$$

$$\therefore \quad a_7 = ar^6 = (-3)^7 = -2187$$

Q16

Let the first term is a and the common ratio is r.

Then

$$ar^2 = 24 \quad ..... (1)$$

and $ar^5 = 192 \quad ..... (2)$

$(2) \div (1)$, we get

$$\frac{ar^5}{ar^2} = \frac{192}{24}$$

$$r^3 = 8$$

$$r = 2$$

Now

$$ar^2 = 24$$

$$a \cdot 2^2 = 24$$

$$a = 6$$

Thus the 10^{th} term will be: $ar^9 = 6 \cdot 2^9 = 3072$

Q17

nth term of GP = ar^{n-1}

pth term = $q = ar^{p-1}$

qth term = $p = ar^{q-1}$

$$\frac{q}{p} = r^{p-q}$$

$$r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

$$a = p\left(\frac{p}{q}\right)^{\frac{1-q}{p-q}}$$

$$p+q \text{ th term} = p\left(\frac{q}{p}\right)^{\frac{1-q}{p-q}}\left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= p\left(\frac{q}{p}\right)^{\frac{1-q+p+q-1}{p-q}}$$

$$= p\left(\frac{q}{p}\right)^{\frac{p}{p-q}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{p}{p-q}-1}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}}$$

$$= \left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$$

Ex 20.2

Q1

Let the three number in G.P be $\frac{a}{r}, a, ar$

Sum of these numbers = $\frac{a}{r} + a + ar = 65$

$$3375 = \text{Product of these numbers}$$

$$3375 = \left(\frac{a}{r}\right)(a)(ar) = a^3$$

$$a^3 = (5)^3 \times (3)^3 = (15)^3$$

$$\Rightarrow \quad a = 15$$

$$a\left(\frac{1}{r} + 1 + r\right) = 65$$

$$15\left(\frac{1}{r} + 1 + r\right) = \frac{65}{15} = \frac{13}{3}$$

$$\frac{1 + r + r^2}{r} = \frac{13}{3}$$

$$3 + 3r + 3r^2 = 13r$$

$$3r^2 - 10r + 3 = 0$$

$$3r^2 - r - 9r + 3 = 0$$

$$r(3r - 1) - 3(3r - 1) = 0$$

$$r = 3, \ \frac{1}{3} \qquad r = \frac{1}{3} \text{ or } r = 3$$

$\therefore$ G.P. is a, ar, ar^2

$\therefore$ G.P. is 45, 15, 5 or 5, 15, 45

Q2

Let the three numbers be a, ar, ar^2 in G.P., where a is first teror and r is the common ratio.

Then,

$$a + ar + ar^2 = 38$$
$$a\left(1 + r + r^2\right) = 38 \qquad \text{---(i)}$$

and

$$(a)(ar)(ar)^2 = 1728$$
$$a^3r^3 = 1728 = 4^3 3^3 = (12)^3$$
$$a^3 = \frac{12^3}{r^3} \Rightarrow \frac{12}{r} = a$$

Putting $a = \frac{12}{r}$ in (i)

$$\frac{12}{r}\left(1 + r + r^2\right) = 38$$
$$12 + 12r + 12r^2 = 38r$$
$$12r^2 - 26r + 12 = 0$$
$$6r^2 - 13r + 6 = 0$$
$$6r^2 - 9r - 4r + 6 = 0$$
$$3r(3r - 3) - 2(3r - 3) = 0$$
$$r = \frac{3}{2}, \ \frac{2}{3}$$
$$a = \frac{12}{\frac{3}{2}} = 8 \text{ or } \frac{12}{\frac{2}{3}} = 18$$

$\therefore$ G.P. is 8, 12, 18.

Q3

Let the first three terms of G.P. are $\frac{a}{r}$, a, ar

Here,

$$\frac{a}{r} + a + ar = \frac{13}{12} \qquad \text{---(i)}$$

and $\quad \frac{a}{r} \times a \times ar = -1$

$\Rightarrow \quad a^3 = -1$

$\Rightarrow \quad a = -1$

Put $a = -1$ in equation (i),

$$\frac{-1}{r} + (-1) - r = \frac{13}{12}$$

$\Rightarrow \quad -1 - r - r^2 = \frac{13}{12} r$

$\Rightarrow \quad -12 - 12r - 12r^2 = 13r$

$\Rightarrow \quad 12r^2 + 12r + 13r + 12 = 0$

$\Rightarrow \quad 12r^2 + 25r + 12 = 0$

$\Rightarrow \quad 12r^2 + 16r + 9r + 12 = 0$

$\Rightarrow \quad 4r(3r + 4) + 3(3r + 4) = 0$

$\Rightarrow \quad (4r + 3)(3r + 4) = 0$

$$r = \frac{-3}{4}, \ \frac{-4}{3}$$

So,

Required G.P. is, $\frac{4}{3}, \ -1, \ \frac{3}{4}, \ \ldots$

or $\quad \frac{3}{4}, \ -1, \ \frac{4}{3}, \ \ldots$

Q4

Let the three numbers in G.P. be $\frac{a}{r}, a, ar$ then product of these numbers $\left(\frac{a}{r}\right)(a)(ar)$

$\Rightarrow \quad a^3 = 125 = 5^3$

$a = 5$

Also, sum of these products in pair

$$\left(\frac{a}{r}\right)(a) + (a)(ar) + \left(\frac{a}{r}\right)(ar) = 87\frac{1}{2} = \frac{195}{2}$$

$$\frac{a^2}{r} + a^2r + a^2 = a^2\left(\frac{1}{r} + r + 1\right)$$

$$= (5)^2\left(\frac{1 + r^2 + r}{r}\right) = \frac{195}{2}$$

$$1 + r^2 + r = \left(\frac{195}{2 \times 25}\right)^r$$

$$2\left(1 + r^2 + r\right) = \frac{39}{5}r$$

$$10 + 10r^2 + 10r = 39r$$

$$10r^2 - 29r + 10 = 0$$

$$10r^2 - 25r - 4r + 10 = 0$$

$$5r(2r - 5) - 2(2r - 5) = 0$$

$$r = \frac{5}{2}, \ \frac{2}{5}$$

$\therefore$ G.P. is $\frac{a}{r}, a, ar$

$$10, 5, \frac{5}{2}, \ldots \text{ or } \frac{5}{2}, 5, 10, \ldots$$

Q5

Let the three numbers in G.P. be $\frac{a}{r}, a, ar$

then product of them is $\left(\frac{a}{r}\right)(a)(ar) = 21$ ---(i)

$$= \frac{a}{r}\left(1 + r + r^2\right) = 21$$

and sum of their squares

$$\frac{a^2}{r^2} + a^2 + a^2r^2 = a^2\frac{\left(1 + r^2 + r^4\right)}{r^2} = 189 \quad \text{---(ii)}$$

Now,

$$a\left(1 + r + r^2\right) = 21r \quad \text{---(iii)}$$

Then, $a^2\left(1 + r + r^2\right)^2 = 441r^2$ [suqaring]

$$a^2\left(1 + r^2 + r^4\right) + 2a^2r\left(1 + r + r^2\right) = 441r$$

$$189r^2 + 2ar \times 21r = 441r^2$$

Dividing both sides by $21r^2$

$$9 + 2a = 21$$

$$2a = 21 - 9 = 12$$

$$a = 6 \Rightarrow a = 6$$

Putting in (iii)

$$6\left(1 + r + r^2\right) = 21r$$

$$6 + 6r + 6r^2 - 21r = 0$$

$$6r^2 - 15r + 6 = 0$$

$$6r^2 - 12r - 3r + 6 = 0$$

$\Rightarrow$ $6r(r - 2) - 3(r - 2) = 0$

$$r = 2, \ \frac{1}{2}$$

$\therefore$ G.P. is 3, 6, 12 or 12, 6, 3.

Q6

Let the numbers are: $\frac{a}{r}, a$ and ar.

Then

$$\frac{a}{r}+a+ar=14$$

Again the numbers $a+1, ar+1$ and ar^2-1 are in A.P, therefore

$$2(a+1)=(ar-1)+\left(\frac{a}{r}+1\right)$$

$$2(a+1)=ar+\frac{a}{r}$$

$$2(a+1)=14-a$$

$$3a=12$$

$$a=4$$

Now we have

$$\frac{4}{r}+4+4r=14$$

$$2-5r+2r^2=0$$

$$2r^2-4r-r+2=0$$

$$2r(r-2)-1(r-2)=0$$

$$(r-2)(2r-1)=0$$

$$r=2,\frac{1}{2}$$

Thus the numbers are: 2, 4, 8 or 8, 4, 2.

Q7

Let the number in G.P. are $\frac{a}{r}$, a, ar

So,

$$\frac{a}{r} \times a \times ar = 216$$

$\Rightarrow \quad a^3 = 216$

$\Rightarrow \quad a = 6$

And also given,

$$\frac{a}{r} + 2,\ a + 8,\ ar + 6 \text{ are in A.P.}$$

$$2(a + 8) = \left(\frac{a}{r} + 2\right) + (ar + 6)$$

$\Rightarrow \quad 2(6 + 8) = \left(\frac{6 + 2r}{r}\right) + 6r + 6$

$\Rightarrow \quad 28r = 6 + 2r + 6r^2 + 6r$

$\Rightarrow \quad 6r^2 - 20r + 6 = 0$

$\Rightarrow \quad 6r^2 - 18r - 2r + 6 = 0$

$\Rightarrow \quad 6r(r - 3) - 2(r - 3) = 0$

$\Rightarrow \quad (r - 3)(6r - 2) = 0$

$$r = 3,\ r = \frac{1}{3}$$

So,

Required G.P. is 18, 6, 2, ...

or, 2, 6, 18, ...

Q8

Let three numbers in G.P. are $\frac{a}{r}$, a, ar

Here,

$$\frac{a}{r} \times a \times ar = 729$$

$\Rightarrow \quad a^3 = 729$

$\Rightarrow \quad a = 9$

And,

$$\left(\frac{a}{r} \times a\right) + (a \times ar) + \left(\frac{a}{r} \times ar\right) = 819$$

$\Rightarrow \quad \frac{81}{r} + 81r + 81 = 819$

$\Rightarrow \quad \frac{9}{r} + 9r + 9 = 91$

$\Rightarrow \quad 9 + 9r^2 + 9r = 91r$

$\Rightarrow \quad 9r^2 - 82r + 9 = 0$

$\Rightarrow \quad 9r^2 - 81r - r + 9 = 0$

$\Rightarrow \quad 9r(r - 9) - 1(r - 9) = 0$

$\quad r = 9, \ \frac{1}{9}$

So, required G.P. are

$\quad 81, 9, 1, \ldots$

or, $\quad 1, 9, 81, \ldots$

Q9

Let the numbers are $\frac{a}{r}, a$ and ar. Then we have

$$\frac{a}{r} + a + ar = \frac{39}{10}$$

And

$$\frac{a}{r} \cdot a \cdot ar = 1$$

$$a^3 = 1$$

$$a = 1$$

Now we have

$$\frac{1}{r} + 1 + r = \frac{39}{10}$$

$$1 + r + r^2 = \frac{39}{10} r$$

$$r^2 - \frac{29}{10} r + 1 = 0$$

$$10r^2 - 29r + 10 = 0$$

$$(2r - 5)(5r - 2) = 0$$

$$r = \frac{5}{2}, \frac{2}{5}$$

Thus the numbers are: either $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$.

Ex 20.3

Q1

2, 6, 18, ... to 7 term

$$a = 2,\ r = \frac{6}{2} = 3,\ n = 7$$

$$S_n = a\frac{\left(r^n - 1\right)}{r - 1}$$

$$S_7 = 2\frac{\left(3^7 - 1\right)}{3 - 1} = \frac{2}{2}\left(3^7 - 1\right)$$

$$= 2187 - 1 = 2186$$

1, 3, 9, 27, ... to 8 terms

$$a = 1,\ r - \frac{3}{1} = 3,\ n = 8$$

$$S_n = a\frac{\left(r^n - 1\right)}{r - 1}$$

$$S_8 = 1\frac{\left(3^8 - 1\right)}{3 - 1} = 3280$$

$1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \ldots, 9$ terms

$$a = 1,\ r = \frac{\frac{-1}{2}}{1} = \frac{-1}{2},\ n = 9$$

$$S_n = a\frac{\left(r^n - 1\right)}{r - 1}$$

$$S_9 = 1\frac{\left(\frac{-1}{2}\right)^9 - 1}{\frac{-1}{2} - 1}$$

$$= \frac{\frac{-1}{512} - 1}{\frac{-1}{2} - 1}$$

$$= \frac{\frac{-1 - 512}{512}}{\frac{-1 - 2}{2}}$$

$$= \frac{-513}{512} \times \frac{2}{-3}$$

$$= \frac{171}{256}$$

$\left(a^2 - b^2\right),\ \left(a - b\right),\ \left(\dfrac{a-b}{a+b}\right), \ldots n$ terms

$$a = a^2 - b^2,\ r = \frac{a-b}{a^2-b^2} = \frac{1}{a+b},\ n = n$$

$$S_n = a\frac{\left(1-r^n\right)}{1-r} \qquad [\because r < 1]$$

$$S_n = \left(a^2-b^2\right)\frac{\left(1-\dfrac{1}{\left(a+b\right)^n}\right)}{1-\dfrac{1}{a+b}}$$

$$= \frac{\left(a-b\right)\left(\left(a+b\right)^n - 1\right)}{\dfrac{\left(a+b\right)^{-1}\left(a+b\right)^n\left(a+b\right)-1}{\left(a+b\right)}}$$

$$= \frac{a-b}{\left(a+b\right)^n}\frac{\left(\left(a+b\right)^n - 1\right)}{\left(a+b\right)-1}$$

$4, 2, 1, \frac{1}{2}, \ldots 10$ terms

$$a = 4,\ r = \frac{2}{4} = \frac{1}{2},\ n = 10$$

$$S_n = a\frac{\left(1-r^n\right)}{1-r}$$

$$= 4\frac{1-\left(\dfrac{1}{2}\right)^{10}}{1-\dfrac{1}{2}}$$

$$= 8\left(1-\frac{1}{2^{10}}\right)$$

$$= 8\left(1-\frac{1}{1024}\right)$$

Q2

$$0.15+0.015+0.0015+\ldots\text{upto 8 terms}$$

$$=15(0.1+0.01+0.001+\ldots\text{upto 8 terms})$$

$$=15\left(\frac{1}{10}+\frac{1}{100}+\ldots\right)$$

$$r=\frac{1}{10},a=\frac{1}{10}$$

$$Sum=15\left(\frac{\frac{1}{10}\left(1-\frac{1}{10^8}\right)}{1-\frac{1}{10}}\right)$$

$$=\frac{5}{3}\left(1-\frac{1}{10^8}\right)$$

Here the first term of the series is $a=\sqrt{2}$ and the common ration is $r=\frac{\frac{1}{\sqrt{2}}}{\sqrt{2}}=\frac{1}{2}$

Thus the sum of the G.P up to 8^{th} terms is:

$$S_8=\frac{a(1-r^8)}{1-r}=\frac{\sqrt{2}\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}}=2\sqrt{2}\left(1-\frac{1}{256}\right)=\frac{255\sqrt{2}}{128}$$

$\frac{2}{9}-\frac{1}{3}+\frac{1}{2}-\frac{3}{4}+\ldots$ to 5 terms.

$$a=\frac{2}{9},\ r=\frac{\frac{-1}{3}}{\frac{2}{9}}=\frac{-1}{3}\times\frac{9}{2}=\frac{-3}{2},\ n=5$$

$$S_5=a\frac{\left(1-r^5\right)}{1-r}$$

$$=\frac{2}{9}\frac{\left(1-\left(\frac{-3}{2}\right)^5\right)}{1-\left(\frac{-3}{2}\right)}$$

$$=\frac{2}{9}\frac{\left(1+\frac{243}{32}\right)}{1+\frac{3}{2}}$$

$$=\frac{2}{9}\frac{(275)}{32}\times\frac{2}{5}$$

$$=\frac{55}{72}$$

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots\ldots$$

$$=\frac{1}{x-y}\{(x^2-y^2)+(x^3-y^3)+\ldots\ldots\ldots to\ \infty\}\ldots\ldots\left[\because \frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+\ldots+y^{n-1}\right]$$

$$=\frac{1}{x-y}\{(x^2+x^3+\ldots\ldots to\ \infty)-(y^2+y^3+\ldots\ldots to\ \infty)\ \}$$

$$=\frac{1}{x-y}\left\{\frac{x^2}{1-x}-\frac{y^2}{1-y}\right\}$$

$$=\frac{1}{x-y}\left\{\frac{x^2-x^2y-y^2+xy^2}{(1-x)(1-y)}-\right\}$$

$$=\frac{x+y-xy}{(1-x)(1-y)}$$

The series can be written as:

$$3\left(\frac{1}{5}+\frac{1}{5^3}+\frac{1}{5^5}+\cdots n \text{ terms}\right)+4\left(\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots n \text{ terms}\right)$$

For the first part $a=\frac{1}{5}$ and the common ratio $r=\frac{1}{5^2}=\frac{1}{25}$

Thus the sum is:

$$3\left(\frac{1}{5}+\frac{1}{5^3}+\frac{1}{5^5}+\cdots n \text{ terms}\right)=3\cdot\frac{\frac{1}{5}\left(1-\left(\frac{1}{25}\right)^n\right)}{1-\frac{1}{25}}$$

$$=\frac{5}{8}\left(1-\frac{1}{5^{2n}}\right)$$

For the second part $a=\frac{1}{25}$ and common ratio $r=\frac{1}{25}$ then

$$4\left(\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots n \text{ terms}\right)=4\cdot\frac{\frac{1}{25}\left(1-\left(\frac{1}{25}\right)^n\right)}{1-\frac{1}{25}}$$

$$=\frac{1}{6}\left(1-\frac{1}{5^{2n}}\right)$$

Thus the sum is:

$$\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\cdots 2n \text{ terms}=\frac{5}{8}\left(1-\frac{1}{5^{2n}}\right)+\frac{1}{6}\left(1-\frac{1}{5^{2n}}\right)$$

$$\frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots+\frac{a}{(1+i)^n}$$

$$a=\frac{a}{1+i},\quad r=\frac{\frac{a}{(1+i)^2}}{\frac{a}{1+i}}=\frac{1}{1+i}$$

$$S_n=a\frac{\left(1-r^n\right)}{1-r}$$

$$=\frac{a}{1+i}\frac{\left(1-\left(\frac{1}{1+i}\right)^n\right)}{1-\frac{1}{1+i}}$$

$$=\frac{a}{1+i}\times\frac{1+i}{(-i)}\left(1-(1+i)^n\right)$$

$$=-ai\left(1-(1+i)^{-n}\right)$$

Re writing the sequence and sum we get,

Sum$=1-a+a^2-a^3+a^4-a^5+$..........

Here, $r=-a$ and first term $=1$

$$\text{Sum}=\frac{[1-(-a)^n]}{1+a}$$

Here the first term of the G.P is $a=x^3$ and the common ratio is $r=\frac{x^5}{x^3}=x^2$

Thus the sum of the G.P is:

$$x^3+x^5+x^7+\cdots \text{ to } n \text{ terms}=\frac{x^3\left(\left(x^2\right)^n-1\right)}{x^2-1}=\frac{x^3\left(x^{2n}-1\right)}{x^2-1}$$

Here the first term of the G.P is $a=\sqrt{7}$ and the common ratio is $r=\frac{\sqrt{21}}{\sqrt{7}}=\sqrt{3}$

Thus the sum of the G.P is:

$$\sqrt{7}+\sqrt{21}+3\sqrt{7}+\cdots \text{ to } n \text{ terms}=\frac{\sqrt{7}\left(\left(\sqrt{3}\right)^n-1\right)}{\sqrt{3}-1}=\frac{\sqrt{7}\left(3^{\frac{n}{2}}-1\right)}{\sqrt{3}-1}$$

Q3

$$\sum_{n=1}^{11}\left(2+3^n\right)$$

$$= \left(2+3^1\right)+\left(2+3^2\right)+\left(2+3^3\right)+\dots+\left(2+3^{11}\right)$$

$$= 2\times 11+3^1+3^2+3^3+\dots+3^{11}$$

$$= 22+\frac{3\left(3^{11}-1\right)}{(3-1)}$$

$$= 22+\frac{3\left(3^{11}-1\right)}{2}$$

$$= \frac{44+3(177147-1)}{2}$$

$$= \frac{44+3(177146)}{2}$$

$$= 265741$$

So,

$$\sum_{n=1}^{11}\left(2+3^n\right) = 265741$$

$$\sum_{k=1}^{n}\left(2^k-3^{k-1}\right)$$

$$= \left(2+3^0\right)+\left(2^2+3\right)+\left(2^3+3^2\right)+\dots+\left(2^n+3^{n-1}\right)$$

$$= \left(2+2^2+2^3+\dots-2^n\right)+\left(3^0-3^1+3^2+\dots+3^{n-1}\right)$$

$$= S_n + S_m$$

$S_n \to a=2,\ n=n,\ r=\frac{2^2}{2}=2$

$$S_n = \frac{a\left(r^n-1\right)}{r-1} = \frac{2\left(2^n-1\right)}{2-1} = 2\left(2^n-1\right)$$

Also, $S_m = S_{n-1}$

$a=1,\ r=3,\ n=n-1$

$$S_{n-1} = \frac{1\left(3^{n-1}-1\right)}{3-1} = \frac{1}{2}\left(3^n-1\right)$$

$$\therefore \quad \sum_{k=1}^{n}\left(2^k-3^{k-1}\right) = 2\left(2^n-1\right)+\frac{1}{2}\left(3^n-1\right)$$

$$= \frac{1}{2}\left[2^{n+2}+3^n-4-1\right]$$

$$= \frac{1}{2}\left[2^{n+2}+3^n-5\right]$$

$$\sum_{n=2}^{10} 4^n$$

$$= 4^2 + 4^3 + 4^4 + \ldots + 4^{10}$$

$$a = 4^2,\ r = \frac{4^3}{4} = 4,\ n = 9$$

$$S_{10} = \frac{a\left(r^9 - 1\right)}{1 - r}$$

$$= \frac{4^2\left(4^9 - 1\right)}{4 - 1}$$

$$= \frac{1}{3}\left[4^{11} - 16\right]$$

$$= \frac{16}{3}\left[4^9 - 1\right]$$

Q4

$5 + 55 + 555 + \ldots n$ terms

Taking 5 common from each term.

$$5\left[1 + 11 + 111 + \ldots n \text{ terms}\right]$$

Dividing and multiplying by 9

$$= \frac{5}{9}\left[9 + 99 + 999 + \ldots n \text{ terms}\right]$$

$$= \frac{5}{9}\left[(10 - 1) + \left(10^2 - 1\right) + \left(10^3 - 1\right) + \ldots n \text{ terms}\right]$$

$$= \frac{5}{9}\left[\left(10 + 10^2 + 10^3 + \ldots n \text{ terms}\right) - n\right] \text{ this is G.P.}$$

So, $S_n = \dfrac{a\left(r^n - 1\right)}{r - 1}$

$$a = 10,\ r = 10,\ n = n$$

$$= \frac{5}{9}\left[\frac{10\left(10^n - 1\right)}{10 - 1} - n\right]$$

$$= \frac{5}{9 \times 9}\left(10^{n+1} - 10 - 9n\right)$$

$$= \frac{5}{81}\left(10^{n+1} - 9n - 10\right)$$

Now we have

$$7+77+777+\cdots \text{ to } n \text{ terms} = 7[1+11+111+\cdots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}[9+99+999+\cdots \text{ to } n \text{ terms}]$$

$$= \frac{7}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\cdots \text{ to } n \text{ terms}\right]$$

$$= \frac{7}{9}\left[10+10^2+10^3+\cdots \text{ to } n \text{ terms}\right] - \frac{7}{9}(1+1+1+\cdots\cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\cdot\frac{10(10^n-1)}{10-1} - \frac{7n}{9}$$

$$= \frac{7}{81}(10^{n+1}-9n-10)$$

$9+99+999+\ldots n$ term

This can be written as

$$= (10-1)+(100-1)+(1000-1)+\ldots n \text{ term}$$

$$= (10+10^2+10^3+\ldots n \text{ term}) - n$$

$$\Rightarrow \quad S_n = \frac{a(r^n-1)}{r-1},\ a=10,\ r=10,\ n=n$$

$$= \frac{10(10^n-1)}{10-1} - n$$

$$= \frac{10}{9}(10^n-1) - n$$

$$= \frac{1}{9}\left[10^{n+1}-10-9n\right]$$

$$= \frac{1}{9}\left[10^{n+1}-9n-10\right]$$

$0.5+0.55+0.555+\ldots \text{ to } n$

$= 5\times 0.1+5\times 0.11+5\times 0.111+$

$$= \frac{5}{9}\left\{\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+ \ldots + \ldots\right\}$$

$$= \frac{5}{9}\left\{(1-\frac{1}{10})+(1-\frac{1}{100})+ \ldots + \right\}$$

$$= \frac{5}{9}\left\{n - \left(\frac{1}{10}+\frac{1}{10^2}+\ldots+\frac{1}{10^n}\right)\right\}$$

$$= \frac{5}{9}\left[n - \frac{1}{10}\frac{\left\{1-(\frac{1}{10})^n\right\}}{\left(1-\frac{1}{10}\right)}\right]$$

$$= \frac{5}{9}\left[n - \frac{1}{9}(1-\frac{1}{10^n})\right]$$

$0.6 + 0.66 + 0.666 + \&..$ to n

$=6\times 0.1+6\times 0.11+6\times 0.111+.......$

$=\frac{6}{9}\left\{\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+........+-\right\}$

$=\frac{6}{9}\{(1-\frac{1}{10})+(1-\frac{1}{100})+........+\ \}$

$=\frac{6}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+......+\frac{1}{10^n}\right)\right\}$

$=\frac{6}{9}\left[n-\frac{1}{10}\frac{\left\{1-(\frac{1}{10})^n\right\}}{\left(1-\frac{1}{10}\right)}\right]$

$=\frac{6}{9}\left[n-\frac{1}{9}(1-\frac{1}{10^n})\right]$

Q5

Here,

$3,\ \frac{3}{2},\ \frac{3}{4},\ ...$ is a G.P.

and $S_n = \frac{3069}{512},\ a = 3, r = \frac{1}{2}$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\frac{3069}{512} = \frac{3\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$

$$\frac{3069}{512} = \frac{3\left(2^n-1\right)}{2^n\times\frac{1}{2}}$$

$$\frac{1023}{512} = \frac{2\left(2^n-1\right)}{2^n}$$

$$10232^n = 1024.2^n - 1024$$

$$1024 = 2^n$$

$\Rightarrow \quad 2^{10} = 2^n$

$\Rightarrow \quad n = 10$

Q6

$2 + 6 + 18 + \ldots$

$$S_n = 728$$

Now,

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$a = 2,\ r = \frac{6}{2} = 3$$

$$728 = \frac{2\left(3^n - 1\right)}{3 - 1}$$

$$728 = \frac{2\left(3^n - 1\right)}{2} = \left(3^n - 1\right)$$

$$728 + 1 = 3^n$$

$$729 = 3^n$$

$$(3)^6 = 3^n$$

$\Rightarrow \quad n = 6$

Q7

$\sqrt{3}, 3, 3\sqrt{3}, \ldots$

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$a = \sqrt{3},\ r = \frac{3}{\sqrt{3}} = \sqrt{3},\ S_n = 39 + 13\sqrt{3}$$

Putting into formula

$$39 + 13\sqrt{3} = \frac{\sqrt{3}\left(\left(\sqrt{3}\right)^n - 1\right)}{\sqrt{3} - 1}$$

$$39 + 13\sqrt{3} = \frac{\left(\sqrt{3}\right)^{n+1} - \sqrt{3}}{\sqrt{3} - 1}$$

$$\left(39 + 13\sqrt{3}\right)\left(\sqrt{3} - 1\right) = \left(\sqrt{3}\right)^{n+1} - \sqrt{3}$$

$$39\sqrt{3} - 39 + 39 - 13\sqrt{3} = \left(\sqrt{3}\right)^{n+1} - \sqrt{3}$$

$$26\sqrt{3} + \sqrt{3} = \left(\sqrt{3}\right)^{n+1}$$

$$\left(27\sqrt{3}\right)^1 = \left(\sqrt{3}\right)^{n+1}$$

$$\left(\sqrt{3}\right)^6\left(\sqrt{3}\right)^1 = \left(\sqrt{3}\right)^{n+1}$$

$$7 = n + 1$$

$\Rightarrow \quad n = 6$

Q8

$3, 6, 12, \ldots n\ 381$

$$a = 3,\ r = \frac{6}{3} = 2,\ n = ?\ S_n = 381$$

We know that

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$381 = \frac{3(2)^n - 1}{2 - 1}$$

$$\frac{381}{3} = 2^n - 1$$

$$127 = 2^n - 1$$

$$128 = 2^n$$

$$2^7 = 2^n$$

$$n = 7$$

Q9

$r = 3$, last term is 486

Sum of terms $= S_n = 728,\ a = ?$

We know that

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$728 = \frac{a\left(3^n - 1\right)}{3 - 1}$$

Also, $t_n = ar^{n-1}$

$t_n = 486$

$\therefore\quad 486 = a(3)^{n-1}$

$a\left(3^{n-1}\right) = 3^5 \times 2$

$3^{n-1} = 3^5$

$n = 6$

and $\quad a = 2$

Q10

Let Sum of first three terms $= a + ar + ar^2$

The ratio $= \dfrac{a + ar + ar^2}{a + ar + ar^2 + ar^3 + ar^4 + ar^5}$

$$= \frac{1+r+r^2}{1+r+r^2+r^3+r^4+r^5}$$

$$= \frac{1+r+r^2}{1+r+r^2+r^3(1+r+r^2)} \text{ .... (1)}$$

Let $A = 1+r+r^2$ ....(2)

$$Ratio = \frac{A}{A+r^3A} = \frac{125}{152}$$

$$\frac{1}{1+r^3} = \frac{125}{152}$$

$$152 = 125 + 125r^3$$

$$r^3 = \frac{27}{125}$$

$$r = \frac{3}{5}$$

Q11

$t_4 = \frac{1}{27}$, $t_7 = \frac{1}{729}$, $t_n = ar^{n-1}$

Where $t_n = n^{th}$ term, r = common difference, n = number of terms.

$t_4 = ar^3 = \frac{1}{27}$ ---(i)

$t_7 = ar^6 = \frac{1}{729}$ ---(ii)

Dividing (ii) by (i), we get

$$\frac{t_7}{t_4} = \frac{ar^6}{ar^3} = r^3 = \frac{27}{729} = \frac{1}{27},\ r = \frac{1}{3}$$

Sum of n term $= S_n = \dfrac{a(1-r^n)}{1-r}$ ---(iii)

When, $r = 3$, $t_4 = ar^3 = \frac{1}{27}$

$$a\left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$a = 1$$

Substituting $a = 1$, $r = \frac{1}{3}$ in (iii)

$$S_n = \frac{1\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}}$$

$$= \frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}}$$

$$= \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$$

Q12

$$\sum_{n=1}^{10}\left\{\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{5}\right)^{n+1}\right\}$$

$$=\sum_{n=1}^{10}\left(\frac{1}{2}\right)^{n-1}+\sum_{n=1}^{10}\left(\frac{1}{5}\right)^{n+1}$$

$$=1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\ldots\ldots$$

$$=\frac{(1-\frac{1}{2^{10}})}{1-\frac{1}{2}}+\frac{\frac{1}{5}(1-\frac{1}{5^{10}})}{1-\frac{1}{5}}$$

$$=\frac{2^{10}-1}{2^9}+\frac{5^{10}-1}{5^{11}}$$

Q13

Fifth term of series is

$ar^{5-1}=81\ldots\ldots\ldots\ldots(1)$

Second term of series is

$ar=24\ \ldots\ldots\ldots\ldots(2)$

Dividing (2) by (1) we get,

$$\frac{ar}{ar^4}=\frac{24}{81}=\frac{8}{27}$$

$$r^3=\frac{27}{8}$$

$$r=\frac{3}{2}$$

Substituting r in (2), we get,

$$a=\frac{24\times 2}{3}=16$$

$$Sum=\frac{16\left[(\frac{3}{2})^8-1\right]}{\frac{3}{2}-1}$$

$$=\frac{16\left[3^8-2^8\right]}{2^7}$$

$$=\frac{6305}{8}$$

Q14

S_1 = sum of n terms,
S_1 = sum of $2n$ terms,
S_1 = sum of $3n$ terms.

Then, $S_1^2 + S_2^2$

$$= (S_n)^2 + (S_{2n})^2$$
$$= \left(\frac{a(1-r^n)}{1-r}\right)^2 + \left(\frac{a(1-r^{2n})}{1-r}\right)^2$$
$$= \frac{a^2}{(1-r)^2}\left[\left(1-(r)^n\right)^2 + \left(1-r^{2n}\right)^2\right]$$
$$= \frac{a^2}{(1-r)^2}\left[1 + r^{2n} - 2r^n + 1 + r^{4n} - 2r^{2n}\right]$$
$$= \frac{a^2}{(1-r)^2}\left[2 - r^{2n} - 2r^n + r^{4n}\right] \quad \text{---(i)}$$

Also, $S_1(S_2 + S_3)$

$$= \frac{a(1-r^n)}{1-r}\left(\frac{a(1-r^{2n})}{1-r} + \frac{a(1-r^{3n})}{1-r}\right)$$
$$= \frac{a^2}{(1-r)^2}\left[(1-r)^n\left(1-r^{2n}\right) + \left(1-r^n\right)\left(1-r^{3n}\right)\right]$$
$$= \frac{a^2}{(1-r)^2}\left[1 - r^{2n} - r^n + r^{3n} - r^{3n} - r^n + 1 + r^{4n}\right]$$
$$= \frac{a^2}{(1-r)^2}\left[2 - r^{2n} - 2r^n + r^{4n}\right] \quad \text{---(ii)}$$

(i) = (ii) Hence, $S_1^2 + S_2^2 = S_1(S_2 + S_3)$

Q15

$S_1, S_2, \ldots, S_n$ are the sums of n terms of G.P. $a = 1$, $r = 1, 2, 3, \ldots n$

Then, $S_1 + S_2 + 2S_3 + 3S_4 + \ldots + (n-1)S_n$

$$\frac{1(1^n-1)}{1-1} + \frac{1(2^n-1)}{2-1} + \frac{2(3^n-1)}{3-1} + \ldots + (n-1)1\left(\frac{1^n-1}{n-1}\right)$$
$$= 2^n - 1 + 23^n - 1 + 3.4^n - 1 + \ldots$$
$$= 2^n + 3^n + 4^n + \ldots + n^n$$

Q16

Let the G.P. be $2n, 2, 2n+4, \dots$

Then, $S_n = \frac{a(r^n - 1)}{r-1}$, $a = 2n$, $r = 2$

$\therefore \quad S_n = \frac{2n(2^n - 1)}{2-1} = 2n^{n+1} - 2n$

Then the G.P. of odd term

$a_1 + a_3 + a_5 + \dots a_{2n-1}$

According to the question

Sum of all terms = 5 (sum of terms occupying the odd places)

$a_1 + a_2 + a_3 + \dots + a_{2n} = 5(a_1 + a_3 + a_5 + \dots + a_{2n-1})$

$a + ar + ar^2 + \dots + ar^{2n-1} = 5(a + ar^2 + ar^4 + \dots + ar^{2n-2})$

$\frac{a(1-r^{2n})}{1-r} = 5\left(\frac{a(1-(r)^2)^n}{1-r^2}\right)$

$\frac{a}{1-r}$ is cancelled on both side

$1 - r^{2n} = \frac{5(1-r^{2n})}{1-r}$

$1 + r - r^{2n} - r^{2n+1} = 5 - 5r^{2n}$

$r^{2n+1} - 4r^{2n} - r + 4 = 0$

$r^{2n}(r-4) - 1(r-4) = 0$

$r^{2n} = 1, r = 4$

$\Rightarrow \quad r = 4$

Q17

Given $\sum_{n=1}^{100} a_{2n} = \alpha$

$\Rightarrow \quad a_2 + a_4 + a_6 + \dots + a_{200} = \alpha$ ---(i)

also, $\sum_{n=1}^{100} a_{2n-1} = \beta$

$\Rightarrow \quad a_1 + a_3 + a_5 + \dots + a_{199} = \beta$ ---(ii)

Sum of G.P.,

$S_n = \frac{a(1-r^n)}{1-n}$

$= a = a_2, r = r^2, n = 100$

$ar + ar^3 + ar^5 + \dots + ar^{199} = \alpha$

$ar\frac{(1-(r^2)^{100})}{1-r^2} = \alpha$ ---(iii)

$a + ar^2 + ar^4 + \dots + ar^{198} = \beta$

$\frac{a(1-(r^2)^{100})}{1-r^2} = \beta$ ---(iv)

$r(\beta) = \alpha$

$r = \frac{\alpha}{\beta}$ [From (iv) and (v)]

Q18

Let the seried be $a_1 + a_2 + a_3 + \ldots + a_{2n}$

It is given that $a_1 = 1, a_2 = a, a_3 = ac, a_4 = a^2c, a_5 = a^2c^2, \ldots$

$\therefore$ Sum of $2n$ term

$$a_1 + a_2 + a_3 + \ldots + a_{2n}$$
$$= 1 + a + ac + a^2c + a^2c^2 + \ldots + 2n \text{ term}$$
$$= (1+a) + ac(1+a) + a^2c^2(1+a) + \ldots + n \text{ term}$$
$$= (1+a)\frac{\left(1-(ac)^n\right)}{1-ac}$$
$$= (a+1)\frac{\left((ac)^n - 1\right)}{ac-1}.$$

Q19

Sum of first n term of G.P.

$$= a + a_2 + a_3 + \ldots + a_n$$
$$= a + ar + ar^2 + \ldots + ar^{n-1} \qquad \left[\because t_n = ar^{n-1}\right] \text{ --- (i)}$$

Also sum of term from

$(n+1)^{th}$ to $(2n)^{th}$ term is

$$= a_{n+1} + a_{n+2} + \ldots + a_{2n}$$
$$= ar^n + ar^{n-1} + \ldots + ar^{2n-1} \qquad \text{--- (ii)}$$

Ratio of (i) and (ii) is

$$= \frac{a + ar + ar^2 + \ldots ar^{n-1}}{ar^n + ar^{n-1} + \ldots + ar^{2n-1}} \qquad \left[\because S_n = \frac{a\left(1-r^n\right)}{1-r}\right]$$
$$= \frac{\frac{a\left(1-r^n\right)}{1-r}}{\frac{ar^n\left(1-r^n\right)}{1-r}}$$
$$= \frac{1}{r^n}$$

Q20

Given,

$a,\ b$ are roots of the equation $x^2 - 3x + p = 0$

$\Rightarrow \quad a + b = 3,\ ab = p$

and $\quad c,\ d$ are roots of the equation $x^2 - 12x + q = 0$

$\Rightarrow \quad c + d = 12,\ cd = q$

Let $b = ar,\ c = ar^2$ and $d = ar^3$, then $a + b = 3$ and $c + d = 12$

$$a(1+r) = 3 \text{ and } ar^2(1+r) = 12$$

$$\Rightarrow \quad \frac{ar^2(1+r)}{a(1+r)} = \frac{12}{3}$$

$\Rightarrow \quad r = 2$

and $a(r+1) = 3$

$$\Rightarrow \quad a = 1$$

$$p = ab$$
$$= a \times ar$$
$$p = 2$$
$$q = cd$$
$$= ar^2 \times ar^3$$
$$= 2^5$$
$$a = 32$$
$$\frac{q+p}{q-p} = \frac{32+2}{32-2}$$
$$= \frac{34}{30}$$
$$(q+p) : (q-p) = 17 : 15$$

Q21

$$\text{Sum} = \frac{3069}{512} = \frac{3(1 - \frac{1}{2^n})}{\frac{1}{2}}$$

$$1 - \frac{1}{2^n} = \frac{3069}{512 \times 6} = \frac{1023}{512 \times 2}$$

$$1 - \frac{1023}{1024} = \frac{1}{2^n}$$

$$\frac{1}{2^n} = \frac{1}{1024}$$

$$n = 10$$

Q22

To find number of ancestors, we will find the sum of $2, 2^2, 2^3, \ldots\ldots$

Number of ancestors$=\dfrac{2(2^{10}-1)}{2-1}$

$= 2(1024-1)$

$= 2\times1023$

$= 2046$

Ex 20.4

Q1

$$S_\infty = 1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \dots$$

$\Rightarrow \quad a = 1,\ r = -\frac{1}{3}$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1+\frac{1}{3}}$$

$$S_\infty = \frac{3}{4}$$

$$S_\infty = 8 + 4\sqrt{2} + 4 + \dots$$

$\Rightarrow \quad a = 8,\ r = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{8}{1-\frac{1}{\sqrt{2}}}$$

$$= \frac{8\sqrt{2}}{\sqrt{2}-1} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= \frac{8(2+\sqrt{2})}{2-1}$$

$$S_\infty = 8(2+\sqrt{2})$$

$$S_\infty = \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \dots$$

$$= \left(\frac{2}{5} + \frac{2}{5^3} + \dots\right) + \left(\frac{3}{5^2} + \frac{3}{5^4} + \dots\right)$$

$$S_\infty = S'_\infty + S''_\infty$$

For

$$S'_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{2}{5}}{1-\frac{1}{25}}$$

$$= \frac{2}{5} \times \frac{25}{24}$$

$$S'_\infty = \frac{5}{12}$$

$$S''_\infty = \frac{\frac{3}{25}}{1-\frac{1}{25}}$$

$$= \frac{3}{25} \times \frac{25}{24}$$

$$= \frac{3}{24}$$

$$S_\infty = S'_\infty + S''_\infty$$

$$= \frac{5}{12} + \frac{3}{24}$$

$$= \frac{13}{24}$$

$$S_\infty = \frac{13}{24}$$

This infinite G.P has first term $a = 10$ and common ratio $r = -\frac{9}{10} = -0.9$

Thus the sum of the infinite G.P will be:

$$10 - 9 + 8.9 - 7.29 + \cdots \infty = \frac{a}{1-r} \quad \left[\text{Since } |r| < 1\right]$$
$$= \frac{10}{1-(-0.9)}$$
$$= \frac{10}{1.9}$$
$$= \frac{100}{19}$$

The G.P can be written as follows:

$$\frac{1}{3}+\frac{1}{5^2}+\frac{1}{3^3}+\frac{1}{5^4}+\frac{1}{3^5}+\frac{1}{5^6}+\cdots\infty = \left(\frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^5}+\cdots\infty\right)+\left(\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots\infty\right)$$
$$= \frac{\frac{1}{3}}{1-\frac{1}{3^2}} + \frac{\frac{1}{5^2}}{1-\frac{1}{5^2}}$$
$$= \frac{3}{8} + \frac{1}{24}$$
$$= \frac{10}{24}$$
$$= \frac{5}{12}$$

Q2

$$9^{\frac{1}{3}} \times 9^{\frac{1}{9}} \times 9^{\frac{1}{27}} \ldots \infty$$
$$= 9^{\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\infty\right)}$$
$$= 9^{\left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right)} \qquad \left[\text{Using } S_\infty = \frac{a}{1-r}\right]$$
$$= 9^{\left(\frac{1}{3}\times\frac{3}{2}\right)}$$
$$= 9^{\frac{1}{2}}$$
$$= 3$$

So,

$$9^{\frac{1}{3}} \times 9^{\frac{1}{9}} \times 9^{\frac{1}{27}} \ldots \infty = 3$$

Q3

$$2^{\frac{1}{4}}.4^{\frac{1}{8}}.8^{\frac{1}{16}}.16^{\frac{1}{32}}......\infty$$

$$= 2^{\frac{1}{4}}.2^{\frac{2}{8}}.2^{\frac{3}{16}}.2^{\frac{4}{32}}......\infty$$

$$= \left(\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+.....\infty\right)$$

$$= 2$$

$$= 2^{5} ------(1)$$

$$S = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+.....\infty$$

$$S = \left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+.....\infty\right)2$$

$$\frac{S}{2} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+.....\infty$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{2}}$$

$$= \frac{1}{4}\times\frac{2}{1}$$

$$S = \frac{1}{2}$$

$$S = 1$$

Thus $\quad 2^{\frac{1}{4}}.4^{\frac{1}{8}}.8^{\frac{1}{16}}.16^{\frac{1}{32}}....... = 2^{1} = 2$

Q4

$$S_p = 1 + r^p + r^{2p} + ... + \infty$$

$$S_p = \frac{1}{1-r^p}$$

$$s_p = 1 - r^p + r^{2p} + ... + \infty$$

$$s_p = \frac{1}{1+r^p}$$

Now,

$$S_p + s_p = \frac{1}{1-r^p} + \frac{1}{1+r^p}$$

$$= \frac{2}{1-r^{2p}}$$

$$S_p + s_p = 2 \times S_{2p}$$

Q5

Here,

$$a = 4$$

$$A_3 - a_5 = \frac{31}{81}$$

$$ar^2 - ar^4 = \frac{32}{81}$$

$$r^2 4\left(1 - r^2\right) = \frac{32}{81}$$

$$r^2\left(1 - r^2\right) = \frac{8}{81}$$

Let $r^2 = A$

$$A\left(1 - A\right) = \frac{8}{81}$$

$$A - A^2 = \frac{8}{81}$$

$$81A - 81A^2 = 8$$

$$81A^2 - 81A + 8 = 0$$

$$A = \frac{81 \pm \sqrt{(81)^2 - 4 \times 81 \times 8}}{81 \times 2}$$

$$= \frac{81 \pm \sqrt{6561 - 2592}}{162}$$

$$= \frac{81 \pm \sqrt{3969}}{162}$$

$$= \frac{81 \pm 63}{162}$$

$$= \frac{81 + 63}{162} \text{ or } \frac{81 - 63}{162}$$

$$= \frac{144}{162} \text{ or } \frac{18}{162}$$

$$r^2 = \frac{8}{9} \text{ or } \frac{1}{9}$$

$$r = \pm\frac{2\sqrt{2}}{3} \text{ or } \pm\frac{1}{3}$$

Since it is a decreasing G.P.

$$r = \frac{2\sqrt{2}}{3}, \frac{1}{3}$$

$$S_\infty = \frac{4}{1 - \frac{2\sqrt{2}}{3}} \text{ and } S_\infty = \frac{4}{1 - \frac{1}{3}}$$

$$S_\infty = \frac{12}{3 - 2\sqrt{2}}, 6$$

Q6

$a = 1$

$$a_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots\ldots\ldots$$
$$ar^{n-1} = ar^n + ar^{n+1} + ar^{n+2} + \ldots\ldots\ldots$$
$$ar^{n-1} = ar^n\left(1 + r + r^2 + \ldots\ldots\infty\right)$$
$$1 = r\left(\frac{1}{1-r}\right)$$
$$1 - r = r$$
$$1 = 2r$$
$$r = \frac{1}{2}$$

G.P. is 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots\ldots$

Q7

$a + ar = 5$

$a(1+r) = 5 - - - - - (1)$

$$a_n = 3\left(a_{n+1} + a_{n+2} + a_{n+3} + \ldots\ldots\ldots\right)$$
$$ar^{n-1} = 3\left(ar^n + ar^{n+1} + ar^{n+2} + \ldots\ldots\ldots\right)$$
$$ar^{n-1} = 3ar^n\left(1 + r + r^2 + \ldots\ldots\infty\right)$$
$$1 = 3r\left(\frac{1}{1-r}\right)$$
$$1 - r = 3r$$
$$1 = 4r$$
$$r = \frac{1}{4}$$
$$a(1+r) = 5$$
$$a\left(\frac{5}{4}\right) = 5$$
$$a = 4$$

G.P. is $4, 1, \frac{1}{4}, \frac{1}{16}, \ldots\ldots\ldots$

Q8

$$0.125125125\ldots\ldots\ldots = 0.\overline{125}$$

$$= 0.125 + 0.000125 + 0.000000125 + \ldots..$$

$$= \frac{125}{10^3} + \frac{125}{10^6} + \frac{125}{10^9} + \ldots..$$

$$= \frac{125}{10^3}\left(1 + \frac{1}{10^3} + \frac{1}{10^6} + \ldots..\right)$$

$$= \frac{125}{10^3}\left(\frac{1}{1 - \frac{1}{1000}}\right)$$

$$= \frac{125}{1000}\left(\frac{1000}{999}\right)$$

$$0.125125125\ldots\ldots\ldots = \frac{125}{999}$$

Q9

$$0.4\overline{23} = 0.4 + 0.0232323\ldots\ldots\ldots$$

$$= 0.4 + 0.023 + 0.00023 + \ldots\ldots\ldots$$

$$= 0.4 + \frac{23}{10^3} + \frac{23}{10^5} + \ldots\ldots\ldots$$

$$= 0.4 + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\ldots\right)$$

$$= 0.4 + \frac{23}{1000}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= 0.4 + \frac{23}{1000}\left(\frac{100}{99}\right)$$

$$= \frac{4}{10} + \frac{23}{990}$$

$$= \frac{396 + 23}{990}$$

$$0.4\overline{23} = \frac{419}{990}$$

Q10

Let a be first term and r be common ratio of G.P. Here,

$$\frac{a_n}{(a_{n+1}+a_{n+2}+\ldots\infty)} = \frac{ar^{n-1}}{ar^n + ar^{n+1} + \ldots}$$

$$= \frac{ar^{n-1}}{ar^n\left(1+r+r^2+\ldots\infty\right)}$$

$$= \frac{ar^{n-1}}{ar^n\left(\frac{1}{1-r}\right)}$$

$$= \left(\frac{1-r}{r}\right)$$

Since r is a constant, so

$$\left(\frac{a_n}{a_{n+1}+a_{n+2}+\ldots\infty}\right) = k \text{ (constant)}$$

Such that $k = \left(\frac{1-r}{r}\right)$

Q11

$$0.\bar{3} = 0.3333\ldots$$

$$= 0.3 + 0.03 + 0.003 + \ldots$$

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$$

$$= \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right)$$

$$= \frac{3}{10}\left(\frac{1}{1-\frac{1}{10}}\right)$$

$$= \frac{3}{10} \times \frac{10}{9}$$

$$= \frac{3}{9}$$

$$0.\bar{3} = \frac{1}{3}$$

$$0.\overline{231} = 0.231231231\ldots$$

$$= 0.231 + 0.000231 + 0.000000231$$

$$= \frac{231}{10^3} + \frac{231}{10^6} + \frac{231}{10^9} + \ldots$$

$$= \frac{231}{10^3}\left(1 + \frac{1}{10^3} + \frac{1}{10^6} + \ldots\right)$$

$$= \frac{231}{1000}\left(\frac{1}{1 - \frac{1}{1000}}\right)$$

$$0.\overline{231} = \frac{231}{999}$$

$$3.5\bar{2} = 3 + 0.52222\ldots$$

$$= 3 + 0.5 + 0.02 + 0.002 + 0.0002 + \ldots$$

$$= 3.5 + \frac{2}{10^2} + \frac{2}{10^3} + \frac{2}{10^4} + \ldots$$

$$= 3.5 + \frac{2}{10^2}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right)$$

$$= \frac{35}{10} + \frac{2}{100}\left(\frac{1}{1 - \frac{1}{10}}\right)$$

$$= \frac{35}{10} + \frac{2}{100} \times \left(\frac{10}{9}\right)$$

$$= \frac{35}{10} + \frac{2}{90}$$

$$= \frac{315 + 2}{90}$$

$$3.5\bar{2} = \frac{317}{90}$$

The rational number can be written as:

$$0.6\bar{8} = 0.6 + 0.08 + 0.008 + 0.0008 + \cdots \infty$$

$$= \frac{3}{5} + 8[0.01 + 0.001 + 0.0001 + \cdots \infty]$$

$$= \frac{3}{5} + 8\left[\frac{1}{100} + \frac{1}{1000} + \cdots \infty\right]$$

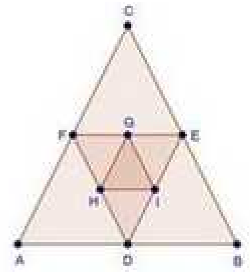
This is an infinite GP with first term $\frac{1}{100}$ and common ratio $\frac{1}{10}$

$$= \frac{3}{5} + 8 \cdot \frac{1}{100} \cdot \frac{1}{1 - \frac{1}{10}}$$

$$= \frac{3}{5} + \frac{4}{45}$$

$$= \frac{31}{45}$$

Q12



Side of triangle $= 18$ cm.

$AD = BD = 9$ cm.

$DE = BD = 9$ cm.

$GI = IF = \frac{9}{2}$ cm.

Sides of the triangles are $18, 9, \frac{9}{2}$.........

(i) sum of perimeters of the equilateral triangle $= \left(54 + 27 + \frac{27}{2} + ...\right)$

$$= \frac{54}{1 - \frac{1}{2}}$$

$$= 54 \times 2$$

Perimeter $= 108$ cm.

(ii) sum of area of equilateral triangle

$$= \left[\frac{\sqrt{3}}{4}(18)^2 + \frac{\sqrt{3}}{4}(9)^2 + \frac{\sqrt{3}}{4}\left(\frac{9}{2}\right)^2 + ...\right]$$

$$= \frac{\sqrt{3}}{4}\left[324 + 81 + \frac{81}{4} + ...\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{324}{1 - \frac{1}{4}}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{324 \times 4}{3}\right]$$

$$= \sqrt{3}\,(108)$$

Q13

$$S = a + ar + ar^2 + ar^3 + \ldots\ldots.$$

$$S = \frac{a}{1-r} - - - - - (1)$$

$$S_1 = a^2 + a^2r^2 + a^2r^4 + a^2r^6 \ldots\ldots$$

$$S_1 = \frac{a^2}{1-r^2} - - - - - (2)$$

$$S^2 = \frac{a^2}{(1-r)^2}$$

$$S^2 = \frac{S_1(1-r^2)}{(1-r^2)}$$

$$(1-r)S^2 = S_1(1+r)$$

$$S^2 - S^2r = S_1 + S_1r$$

$$S_1r + S^2r = S^2 - S_1$$

$$r = \frac{S^2 - S_1}{S_1 + S^2}$$

Put r in equation (1)

$$S(1-r) = a$$

$$a = S\left[1 - \frac{S^2 - S_1}{S^2 + S_1}\right]$$

$$a = S\left[\frac{S^2 + S_1 - S^2 + S_1}{S^2 + S_1}\right]$$

$$a = \frac{2SS_1}{S^2 + S_1}$$

Ex 20.5

Q1

Here,

$a,\ b,\ c$ are in G.P.

$b^2 = ac$ ---(i)

Now,

$$2\log b = \log b^2$$
$$= \log ac$$
$$2\log b = \log a + \log c$$
$$\log b - \log a = \log c - \log b$$

$\Rightarrow$ $\log a,\ \log b,\ \log c$ are in A.P.

Q2

Here,

$a,\ b,\ c$ are in G.P., so

$b^2 = ac$

$$\frac{2}{\log_b m} = 2\log_m b$$
$$= \log_m b^2$$
$$= \log_m ac$$
$$= \log_m a + \log_m c$$
$$\frac{2}{\log_b m} = \frac{1}{\log_a m} + \frac{1}{\log_c m}$$

$$\Rightarrow \quad \frac{1}{\log_b m} - \frac{1}{\log_a m} = \frac{1}{\log_c m} - \frac{1}{\log_b m}$$

$\Rightarrow$ $\dfrac{1}{\log_a m},\ \dfrac{1}{\log_b m},\ \dfrac{1}{\log_c m}$ are in A.P.

Q3

Here,

a, b, c are in A.P.

$2b = a + c$ ---(i)

and a, b, d are in G.P., so

$b^2 = ad$ ---(ii)

Now,

$$(a-b)^2 = a^2 + b^2 - 2ab$$
$$= a^2 + ad - a(a+c)$$

Using equation (i) and (ii)

$$= a^2 + ad - a^2 - ac$$
$$= ad - ac$$
$$(a-b)^2 = a(d-c)$$
$$\frac{(a-b)}{a} = \frac{(d-c)}{(a-b)}$$

$\Rightarrow$ $a,\ (a-b),\ (d-c)$ are in G.P.

Q4

Here, Let R be common ratio,

a_p, a_q, a_r, a_s of AP are in GP

$$R = \frac{a_q}{a_p} = \frac{a_r}{a_q}$$

$$= \frac{a_q - a_r}{a_p - a_q} \qquad \text{(Ratio property)}$$

$$= \frac{[a+(q-1)d]-[a+(r-1)d]}{[a+(p-1)d]-[a+(q-1)d]}$$

$$= \frac{(q-r)d}{(p-q)d}$$

$$R = \frac{q-r}{p-q} \text{-------}(1)$$

Now,

$$R = \frac{a_r}{a_q} = \frac{a_s}{a_r}$$

$$= \frac{a_r - a_s}{a_q - a_r} \qquad \text{(Ratio property)}$$

$$= \frac{[a+(r-1)d]-[a+(s-1)d]}{[a+(q-1)d]-[a+(r-1)d]}$$

$$= \frac{(r-s)d}{(q-r)d}$$

$$R = \frac{r-s}{q-r} \text{-------}(2)$$

From equation as (1) and (2)

$$\frac{q-r}{p-q} = \frac{r-s}{p-r}$$

$\Rightarrow \quad (p-q), (q-r), (r-s)$ are in GP

Q5

$$\frac{1}{a+b},\ \frac{1}{2b},\ \frac{1}{b+c} \text{ are in A.P.}$$

$$\frac{2}{2b} = \frac{1}{(a+b)} + \frac{1}{(b+c)}$$

$$\frac{1}{b} = \frac{b+c+a+b}{(a+b)(b+c)}$$

$$\frac{1}{b} = \frac{2b+c+a}{ab+ac+b^2+bc}$$

$$ab+ac+b^2+bc = 2b^2+bc+ba$$

$$b^2+ac = 2b^2$$

$$b^2 = ac$$

So,

$a,\ b,\ c$ are in G.P.

Q6

$$x^a = x^{\frac{b}{2}} z^{\frac{b}{2}} = z^c = \lambda\ (\text{say})$$

$$x = \lambda^{\frac{1}{a}},\ z = \lambda^{\frac{1}{c}}$$

$$x^{\frac{b}{2}} \times z^{\frac{b}{2}} = \lambda$$

$$\lambda^{\frac{1}{a}\left(\frac{b}{2}\right)} \times \lambda^{\frac{b}{2}\times\frac{1}{c}} = \lambda$$

$$\lambda^{\frac{b}{2a}+\frac{b}{2c}} = \lambda^1$$

$$\frac{b}{2a} + \frac{b}{2c} = 1$$

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

$$\Rightarrow \quad \frac{1}{a},\ \frac{1}{b},\ \frac{1}{c} \text{ are in A.P.}$$

Q7

$k+9,\ k-6,\ 4$ are in G.P.

$(k-6)^2 = (k+9)4$

$k^2 + 36 - 12k = 4k + 36$

$k^2 - 16k = 0$

$k(k-16) = 0$

$k = 0,\ k = 16$

Q8

Let $a-d,\ a,\ a+d$ be numbers in A.P.

Here,

$$a-d+a+a+d = 15$$
$$3a = 15$$
$$a = 5$$

Find

$$[(5-d)+1],\ (5+3),\ [(5+d)+9] \text{ are in G.P.}$$

$\Rightarrow$ $(6-d),\ 8,\ (14+d)$ are in G.P.

$$(8)^2 = (6-d)(14+d)$$
$$64 = 84 + 6d - 14d - d^2$$
$$d^2 + 8d - 20 = 0$$
$$(d+10)(d-2) = 0$$
$$d = 2,\ -10$$

So,

Numbers are 3, 5, 7 or 15, 5, −5

Q9

Let three numbers in A.P. be $a-d,\ a,\ a+d$

Here,

$$a-d+a+a+d = 21$$
$$3a = 21$$
$$a = 7$$

And

$$(7-d),\ (7-1),\ (7+d)+1 \text{ are in G.P.}$$
$$(7-d),\ 6,\ (8+d) \text{ are in G.P.}$$
$$(6)^2 = (7-d)(8+d)$$
$$36 = 56 + 7d - 8d - d^2$$
$$d^2 + d - 20 = 0$$
$$(d+5)(d-4) = 0$$
$$d = 4,\ -5$$

So,

Numbers are 3, 7, 11 or 12, 7, 2.

Q10

Here,

a, b, c are in A.P.

Let $a = A - d,\ b = A,\ c = A + d$

Here,

$$a + b + c = 18$$
$$A - d + A + A + d = 18$$
$$3A = 18$$
$$A = 6$$

And,

$(a + 4),\ (b + 4),\ (c + 36)$ are in G.P.

$(6 - d + 4),\ (6 + 4),\ (6 + d + 36)$ are in G.P.

$(10 - d),\ (10),\ (42 + d)$ are in G.P.

$$(10)^2 = (10 - d)(42 + d)$$
$$100 = 420 + 10d - 42d - d^2$$
$$d^2 + 32d - 320 = 0$$
$$(d + 40)(d - 8) = 0$$
$$d = -40,\ 8$$

So,

Numbers of $-2, 6, 14$ or $46, 6, -34$.

Q11

Let numbers are a, ar, ar^2

$$a + ar + ar^2 = 56 - - - - - (1)$$

$(a-1), (ar-7), (ar^2-21)$ are in AP

$$\Rightarrow \quad 2(ar-7) = a - 1 + ar^2 - 21$$

$$= (ar^2 + a) - 22$$

$$2ar - 14 = (56 - ar) - 22 \qquad [\text{using equation } (1)]$$

$$2ar - 14 = 34 - ar$$

$$3ar = 48$$

$$ar = 16 - - - - - - - (2)$$

$$a = \frac{16}{r}$$

Put a in equation (1),

$$\frac{16 + 16r + 16r^2}{r} = 56$$

$$16 + 16r + 16r^2 = 56r$$

$$16r^2 - 40r + 16 = 0$$

$$2r^2 - 5r + 2 = 0$$

$$2r^2 - 4r - r + 2 = 0$$

$$2r(r-2) - 1(r-2) = 0$$

$$(r-2)(2r-1) = 0$$

$$r = 2, \ \frac{1}{2}$$

Put r in equation (2),

$$ar = 16$$

for $r = \frac{2}{a} = 8$

for $r = \frac{1}{2}, \ a = 32$

thus, there numbers are

8,16,32

in both cases.

Q12

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

$a\left(b^2 + c^2\right) = c\left(a^2 + b^2\right)$

$a\left(a^2r^2 + a^2r^4\right) = ar^2\left(a^2 + a^2r^2\right)$

$a^3r^2\left(1 + r^2\right) = a^3r^2\left(1 + r^2\right)$

LHS = RHS

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

$$\text{LHS} = a^2b^2c^2\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right)$$

$$= a^2 \times a^2r^2 \times a^2r^4\left(\frac{1}{a^3} + \frac{1}{a^3r^3} + \frac{1}{a^3r^6}\right)$$

$$= a^6r^6\left(\frac{r^6 + r^3 + 1}{a^3r^6}\right)$$

$$= a^3\left(r^6 + r^3 + 1\right)$$

$$= a^3 + a^3r^3 + a^3r^6$$

$$= a^3 + (ar)^3 + \left(ar^2\right)^3$$

$$= a^3 + b^3 + c^3$$

= RHS

∴ LHS = RHS

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

$$\text{LHS} = \frac{(a + b + c)^2}{a^2 + b^2 + c^2}$$

$$= \frac{\left(a + ar + ar^2\right)^2}{a^2 + a^2r^2 + a^2r^4}$$

$$= \frac{a^2\left(1 + r + r^2\right)^2}{a^2\left(1 + r^2 + r^4\right)}$$

$$= \frac{a^2\left(1 + r + r^2\right)^2}{a^2\left[\left(1 + 2r^2 + r^4\right) - r^2\right]}$$

$$= \frac{a^2\left(1 + r + r^2\right)^2}{a^2\left[\left(1 + r^2 - r\right)\left(1 + r^2 + r\right)\right]}$$

$$= \frac{a\left(1 + r + r^2\right)}{a\left(1 + r^2 - r\right)}$$

$$= \frac{a - ar + ar^2}{a - ar^2 - ar}$$

$$= \frac{a - b + c}{a - b + c}$$

= RHS

∴ LHS = RHS

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

LHS $= \dfrac{1}{a^2 - b^2} + \dfrac{1}{b^2}$

$= \dfrac{1}{a^2 - a^2r^2} + \dfrac{1}{a^2r^2}$

$= \dfrac{1}{a^2}\left[\dfrac{1}{1 - r^2} + \dfrac{1}{r^2}\right]$

$= \dfrac{1}{a^2}\left[\dfrac{r^2 + 1 - r^2}{\left(1 - r^2\right)r^2}\right]$

$= \dfrac{1}{a^2}\left[\dfrac{1}{r^2 - r^4}\right]$

$= \dfrac{1}{(ar)^2 - \left(ar^2\right)^2}$

$= \dfrac{1}{b^2 - c^2}$

$=$ RHS

$\therefore$ LHS = RHS

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

LHS $= (a + 2b + 2c)(a - 2b + 2c)$

$= \left(a + 2ar + 2ar^2\right)\left(a - 2ar + 2ar^2\right)$

$= a^2\left(1 + 2r + 2r^2\right)\left(1 - 2r + 2r^2\right)$

$= a^2\left[\left(1 + 2r^2\right)^2 - (2r)^2\right]$

$= a^2\left[1 + 4r^4 + 4r^2 - 4r^2\right]$

$= a^2\left[1 + 4r^4\right]$

$= a^2 + 4\left(ar^2\right)^2$

$= a^2 + 4c^2$

$=$ RHS

$\therefore$ LHS = RHS

Q13

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

$$\frac{ab - cd}{b^2 - c^2} = \frac{a + c}{b}$$

$$\frac{a(ar) - (ar^2)(ar^3)}{a^2r^2 - a^2r^4} = \frac{a + ar^2}{ar}$$

$$\frac{a^2r - a^2r^5}{a^2r^2(1 - r^2)} = \frac{a(1 + r^2)}{ar}$$

$$\frac{a^2r(1 - r^4)}{a^2r^2(1 - r^2)} = \frac{a(1 + r^2)}{ar}$$

$$\frac{1 + r^2}{r} = \frac{1 + r^2}{r}$$

LHS = RHS

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

$$(a + b + c + d)^2 = (a + b)^2 + 2(b + c)^2 + (c + d)^2$$

$$\Rightarrow \quad (a + ar + ar^2 + ar^3)^2 = (a + ar)^2 + 2(ar + ar^2)^2 + (ar^2 + ar^3)^2$$

$$\Rightarrow \quad a^2(1 + r + r^2 + r^3)^2 = a^2\left[(1 + r)^2 + 2(r + r^2)^2 + (r^2 + r^3)^2\right]$$

$$\Rightarrow \quad (1 + r + r^2 + r^3)^2 = 1 + r^2 + 2r + 2(r^2 + r^4 + 2r^3) + r^4 + r^6 + 2r^5$$

$$\Rightarrow \quad (1 + r + r^2 + r^3 + r + r^2 + r^3 + r^4 + r^2 + r^3 + r^4 + r^5 + r^3 + r^4 + r^5 + r^6)$$
$$= (1 + r^2 + 2r + 2r^2 + 2r^4 + 4r^3 + r^4 + r^6 + 2r^5)$$

$$\Rightarrow \quad (r^6 + 2r^5 + 3r^4 + 4r^3 + 3r^2 + 2r + 1) = (r^6 + 2r^5 + 3r^4 + 4r^3 + 3r^2 + 2r + 1)$$

LHS = RHS

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

$$(b + c)(b + d) = (c + a)(c + d)$$

$$\Rightarrow \quad (ar + ar^2)(ar + ar^3) = (ar^2 + a)(ar^2 + ar^3)$$

$$\Rightarrow \quad a^2(r + r^2)(r + r^3) = a^2(r^2 + 1)(r^2 + r^3)$$

$$\Rightarrow \quad r^2(1 + r)(1 + r^2) = r^2(1 + r^2)(1 + r)$$

$\therefore$ LHS = RHS

a, b, c are in G.P.

$\Rightarrow \quad b^2 = ac \qquad$ ---(i)

$\left(b^2\right)^2 = (ac)^2$

$\left(b^2\right)^2 = a^2c^2$

$\Rightarrow \quad a^2, b^2, c^2$ are in G.P.

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

$\left(b^3\right)^2 = a^3c^3$

$\left((ar)^3\right)^2 = a^3\left(ar^2\right)^3$

$a^6r^6 = a^3\left(a^3r^6\right)$

$a^6r^6 = a^6r^6$

LHS = RHS

$\Rightarrow \quad \left(b^3\right)^2 = a^3c^3$

So,

a^3, b^3, c^3 are in G.P.

a, b, c are in G.P.

$a,\ b = ar,\ c = ar^2$

$(ab + bc)^2 = \left(a^2 + b^2\right)\left(b^2 + c^2\right)$

$\left(a \times ar + ar \times ar^2\right)^2 \left(a^2 + (ar)^2\right)\left((ar)^2 + \left(ar^2\right)^2\right)$

$\left(a^2r + a^2r^3\right)^2 = \left(a^2 + a^2r^2\right)\left(a^2r^2 + a^2r^4\right)$

$a^4\left(r + r^3\right)^2 = a^4\left(1 + r^2\right)\left(r^2 + r^4\right)$

$a^4r^2\left(1 + r^2\right)^2 = a^4\left(1 + r^2\right)r^2\left(1 + r^2\right)$

$a^4r^2\left(1 + r^2\right)^2 = a^4r^2\left(1 + r^2\right)^2$

LHS = RHS

$\Rightarrow \quad (ab + bc)^2 = \left(a^2 + b^2\right)\left(b^2 + c^2\right)$

$\Rightarrow \quad \left(a^2 + b^2\right),\ (ab + bc),\ \left(b^2 + c^2\right)$ are in G.P.

Q14

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

Now,

$$\left(b^2 + c^2\right)^2 = \left(a^2 + b^2\right)\left(c^2 + d^2\right)$$

$$\left(a^2r^2 + a^2r^4\right)^2 = \left(a^2 + a^2r^2\right)\left(a^2r^4 + a^2r^6\right)$$

$$a^4\left(r^2 + r^4\right)^2 = a^2\left(1 + r^2\right)a^2r^4\left(1 + r^2\right)$$

$$a^4r^4\left(1 + r^2\right)^2 = a^4r^4\left(1 + r^2\right)^2$$

LHS = RHS

$\Rightarrow \quad \left(b^2 + c^2\right)^2 = \left(a^2 + b^2\right)\left(c^2 + d^2\right)$

$\Rightarrow \quad \left(a^2 + b^2\right),\ \left(b^2 + c^2\right),\ \left(c^2 + d^2\right)$ are in G.P.

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

Now,

$$\left(b^2 - c^2\right)^2 = \left(a^2 - b^2\right)\left(c^2 - d^2\right)$$

$$\left(a^2r^2 - a^2r^4\right)^2 = \left(a^2 - a^2r^2\right)\left(a^2r^4 - a^2r^6\right)$$

$$a^4\left(r^2 - r^4\right)^2 = a^2\left(1 - r^2\right)a^2r^4\left(1 - r^2\right)$$

$$a^4r^4\left(1 - r^2\right)^2 = a^4r^4\left(1 - r^2\right)^2$$

LHS = RHS

$\Rightarrow \quad \left(b^2 - c^2\right)^2 = \left(a^2 - b^2\right)\left(c^2 - d^2\right)$

$\Rightarrow \quad \left(a^2 - b^2\right),\ \left(b^2 - c^2\right),\ \left(c^2 - d^2\right)$ are in G.P.

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

Now,

$$\left(\frac{1}{b^2+c^2}\right)^2 = \left(\frac{1}{a^2+b^2}\right)\left(\frac{1}{c^2+d^2}\right)$$

$$\left(\frac{1}{a^2r^2+a^2r^4}\right)^2 = \left(\frac{1}{a^2+a^2r^2}\right)\left(\frac{1}{a^2r^4+a^2r^6}\right)$$

$$\frac{1}{a^4\left(r^2+r^4\right)^2} = \frac{1}{a^2\left(1+r^2\right)} \times \frac{1}{a^2\left(r^4+r^6\right)}$$

$$\frac{1}{a^4r^4\left(1+r^2\right)^2} = \frac{1}{a^2r^4\left(1+r^2\right)\left(1+r^2\right)}$$

$$\frac{1}{a^4r^4\left(1+r^2\right)^2} = \frac{1}{a^2r^4\left(1+r^2\right)^2}$$

LHS = RHS

$\Rightarrow \quad \left(\frac{1}{b^2+c^2}\right)^2 = \left(\frac{1}{a^2+b^2}\right)\left(\frac{1}{c^2+d^2}\right)$

$\Rightarrow \quad \left(\frac{1}{a^2+b^2}\right),\ \left(\frac{1}{b^2+c^2}\right),\ \left(\frac{1}{c^2+d^2}\right)$ are in G.P.

---

a, b, c, d are in G.P.

$a,\ b = ar,\ c = ar^2,\ d = ar^3$

Now,

$$(ab+bc+cd)^2 = \left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right)$$

$$\left(a^2r+a^2r^3+a^2r^5\right)^2 = \left(a^2+a^2r^2+a^2r^4\right)\left(a^2r^2+a^2r^4+a^2r^6\right)$$

$$a^4\left(r+r^3+r^5\right)^2 = a^2\left(1+r^2+r^4\right)a^2r^2\left(1+r^2+r^4\right)$$

$$a^4r^2\left(1+r^2+r^4\right)^2 = a4r^2\left(1+r^2+r^4\right)^2$$

LHS = RHS

$\Rightarrow \quad (ab+bc+cd)^2 = \left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right)$

$\Rightarrow \quad \left(a^2+b^2+c^2\right),\ (ab+bc+cd),\ \left(b^2+c^2+d^2\right)$ are in G.P.

Q15

$(a-b),\ (b-c),\ (c-a)$ are in G.P.

$(b-c)^2 = (a-b)(c-a)$

$b^2 + c^2 - 2bc = ac - a^2 - bc + ab$

$b^2 + c^2 + a^2 = ac + bc + ab$ ---(i)

Now,

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$= ac + bc + ab + 2ab + 2bc + 2ca$

Using equation (i)

$= 3ab + 3bc + 3ca$

$(a+b+c)^2 = 3(ab + bc + ca)$

Q16

a, b, c are in A.P. $\Rightarrow 2b = a + c$ ---(i)

b, c, d are in G.P. $\Rightarrow c^2 = bd$ ---(ii)

$\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. $\Rightarrow \frac{2}{d} = \frac{1}{c} + \frac{1}{e}$ ---(iv)

We need to prove that

a, b, c are in G.P.

$\Rightarrow$ $c^2 = ae$

Now,

$c^2 = bd = 2b \times \frac{d}{2}$

$\Rightarrow$ $c^2 = (a+c) \times \frac{ce}{c+e}$ $\left[\because \frac{2}{d} = \frac{e+c}{ce}\right]$

$\Rightarrow$ $c^2 = \frac{(a+c)ce}{c+e}$

$\Rightarrow$ $c^2(c+e) = ace + c^2e$

$\Rightarrow$ $c^3 + c^2e = ace + c^2e$

$\Rightarrow$ $c^3 = ace$

$\Rightarrow$ $c^2 = ae$

Hence proved.

Q17

a, b, c are in A.P.

$\Rightarrow \quad 2b = a + c - - - - - (1)$

a, x, b are in GP

$\Rightarrow \quad x^2 = ab - - - - - - (2)$

b, y, c are in G.P.

$\Rightarrow \quad y^2 = bc - - - - - - - (3)$

Now

$$2b^2 = x^2 + y^2$$

$$= (ab) + (bc) \qquad [\text{Using (2) and (3)}]$$

$$2b^2 = b(a + c)$$

$$2b^2 = b(2b) \qquad [\text{Using (1)}]$$

$$2b^2 = 2b^2$$

$$LHS = RHS$$

$\Rightarrow \quad 2b^2 = x^2 + y^2$

$\Rightarrow \quad x^2, b^2, y^2$ are in A.P.

Q18

a, b, c are in A.P.

$\Rightarrow \quad 2b = a + c - - - - - (1)$

a, b, d are in GP

$\Rightarrow \quad b^2 = ad - - - - - - (2)$

Now

$$(a - b)^2 = a(d - c)$$

$$[\text{Using (2)}]$$

$$a^2 - 2ab = -ac$$

$$a^2 - 2ab = ab - ac$$

$$a(a - b) = a(b - c)$$

$$a - b = a - c$$

$$2b = a + c$$

$$a + c = a + c, \qquad [\text{Using equation (1)}]$$

$$LHS = RHS$$

$\Rightarrow \quad a, (a - b), (d - c)$ are in G.P.

Q19

a, b, c are in G.P.

$$a, b = ar, \qquad c = ar^2$$

$$\frac{a^2 + ab + b^2}{bc + ca + ab} = \frac{b + a}{c + b}$$

$$\frac{a^2 + a(ar) + a^2r^2}{(ar)(ar^2) + (ar^2)a + a(ar)} = \frac{ar + a}{ar^2 + ar}$$

$$\frac{a^2(1 + r + r^2)}{a^2(r^3 + r^2 + r)} = \frac{a(1 + r)}{a(r^2 + r)}$$

$$\frac{1 + r + r^2}{r(1 + r + r^2)} = \frac{1 + r}{r(1 + r)}$$

$$\frac{1}{r} = \frac{1}{r}$$

$$LHS = RHS$$

so,

$$\frac{a^2 + ab + b^2}{bc + ca + ab} = \frac{b + a}{c + b}$$

Q20

Let r be the common ratio of G.P.

$$a, \qquad b = ar, c = ar^2$$

$$a + b + c = xb$$

$$a + ar + ar^2 = x(ar)$$

$$a(1 + r + r^2) = xar$$

$$r^2 + (1 - x)r + 1 = 0$$

Here, r is real, so

$$D \geq 0$$

$$(1 - x)^2 - 4(1)(1) \geq 0$$

$$1 + x^2 - 2x - 4 \geq 0$$

$$x^2 - 2x - 3 \geq 0$$

$$(x - 3)(x + 1) \geq 0$$

$\Rightarrow \qquad x < -1$ or $x > 3$

Q21

Let the 4th term be ar^3

10th term be ar^9

16th term be ar^{15}

$$ar^9 = \sqrt{(ar^3)(ar^{15})} = ar^9$$

$\therefore$ *4th, 10th, 16th terms are also in GP*

Hence Proved

Q22

Let the A.P. be A, A +D, A +2 D, ... and G.P. be x, xR, xR^2, ... then

$a = A + (p-1)D$, $b = A + (q-1)D$, $c = A + (r-1)D$

=> $a - b = (p-q)D$, $b - c = (q-r)D$, $c - a = (r-p)D$

Also $a = xR^{p-1}$, $b = xR^{q-1}$, $c = xR^{r-1}$

Hence $a^{b-c}.b^{c-a}.c^{a-b} = (xR^{p-1})^{(q-r)D}.(xR^{q-1})^{(r-p)D}.(xR^{r-1})^{(p-q)D}$

$= x^{(q-r+r-p+p-q)D}.\ R^{[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]D}$

$= x^0.\ R^0 = 1.1 = 1$

Ex 20.6

Q1

6 Geometric means between 27 and $\frac{1}{81}$

Let $G_1, G_2, G_3, G_4, G_5, G_6$ be 6 geometric means between $a = 27$ and $b = \frac{1}{81}$.

Then, $27, G_1, G_2, G_3, G_4, G_5, G_6, \frac{1}{81}$ is a G.P. with common ratio r given by

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$= \left(\frac{\frac{1}{81}}{27}\right)^{\frac{1}{6+1}} = \left(\frac{1}{81 \times 27}\right)^{\frac{1}{7}} = \left(\frac{1}{3^7}\right)^{\frac{1}{7}}$$

$\therefore$

$$G_1 = ar = 27\left(\frac{1}{3}\right) = 9$$

$$G_2 = ar^2 = 27 \times \frac{1}{9} = 3$$

$$G_3 = ar^3 = 27 \times \frac{1}{27} = 1$$

$$G_4 = ar^4 = 27 \times \frac{1}{27 \times 3} = \frac{1}{3}$$

$$G_5 = ar^5 = 27 \times \frac{1}{3^5} = \frac{1}{9}$$

$$G_6 = ar^6 = 27 \times \frac{1}{36} = \frac{1}{27}$$

Hence, $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ are 6 geometric means between 27 and $\frac{1}{81}$.

Q2

5 Geometric means between 16 and $\frac{1}{4}$

Let $G_1, G_2, G_3, G_4, G_5,$ be five geometric means between 16 and $\frac{1}{4}$.

$16, G_1, G_2, G_3, G_4, G_5, \frac{1}{4}$ is a G.P. with $a = 16, b = \frac{1}{4}$.

Then,

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$= \left(\frac{\frac{1}{4}}{16}\right)^{\frac{1}{5+1}} = \left(\frac{1}{26}\right)^{\frac{1}{6}} = \frac{1}{2}$$

$\therefore$ $G_1 = ar = 16 \times \frac{1}{2} = 8$

$$G_2 = ar^2 = 16 \times \frac{1}{4} = 4$$

$$G_3 = ar^3 = 16 \times \frac{1}{8} = 2$$

$$G_4 = ar^4 = 16 \times \frac{1}{16} = 1$$

$$G_5 = ar^5 = 16 \times \frac{1}{2^5} = \frac{1}{2}$$

Hence, $8, 4, 2, 1, \frac{1}{2}$ are five geometric means between 16 and $\frac{1}{4}$.

Q3

5 Geometric means between $\frac{32}{9}$ and $\frac{81}{2}$

Let G_1, G_2, G_3, G_4, G_5, be five geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.

Then, $\frac{32}{9}, G_1, G_2, G_3, G_4, G_5, \frac{81}{2}$ is a G.P. with $a = \frac{32}{9}, b = \frac{81}{2}$.

Then,

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$= \left(\frac{\frac{81}{2}}{\frac{32}{9}}\right)^{\frac{1}{5+1}} = \left(\frac{81}{2} \times \frac{9}{32}\right)^{\frac{1}{6}} = \left(\frac{3^6}{2^6}\right) = \frac{3}{2}$$

Thus, $G_1 = ar = \frac{32}{9} \times \frac{3}{2} = \frac{16}{3}$

$$G_2 = ar^2 = \frac{32}{9} \times \frac{9}{4} = 8$$

$$G_3 = ar^3 = \frac{32}{9} \times \frac{27}{8} = 12$$

$$G_4 = ar^4 = \frac{32}{9} \times \frac{3^4}{2^4} = 2 \times 9 = 18$$

$$G_5 = ar^5 = \frac{32}{9} \times \frac{3^5}{2^5} = 27$$

Hence, $\frac{16}{3}, 8, 12, 18, 27$ are five geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.

Q4

(i) 2 and 8

Geometric means between a and $b = \sqrt{ab}$ ---(i)

Here, $a = 2, b = 8$

$\therefore$ Geometric means $= \sqrt{2 \times 8} = \sqrt{16} = 4$

(ii) a^3b and ab^3

Using (i)

$a = a^3b, b = ab^3$

Geometric means $= \sqrt{a^3b \times ab^3} = \sqrt{a^4b^4} = a^2b^2$

(iii) -8 and -2

Using (i)

$a = -8, b = -2$

Geometric means $= \sqrt{-8 \times -2} = \sqrt{16} = 4, -4$

Q5

a is geometric means between 2 and $\frac{1}{4}$.

Then, $a = \sqrt{2 \times \frac{1}{4}}$

$a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Q6

Let the first term of a GP is a and common ratio of the series is r.

The $(n+2)$th term is ar^{n+1}.

The GM of a and ar^{n+1} will be:

$$G_1 = \sqrt{a \cdot ar^{n+1}} = \left(a^2 r^{n+1}\right)^{\frac{1}{2}}$$

Now the n GM in between a and ar^{n+1} are:

$$ar, ar^2, \cdots, ar^n$$

Therefore the product of n GM will be:

$$ar \times ar^2 \times \cdots \times ar^n = a^n r^{1+2+3+\cdots+n}$$

$$= a^n r^{\frac{n(n+1)}{2}}$$

$$= \left(a^2 r^{n+1}\right)^{\frac{n}{2}}$$

$$= G_1^n$$

Hence it is proved.

Q7

Given,

A.M = 25

G.M = 20

Now, $A.M = \frac{a+b}{2} = 25$

and, $G.M = \sqrt{ab} = 20$

$a + b = 50, ab = 400$

$(a-b) = \sqrt{(a+b)^2 - 4ab}$

$= \sqrt{(50)^2 - 1600}$

$= \sqrt{2500 - 1600}$

$= \pm 30$

$a - b = \pm 30$

$\underline{a + b = 50}$

$2a = 80$

$a = 40$

Also, $2b = 20$

$b = 10$

$\therefore$ The numbers are 40,10

Q8

A.M. between two numbers a and b $(a > b)$ is $\frac{a+b}{2}$

Also, geometric mean between 2 numbers is $\sqrt{ab}$

Given,

$$A.M = 2G.M$$

$$\Rightarrow \quad \frac{a+b}{2} = 2\sqrt{ab}$$

$$a + b = 4\sqrt{ab}$$

$$\frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$$

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{2+1}{2-1} = \frac{3}{1} \qquad \text{[By componendo and dividendo]}$$

$$\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2} = \frac{\left(\sqrt{3}\right)^2}{(1)^2}$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$$

By componendo and dividendo, we get

$$\frac{\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{a}{b} = \frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}-1\right)^2} = \frac{3+1+2\sqrt{3}}{3+1-2\sqrt{3}}$$

$$= \frac{4+2\sqrt{3}}{4-2\sqrt{3}}$$

$$\frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

Thus, $a : b = \left(2+\sqrt{3}\right) : \left(2-\sqrt{3}\right)$.

Q9

Let A.M = A between a and b

G.M = G_1 and G_2 between a and b

$\Rightarrow \quad A = \frac{a+b}{2}$

$a, G_1 G_2, b$ is G.P. with common ratio $r = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{3}} = a^{\frac{1}{3}} b^{\frac{2}{3}}$$

Now,

$$G_1^{\,2} = a^2\left(\frac{b}{a}\right)^{\frac{2}{3}}$$

$$G_2^{\,2} = a^{\frac{2}{3}} b^{\frac{4}{3}}$$

Then,

$$\frac{G^2_1}{G_2} + \frac{G^2_2}{G_1} = \frac{a^2\left(\frac{b}{a}\right)^{\frac{2}{3}}}{a^{\frac{1}{3}} b^{\frac{2}{3}}} + \frac{a^{\frac{2}{3}} b^{\frac{4}{3}}}{a^2\left(\frac{b}{a}\right)^{\frac{2}{3}}}$$

$$= a^{2-\frac{2}{3}-\frac{1}{3}} b^{\frac{2}{3}-\frac{2}{3}} + a^{\frac{2}{3}-2+\frac{2}{3}} b^{\frac{4}{3}-\frac{2}{3}}$$

$$= a^{\frac{3}{3}} b^0 + a^0 b$$

$$= a + b$$

$$= 2a$$

$$= \text{RHS}$$

Q10

A.M. of root of quadratic equation is A.

G.M. of root of quadretic equation is G.

Then,

$$\frac{a+b}{2} = A, \; F = \sqrt{ab}$$

The equation having a and b as roots of quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - (a+b)x + ab = 0$$

$$x^2 - 2Ax + G^2 = 0$$

Q11

Let a, b be the numbers.

$$a + b = 6\,(\text{G.M of } a, b)$$

$$a + b = 6\sqrt{ab}$$

$$\frac{a+b}{2\sqrt{ab}} = \frac{3}{1}$$

Applying components and dividens,

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3+1}{3-1}$$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2 = \frac{4}{2}$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{2}}{1}$$

Again applying components and dividends,

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}-\sqrt{a}+\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\left(\frac{a}{b}\right) = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^2$$

$$= \frac{2+1+2\sqrt{2}}{2+1-2\sqrt{2}}$$

$$= \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

$$a : b = \left(3+2\sqrt{2}\right) : \left(3-2\sqrt{2}\right)$$

Q12

Let quadratic equation be $(x-\alpha)(x-\beta) = 0$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

Roots are α, β

Here,

$$\frac{\alpha+\beta}{2} = 8, \quad \sqrt{\alpha\beta} = 5$$

$$\alpha+\beta = 16, \quad \alpha\beta = 25$$

$\therefore$ Required quadratic equation is,

$$x^2 - 16x + 25 = 0$$

Q13

The AM and GM of a and b will be:

$$\frac{a+b}{2}=10 \Rightarrow a+b=20 \qquad \text{......(1)}$$

$$\sqrt{ab}=8 \Rightarrow ab=64$$

Now

$$\begin{aligned} a-b &= \sqrt{(a+b)^2-4ab} \\ &= \sqrt{20^2-4\cdot 64} \\ &= \sqrt{400-256} \\ &= \sqrt{144} \end{aligned}$$

$$a-b=12 \qquad \text{......(2)}$$

Adding (1) and (2)

$$2a=32$$

$$a=16$$

From (1)

$$b=20-16=4$$

Thus the numbers are $a=16$ and $b=4$.