Ex 14.1

Differentials Errors and Approximation Ex 14.1 Q1

Let $x = \frac{\pi}{2}$, $x + \Delta x = \frac{22}{14}$

$$\Delta x = \frac{22}{14} - x$$
$$\Delta x = \left(\frac{22}{14} - \frac{\pi}{2}\right)$$

$$y = \sin x$$
$$\frac{dy}{dx} = \cos x$$
$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = \frac{\cos \pi}{2}$$
$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = 0$$

$$\therefore \quad \Delta y = \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} \times \Delta ax$$
$$= 0 \times \left(\frac{22}{14} - \frac{\pi}{2}\right)$$
$$\Delta y = 0$$

So, there is no change in y.

Differentials Errors and Approximation Ex 14.1 Q2

Let $x = 10,\ x + \Delta x = 9.8$

$$\Delta x = 9.8 - x$$
$$= 9.8 - 10$$
$$\Delta x = -0.2$$

$$y = \frac{4}{3}\pi x^3 \qquad \text{[volume of sphere]}$$
$$\frac{dy}{dx} = 4\pi r^2$$
$$\left(\frac{dy}{dx}\right)_{x=10} = 4\pi\,(10)^2$$
$$\left(\frac{dy}{dx}\right)_{x=10} = 400\pi \ \text{cm}^2$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=10} \times \Delta x$$
$$= 400\pi \times (-0.2)$$
$$\Delta y = -80\pi \ \text{cm}^3$$

So, approximate diecocase in volume is $80\pi \ \text{cm}^3$.

Differentials Errors and Approximation Ex 14.1 Q3

Let $x = 10,\ x + \Delta x = 10 + \frac{k}{100} \times 10$

$$x + \Delta x = 10 + 0.k$$
$$\Rightarrow \quad \Delta x = 10 + 0.k - 10$$
$$\Delta x = 0.k$$

$$y = \pi r^2$$
$$\frac{dy}{dx} = 2\pi r$$
$$\left(\frac{dy}{dx}\right)_{x=10} = 2\pi\,(10)$$
$$\left(\frac{dy}{dx}\right)_{x=10} = 20\pi \ \text{cm}$$

So,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=10} \times \Delta x$$
$$= (20\pi) \times (0.k)$$
$$\Delta y = 2k\pi \text{cm}^2$$

Area of the plate increases by $2k\pi \ \text{cm}^2$.

Differentials Errors and Approximation Ex 14.1 Q4

Let $\quad \text{length}(L) = x$

$$x + \Delta x = x + \frac{x}{100}$$
$$\Delta x = 0.01x$$

Now,

$$y = 6x^2$$
$$\frac{dy}{dx} = 12x \ \text{cm}$$

So,

$$\Delta y = \left(\frac{dy}{dx}\right) \times \Delta x$$
$$= (12x)(0.01x)$$
$$\Delta y = 0.12x^2 \ \text{cm}^2$$
$$= 6(0.02)x^2$$
$$= 2\% \text{ of } 6x^2$$

Percentage error in area is 2%.

Differentials Errors and Approximation Ex 14.1 Q5

Let x be the radius of sphere,

$$\Delta x = 0.1\% \text{ of } x$$
$$\Delta x = 0.001x$$

Now,

Let y = volume of sphere

$$y = \frac{4}{3}\pi x^3$$
$$\frac{dy}{dx} = 4\pi x^2$$
$$\Delta y = \left(\frac{dy}{dx}\right) \times \Delta x$$
$$= 4\pi x^2 \times 0.001x$$
$$= \frac{4}{3}\pi x^3 (0.003)$$
$$= \frac{0.3}{100} \times y$$
$$\Delta y = 0.3\% \text{ of y}$$

So, percentage error in volume of error = 0.3%.

Differentials Errors and Approximation Ex 14.1 Q6

Given, $\Delta v = -\frac{1}{2}\%$

$$= -0.5\%$$
$$\Delta v = -0.005$$

Here,

$$pv^{1.4} = k$$

Taking log on both the sides,

$$\log\left(pv^{1.4}\right) = \log k$$
$$\log p + 1.4 \log v = \log k$$

Differentiate it with respect to v,

$$\frac{1}{p}\frac{dp}{dv} + \frac{1.4}{v} = 0$$
$$\frac{dp}{dv} = -\frac{1.4}{v}p$$

$$\Delta p = \left(\frac{dp}{dv}\right)\Delta v$$
$$= -\frac{1.4p}{v} \times (-0.005)$$
$$\Delta p = \frac{1.4p(0.005)}{v}$$
$$\Delta p \text{ in } \% = \frac{\Delta p}{p} \times 100$$
$$= \frac{1.4p(0.005)}{p} \times 100$$
$$= 0.7\%$$

So, percentage error in p = 0.7%.

Differentials Errors and Approximation Ex 14.1 Q7

Let h be the height of the cone, and α be the semivertide angle.

Here vertgide angle α is fixed.

$$\Delta h = k\% \text{ of } h$$
$$= \frac{k}{100} \times h$$
$$\Delta h = (0.0k)h$$

(i) $$A = \pi r(r+l)$$
$$= \pi(r^2 + rl)$$
$$= \pi(r^2) + r\sqrt{h^2+r^2} \qquad \left[\text{Since, in a cone } l^2 = h^2 + r^2\right]$$

$$r = h\tan\alpha \qquad [\text{from figure}]$$
$$A = \pi\left[h^2\tan^2\alpha + h\tan\alpha\sqrt{h^2 + h^2\tan^2\alpha}\right]$$
$$= \pi\left[h^2\tan^2\alpha + h\tan\alpha\sqrt{h^2(1+\tan^2\alpha)}\right]$$
$$= \pi\left[h^2\tan^2\alpha + h\tan\alpha \times h\sec\alpha\right]$$
$$= \pi h^2\left[\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\sin\alpha}{\cos^2\alpha}\right]$$
$$A = \pi h^2\frac{\sin\alpha(\sin\alpha+1)}{\cos^2\alpha}$$

Differentiating with respect to h as α is fixed.

$$\frac{dA}{dh} = 2\pi h\frac{\sin\alpha(\sin\alpha+1)}{\cos^2\alpha}$$

So,

$$\Delta A = \frac{dA}{dh} \times \Delta h$$
$$\Delta a = \frac{2\pi h(0.0kh)\sin\alpha(\sin\alpha+1)}{\cos^2\alpha}$$
$$\Delta A \text{ in \% of } A = \frac{2\pi h(0.0kh)\sin\alpha(\sin\alpha+1)}{\cos^2\alpha} \times \frac{100}{A}$$
$$= \frac{2\pi kh^2 \times \sin\alpha(\sin\alpha+1)}{\cos^2\alpha} \times \frac{\cos^2\alpha}{\pi h^2\sin\alpha(\sin\alpha+1)}$$
$$= 2k\%$$

So, percentage increase in area $= 2k\%$.

(ii)

Let $$v = \text{volume of cone}$$
$$= \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(h\tan\alpha)^2 h$$
$$v = \frac{\pi}{3}\tan^2\alpha h^2$$

Differentiating it with respect to h treating α as constant,

$$\frac{dv}{dh} = \pi\tan^2\alpha \times h^2$$
$$\Delta v = \left(\frac{dv}{dh}\right)\Delta h$$
$$= \pi\tan^2\alpha h^2 \times (0.0kh)$$
$$\Delta v = 0.0k\pi h^3\tan^2\alpha$$

Percentage increase in $v = \dfrac{\Delta v \times 100}{v}$

$$= \frac{0.0k\pi h^3\tan^2\alpha \times 100}{\frac{\pi}{3}\tan^2\alpha \times h^3}$$
$$= 3k\%$$

So, percentage increase in volume $= 3k\%$.

Differentials Errors and Approximation Ex 14.1 Q8

Let $\quad$ error in radius$(r) = x\%$ of r

$$\Delta r = 0.0xr$$

Let $\quad v =$ volume of sphere

$$v = \frac{4}{3}\pi r^3$$

Differentiating it with respect to r,

$$\frac{dv}{dr} = 4\pi r^2$$

So,

$$\Delta v = \left(\frac{dv}{dr}\right) \times \Delta r$$
$$= \left(4\pi r^2\right)(0.0x)r$$
$$\Delta v = 0.0x \times 4\pi r^3$$

$$\text{Percentage of error in volume} = \frac{\Delta v \times 100}{v}$$
$$= \frac{(0.0x)\,4\pi r^3 \times 100}{\frac{4}{3}\pi r^3}$$
$$= 3x\%$$

Percentage of error in volume $= 3$(percentage of error in radius).

Differentials Errors and Approximation Ex 14.1 Q9(i)

Let $\quad x = 25,\ x + \Delta x = 25.02$

$$\Delta x = 25.02 - 25$$
$$\Delta x = 0.02$$

Let $\quad y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$
$$\left(\frac{dy}{dx}\right)_{x=25} = \frac{1}{2\sqrt{25}}$$
$$\left(\frac{dy}{dx}\right)_{x=25} = \frac{1}{10}$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=25} \times x$$
$$= \frac{1}{10}(0.02)$$
$$\Delta y = 0.002$$
$$\sqrt{25.02} = y + \Delta y$$
$$= \sqrt{25} + 0.002$$
$$= 5 + 0.002$$

$$\sqrt{25.02} = 5.002$$

Differentials Errors and Approximation Ex 14.1 Q9(ii)

Let $x = 0.008,\ x + \Delta x = 0.009$

$\Delta x = 0.009 - 0.008$

$\Delta x = 0.001$

Let $y = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}$$

$$\left(\frac{dy}{dx}\right)_{x=0.008} = \frac{1}{3(0.008)^{\frac{2}{3}}}$$

$$= \frac{1}{3(0.04)}$$

$$= \frac{100}{12}$$

$$= 0.8333$$

So,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=0.008} \times \Delta x$$

$$= (0.8333)(0.001)$$

$$\Delta y = 0.008333$$

$$(0.009)^{\frac{1}{3}} = y + \Delta y$$

$$= (x)^{\frac{1}{3}} + 0.008333$$

$$= (0.008)^{\frac{1}{3}} + 0.008333$$

$$= 0.52 + 0.008333$$

$$(0.009)^{\frac{1}{3}} = 0.208333$$

Differentials Errors and Approximation Ex 14.1 Q9(iii)

Let $x = 0.008,\ x + \Delta x = 0.007$

$\Delta x = 0.007 - 0.008$

$\Delta x = -0.001$

Let $y = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3(x)^{\frac{2}{3}}}$$

$$\left(\frac{dy}{dx}\right)_{x=0.008} = \frac{1}{3(0.008)^{\frac{2}{3}}}$$

$$= \frac{100}{12}$$

$$\left(\frac{dy}{dx}\right)_{x=0.008} = 8.333$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=0.008} \times \Delta x$$

$$= (8.333)(-0.001)$$

$$\Delta y = -0.008333$$

$$(0.007)^{\frac{1}{3}} = y + \Delta y$$

$$= x^{\frac{1}{3}} - 0.008333$$

$$= (0.008)^{\frac{1}{3}} - 0.008333$$

$$= 0.2 - 0.008333$$

$$(0.007)^{\frac{1}{3}} = 0.191667$$

Differentials Errors and Approximation Ex 14.1 Q9(iv)

Let $x = 400,\ x + \Delta x = 401$

$\Delta x = 401 - 400$

$\Delta x = 1$

Let $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)_{x=400} = \frac{1}{2\sqrt{400}}$$

$$= \frac{1}{40}$$

$$\left(\frac{dy}{dx}\right)_{x=400} = 0.025$$

So,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=400} \times \Delta x$$

$$= (0.025)(1)$$

$$= 0.025$$

$$\sqrt{401} = y + \Delta y$$

$$= \sqrt{x} + 0.025$$

$$= \sqrt{400} + 0.025$$

$$= 20 + 0.025$$

$$\sqrt{401} = 20.025$$

Differentials Errors and Approximation Ex 14.1 Q9(v)

Let $x = 16,\ x + \Delta x = 15$

$\Delta x = 15 - 16$

$\Delta x = -1$

Let $y = x^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$

$$\left(\frac{dy}{dx}\right)_{x=16} = \frac{1}{4(16)^{\frac{3}{4}}}$$

$$= \frac{1}{32}$$

$$= 0.03125$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=16} \times \Delta x$$

$$= (0.03125)(-1)$$

$$\Delta y = -0.03125$$

$$(15)^{\frac{1}{4}} = y + \Delta y$$

$$= (x)^{\frac{1}{4}} - 0.03125$$

$$= (16)^{\frac{1}{4}} - 0.03125$$

$$= 2 - 0.03125$$

$$(15)^{\frac{1}{4}} = 1.96875$$

Differentials Errors and Approximation Ex 14.1 Q9(vi)

Let $x = 256,\ x + \Delta x = 255$

$\Delta x = 255 - 256$

$\Delta x = -1$

Let $y = x^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$

$$\left(\frac{dy}{dx}\right)_{x=256} = \frac{1}{4(256)^{\frac{3}{4}}} = \frac{1}{256} = 0.00391$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=256} \times \Delta x = (0.00391)(-1)$$

$\Delta y = -0.00391$

$$(255)^{\frac{1}{4}} = y + \Delta y = (x)^{\frac{1}{4}} + (-0.00391) = (256)^{\frac{1}{4}} - 0.00391 = 4 - 0.00391$$

$$(255)^{\frac{1}{4}} = 3.99609$$

Differentials Errors and Approximation Ex 14.1 Q9(vii)

Let $x = 2,\ x + \Delta x = 2.002$

$\Delta x = 2.002 - 2$

$\Delta x = 0.002$

Let $y = \frac{1}{x^2}$

$$\frac{dy}{dx} = -\frac{2}{x^3}$$

$$\left(\frac{dy}{dx}\right)_{x=2} = -\frac{2}{8} = -0.25$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=2} \times \Delta x = (-0.25)(0.002)$$

$\Delta y = -0.0005$

Now,

$$\frac{1}{(2.002)^3} = y + \Delta y = \frac{1}{x^2} + (-0.0005) = \frac{1}{4} - 0.0005 = 0.25 - 0.0005$$

$$\frac{1}{(2.002)^3} = 0.2495$$

Differentials Errors and Approximation Ex 14.1 Q9(viii)

Let $x = 4,\ x + \Delta x = 4.04$
$\Delta x = 4.04 - 4$
$\Delta x = 0.04$

Let $y = \log x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\left(\frac{dy}{dx}\right)_{x=4} = \frac{1}{4} = 0.25$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=4} \times \Delta x = (0.25)(0.04)$$

$\Delta y = 0.01$

$$\begin{aligned} \log_e 4.04 &= y + \Delta y \\ &= \log x + (0.01) \\ &= \log_e 4 + 0.01 \\ &= \frac{\log_e 4}{\log_{10} e} + 0.01 \\ &= \frac{0.6021}{0.4343} + 0.01 \qquad \left[\text{Since, } \log_a b = \frac{\log_c b}{\log_c a}\right] \\ &= 1.38637 + 0.01 \end{aligned}$$

$\log_e 4.04 = 1.39637$

Differentials Errors and Approximation Ex 14.1 Q9(ix)

Let $x = 10,\ x + \Delta x = 10.02$
$\Delta x = 10.02 - 10$
$\Delta x = 0.02$

Let $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\left(\frac{dy}{dx}\right)_{x=10} = \frac{1}{10}$$

$$\left(\frac{dy}{dx}\right)_{x=10} = 0.1$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=10} \times \Delta x = (0.1)(0.02)$$

$\Delta y = 0.002$

$$\begin{aligned} \log_e (10.02) &= y + \Delta y \\ &= \log_e x + 0.002 \\ &= \log_e 10 + 0.002 \\ &= 2.3026 + 0.002 \end{aligned}$$

$\log_e (10.02) = 2.3046$

Differentials Errors and Approximation Ex 14.1 Q9(x)

Let $x = 10,\ x + \Delta x = 10.1$

$\Delta x = 10.1 - 10$

$\Delta x = 0.1$

Let $y = \log_{10} x$

$= \dfrac{\log_e x}{\log_e 10}$ $\left[\text{Since, } \log_a b = \dfrac{\log_c a}{\log_c b}\right]$

$\left(\dfrac{dy}{dx}\right) = \dfrac{1}{x \log_e 10}$

$\left(\dfrac{dy}{dx}\right)_{x=10} = \dfrac{1}{10 \log_e 10}$

$\Delta y = \left(\dfrac{dy}{dx}\right)_{x=10} \times \Delta x$

$= \dfrac{1}{10(\log_e 10)} \times 0.1$

$\Delta y = \dfrac{0.01}{(\log_e 10)}$

$\log_{10}(10.1) = y + \Delta y$

$= \log_{10} x + \dfrac{0.01}{\log_e 10}$

$= \log_{10} 10 + 0.01 \log_{10} e$ $\left[\text{Since, } \log_a b = \dfrac{1}{\log_b a}\right]$

$= 1 + (0.01)(0.4343)$

$\log_{10}(10.1) = 1.004343$

Differentials Errors and Approximation Ex 14.1 Q9(xi)

Let $x = 60°,\ x + \Delta x = 61°$

$\Delta x = 61° - 60°$

$\Delta x = 1° = \dfrac{\pi}{18°} = 0.01745$

Let $y = \cos x$

$\dfrac{dy}{dx} = -\sin x$

$\left(\dfrac{dy}{dx}\right)_{x=60°} = -\sin(60°)$

$= -\dfrac{\sqrt{3}}{2}$

$= -0.866$

$\Delta y = \left(\dfrac{dy}{dx}\right)_{x=60°} \times (\Delta x)$

$= (-0.866)(0.01745)$

$= -0.01511$

So,

$\cos 61° = y + \Delta y$

$= \cos 60° - 0.01511$

$= \dfrac{1}{2} - 0.01511$

$= 0.5 - 0.01511$

$\cos 61° = 0.48489$

Differentials Errors and Approximation Ex 14.1 Q9(xii)

Let $\quad x = 25,\ x + \Delta x = 25.1$

$\Delta x = 25.1 - 25$

$\Delta x = 0.1$

Let $\quad y = \frac{1}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{2}{2x^{\frac{3}{2}}}$$

$$\left(\frac{dy}{dx}\right)_{x=25} = -\frac{1}{2(25)^{\frac{3}{2}}}$$

$$= -\frac{1}{250}$$

$$= -0.004$$

Now,

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=25} \times (\Delta x)$$

$$= (-0.004)(0.1)$$

$$= -0.0004$$

$$\frac{1}{\sqrt{25.1}} = y + \Delta y$$

$$= \frac{1}{\sqrt{x}} + (-0.0004)$$

$$= \frac{1}{\sqrt{25}} - 0.0004$$

$$= \frac{1}{5} - 0.0004$$

$$= 0.2 - 0.0004$$

$$\frac{1}{\sqrt{25.1}} = 0.1996$$

Differentials Errors and Approximation Ex 14.1 Q9(xiii)

Let $\quad x = \frac{\pi}{2},\ x + \Delta x = \frac{22}{14}$

$$\Delta x = \left(\frac{22}{14} - \frac{\pi}{2}\right)$$

$$\Delta x = \sin x$$

Let $\quad y = \sin x$

$$\frac{dy}{dx} = \cos x$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = \cos\frac{\pi}{2}$$

$$\left(\frac{dy}{dx}\right)_{\frac{\pi}{2}} = 0$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} \times (\Delta x)$$

$$= 0 \times \left(\frac{22}{14} - \frac{\pi}{2}\right)$$

$$= 0$$

So,

$$\sin\left(\frac{22}{14}\right) = y + \Delta y$$

$$= \sin x + 0$$

$$= \sin\left(\frac{\pi}{2}\right)$$

$$\sin\left(\frac{22}{14}\right) = 1$$

Differentials Errors and Approximation Ex 14.1 Q9(xiv)

Let $x = \frac{\pi}{3},\ x + \Delta x = \frac{11\pi}{36}$

$$\Delta x = \frac{11\pi}{36} - \frac{\pi}{3}$$
$$= -\frac{\pi}{36}$$
$$= -\frac{22}{7 \times 36}$$
$$= -0.0873$$

Let $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$
$$\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} = -\sin\frac{\pi}{3}$$
$$= -\frac{\sqrt{3}}{2}$$
$$= -0.866$$
$$\Delta y = \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}} \times (\Delta x)$$
$$= (-0.866)(-0.0873)$$
$$= 0.0756$$

$$\cos\left(\frac{11\pi}{36}\right) = y + \Delta y$$
$$= \cos x + (0.0756)$$
$$= \cos\frac{\pi}{3} + 0.0756$$
$$= \frac{1}{2} + 0.0756$$
$$= 0.5 + 0.0756$$

$$\cos\frac{11\pi}{36} = 0.7546$$

Differentials Errors and Approximation Ex 14.1 Q9(xv)

Let $x = 36,\ x + \Delta x = 37$

$$\Delta x = 37 - 36$$
$$= 1$$

Let $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$
$$\left(\frac{dy}{dx}\right)_{x=36} = \frac{1}{2\sqrt{36}}$$
$$= \frac{1}{12}$$
$$= 0.0833$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=36} \times (\Delta x)$$
$$= (0.0833)(1)$$
$$= 0.0833$$

$$\sqrt{37} = y + \Delta y$$
$$= \sqrt{x} + 0.0833$$
$$= \sqrt{36} + 0.0833$$

$$\sqrt{37} = 6.0833$$

Differentials Errors and Approximation Ex 14.1 Q9(xvi)

Let $x = 81,\ x + \Delta x = 80$

$\Delta x = 80 - 81$

$= -1$

Let $y = x^{\frac{1}{4}}$

$\frac{dy}{dx} = \frac{1}{4(81)^{\frac{3}{4}}}$

$= \frac{1}{108}$

$= 0.00926$

$\Delta y = \left(\frac{dy}{dx}\right)_{x=81} \times (\Delta x)$

$= (0.00926)(-1)$

$= -0.00926$

$(80)^{\frac{1}{4}} = y + \Delta y$

$= x^{\frac{1}{4}} - 0.00926$

$= (81)^{\frac{1}{4}} - 0.00926$

$= 3 - 0.00926$

$(80)^{\frac{1}{4}} = 2.99074$

Differentials Errors and Approximation Ex 14.1 Q9(xvii)

Let $x = 27,\ x + \Delta x = 29$

$\Delta x = 29 - 27$

$= 2$

Let $y = x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3(x)^{\frac{2}{3}}}$

$\left(\frac{dy}{dx}\right)_{x=27} = \frac{1}{3(27)^{\frac{2}{3}}}$

$= \frac{1}{27}$

$= 0.03704$

$\Delta y = \left(\frac{dy}{dx}\right)_{x=27} \times (\Delta x)$

$= (0.03704)(2)$

$\Delta y = 0.07408$

$(28)^{\frac{1}{3}} = y + \Delta y$

$= x^{\frac{1}{3}} + 0.07408$

$= (27)^{\frac{1}{3}} + 0.07408$

$= 3 + 0.07408$

$(29)^{\frac{1}{3}} = 3.07408$

Differentials Errors and Approximation Ex 14.1 Q9(xviii)

Let $x = 64,\ x + \Delta x = 66$

$\Delta x = 66 - 64$

$= 2$

Let $y = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3(x)^{\frac{2}{3}}}$$

$$\left(\frac{dy}{dx}\right)_{x=64} = \frac{1}{3(64)^{\frac{2}{3}}}$$

$$= \frac{1}{48}$$

$$= 0.020833$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=64} \times (\Delta x)$$

$$= (0.020833)(2)$$

$$= 0.041666$$

$$(66)^{\frac{1}{3}} = y + \Delta y$$

$$= x^{\frac{1}{3}} + 0.041666$$

$$= (64)^{\frac{1}{3}} + 0.041666$$

$$= 4 + 0.041666$$

$$(66)^{\frac{1}{3}} = 4.041666$$

Differentials Errors and Approximation Ex 14.1 Q9(xix)

Let $x = 25,\ x + \Delta x = 26$

$\Delta x = 26 - 25$

$= 1$

Let $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)_{x=25} = \frac{1}{2\sqrt{25}}$$

$$= \frac{1}{10}$$

$$= 0.1$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=25} \times (\Delta x)$$

$$= (0.1)(1)$$

$$= 0.1$$

$$\sqrt{26} = y + \Delta y$$

$$= \sqrt{x} + 0.01$$

$$= \sqrt{25} + 0.1$$

$$\sqrt{26} = 5.1$$

Differentials Errors and Approximation Ex 14.1 Q9(xx)

Let $x = 0.49,\ x + \Delta x = 0.487$

$\Delta x = 0.48 - 0.49$

$= -0.01$

Let $y = \sqrt{x}$

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{2\sqrt{x}}$

$\left(\frac{dy}{dx}\right)_{x=0.49} = \frac{1}{2\sqrt{0.49}}$

$= \frac{1}{1.4}$

$= 0.71428$

$\Delta y = \left(\frac{dy}{dx}\right)_{x=0.49} \times (\Delta x)$

$= (0.71428)(-0.01)$

$\Delta y = -0.0071428$

$\sqrt{37} = y + \Delta y$

$= \sqrt{0.49} - 0.0071428$

$= 0.7 - 0071428$

$\sqrt{0.48} = 0.6928572$

Differentials Errors and Approximation Ex 14.1 Q9(xxi)

Let $x = 81,\ x + \Delta x = 82$

$\Delta x = 82 - 81$

$= 1$

Let $y = x^{\frac{1}{4}}$

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{4x^{\frac{3}{4}}}$

$\left(\frac{dy}{dx}\right)_{x=81} = \frac{1}{4(81)^{\frac{3}{4}}}$

$= \frac{1}{108}$

$= 0.009259$

$\Delta y = \left(\frac{dy}{dx}\right)_{x=81} \times (\Delta x)$

$= (0.008259)(1)$

$= 0.009259$

$(82)^{\frac{1}{4}} = y + \Delta y$

$= x^{\frac{1}{4}} + 0.009259$

$= (81)^{\frac{1}{4}} + 0.009259$

$(82)^{\frac{1}{4}} = 3.009259$

Differentials Errors and Approximation Ex 14.1 Q9(xxii)

Let $x = \frac{16}{81}$, $x + \Delta x = \frac{17}{81}$

$$\Delta x = \frac{17}{81} - \frac{16}{81}$$
$$= \frac{1}{81}$$

Let $y = x^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$
$$\left(\frac{dy}{dx}\right)_{x=\frac{16}{81}} = \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}$$
$$= \frac{27}{32}$$
$$= 0.84375$$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=\frac{16}{81}} \times (\Delta x)$$
$$= (0.84375)\left(\frac{1}{81}\right)$$
$$= 0.01041$$

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} = y + \Delta y$$
$$= \left(\frac{16}{81}\right)^{\frac{1}{4}} + 0.01041$$
$$= 0.6666 + 0.01041$$

$$\left(\frac{17}{81}\right)^{\frac{1}{4}} = 0.67707$$

Differentials Errors and Approximation Ex 14.1 Q9(xxiii)

Let $x = 32$, $x + \Delta x = 33$

$$\Delta x = 33 - 32$$
$$= 1$$

Let $y = x^{\frac{1}{5}}$

$$\frac{dy}{dx} = \frac{1}{5x^{\frac{4}{5}}}$$
$$\left(\frac{dy}{dx}\right)_{x=32} = \frac{1}{5(32)^{\frac{4}{5}}}$$
$$= \frac{1}{80}$$
$$= 0.0125$$

$$\therefore \quad \Delta y = \left(\frac{dy}{dx}\right)_{x=32} \times (\Delta x)$$
$$= (0.0125)(1)$$
$$\Delta y = 0.0125$$

$$(33)^{\frac{1}{5}} = y + \Delta y$$
$$= x^{\frac{1}{5}} + 0.0125$$
$$= (32)^{\frac{1}{5}} + 0.0125$$

$$(33)^{\frac{1}{5}} = 2.0125$$

Differentials Errors and Approximation Ex 14.1 Q9(xxiv)

Let $x = 36,\ x + \Delta x = 36.6$

$\Delta x = 36.6 - 36$

$= 0.6$

Let $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)_{x=36} = \frac{1}{2\sqrt{36}} = \frac{1}{12} = 0.0833$$

$$\therefore \quad \Delta y = \left(\frac{dy}{dx}\right)_{x=36} \times (\Delta x) = (0.0833)(0.6) = 0.04998$$

$$\sqrt{36.6} = y + \Delta y = \sqrt{x} + 0.04998 = \sqrt{36} + 0.04998$$

$$\sqrt{36.6} = 6.04998$$

Differentials Errors and Approximation Ex 14.1 Q9(xxv)

Let $x = 27,\ x + \Delta x = 25$

$\Delta x = 25 - 27$

$= -2$

Let $y = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}$$

$$\left(\frac{dy}{dx}\right)_{x=27} = \frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{27} = 0.037$$

$$\therefore \quad \Delta y = \left(\frac{dy}{dx}\right)_{x=27} \times (\Delta x) = (0.037)(-2) = -0.074$$

$$(25)^{\frac{1}{3}} = y + \Delta y = x^{\frac{1}{3}} + (-0.074) = (27)^{\frac{1}{3}} - 0.074 = 3 - 0.074$$

$$(25)^{\frac{1}{3}} = 2.926$$

Differentials Errors and Approximation Ex 14.1 Q9(xxvi)

Let $y = f(x) = \sqrt{x}$, $x = 49$ and $x + \Delta x = 49.5$
Then $\Delta x = 0.5$
For $x = 49$ we have
$y = \sqrt{49} = 7$

$dx = \Delta x = 0.5$
$y = \sqrt{x}$
$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$
$\Rightarrow \left(\frac{dy}{dx}\right)_{x=49} = \frac{1}{2 \times 7} = \frac{1}{14}$
$\therefore dy = \frac{dy}{dx} dx$
$\Rightarrow dy = \frac{1}{14}(0.5) = \frac{5}{140}$
$\Rightarrow \Delta y = \frac{1}{28}$
Hence,
$\sqrt{49.5} = y + \Delta y = 7 + \frac{1}{28} = 7 + 0.0357 = 7.0357$

Differentials Errors and Approximation Ex 14.1 Q9(xxvii)

Define a function $y = x^{3/2}$
For $x = 4$, $y = 8$
$x + \Delta x = 3.968 \Rightarrow \Delta x = 3.968 - 4 = -0.032$
$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$
$\Rightarrow dy = \left(\frac{3}{2} x^{1/2}\right) dx$
$\Rightarrow \Delta y|_{x=4} \simeq (3)\Delta x$
$\Rightarrow \Delta y|_{x=4} \simeq 3 \times (-0.032) = -0.096$
$(3.968)^{3/2} = y + \Delta y = 8 - 0.096$
$= 7.904$

Differentials Errors and Approximation Ex 14.1 Q9(xxviii)

Let $y = f(x) = x^5$, $x = 2$ and $x + \Delta x = 1.999$
Then $\Delta x = -0.001$
For $x = 2$ we have
$y = (2)^5 = 32$

$dx = \Delta x = -0.001$
$y = x^5$
$\Rightarrow \frac{dy}{dx} = 5x^4$
$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 5(2)^4 = 80$
$\therefore dy = \frac{dy}{dx} dx$
$\Rightarrow dy = 80(-0.001) = -0.080$
$\Rightarrow \Delta y = -0.080$
Hence,
$(1.999)^5 = y + \Delta y = 32 - 0.080 = 31.920$

Differentials Errors and Approximation Ex 14.1 Q9(xxix)

Let $y = f(x) = \sqrt{x}$, $x = 0.09$ and $x + \Delta x = 0.082$
Then $\Delta x = -0.008$
For $x = 0.09$ we have
$y = \sqrt{0.09} = 0.3$

$dx = \Delta x = -0.008$
$y = \sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2 \times \sqrt{0.09}} = \frac{1}{2 \times 0.3} = \frac{1}{0.6}$$

$$\therefore dy = \frac{dy}{dx} dx$$

$$\Rightarrow dy = \frac{1}{0.6}(-0.008)$$

$$\Rightarrow \Delta y = -\frac{8}{600}$$

Hence,

$$\sqrt{0.082} = y + \Delta y = 0.3 - \frac{8}{600} = 0.3 - 0.0133 = 0.2867$$

Differentials Errors and Approximation Ex 14.1 Q10

Let $x = 2$ and $\Delta x = 0.01$. Then, we have:

$$f(2.01) = f(x + \Delta x) = 4(x + \Delta x)^2 + 5(x + \Delta x) + 2$$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$$\therefore f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x) \cdot \Delta x \qquad (\text{as } dx = \Delta x)$$

$$\begin{aligned}
\Rightarrow f(2.01) &\approx (4x^2 + 5x + 2) + (8x + 5)\Delta x \\
&= \left[4(2)^2 + 5(2) + 2\right] + \left[8(2) + 5\right](0.01) \qquad [\text{as } x = 2,\ \Delta x = 0.01] \\
&= (16 + 10 + 2) + (16 + 5)(0.01) \\
&= 28 + (21)(0.01) \\
&= 28 + 0.21 \\
&= 28.21
\end{aligned}$$

Hence, the approximate value of $f(2.01)$ is 28.21.

Differentials Errors and Approximation Ex 14.1 Q11

Let $x = 5$ and $\Delta x = 0.001$. Then, we have:

$$f(5.001) = f(x + \Delta x) = (x + \Delta x)^3 - 7(x + \Delta x)^2 + 15$$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$$\begin{aligned}
\therefore f(x + \Delta x) &= f(x) + \Delta y \\
&\approx f(x) + f'(x) \cdot \Delta x \qquad (\text{as } dx = \Delta x)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow f(5.001) &\approx (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x \\
&= \left[(5)^3 - 7(5)^2 + 15\right] + \left[3(5)^2 - 14(5)\right](0.001) \qquad [x = 5, \Delta x = 0.001] \\
&= (125 - 175 + 15) + (75 - 70)(0.001) \\
&= -35 + (5)(0.001) \\
&= -35 + 0.005 \\
&= -34.995
\end{aligned}$$

Hence, the approximate value of $f(5.001)$ is -34.995.

Differentials Errors and Approximation Ex 14.1 Q12

Let $x = 1000,\ x + \Delta x = 1005$

$$\Delta x = 1005 - 1000 = 5$$

Let $y = \log_{10} x$

$$\frac{dy}{dx} = \frac{\log_e x}{\log_e 10} \qquad \left[\because \log_a b = \frac{\log_e b}{\log_e a}\right]$$

$$\frac{dy}{dx} = \frac{1}{x \log_e 10}$$

$$\left(\frac{dy}{dx}\right)_{x=1000} = \frac{\log_{10} e}{1000} \qquad \left[\because \log_a b = \frac{1}{\log_b a}\right]$$

$$= \frac{0.4343}{1000}$$

$$= (0.0004343)$$

$$\therefore \quad \Delta y = \left(\frac{dy}{dx}\right)_{x=1000} \times (\Delta x)$$

$$= (0.0004343)(5)$$

$$= 0.0021715$$

$$\log_{10} 1005 = y + \Delta y$$

$$= \log_{10} x + 0.0021715$$

$$= \log_{10} 1000 + 0.0021715$$

$$= \log_{10} 10^3 + 0.0021715$$

$$= 3\log_{10} 10 + 0.0021715$$

$$\log_{10} 1005 = 3.0021715$$

Differentials Errors and Approximation Ex 14.1 Q13

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 9$ m and $\Delta r = 0.03$ m

Now, the surface area of the sphere (S) is given by,

$$S = 4\pi r^2$$

$$\therefore \frac{dS}{dr} = 8\pi r$$

$$\therefore dS = \left(\frac{dS}{dr}\right)\Delta r$$

$$= (8\pi r)\Delta r$$

$$= 8\pi(9)(0.03)\ \text{m}^2$$

$$= 2.16\pi\ \text{m}^2$$

Hence, the approximate error in calculating the surface area is $2.16\pi\ \text{m}^2$.

Differentials Errors and Approximation Ex 14.1 Q14

The surface area of a cube (S) of side x is given by $S = 6x^2$.

$$\therefore \frac{dS}{dx} = \left(\frac{dS}{dx}\right)\Delta x$$

$$= (12x)\Delta x$$

$$= (12x)(0.01x) \qquad [\text{as } 1\% \text{ of } x \text{ is } 0.01x]$$

$$= 0.12x^2$$

Hence, the approximate change in the surface area of the cube is $0.12x^2\ \text{m}^2$.

Differentials Errors and Approximation Ex 14.1 Q15

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 7$ m and $\Delta r = 0.02$ m

Now, the volume V of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$
$$\therefore \frac{dV}{dr} = 4\pi r^2$$
$$\therefore dV = \left(\frac{dV}{dr}\right)\Delta r$$
$$= \left(4\pi r^2\right)\Delta r$$
$$= 4\pi(7)^2(0.02)\ \text{m}^3 = 3.92\pi\ \text{m}^3$$

Hence, the approximate error in calculating the volume is $3.92\ \pi\ \text{m}^3$.

Differentials Errors and Approximation Ex 14.1 Q16

The volume of a cube (V) of side x is given by $V = x^3$.

$$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$$
$$= \left(3x^2\right)\Delta x$$
$$= \left(3x^2\right)(0.01x) \qquad [\text{as } 1\% \text{ of } x \text{ is } 0.01x]$$
$$= 0.03x^3$$

Hence, the approximate change in the volume of the cube is $0.03x^3$ m^3.