

Exercise – 4.1

1. Evaluate each of the following using identities:

(i) $\left(2x - \frac{1}{x}\right)^2$

(ii) $(2x + y)(2x - y)$

(iii) $(a^2b - ab^2)^2$

(iv) $(a - 0.1)(a + 0.1)$

(v) $[1.5x^2 - 0.3y^2][1.5x^2 + 0.3y^2]$

Sol:

(i) We have,

$$\left(2x - \frac{1}{x}\right)^2 = (2x)^2 + \left(\frac{1}{x}\right)^2 - 2 \cdot 2x \cdot \frac{1}{x}$$

$$\left(2x - \frac{1}{x}\right)^2 = 4x^2 + \frac{1}{x^2} - 4 \quad \left[\because (a-b)^2 = a^2 + b^2 - 2ab \text{ where } a = 2x \text{ and } b = \frac{1}{x} \right]$$

$$\therefore \left(2x - \frac{1}{x}\right)^2 = 4x^2 + \frac{1}{x^2} - 4.$$

(ii) We have

$$(2x + y)(2x - y)$$

$$= (2x)^2 - (y)^2 \quad \left[\because (a+b)(a-b) = a^2 - b^2 \right] \text{ where } a = 2x \text{ and } b = y.$$

$$= 4x^2 - y^2$$

$$(2x + y)(2x - y) = 4x^2 - y^2$$

(iii) We have

$$(a^2b - ab^2)^2$$

$$= (a^2b)^2 + (ab^2)^2 - 2 \times a^2b \times ab^2 \quad \left[\because (a-b)^2 = a^2 + b^2 - 2ab \right]$$

$$= a^4b^2 + b^4a^2 - 2a^3b^3 \quad \text{where } a^2b \text{ and } b = ab^2$$

$$\therefore (a^2b - ab^2)^2 = a^4b^2 + b^4a^2 - 2a^3b^3$$

(iv) We have

$$(a - 0.1)(a + 0.1) = a^2 - (0.1)^2 \quad \left[\because (a-b)(a+b) = a^2 - b^2 \right]$$

$$= a^2 - 0.01 \quad [a = a; b = 0.1]$$

$$(a - 0.1)(a + 0.1) = a^2 - 0.01$$

(v) We have

$$\begin{aligned}
 & [1 \cdot 5x^2 - 0 \cdot 3y^2][1 \cdot 5x^2 + 0 \cdot 3y^2] \\
 &= [1 \cdot 5x^2]^2 - [0 \cdot 3y^2]^2 \quad \left[\because (a+b)(a-b) = a^2 - b^2 \right] \\
 &= 2 \cdot 25x^4 - 0 \cdot 09y^4 \quad \left[\because a = 1 \cdot 5x^2 \text{ and } b = 0 \cdot 3y^2 \right] \\
 & [1 \cdot 5x^2 - 0 \cdot 3y^2][1 \cdot 5x^2 + 0 \cdot 3y^2] = 2 \cdot 25x^4 - 0 \cdot 09y^4.
 \end{aligned}$$

2. Evaluate each of the following using identities:

(i) $(399)^2$

(ii) $(0 \cdot 98)^2$

(iii) 991×1009

(iv) 117×83

Sol:

(i) We have

$$\begin{aligned}
 & (399)^2 = (400 - 1)^2 \\
 &= (400)^2 + (1)^2 - 2(400)(1) \quad \left[\because (a-b)^2 = a^2 + b^2 - 2ab \right] \\
 &= 1,60,000 + 1 - 8,000 \quad \left[\because a = 400 \text{ \& } b = 1 \right] \\
 &= 159201 \\
 & (399)^2 = 159201
 \end{aligned}$$

(ii) We have

$$\begin{aligned}
 & (0 \cdot 98)^2 = [1 - 0 \cdot 02]^2 \\
 &= (1)^2 + (0 \cdot 02)^2 - 2 \times 1 \times 0 \cdot 02 \\
 &= 1 + 0 \cdot 0004 - 0 \cdot 04 \quad \left[\because a = 1; b = 0 \cdot 02 \right] \\
 &= 1 \cdot 0004 - 0 \cdot 04 \\
 &= 0 \cdot 9604 \\
 & \therefore (0 \cdot 98)^2 = 0 \cdot 9604.
 \end{aligned}$$

(iii) We have

$$\begin{aligned}
 & 991 \times 1009 \\
 &= (1000 - 9)(1000 + 9) \\
 &= (1000)^2 - (9)^2 \quad \left[\because (a-b)(a+b) = a^2 - b^2 \right] \\
 &= 1000000 - 81 \quad \left[\because a = 1000; b = 9 \right] \\
 &= 999919 \\
 & \quad \quad \quad 991 \times 1009 = 999919
 \end{aligned}$$

(iv) We have

$$117 \times 83$$

$$= (100 + 17)(100 - 17)$$

$$= (100)^2 - (17)^2 \quad \left[\because (a+b)(a-b) = a^2 - b^2 \right]$$

$$= 10000 - 289 \quad \left[\because a = 100; b = 17 \right]$$

$$= 9711$$

$$117 \times 83 = 9711$$

3. Simplify each of the following:

(i) $175 \times 175 + 2 \times 175 \times 25 + 25 \times 25$

(ii) $322 \times 322 - 2 \times 322 \times 22 + 22 \times 22$

(iii) $0.76 \times 0.76 + 2 \times 0.76 \times 0.24 + 0.24 \times 0.24$

(iv)
$$\frac{7.83 + 7.83 - 1.17 \times 1.17}{6.66}$$

Sol:

(i) We have

$$175 \times 175 + 2 \times 175 \times 25 + 25 \times 25 = (175)^2 + 2(175)(25) + (25)^2$$

$$= (175 + 25)^2 \quad \left[\because a^2 + b^2 + 2ab = (a+b)^2 \right]$$

$$= (200)^2 = 40000 \quad \left[\text{here } a = 175 \text{ and } b = 25 \right]$$

$$\therefore 175 \times 175 + 2 \times 175 \times 25 + 25 \times 25 = 40000$$

(ii) We have

$$322 \times 322 - 2 \times 322 \times 22 + 22 \times 22$$

$$= (322 - 22)^2 \quad \left[\because (a-b)^2 = a^2 - 2ab + b^2 \right]$$

$$= (300)^2$$

$$= 90000$$

$$\therefore 322 \times 322 - 2 \times 322 \times 22 + 22 \times 22 = 90000$$

(iii) We have

$$0.76 \times 0.76 + 2 \times 0.76 \times 0.24 + 0.24 \times 0.24$$

$$= [0.76 + 0.24]^2 \quad \left[\because a^2 + b^2 + 2ab = (a+b)^2 \right]$$

$$= [1.00]^2$$

$$= 1$$

$$\therefore 0.76 \times 0.76 + 2 \times 0.76 \times 0.24 + 0.24 \times 0.24 = 1$$

(iv) We have

$$\begin{aligned}
 & \frac{7 \cdot 83 + 7 \cdot 83 - 1 \cdot 17 \times 1 \cdot 17}{6 \cdot 66} \\
 &= \frac{(7 \cdot 83 + 1 \cdot 17)(7 \cdot 83 - 1 \cdot 17)}{6 \cdot 66} \quad \left[\because (a^2 - b^2) = (a + b)(a - b) \right] \\
 &= \frac{(9 \cdot 00)(6 \cdot 66)}{(6 \cdot 66)} \\
 &= 9 \\
 &\therefore \frac{7 \cdot 83 \times 7 \cdot 83 - 1 \cdot 17 \times 1 \cdot 17}{6 \cdot 66} = 9
 \end{aligned}$$

4. If $x + \frac{1}{x} = 11$, find the value of $x^2 + \frac{1}{x^2}$

Sol:

We have $x + \frac{1}{x} = 11$

Now, $\left(x + \frac{1}{x}\right)^2 = x^2 + \left(\frac{1}{x}\right)^2 + 2 \times x \times \frac{1}{x}$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow (11)^2 = x^2 + \frac{1}{x^2} + 2 \quad \left[\because x + \frac{1}{x} = 11 \right]$$

$$\Rightarrow 121 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 119.$$

5. If $x - \frac{1}{x} = -1$, find the value of $x^2 + \frac{1}{x^2}$

Sol:

We have

$$\left(x - \frac{1}{x}\right)^2 = x^2 + \left(\frac{1}{x}\right)^2 - 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow (-1)^2 = x^2 + \frac{1}{x^2} - 2 \quad \left[\because x - \frac{1}{x} = -1 \right]$$

$$\Rightarrow 2 + 1 = x^2 + \frac{1}{x^2}$$

$$\therefore x^2 + \frac{1}{x^2} = 3.$$

6. If $x + \frac{1}{x} = \sqrt{5}$, find the values of $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$

Sol:

We have

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow (\sqrt{5})^2 = x^2 + \frac{1}{x^2} + 2 \quad \left[\because x + \frac{1}{x} = \sqrt{5} \right]$$

$$\Rightarrow 5 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 3 \quad \text{.....(1)}$$

$$\text{Now, } \left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2 \times x^2 \times \frac{1}{x^2}$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2$$

$$\Rightarrow 9 = x^4 + \frac{1}{x^4} + 2 \quad \left[\because x^2 + \frac{1}{x^2} = 3 \right]$$

$$\Rightarrow x^4 + \frac{1}{x^4} = 7$$

$$\text{Hence, } x^2 + \frac{1}{x^2} = 3; x^4 + \frac{1}{x^4} = 7.$$

7. If $x^2 + \frac{1}{x^2} = 66$, find the value of $x - \frac{1}{x}$

Sol:

We have

$$\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x}$$

$$\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 66 - 2 \quad \left[\because x^2 + \frac{1}{x^2} = 66 \right]$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 64$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = (\pm 8)^2$$

$$\Rightarrow x - \frac{1}{x} = \pm 8$$

8. If $x^2 + \frac{1}{x^2} = 79$, find the value of $x + \frac{1}{x}$

Sol:

We have

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = \left(x^2 + \frac{1}{x^2}\right) + 2$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 79 + 2$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 81$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = (\pm 9)^2$$

$$\Rightarrow x + \frac{1}{x} = \pm 9.$$

9. If $9x^2 + 25y^2 = 181$ and $xy = -6$, find the value of $3x + 5y$

Sol:

We have,

$$(3x + 5y)^2 = (3x)^2 + (5y)^2 + 2 \times 3x \times 5y$$

$$\Rightarrow (3x + 5y)^2 = 9x^2 + 25y^2 + 30xy$$

$$= 181 + 30(-6) \quad \left[\because 9x^2 + 25y^2 = 181 \text{ and } xy = -6 \right]$$

$$= 181 - 180$$

$$\Rightarrow (3x + 5y)^2 = 1$$

$$\Rightarrow (3x + 5y)^2 = (\mp 1)^2$$

$$\Rightarrow 3x + 5y = \pm 1$$

10. If $2x + 3y = 8$ and $xy = 2$, find the value of $4x^2 + 9y^2$

Sol:

We have

$$\begin{aligned}(2x+3y)^2 &= (2x)^2 + (3y)^2 + 2(2x)(3y) \\ \Rightarrow (2x+3y)^2 &= 4x^2 + 9y^2 + 12xy \\ \Rightarrow (8)^2 &= 4x^2 + 9y^2 + 24 & [\because 2x+3y=8, xy=2] \\ \Rightarrow 64 - 24 &= 4x^2 + 9y^2 \\ \Rightarrow 4x^2 + 9y^2 &= 40.\end{aligned}$$

11. If $3x - 7y = 10$ and $xy = -1$, find the value of $9x^2 + 49y^2$

Sol:

We have,

$$\begin{aligned}(3x-7y)^2 &= (3x)^2 + (-7y)^2 - 2(3x)(7y) \\ &= 9x^2 + 49y^2 - 42xy \\ \Rightarrow [10]^2 &= 9x^2 + 49y^2 - 42xy & [\because 3x-7y=10] \\ \Rightarrow 100 &= 9x^2 + 49y^2 - 42[-1] & [\because xy=-1] \\ \Rightarrow 100 &= 9x^2 + 49y^2 + 42 \\ \Rightarrow 100 - 42 &= 9x^2 + 49y^2 \\ \Rightarrow 9x^2 + 49y^2 &= 58.\end{aligned}$$

12. Simplify each of the following products:

(i) $\left(\frac{1}{2}a - 3b\right)\left(\frac{1}{2}a + 3b\right)\left(\frac{1}{4}a^2 + 9b^2\right)$

(ii) $\left(m + \frac{n}{7}\right)^3\left(m - \frac{n}{7}\right)$

(iii) $\left(\frac{x}{2} - \frac{2}{5}\right)\left(\frac{2}{5} - \frac{x}{2}\right) - x^2 + 2x$

(iv) $(x^2 + x - 2)(x^2 - x + 2)$

(v) $(x^3 - 3x^2 - x)(x^2 - 3x + 1)$

(vi) $[2x^4 - 4x^2 + 1][2x^4 - 4x^2 - 1]$

Sol:

(i) $\left(\frac{1}{2}a - 3b\right)\left(\frac{1}{2}a + 3b\right)\left(\frac{1}{4}a^2 + 9b^2\right)$

$$\begin{aligned}
&\Rightarrow \left[\left(\frac{1}{2}a \right)^2 - (3b)^2 \right] \left[\frac{1}{4}a^2 + 9b^2 \right] && \left[\because (a+b)(a-b) = a^2 - b^2 \right] \\
&= \left[\left(\frac{1}{4}a^2 \right) - 9b^2 \right] \left[\frac{1}{4}a^2 + 9b^2 \right] && \left[\because (ab)^2 = a^2b^2 \right] \\
&= \left[\frac{1}{4}a^2 \right]^2 - [9b^2]^2 && \left[\because (a-b)(a+b) = a^2 - b^2 \right] \\
&= \frac{1}{16}a^4 - 81b^4 \\
&\therefore \left(\frac{1}{2}a - 3b \right) \left(\frac{1}{2}a + 3b \right) \left(\frac{1}{4}a^2 + 9b^2 \right) = \frac{1}{16}a^4 - 81b^4
\end{aligned}$$

(ii) We have

$$\begin{aligned}
&\left(m + \frac{n}{7} \right) \left(m - \frac{n}{7} \right) \\
&= \left(m + \frac{n}{7} \right) \left(m + \frac{n}{7} \right) \left(m + \frac{n}{7} \right) \left(m - \frac{n}{7} \right) \\
&= \left(m + \frac{n}{7} \right)^2 \left(\left(m \right)^2 - \left(\frac{n}{7} \right)^2 \right) && \left[\because (a+b)(a+b) = (a+b)^2 \text{ \& } (a+b)(a-b) = a^2 - b^2 \right] \\
&= \left(m + \frac{n}{7} \right)^2 \left[m^2 - \frac{n^2}{49} \right] \\
&\therefore \left(m + \frac{n}{7} \right)^3 \left(m - \frac{n}{7} \right) = \left(m + \frac{n}{7} \right)^2 \left[m^2 - \frac{n^2}{49} \right]
\end{aligned}$$

(iii) We have

$$\begin{aligned}
&\left(\frac{x}{2} - \frac{2}{5} \right) \left(\frac{2}{5} - \frac{x}{2} \right) - x^2 + 2x \\
&\Rightarrow - \left(\frac{2}{5} - \frac{x}{2} \right) \left(\frac{2}{5} - \frac{x}{2} \right) - x^2 + 2x \\
&\Rightarrow - \left(\frac{2}{5} - \frac{x}{2} \right)^2 - x^2 + 2x && \left[\because (a-b)(a-b) = (a-b)^2 \right] \\
&\Rightarrow - \left[\left(\frac{2}{5} \right)^2 + \left(\frac{x}{2} \right)^2 - 2 \left(\frac{2}{5} \right) \left(\frac{x}{2} \right) \right] - x^2 + 2x \\
&\Rightarrow - \left[\frac{4}{25} + \frac{x^2}{4} - \frac{2x}{5} \right] - x^2 + 2x \\
&\Rightarrow - \frac{x^2}{4} + \frac{2x}{5} - x^2 + 2x - \frac{4}{25} \Rightarrow - \frac{x^2}{4} - x^2 + \frac{2x}{5} + 2x - \frac{4}{25}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow -\frac{5x^2}{4} + \frac{2x}{5} + 2x - \frac{4}{25} \\
&\Rightarrow -\frac{5x^2}{4} + \frac{2x+10x}{5} - \frac{4}{25} \\
&\Rightarrow \frac{-5x^2}{4} + \frac{12x}{5} - \frac{4}{25} \\
&\therefore \left(\frac{x}{2} - \frac{2}{5}\right)\left(\frac{2}{5} - \frac{x}{2}\right) - x^2 + 2x \\
&= \frac{-5x^2}{4} + \frac{12x}{5} - \frac{4}{25}
\end{aligned}$$

(iv) We have,

$$\begin{aligned}
&(x^2 + x - 2)(x^2 - x + 2) \\
&\left[(x)^2 + (x - 2)\right]\left[x^2 - (x - 2)\right] \\
&\Rightarrow \left[x^2\right]^2 - (x - 2)^2 \quad \left[(a - b)(a + b) = a^2 - b^2\right] \\
&\Rightarrow x^4 - (x^2 + 4 - 4x) \quad \left[\because (a - b)^2 = a^2 + b^2 - 2ab\right] \\
&\Rightarrow x^4 - x^2 - 4 + 4x \\
&\Rightarrow x^4 - x^2 + 4x - 4 \\
&\therefore (x^2 + x - 2)(x^2 - x + 2) = x^4 - x^2 + 4x - 4
\end{aligned}$$

(v) We have,

$$\begin{aligned}
&(x^3 - 3x^2 - x)(x^2 - 3x + 1) \\
&\Rightarrow x(x^2 - 3x - 1)(x^2 - 3x + 1) \\
&\Rightarrow x\left[\left[x^2 - 3x\right]^2 - [1]^2\right] \quad \left[\because (a - b)(a + b) = a^2 - b^2\right] \\
&\Rightarrow x\left[(x^2)^2 + (-3x)^2 - 2(+3x)x^2 - 1\right] \\
&\Rightarrow x\left[x^4 + 9x^2 - 6x^3 - 1\right] \\
&\Rightarrow x^5 - 6x^4 + 9x^3 - x \\
&\therefore (x^3 - 3x^2 - 2)(x^2 - 3x + 1) = x^5 - 6x^4 + 9x^3 - x.
\end{aligned}$$

(vi) We have

$$\begin{aligned}
&\left[2x^4 - 4x^2 + 1\right]\left[2x^4 - 4x^2 - 1\right] \\
&\Rightarrow \left[(2x^4 - 4x^2)^2 - (1)^2\right] \quad \left[\because (a + b)(a - b) = a^2 - b^2\right] \\
&\Rightarrow \left[(2x^4)^2 + (4x^2)^2 - 2(2x^4)(4x^2) - 1\right]
\end{aligned}$$

$$\begin{aligned} &\Rightarrow 4x^8 + 16^4 - 16x^6 - 1 && \left[\because (a-b)^2 = a^2 + b^2 - 2ab \right] \\ &\Rightarrow 4x^8 - 16x^6 + 16x^4 - 1 \\ &\therefore [2x^4 - 4x^2 + 1][2x^4 - 4x^2 - 1] = 4x^8 - 16x^6 + 16x^4 - 1. \end{aligned}$$

13. Prove that $a^2 + b^2 + c^2 - ab - bc - ca$ is always non-negative for all values of a, b and c

Sol:

We have

$$\begin{aligned} &a^2 + b^2 + c^2 - ab - bc - ca \\ &= \frac{2}{2} [a^2 + b^2 + c^2 - ab - bc - ca] && \text{[Multiply and divide by '2']} \\ &= \frac{1}{2} [2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] \\ &= \frac{1}{2} [a^2 + a^2 + b^2 + b^2 + c^2 + c^2 - 2ab - 2bc - 2ca] \\ &= \frac{1}{2} [(a^2 + b^2 - 2ab) + (a^2 + c^2 - 2ac) + (b^2 + c^2 - 2bc)] \\ &= \frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2] && \left[\because (a-b)^2 = a^2 + b^2 - 2ab \right] \\ &= \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \geq 0 \\ &\therefore a^2 + b^2 + c^2 - ab - bc - ca \geq 0 \end{aligned}$$

Hence, $a^2 + b^2 + c^2 - ab - bc - ca$ is always non negative for all values of a, b and c.

Exercise – 4.2

1. Write the following in the expanded form:

(i) $(a + 2b + c)^2$

(ii) $(2a - 3b - c)^2$

(iii) $(-3x + y + z)^2$

(iv) $(m + 2n - 5p)^2$

(v) $(2 + x - 2y)^2$

(vi) $(a^2 + b^2 + c^2)^2$

(vii) $(ab + bc + ca)^2$

(viii) $\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2$

(ix) $\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)^2$

(x) $(x + 2y + 4z)^2$

(xi) $(2x - y + z)^2$

(xii) $(-2x + 3y + 2z)^2$

Sol:

(i) We have, $(a + 2b + c)^2$
 $= a^2 + (2b)^2 + (c)^2 + 2(a)(2b) + 2ac + (2b)2c$
 $\left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right]$
 $\therefore (a + 2b + c)^2 = a^2 + 4b^2 + c^2 + 4ab + 2ac + 4bc.$

(ii) We have
 $(2a - 3b - c)^2 = [(2a) + (-3b) + (-c)]^2$
 $= (2a)^2 + (-3b)^2 + (-c)^2 + 2(2a)(-3b) + 2(-3b)(-c) + 2(2a)(-c)$
 $\left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right]$
 $= 4a^2 + 9b^2 + c^2 - 12ab + 6bc - 4ac$
 $\therefore (2a - 3b - c)^2 = 4a^2 + 9b^2 + c^2 - 12ab + 6bc - 4ca.$

$$\begin{aligned}
 \text{(iii)} \quad & (-3x + y + z)^2 = [(-3x) + y + z]^2 \\
 & = (-3x)^2 + y^2 + z^2 + 2(-3x)y + 2yz + 2(-3x)z \\
 & \left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\
 & = 9x^2 + y^2 + z^2 - 6xy + 2yz - 6xz \\
 & \therefore (-3x + y + z)^2 = 9x^2 + y^2 + z^2 - 6xy + 2yz - 6xz
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \text{We have,} \\
 & (m + 2n - 5p)^2 = m^2 + (2n)^2 + (-5p)^2 + 2(m)(2n) + 2(2n)(-5p) + 2(m)(-5p) \\
 & \left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\
 & = m^2 + 4n^2 + 25p^2 + 4mn - 20np - 10pm \\
 & \therefore (m + n - 5p)^2 = m^2 + 4n^2 + 25p^2 + 4mn - 20np - 10pm.
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad & \text{We have,} \\
 & (2 + x - 2y)^2 = [2 + x + (-2y)]^2 \\
 & = (2)^2 + x^2 + (-2y)^2 + 2(2)(x) + 2(x)(-2y) + 2(2)(-2y) \\
 & \left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\
 & = 4 + x^2 + 4y^2 + 4x - 4xy - 8y \\
 & \therefore (2 + x - 2y)^2 = 4 + x^2 + 4y^2 + 4x - 4xy - 8y
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & \text{We have} \\
 & (a^2 + b^2 + c^2)^2 = (a^2)^2 + (b^2)^2 + (c^2)^2 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2 \\
 & \left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\
 & = a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2 \\
 & \therefore (a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2
 \end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \quad & \text{We have} \\
 & (ab + bc + ca)^2 = (ab)^2 + (bc)^2 + (ca)^2 + 2(ab)(bc) + 2(bc)ca + 2(ab)(ca) \\
 & \left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\
 & = a^2b^2 + b^2c^2 + c^2a^2 + 2ab^2c + 2bc^2a + 2a^2bc \\
 & \therefore (ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2ab^2c + 2bc^2a + 2a^2bc
 \end{aligned}$$

(viii) We have

$$\begin{aligned} \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 &= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 + 2 \cdot \frac{x}{y} \cdot \frac{y}{z} + 2 \cdot \frac{y}{z} \cdot \frac{z}{x} + 2 \cdot \frac{z}{x} \cdot \frac{x}{y} \\ &\left[\because (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\ \therefore \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 &= \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} + 2\frac{x}{z} + 2\frac{y}{x} + 2\frac{z}{y} \end{aligned}$$

(ix) We have

$$\begin{aligned} \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)^2 &= \left(\frac{a}{bc}\right)^2 + \left(\frac{b}{ca}\right)^2 + \left(\frac{c}{ab}\right)^2 + 2\left(\frac{a}{bc}\right)\left(\frac{b}{ca}\right) + 2\left(\frac{b}{ca}\right)\left(\frac{c}{ab}\right) + 2\left(\frac{a}{bc}\right)\left(\frac{c}{ab}\right) \\ &\left[\because (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\ \therefore \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)^2 &= \frac{a^2}{b^2c^2} + \frac{b^2}{c^2a^2} + \frac{c^2}{a^2b^2} + \frac{2}{a^2} + \frac{2}{b^2} + \frac{2}{c^2} \end{aligned}$$

$$\begin{aligned} \text{(x)} \quad (x+2y+4z)^2 &= x^2 + (2y)^2 + (4z)^2 + 2x(2y) + 2(2y)(4z) + 2x(4z) \\ &= x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 4xz. \end{aligned}$$

$$\begin{aligned} \text{(xi)} \quad (2x-y+z)^2 &= [(2x) + (-y) + z]^2 \\ &= (2x)^2 + (-y)^2 + (z)^2 + 2(2x)(-y) + 2(-y)(z) + 2(2x)(z) \\ &= 4x^2 + y^2 + z^2 + 4x(-y) - 2yz + 4xz \\ \therefore (2x-y+z)^2 &= 4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz \end{aligned}$$

$$\begin{aligned} \text{(xii)} \quad (-2x+3y+2z)^2 &= [(-2x) + 3y + 2z]^2 \\ &= (-2x)^2 + (3y)^2 + (2z)^2 + 2(-2x)(3y) + 2(3y)(2z) + 2(2z)(-2x) \\ &= 4x^2 + 9y^2 + 4z^2 - 12xy + 12yz - 8xz \end{aligned}$$

2. Simplify:

- (i) $(a+b+c)^2 + (a-b+c)^2$
- (ii) $(a+b+c)^2 - (a-b+c)^2$
- (iii) $(a+b+c)^2 + (a-b+c)^2 + (a+b-c)^2$
- (iv) $(2x+p-c)^2 - (2x-p+c)^2$
- (v) $(x^2+y^2-z)^2 - (x^2-y^2+z^2)^2$

Sol:

(i) We have

$$\begin{aligned}
& (a+b+c)^2 + (a-b+c)^2 \\
&= (a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) + (a^2 + b^2 + c^2 - 2ab - 2bc + 2ac) \\
& \left[\because (x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \right] \\
&= 2a^2 + 2b^2 + 2c^2 + 4ca \\
&\therefore (a+b+c)^2 + (a-b+c)^2 = 2a^2 + 2b^2 + 2c^2 + 4ca.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
& (a+b+c)^2 - (a-b+c)^2 \\
&= \left[(a+b+c)^2 \right] - \left[(a-b+c)^2 \right] \\
&= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - \left[a^2 + b^2 + c^2 - 2ab - 2bc + 2ca \right] \\
&= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - a^2 - b^2 - c^2 + 2ab + 2bc - 2ca \\
&= 4ab + 4bc \\
&\therefore (a+b+c)^2 - (a-b+c)^2 = 4ab + 4bc
\end{aligned}$$

(iii) We have

$$\begin{aligned}
& (a+b+c)^2 + (a-b+c)^2 + (a+b-c)^2 \\
&= \left[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] + \left[a^2 + b^2 + c^2 - 2bc - 2ab + 2ca \right] \\
& \quad + \left[a^2 + b^2 + c^2 - 2ca - 2bc + 2ab \right] \\
& \left[\because (x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \right] \\
&= 3a^2 + 3b^2 + 3c^2 + 2ab + 2bc + 2ca - 2bc - 2ab + 2ca - 2ca - 2bc + 2ab \\
&= 3a^2 + 3b^2 + 3c^2 + 2ab - 2bc + 2ca \\
&= 3(a^2 + b^2 + c^2) + 2(ab - bc + ca) \\
&\therefore (a+b+c)^2 + (a-b+c)^2 + (a+b-c)^2 = 3(a^2 + b^2 + c^2) + 2(ab - bc + ca)
\end{aligned}$$

(iv) We have

$$\begin{aligned}
& (2x+p-c)^2 - (2x-p+c)^2 \\
&= \left[(2x)^2 + (p)^2 + (-c)^2 + 2(2x)(p) + 2(p)(-c) + 2(2x)(-c) \right] \\
& \quad - \left[(2x)^2 + (-p)^2 + c^2 + 2(2x)(-p) + 2(2x)(c) + 2(-p)c \right] \\
&= \left[4x^2 + p^2 + c^2 + 4xp - 2pc - 4cx \right] - \left[4x^2 + p^2 + c^2 - 4xp - 2pc + 4cx \right] \\
&= 4x^2 + p^2 + c^2 + 4xp - 2pc - 4cx - 4x^2 - p^2 - c^2 + 4xp + 2pc - 4cx
\end{aligned}$$

$$\begin{aligned}
 &= 8xp - 8xc \\
 &= 8x(p - c) \\
 \therefore (2x + p - c)^2 - (2x - p + c)^2 &= 8x(p - c)
 \end{aligned}$$

(v) We have

$$\begin{aligned}
 &(x^2 + y^2 - z)^2 - (x^2 - y^2 + z^2)^2 \\
 &= [x^2 + y^2 + (-z)^2]^2 - [x^2 + (-y^2) + (z^2)]^2 \\
 &= [(x^2)^2 + (y^2)^2 + (-z^2)^2 + 2(x^2)(y^2) + 2(y^2)(-z^2) + 2(x^2)(-z^2)] \\
 &\quad - [(x^2)^2 + (-y^2)^2 + (z^2)^2 + 2(x^2)(-y^2) + 2(-y^2)(z^2) + 2x^2z^2] \\
 &= [\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca] \\
 &= x^4 + y^4 + z^4 + 2x^2y^2 - 2y^2z^2 - 2z^2x^2 - x^4 - y^4 - z^4 + 2x^2y^2 + 2y^2z^2 - 2z^2x^2 \\
 &= 4x^2y^2 - 4z^2x^2 \\
 \therefore (x^2 + y^2 - z^2)^2 - (x^2 - y^2 + z^2)^2 &= 4x^2y^2 - 4z^2x^2
 \end{aligned}$$

3. If $a + b + c = 0$ and $a^2 + b^2 + c^2 = 16$, find the value of $ab + bc + ca$.

Sol:

We know that,

$$\begin{aligned}
 (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\
 \Rightarrow (0)^2 &= 16 + 2(ab + bc + ca) \quad \left[\because a + b + c = 0 \text{ and } a^2 + b^2 + c^2 = 16 \right] \\
 \Rightarrow 2(ab + bc + ca) &= -16 \\
 \Rightarrow ab + bc + ca &= -8
 \end{aligned}$$

4. If $a^2 + b^2 + c^2 = 16$ and $ab + bc + ca = 10$, find the value of $a + b + c$.

Sol:

We know that,

$$\begin{aligned}
 (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\
 \Rightarrow (a + b + c)^2 &= 16 + 2(10) \quad \left[\because a^2 + b^2 + c^2 = 16 \text{ and } ab + bc + ca = 10 \right] \\
 \Rightarrow (a + b + c)^2 &= 16 + 20 \\
 \Rightarrow (a + b + c) &= \sqrt{36} \\
 \Rightarrow a + b + c &= \pm 6
 \end{aligned}$$

5. If $a + b + c = 9$ and $ab + bc + ca = 23$, find the value of $a^2 + b^2 + c^2$.

Sol:

We know that,

$$\begin{aligned}(a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ \Rightarrow (9)^2 &= a^2 + b^2 + c^2 + 2(23) \\ \Rightarrow 81 &= a^2 + b^2 + c^2 + 46 & [\because a + b + c = 9 \text{ and } (ab + bc + ca = 23)] \\ \Rightarrow a^2 + b^2 + c^2 &= 81 - 46 \\ \Rightarrow a^2 + b^2 + c^2 &= 35.\end{aligned}$$

6. Find the value of $4x^2 + y^2 + 25z^2 + 4xy - 10yz - 20zx$ when $x = 4$, $y = 3$ and $z = 2$.

Sol:

We have,

$$\begin{aligned}4x^2 + y^2 + 25z^2 + 4xy - 10yz - 20zx \\ \Rightarrow (2x)^2 + (y)^2 + (-5z)^2 + 2(2x)(y) + 2(y)(-5z) + 2(-5z)(2x) \\ \Rightarrow (2x + y - 5z)^2 \\ \Rightarrow [2[4] + 3 - 5(2)]^2 & [\because x = 4, y = 3 \text{ and } z = 2] \\ = [8 + 3 - 10]^2 \\ = [1]^2 \\ = 1 \\ \therefore 4x^2 + y^2 + 25z^2 + 4xy - 10yz - 20zx = 1.\end{aligned}$$

7. Simplify each of the following expressions:

$$\begin{aligned}\text{(i)} \quad & (x + y + z)^2 + \left(x + \frac{y}{2} + \frac{2}{3}\right)^2 - \left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right)^2 \\ \text{(ii)} \quad & (x + y - 2z)^2 - x^2 - y^2 - 3z^2 + 4xy \\ \text{(iii)} \quad & (x^2 - x + 1)^2 - (x^2 + x + 1)^2\end{aligned}$$

Sol:

(i) We have,

$$\begin{aligned}(x + y + z)^2 + \left(x + \frac{y}{2} + \frac{2}{3}\right)^2 - \left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right)^2 \\ = [x^2 + y^2 + z^2 + 2xy + 2yz + 2zx] + \left[x^2 + \frac{y^2}{4} + \frac{z^2}{9} + 2x \cdot \frac{y}{2} + 2 \frac{zx}{3} + \frac{yz}{3}\right]\end{aligned}$$

$$\begin{aligned}
& - \left[\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{10} + \frac{xy}{3} + \frac{xz}{4} + \frac{yz}{6} \right] \\
& = x^2 + y^2 + z^2 + x^2 + \frac{y^2}{4} + \frac{z^2}{9} - \frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} + 2xy + 2x \cdot \frac{y}{2} - \frac{xy}{3} + 2yz + \frac{yz}{3} - \frac{yz}{6} + 2zx + \frac{2zx}{3} - \frac{xz}{4} \\
& = \frac{8x^2 - x^2}{4} + \frac{36y^2 + 9y^2 - 4y^2}{36} + \frac{144z^2 + 16z^2 - 9z^2}{144} + \frac{6xy + 3xy - xy}{3} + \frac{13yz}{6} + \frac{29xz}{12} \\
& = \frac{7x^2}{4} + \frac{41y^2}{36} + \frac{151z^2}{144} + \frac{8xy}{3} + \frac{13yz}{6} + \frac{29zx}{12} \\
& \therefore (x + y + z)^2 + \left(x + \frac{y}{2} + \frac{z}{3} \right)^2 - \left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4} \right)^2 \\
& = \frac{7x^2}{4} + \frac{41y^2}{36} + \frac{151z^2}{144} + \frac{8xy}{3} + \frac{13yz}{6} + \frac{29zx}{12}
\end{aligned}$$

(ii) We have,

$$\begin{aligned}
& (x + y - 2z)^2 - x^2 - y^2 - 3z^2 + 4xy \\
& = \left[x^2 + y^2 + (-2z)^2 + 2xy + 2(y)(-2z) + 2x(-2z) \right] - x^2 - y^2 - 3z^2 + 4xy \\
& = x^2 + y^2 + 4z^2 + 2xy - 4yz - 4xz - x^2 - y^2 - 3z^2 + 4xy \\
& = z^2 + 6xy - 4yz - 4zx \\
& \therefore (x + y - 2z)^2 - x^2 - y^2 - 3z^2 + 4xy = z^2 + 6xy - 4yz - 4zx
\end{aligned}$$

(iii) We have,

$$\begin{aligned}
& [x^2 - x + 1]^2 - [x^2 + x + 1]^2 \\
& = \left[(x^2)^2 + (-x)^2 + 1^2 + 2(x^2)(-x) + 2(-x)(1) + 2x^2(1) \right] \\
& \quad - \left[(x^2)^2 + (x)^2 + (1)^2 + 2x^2(x) + 2(x)(1) + 2(x^2)(1) \right] \\
& = x^4 + x^2 + 1 - 2x^3 - 2x + 2x^2 - x^2 - x^4 - 1 - 2x^3 - 2x - 2x^2 \\
& \left[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \right] \\
& = -4x^3 - 4x \\
& = -4x[x^2 + 1] \\
& \therefore [x^2 - x + 1]^2 - [x^2 + x + 1]^2 = -4x[x^2 + 1]
\end{aligned}$$