Ex 12.1

Q1

$P(n): n(n+1)$ is even

$P(3): 3.(3+1)$ is even

Q2

$P(n): n^3+n$ is divisible by 3

$P(3): 3^3+3$ is divisible by 3

$\Rightarrow P(3): 30$ is divisible by 3.

$\therefore$ $P(3)$ is true

Now,

$P(4): 4^3+3=67$ is divisible by 3

Since, 67 is not divisible by 3

So, $P(4)$ is not true

Q3

$P(n) : 2^n \geq 3n$

Given that $P(r)$ is true

$\Rightarrow \quad 2^r \geq 3r$

Multiplying both the sides by 2,

$2.2^r \geq 2.3r$

$2^{r+1} \geq 6r$

$\geq 3r + 3r$

$\geq 3 + 3r,$ $\quad$ [Since $3r \geq 3 \Rightarrow \quad 3r + 3r \geq 3 + 3r$]

$2^{r+1} \geq 3(r+1)$

$\Rightarrow \quad P(r+1)$ is true

Q4

Here, $P(n): n^2 + n$ is even

Given, $P(r)$ is true

$\Rightarrow \quad r^2 + r$ is even

$\Rightarrow \quad r^2 + r = 2\lambda \qquad ---(1)$

Now,

$(r+1)^2 + (r+1)$

$= r^2 + 2r + 1 + r + 1$

$= (r^2 + r) + 2r + 2$

$= 2\lambda + 2r + 2 \qquad$ [Using equation (1)]

$= 2(\lambda + r + 1)$

$= 2\mu$

$\Rightarrow \quad (r+1)^2 + (r+1)$ is even

$\Rightarrow \quad P(r+1)$ is true

Q5

$P(n): 1 + 2 + 3 + --- + n = \frac{n(n+1)}{2}$ is true for all $n \in N$

Q6

$P(n): n^2 - n + 41$ is prime

$P(1): 1 - 1 + 41$ is prime
$\Rightarrow P(1): 41$ is prime
$\therefore P(1)$ is true.

$P(2): 2^2 - 2 + 41$ is prime
$\Rightarrow P(2): 43$ is prime
$\therefore P(2)$ is true.

$P(3): 3^2 - 3 + 41$ is prime
$\Rightarrow P(3): 47$ is prime
$\therefore P(3)$ is true.

$P(41): (41)^2 - 41 + 41$ is prime
$P(41): (41)^2$ is prime
$\Rightarrow \quad P(41)$ is not true

Ex 12.2

Q1

Let $P(n): 1+2+3+---+n = \frac{n(n+1)}{2}$

For $n = 1$,
LHS of P(n) = 1

RHS of P(n) $= \frac{1(1+1)}{2} 1 = 1$

Since, LHS = RHS

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ be true for $n = k$, so

$$1+2+3+---+k = \frac{k(k+1)}{2} \qquad ---(1)$$

Now

$$(1+2+3+---+k)+(k+1)$$
$$= \frac{k(k+1)}{2}+(k+1)$$
$$= (k+1)\left(\frac{k}{2}+1\right)$$
$$= \frac{(k+1)(k+2)}{2}$$
$$= \frac{(k+1)[(k+1)+1]}{2}$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$

So, by the principle of mathematical induction

$P(n): 1+2+3+---+n = \frac{n(n+1)}{2}$ is true for all $n \in N$

Q2

Let $P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

For $n = 1$

$$P(1): 1 = \frac{1(1+1)(2+1)}{6}$$

$$1 = 1$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$P(k): 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad ---(1)$$

We have to show that $P(n)$ is true for $n = k + 1$

$$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

So, $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \left[\text{Using equation } (1)\right]$$

$$= (k+1)\left[\frac{2k^2 + k}{6} + \frac{(k+1)}{1}\right]$$

$$= (k+1)\left[\frac{2k^2 + k + 6k + 6}{6}\right]$$

$$= (k+1)\left[\frac{2k^2 + 7k + 6}{6}\right]$$

$$= (k+1)\left[\frac{2k^2 + 4k + 3k + 6}{6}\right]$$

$$= (k+1)\left[\frac{2k(k+2) + 3(k+2)}{6}\right]$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q3

Let $P(n): 1+3+3^2+\ldots+3^{n-1} = \frac{3^n - 1}{2}$

For $n = 1$

$P(1): 1 = \frac{3^1 - 1}{2}$

$1 = 1$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$1+3+3^2+\ldots+3^{k-1} = \frac{3^k - 1}{2} \qquad ---(1)$$

We have to show $P(n)$ is true for $n = k+1$

i.e. $1+3+3^2+\ldots+3^k = \frac{3^{k+1} - 1}{2}$

Now,

$\left\{1+3+3^2+\ldots+3^{k-1}\right\}+3^{k+1-1}$

$= \frac{3k - 1}{2} + 3^k \qquad$ [Using equation (1)]

$= \frac{3^k - 1 + 2.3^k}{2}$

$= \frac{3.3^k - 1}{2}$

$= \frac{3^{k+1} - 1}{2}$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q4

Let $P(n): \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

For $n=1$

$$P(1): \frac{1}{1.2}=\frac{1}{1+1}$$
$$\frac{1}{2}=\frac{1}{2}$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad ---(1)$$

We have to show that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}+\frac{k}{(k+1)(k+2)}=\frac{k+1}{(k+2)}$$

Now,

$$\left\{\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}\right\}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \qquad \left[\text{Using equation } (1)\right]$$

$$=\frac{1}{k+1}\left[\frac{k(k+2)+1}{(k+2)}\right]$$

$$=\frac{1}{k+1}\left[\frac{k^2+2k+1}{(k+2)}\right]$$

$$=\frac{1}{k+1}\left[\frac{(k+1)(k+1)}{(k+2)}\right]$$

$$=\frac{(k+1)}{(k+2)}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q5

Let $P(n): 1+3+5+\ldots+(2n-1)=n^2$

For $n=1$

$P(1): 1=1^2$

$1=1$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$P(k): 1+3+5+\ldots+(2k-1)=k^2 \qquad ---(1)$

We have to show that

$1+3+5+\ldots+(2k-1)+2(k+1)-1=(k+1)^2$

Now,

$\{1+3+5+\ldots+(2k-1)\}+(2k+1)$

$=k^2+(2k+1) \qquad$ [Using equation (1)]

$=k^2+2k+1$

$=(k+1)^2$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q6

Let $P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$

Put $n=1$

$$P(1): \frac{1}{2.5}=\frac{1}{6+4}$$
$$\frac{1}{10}=\frac{1}{10}$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{(6k+4)} \qquad ---(1)$$

We have to show that,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}=\frac{(k+1)}{(6k+10)}$$

Now,

$$\left\{\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\frac{1}{(3k-1)(3k+2)}\right\}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$
$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k(3k+5)+2}{2(3k+2)(3k+5)}$$
$$=\frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$
$$=\frac{3k^2+3k+2k+2}{2(3k+2)(3k+5)}$$
$$=\frac{3k(k+1)+2(k+1)}{2(3k+2)(3k+5)}$$
$$=\frac{(k+1)\cancel{(3k+2)}}{2\cancel{(3k+2)}(3k+5)}$$
$$=\frac{(k+1)}{2(3k+5)}$$

$P(n)$ is true for $n=k+1$

$P(n)$ is true for all $n \in N$ by PMI

Q7

Let $P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Put $n=1$

$$P(1): \frac{1}{1.4}=\frac{1}{4}$$
$$\frac{1}{4}=\frac{1}{4}$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \qquad ---(1)$$

We have to show that,

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}=\frac{(k+1)}{(3k+4)}$$

Now,

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{(3k+1)(3k+4)}$$

Now,

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$
$$=\frac{1}{(3k+1)}\left[\frac{k}{1}+\frac{1}{(3k+4)}\right]$$

$$=\frac{1}{(3k+1)}\left[\frac{k(3k+4)+1}{(3k+4)}\right]$$
$$=\frac{1}{(3k+1)}\left[\frac{3k^2+4k+1}{(3k+4)}\right]$$
$$=\frac{1}{(3k+1)}\frac{\left(3k^2+3k+k+1\right)}{(3k+4)}$$
$$=\frac{3k(k+1)+(k+1)}{(3k+1)(3k+4)}$$
$$=\frac{(k+1)(3k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{(3k+4)}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q8

Let $P(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

Put $n=1$

$\frac{1}{3.5}=\frac{1}{3(5)}$

$\frac{1}{15}=\frac{1}{15}$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \qquad ---(1)$$

We have to show that,

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}=\frac{(k+1)}{3(2k+5)}$$

Now,

$$\left\{\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}\right\}+\frac{1}{(2k+3)(2k+5)}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad [\text{Using equation } (1)]$$

$$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{(k+1)(2k+3)}{(2k+5)}\right]$$

$$=\frac{(k+1)}{2k+5}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q9

Let $P(n): \frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4n-1)(4n+3)}=\frac{n}{3(4n+3)}$

For $n=1$

$$\frac{1}{3.7}=\frac{1}{3(7)}$$

$$\frac{1}{21}=\frac{1}{21}$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4n-1)(4n+3)}=\frac{k}{3(4k+3)} \qquad ---(1)$$

We have to show that,

$$\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4k-1)(4k+3)}+\frac{1}{(4k+3)(4k+7)}=\frac{(k+1)}{3\quad(4k+7)}$$

Now,

$$\left\{\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4k-1)(4k+3)}\right\}+\frac{1}{(4k+3)(4k+7)}$$

Now,

$$\left\{\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4k-1)(4k+3)}\right\}+\frac{1}{(4k+3)(4k+7)}$$

$$=\frac{k}{3(4k+3)}+\frac{1}{(4k+3)(4k+7)}$$

$$=\frac{1}{(4k+3)}\left[\frac{k}{3}+\frac{1}{4k+7}\right]$$

$$=\frac{1}{(4k+3)}\left[\frac{k(4k+7)+3}{3(4k+7)}\right]$$

$$=\frac{1}{(4k+3)}\left[\frac{4k^2+7k+3}{3(4k+7)}\right]$$

$$=\frac{1}{(4k+3)}\left[\frac{4k^2+4k+3k+3}{3(4k+7)}\right]$$

$$=\frac{1}{(4k+3)}\left[\frac{4k(k+1)+3(k+1)}{3(4k+7)}\right]$$

$$=\frac{1}{(4k+3)}\left[\frac{(4k+3)(k+1)}{3(4k+7)}\right]$$

$$=\frac{(k+1)}{3(4k+7)}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q10

Let $P(n): 1.2 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n-1)2^{n+1} + 2$

For $n = 1$

$1.2 = 0.2^0 + 2$

$2 = 2$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k = (k-1)2^{k+1} + 2 \qquad ---(1)$$

We have to show that,

$$\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\} + (k+1)2^{k+1} = k2^{k+2} + 2$$

Now,

$\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\} + (k+1)2^{k+1}$

$= [(k-1)2^{k+1} + 2] + (k+1)2^{k+1}$ [Using equation (1)]

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}(k - 1 + k + 1) + 2$

$= 2^{k+1}.2k + 2$

$= k2^{k+2} + 2$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q11

Let $P(n): 2+5+8+11+\ldots+(3n-1)=\frac{1}{2}n(3n+1)$

For $n=1$

$P(1)\ 2=\frac{1}{2}.1.(4)$

$2=2$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$2+5+8+11+\ldots+(3k-1)=\frac{1}{2}k(3k+1) \qquad ---(1)$$

We have to show that,

$$2+5+8+11+\ldots+(3k-1)+(3k+2)=\frac{1}{2}(k+1)(3k+4)$$

Now,

$\{2+5+8+11+\ldots+(3k-1)\}+(3k+2)$

$=\frac{1}{2}k(3k+1)+(3k+2)$

$=\frac{3k^2+k+2(3k+2)}{2}$

$=\frac{3k^2+k+6k+4}{2}$

$=\frac{3k^2+7k+4}{2}$

$=\frac{3k^2+3k+4k+k}{2}$

$=\frac{3k(k+1)+4(k+1)}{2}$

$=\frac{(k+1)(3k+4)}{2}$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q12

Let $1.3+2.4+3.5+\ldots+n(n+2)=\frac{1}{6}n(n+1)(2n+7)$

For $n=1$

$$1.3=\frac{1}{6}.1.(2)(9)$$
$$3=3$$

$\Rightarrow$ $P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$1.3+2.4+3.5+\ldots+k(k+2)=\frac{1}{6}k(k+1)(2k+7) \qquad ---(1)$$

We have to show that,

$$1.3+2.4+3.5+\ldots+k(k+2)+(k+1)(k+3)=\frac{(k+1)}{6}(k+2)(2k+9)$$

Now,

$$\{1.3+2.4+3.5+\ldots+k(k+2)\}+(k+1)(k+3)$$

$$=\frac{1}{6}k(k+1)(2k+7)+(k+1)(k+3) \qquad [\text{Using equation } (1)]$$

$$=(k+1)\left[\frac{k(2k+7)}{6}+\frac{k+3}{1}\right]$$

$$=(k+1)\left[\frac{2k^2+7k+6k+18}{6}\right]$$

$$=(k+1)\left(\frac{2k^2+13k+18}{6}\right)$$

$$=(k+1)\left[\frac{2k^2+4k+9k+18}{6}\right]$$

$$=(k+1)\left[\frac{2k(k+2)+9(k+2)}{6}\right]$$

$$=(k+1)\left[\frac{(2k+9)(k+2)}{6}\right]$$

$$=\frac{1}{6}(k+1)(k+2)(2k+9)$$

$\Rightarrow$ $P(n)$ is true for $n=k+1$

$\Rightarrow$ $P(n)$ is true for all $n\in N$ by PMI

Q13

Let $P(n): 1.3+3.5+5.7+\ldots+(2n-1)(2n+1)=\dfrac{n(4n^2+6n-1)}{3}$

For $n=1$

$$1.3=\frac{1(4+6-1)}{3}$$

$$3=3$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$1.3+3.5+5.7+\ldots+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3} \qquad ---(1)$$

We have to show that,

$$1.3+3.5+5.7+\ldots+(2k-1)(2k+1)+(2k+1)(2k+3)=\frac{(k+1)\left[4(k+1)^2+6(k+1)-1\right]}{3}$$

Now,

$$\{1.3+3.5+5.7+\ldots+(2k-1)(2k+1)\}+(2k+1)(2k+3)$$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+1)(2k+3) \qquad \text{[Using equation (1)]}$$

$$=\frac{k(4k^2+6k-1)+3(4k^2+6k+2k+3)}{3}$$

$$=\frac{4k^3+6k^2-k+12k^2+18k+6k+9}{3}$$

$$=\frac{4k^3+18k^2+23k+9}{3}$$

$$=\frac{4k^3+4k^2+14k^2+14k+9k+9}{3}$$

$$=\frac{(k+1)(4k^2+8k+4+6k+6-1)}{3}$$

$$=\frac{(k+1)\left[4(k+1)^2+6(k+1)-1\right]}{3}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q14

Let $P(n): 1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

For $n = 1$

$$1.2 = \frac{1(1+1)(1+2)}{3}$$

$$2 = 2$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$\Rightarrow \quad 1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3} \qquad ---(1)$$

We have to show that,

$$1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Now,

$$\{1.2 + 2.3 + 3.4 + \ldots + k(k+1)\} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{(k+1)(k+2)}{1}$$

$$= (k+1)(k+2)\left[\frac{k}{3} + 1\right]$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q15

Let $P(n): \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

For $n=1$

$$\frac{1}{2}=1-\frac{1}{2^1}$$

$$\frac{1}{2}=\frac{1}{2}$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \qquad ---(1)$$

We have to show that,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}$$

Now,

$$\left\{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}\right\}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^{k+1}} \qquad \left[\text{Using equation } (1)\right]$$

$$=1-\left(\frac{2-1}{2^{k+1}}\right)$$

$$=1-\frac{1}{2^{k+1}}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q16

Let $P(n): 1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{1}{3}n(4n^2-1)$

For $n=1$

$$1=\frac{1}{3}.1.(4-1)$$

$$1=1$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{1}{3}k(4k^2-1) \qquad ---(1)$$

We have to show that,

$$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2=\frac{1}{3}(k+1)\left[4(k+1)^2-1\right]$$

Now,

$$\left\{1^2+3^2+5^2+\ldots+(2k-1)^2\right\}+(2k+1)^2$$

$$=\frac{1}{3}k(4k^2-1)+(2k+1)^2 \qquad [\text{Using equation }(1)]$$

$$=\frac{1}{3}k(2k+1)(2k-1)+(2k+1)^2$$

$$=(2k+1)\left[\frac{k(2k-1)}{3}+(2k+1)\right]$$

$$=(2k+1)\left[\frac{2k^2-k+3(2k+1)}{3}\right]$$

$$=(2k+1)\left[\frac{2k^2-k+6k+3}{3}\right]$$

$$=\frac{(2k+1)(2k^2+5k+3)}{3}$$

$$=\frac{(2k+1)(2k^2+5k+3)}{3}$$

$$=\frac{(2k+1)(2k(k+1)+3(k+1))}{3}$$

$$=\frac{(2k+1)(2k+3)(k+1)}{3}$$

$$=\frac{(k+1)}{2}\left[4k^2+6k+2k+3\right]$$

$$=\frac{(k+1)}{2}\left[4k^2+8k+4-1\right]$$

$$=\frac{(k+1)}{2}\left[4(k+1)^2-1\right]$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n\in N$ by PMI

Q17

Let $P(n): a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right), r \neq 1$

For $n = 1$

$$a = a\left(\frac{r^1 - 1}{r - 1}\right)$$

$$a = a$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$a + ar + ar^2 + \ldots + ar^{k-1} = a\left(\frac{r^k - 1}{r - 1}\right), r \neq 1 \qquad ---(1)$$

We have to show that,

$$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = a\left(\frac{r^{k+1} - 1}{r - 1}\right)$$

Now,

$$\left\{a + ar + ar^2 + \ldots + ar^{k-1}\right\} + ar^k$$

$$= a\left(\frac{r^k - 1}{r - 1}\right) + ar^k \qquad \left[\text{Using equation } (1)\right]$$

$$= \frac{a\left[r^k - 1 + r^k(r - 1)\right]}{r - 1}$$

$$= \frac{a\left[r^k - 1 + r^{k+1} - r^k\right]}{r - 1}$$

$$= \frac{a\left(r^{k+1} - 1\right)}{r + 1}$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q18

Let $P(n): a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)=\frac{n}{2}[2a+(n-1)d]$

For $n=1$

$$a=\frac{1}{2}[2a+(1-1)d]$$
$$a=a$$

$\Rightarrow \quad P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$a+(a+d)+(a+2d)+\ldots+(a+(k-1)d)=\frac{k}{2}[2a+(k-1)d] \qquad ---(1)$$

We have to show that,

$$a+(a+d)+(a+2d)+\ldots+(a+(k-1)d)+(a+(k)d)=\frac{(k+1)}{2}[2a+kd]$$

Now,

$$\{a+(a+d)+(a+2d)+\ldots+(a+(k-1)d)\}+(a+kd)$$

$$=\frac{k}{2}[2a+(k-1)d]+(a+kd) \qquad [\text{Using equation } (1)]$$

$$=\frac{2ka+k(k-1)d+2(a+kd)}{2}$$

$$=\frac{2ka+k^2d-kd+2a+2kd}{2}$$

$$=\frac{2ka+2a+k^2d+kd}{2}$$

$$=\frac{2a(k+1)+d(k^2+k)}{2}$$

$$=\frac{(k+1)}{2}[2a+kd]$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n\in N$ by PMI

Q19

Let $P(n) : \left(5^{2n} - 1\right)$ is divisible by 24

For $n = 1$

$5^2 - 1 = 24$

Which is divisible by 24

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$\Rightarrow \quad \left(5^{2k} - 1\right)$ is divisible by 24

$\Rightarrow \quad 5^{2k} - 1 = 24\lambda \qquad\qquad --- (1)$

We have to show that,

$\left(5^{2k} - 1\right)$ is divisible by 24

$5^{2(k+1)} - 1 = 24\mu$

Now,

$5^{2(k+1)} - 1$

$= 5^{2k}.5^2 - 1$

$= 25.5^{2k} - 1$

$= 25\left(24\lambda + 1\right) - 1 \qquad$ [Using equation (1)]

$= 25.24\lambda + 24$

$= 24\left(25\lambda + 1\right)$

$= 24\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q20

Let $P(n): 3^{2n} + 7$ is divisible by 8

For $n = 1$

$3^2 + 7 = 16$

Which is divisible by 8

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$3^{2k} + 7$ is divisible by 8

$\Rightarrow \quad 3^{2k} + 7 = 8\lambda \qquad ---(1)$

We have to show that,

$3^{2(k+1)} + 7$ is divisible by 8

$3^{2(k+1)} + 7 = 8\mu$

Now,

$3^{2(k+1)} + 7$

$= 3^{2k}.3^2 + 7$

$= 9.3^{2k} + 7$

$= 9.(8\lambda - 7) + 7$

$= 72\lambda - 56$

$= 8(9\lambda - 7)$

$= 8\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q21

Let $P(n): 5^{2n+2} - 24n - 25$ is divisible by 576

For $n = 1$

$5^4 - 24 - 25$

$= 625 - 49$

$= 576$

Which is divisible by 576

Let $P(n)$ is true for $n = k$, so

$5^{2k+2} - 24k - 25$ is divisible by 576

$$5^{2k+2} - 24k - 25 = 576\lambda \qquad ---(1)$$

We have to show that,

$5^{2k+4} - 24(k+1) - 25$ is divisible by 576

$5^{(2k+2)+2} - 24(k+1) - 25 = 576\mu$

Now,

$5^{(2k+2)+2} - 24(k+1) - 25$

$= 5^{(2k+2)}.5^2 - 24k - 24 - 25$

$= (576\lambda + 24k + 25)25 - 24k - 49 \qquad$ [Using equation (1)]

$= 25.576\lambda + 600k + 625 - 24k - 49$

$= 25.576\lambda + 576k + 576$

$= 576(25\lambda + k + 1)$

$= 576\mu$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q22

Let $P(n): 3^{2n+2} - 8n - 9$ is divisible by 8

For $n = 1$

$3^{2+2} - 8 - 9$

$= 81 - 17$

$= 64$

It is divisible by 8

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$\left(3^{2k+2} - 8k - 9\right)$ is divisible by 8

$\Rightarrow \quad 3^{2k+2} - 8k - 9 = 8\lambda \qquad ---(1)$

We have to show that,

$3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8

$3^{2(k+1)}.3^2 - 8(k+1) - 9 = 8\mu$

Now,

$3^{2(k+1)}.9 - 8k - 8 - 9$

$= (8\lambda + 8k + 9)9 - 8k - 8 - 9$

$= 72\lambda + 72k + 81 - 8k - 17$

$= 72\lambda + 64k + 64$

$= 8(9\lambda + 8k + 8)$

$= 8\mu$

$\Rightarrow \quad P(n)$ is true for $n = 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q23

Let $P(n):(ab)^n = a^n b^n$

For $n = 1$

$(ab)^1 = a^1 b^1$

$ab = ab$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$$(ab)^k = a^k b^k \qquad ---(1)$$

We have to show that,

$$(ab)^{k+1} = a^{k+1} b^{k+1}$$

Now,

$$(ab)^{k+1}$$
$$= (ab)^k (ab)$$
$$= \left(a^k b^k\right)(ab) \qquad \left[\text{Using equation } (1)\right]$$
$$= \left(a^{k+1}\right)\left(b^{k+1}\right)$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q24

Let $P(n): n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$

For $n = 1$
$1.(1+1)(1+5)$
$= (2)(6)$

$= 12$

it is a multiple of 3

Let $P(n)$ is true for $n = k$
$k(k+1)(k+5)$ is a multiple of 3

$k(k+1)(k+5) = 3\lambda$ $\qquad ---(1)$

We have to show that,

$(k+1)[(k+1)+1][(k+1)+5]$ is a multiple of 3
$(k+1)[(k+1)+1][(k+1)+5] = 3\mu$

Now,

$(k+1)(k+2)[(k+1)+5]$
$= [k(k+1)+2(k+1)][(k+5)+1]$
$= k(k+1)(k+5)+k(k+1)+2(k+1)(k+5)+2(k+1)$
$= 3\lambda + k^2 + k + 2(k^2+6k+5)+2k+2$ $\qquad$ [Using equation (1)]
$= 3\lambda + k^2 + k + 2k^2 + 12k + 10 + 2k + 2$
$= 3\lambda + 3k^2 + 15k + 12$
$= 3(\lambda + k^2 + 5k + 4)$

$= 3\mu$

$\Rightarrow$ $P(n)$ is true for $n = k+1$
$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Q25

Let $P(n): 7^{2n} + 2^{3n-3}.3^{n-1}$ is divisible by 25

For $n = 1$

$7^2 + 2^0.3^0$

$= 49 + 1$

$= 50$

it is divisible of 25

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$7^{2k} + 2^{3k-3}.3^{k-1}$ is divisible by 25

$\Rightarrow \quad 7^{2k} + 2^{3k-3}.3^{k-1} = 25\lambda \qquad ---(1)$

We have to show that,

$7^{2(k+1)} + 2^{3k}3^k$ is divisible by 25

$7^{2(k+1)} + 2^{3k}.3^k = 25\mu$

Now,

$7^{2(k+1)} + 2^{3k}.3^k$

$= 7^{2k}.7^2 + 2^{3k}.3^k$

$= \left(25\lambda - 2^{3k-3}.3^{k-1}\right)49 + 2^{3k}.3k \qquad$ [Using equation (1)]

$= 25\lambda.49 - \frac{2^{3k}}{8}.\frac{3^k}{3}.49 + 2^{3k}.3^k$

$= 24.25.49\lambda - 2^{3k}.3^k.49 + 24.2^{3k}.3^k$

$= 24.25.49\lambda - 25.2^{3k}.3^k$

$= 25\left(24.49\lambda - 2^{3k}.3^k\right)$

$= 25\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q26

Let $P(n): 2.7^n + 3.5^n - 5$ is divisible by 24

For $n = 1$

$2.7 + 3.5 - 5$

$= 24$

it is divisible of 24

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$2.7^k + 3.5^k - 5$ is divisible by 24

$2.7^k + 3.5^k - 5 = 24\lambda \qquad ---(1)$

We have to show that,

$2.7^{(k+1)} + 3.5^{(k+1)} - 5$

$= 2.7^k.7 + 3.5^k.5 - 5$

$= \left(24\lambda - 3.5^k + 5\right)7 + 15.5^k - 5$

$= 24.7\lambda - 21.5^k + 35 + 15.5^k - 5$

$= 24.7\lambda - 6.5^k + 30$

$= 24.7\lambda - 6\left(5^k - 5\right)$

$= 24.7\lambda - 6.\left(20\nu\right) \qquad \left[\text{Since } 5^k - 5 \text{ is multiple of } 20\right]$

$= 24\left(7\lambda - 5\nu\right)$

$= 24\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q27

Let $P(n): 11^{n+2} + 12^{2n+1}$ is divisible by 133

For $n = 1$

$11^3 + 12^3$

$= 1331 + 1728$

$= 3059$

it is divisible of 133

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$11^{k+2} + 12^{2k+1}$ is divisible by 133

$11^{k+2} + 12^{2k+1} = 133\lambda \qquad\qquad --- (1)$

We have to show that,

$11^{k+3} + 12^{2k+3}$ is divisible by 133

Now,

$11^{k+2}.11 + 12^{2k+1}.12^2$

$= \left(133\lambda - 12^{2k+1}\right)11 + 12^{2k+1}.144$

$= 11.133\lambda - 11.12^{2k+1} + 144.12^{2k+1}$

$= 11.133\lambda + 133.12^{2k+1}$

$= 133\left(11\lambda + 12^{2k+1}\right)$

$= 133\mu$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q28

Consider equation

$1\times 1! + 2\times 2! + 3\times 3! + ...n\times n!$

Lets take $(n+1)!-n! = n!\,(n+1-1) = n\times n!$

Now substitue n=1,2,3,4,...n in above equation we get

$2!-1! = 1\times 1!$

$3!-2! = 2\times 2!$

$4!-3! = 3\times 3!$

....

$(n+1)!-n! = n\times n!$

Adding all the above terms gives

$1\times 1! + 2\times 2! + 3\times 3! + ...n\times n! = 2!-1!+3!-2!+4!-3!\ .....+(n+1)!-n!$

$1\times 1! + 2\times 2! + 3\times 3! + ...n\times n! = (n+1)!-1$

Q29

Let P(n) be the statement given by

$P(n): n^3 - 7n + 3$ is divisible by 3.

Step I:

$P(1): 1^3 - 7(1) + 3$ is divisible by 3

$\because 1 - 7 + 3 = -3$ is divisible by 3

$\therefore P(1)$ is true.

Step II:

Let P(m) is true. Then,

$m^3 - 7m + 3$ is divisible by 3

$\Rightarrow m^3 - 7m + 3 = 3\lambda$ for some $\lambda \in N$ ....(i)

We have to prove that $P(m+1)$ is true.

$$\begin{aligned}(m+1)^3 - 7(m+1) + 3 &= m^3 + 3m^2 + 3m + 1 - 7m - 7 + 3\\ &= m^3 - 7m + 3 + 3m^2 + 3m + 1 - 7\\ &= \left(m^3 - 7m + 3\right) + 3\left(m^2 + m - 2\right)\\ &= 3\lambda + 3\left(m^2 + m - 2\right)..................\left[\text{Using (i)}\right]\\ &= 3\left[\lambda + \left(m^2 + m - 2\right)\right] \text{ which is divisible by 3}\end{aligned}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q30

Let P(n) be the statement given by

$P(n): 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in N$.

Step I:

$P(1): 1 + 2^1 = 2^{1+1} - 1$

$\Rightarrow 1 + 2 = 4 - 1$

$\Rightarrow 3 = 3$

$\therefore P(1)$ is true.

Step II:

Let P(m) is true. Then,

$1 + 2 + 2^2 + \cdots + 2^m = 2^{m+1} - 1$ ....(i)

We have to prove that $P(m+1)$ is true.

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^{m+1} &= 1 + 2 + 2^2 + \cdots + 2^m + 2^{m+1} \\ &= \left(2^{m+1} - 1\right) + 2^{m+1} \text{..................[Using (i)]} \\ &= \left(2^{m+1} + 2^{m+1}\right) - 1 \\ &= 2 \times 2^{m+1} - 1 \\ &= 2^{m+2} - 1 \end{aligned}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q31

Let P(n) be the statement given by

$$P(n): 7 + 77 + 777 + \cdots + \underbrace{777\ldots\ldots\ldots.7}_{n-\text{digits}} = \frac{7}{81}\left[10^{n+1} - 9n - 10\right] \text{ for all } n \in N.$$

Step I:

$$P(1): 7 = \frac{7}{81}\left[10^{1+1} - 9(1) - 10\right]$$

$$\Rightarrow 7 = \frac{7}{81} \times (100 - 9 - 10)$$

$$\Rightarrow 7 = \frac{7}{81} \times 81$$

$$\Rightarrow 7 = 7 \times (1)$$

$\therefore$ P(1) is true.

Step II:

Let P(m) is true. Then,

$$7 + 77 + 777 + \cdots + \underbrace{777\ldots\ldots\ldots.7}_{m-\text{digits}} = \frac{7}{81}\left[10^{m+1} - 9m - 10\right] \ldots\ldots\ldots(i)$$

We have to prove that P(m + 1) is true.

$$7 + 77 + 777 + \cdots + \underbrace{777\ldots\ldots\ldots.7}_{m+1-\text{digits}} = 7 + 77 + 777 + \cdots + \underbrace{777\ldots\ldots\ldots.7}_{m-\text{digits}} + \underbrace{777\ldots\ldots\ldots.7}_{m+1-\text{digits}}$$

$$= \frac{7}{81}\left[10^{m+1} - 9m - 10\right] + 7\left[\underbrace{1111\ldots\ldots\ldots.1}_{m+1-\text{digits}}\right] \qquad \left[\text{Using (i)}\right]$$

$$= \frac{7}{81}\left[10^{m+1} - 9m - 10\right] + \frac{7}{9}\left[\underbrace{9999\ldots\ldots\ldots.9}_{m+1-\text{digits}}\right]$$

$$= \frac{7}{81}\left[10^{m+1} - 9m - 10\right] + \frac{7}{9}\left[10^{m+1} - 1\right]$$

$$= \frac{7}{81}\left[(1+9)10^{m+1} - 9m - 19\right]$$

$$= \frac{7}{81}\left[10 \times 10^{m+1} - 9(m+1) - 10\right]$$

$$= \frac{7}{81}\left[10^{m+2} - 9(m+1) - 10\right]$$

$\Rightarrow$ P(m+1) is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q32

Let $P(n): \frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}$ n is a positive integer

For $n=1$

$$\frac{1}{7}+\frac{1}{5}+\frac{1}{3}+\frac{1}{2}-\frac{37}{210}$$
$$=\frac{30+42+70+105-37}{210}$$
$$=\frac{247-37}{210}$$

It is a positive integer

$\Rightarrow$ $P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$,

$$\frac{k^7}{7}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{k^2}{2}-\frac{37}{210}k \text{ is positive integer}$$
$$\frac{k^7}{7}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{k^2}{2}-\frac{37}{210}k=\lambda$$

For $n=k+1$,

$$\frac{(k+1)^7}{7}+\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{(k+1)^2}{2}-\frac{37}{210}(k+1)$$
$$=\frac{1}{7}\left[k^7+7k^6+21k^5+35k^4+35k^3+21k^2+7k+1\right]+\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]$$
$$+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\frac{1}{2}\left[k^2+2k+1\right]-\frac{37k}{210}-\frac{37}{210}$$
$$=\left[\frac{k^7}{7}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{k^2}{2}-\frac{37k}{210}\right]+\left[k^6+3k^5+5k^4+5k^3+3k^2+k+\frac{1}{7}+k^4+2k^3+2k^2+\frac{1}{5}+k^2+k+\frac{1}{3}+k+\frac{1}{2}-\frac{37}{210}\right]$$
$$=\lambda+k^6+3k^5+6k^4+7k^3+6k^2+3k+\frac{1}{7}+\frac{1}{5}+\frac{1}{3}+\frac{1}{2}-\frac{37}{210}$$
$$=\lambda+k^6+3k^5+6k^4+7k^3+6k^2+3k+1$$
$$=\text{Positive integer}$$

$\Rightarrow$ $P(n)$ is true for $n=k+1$

$\Rightarrow$ $P(n)$ is true for all $n\in N$ by PMI

Q33

$P(n):\quad \frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer

For $n = 1$

$$\frac{1}{11}+\frac{1}{5}+\frac{1}{3}+\frac{62}{165}$$
$$=\frac{15+33+55+62}{165}$$
$$=\frac{165}{165}$$

Which is a positive integer

Let $P(n)$ is true for $n = k$, so

$\frac{k^{11}}{11}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{62k}{165}$ is a positive integer

$$\frac{k^{11}}{11}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{62k}{165}=\lambda \qquad \text{---(i)}$$

For $n = k+1$

$$\frac{(k+1)^{11}}{11}+\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{62}{165}(k+1)$$
$$=\frac{1}{11}\left[k^{11}+11k^{10}+55k^9+165k^8+330k^7+462k^6+462k^5+330k^4+165k^3+55k^2+11k+1\right]$$
$$+\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\frac{62}{165}[k+1]$$
$$=\left[\frac{k^{11}}{11}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{62k}{165}\right]+k^{10}+5k^9+15k^8+30k^7+42k^6+42k^5+30k^4+15k^3+5k^2+1+\frac{1}{11}$$
$$+k^4+2k^3+2k^2+k+\frac{1}{5}+k^2+k+\frac{1}{3}+\frac{62}{165}$$
$$=\lambda+k^{10}+5k^9+15k^8+30k^7+42k^6+42k^5+31k^4+17k^3+8k^2+2k+1$$
$$=\text{An integer}$$

$\Rightarrow\quad P(n)$ is true for $n = k+1$

$\Rightarrow\quad P(n)$ is true for all $n \in N$ by PMI

Q34

Let $P(n): \frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)-\cot x$

For $n=1$

$$\frac{1}{2}\tan\frac{x}{2}=\frac{1}{2}\cot\left(\frac{x}{2}\right)-\cot x$$

$$=\frac{1}{2}\frac{1}{\tan\frac{x}{2}}-\frac{1}{\tan x}$$

$$=\frac{1}{2\tan\frac{x}{2}}-\frac{1}{\left(\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right)}$$

$$=\frac{1}{2\tan\frac{x}{2}}-\frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$$=\frac{1-1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$$=\frac{\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$$=\frac{1}{2}\tan\frac{x}{2}$$

$\Rightarrow$ $P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)=\frac{1}{2^k}\cot\left(\frac{x}{2^k}\right)-\cot x \qquad ---(1)$$

We have to show that,

$$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)+\frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)=\frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right)-\cot x$$

Now,

$$\left\{\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)\right\}+\frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

$$=\frac{1}{2^k}\cot\left(\frac{x}{2^k}\right)-\cot x+\frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

$$=\frac{1}{2^k}\cot\left(\frac{x}{2^k}\right)-\cot x+\frac{1}{2.2^k}\frac{1}{\cot\left(\frac{x}{2^k}\cdot\frac{1}{2}\right)}$$

$$=\frac{1}{2^k}\left[\frac{1}{\tan\left(\frac{x}{2^k}\right)}+\frac{1}{2}.\tan\left\{\left(\frac{x}{2^k}\right)\cdot\frac{1}{2}\right\}\right]-\cot x$$

$$=\frac{1}{2^k}\left[\frac{1-\tan^2\left(\frac{x}{2^{k+1}}\right)}{2\tan\left(\frac{x}{2^{k+1}}\right)}+\frac{1}{2}\tan\left(\frac{x}{2.2^k}\right)\right]-\cot x$$

$$=\frac{1}{2^k}\left[\frac{1-\tan^2\left(\frac{x}{2^{k+1}}\right)+\tan^2\left(\frac{x}{2^{k+1}}\right)}{2\tan\left(\frac{x}{2^{k+1}}\right)}\right]-\cot x$$

$$=\frac{1}{2^{k+1}}\left[\frac{1}{\tan\left(\frac{x}{2^{k+1}}\right)}\right]-\cot x$$

$$=\frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right)-\cot x$$

$\Rightarrow$ $P(n)$ is true for $n=k+1$

$\Rightarrow$ $P(n)$ is true for all $n\in N$ by PMI

Q35

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)$$

Above can be written as

$$=\left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\left(\frac{4^2-1}{4^2}\right)\ldots\left(\frac{n^2-1}{n^2}\right)$$

$$=\left(\frac{(2+1)(2-1)}{2^2}\right)\left(\frac{(3+1)(3-1)}{3^2}\right)$$

$$\left(\frac{(4+1)(4-1)}{4^2}\right)\ldots\ldots\left(\frac{(n+1)(n-1)}{n^2}\right)$$

$$=\left(\frac{3\cdot 1}{2^2}\right)\left(\frac{4\cdot 2}{3^2}\right)\left(\frac{5\cdot 3}{4^2}\right)\ldots\ldots\left(\frac{(n+1)\cdot(n-1)}{n^2}\right)$$

In the above product, there are two series in numerator

$3\cdot 4\cdot 5\ldots\ldots(n+1)\, and\, 1\cdot 2\cdot 3\ldots.(n-1)$

All numbers from 3 to (n-1) are repeated twice

and 1, 2, n are appeared once in numerator

So after cancelling like terms we get

$$=\frac{(n+1)}{2n}$$

Q36

$$P(n): \frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{\sqrt{3n+1}}$$

For $n = 1$

$$\frac{2!}{2^2.1} \le \frac{1}{\sqrt{4}}$$

$$= \frac{1}{2} \le \frac{1}{2}$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{(2k)!}{2^{2k}(k!)^2} \le \frac{1}{\sqrt{3k+1}} \qquad ---(1)$$

We have to show that,

$$\frac{2(k+1)!}{2^{2(k+1)}\left[(k+1)!\right]^2} \le \frac{1}{\sqrt{3k+4}}$$

Now,

$$\frac{2(k+1)!}{2^{2(k+1)}\left[(k+1)!\right]^2}$$

$$= \frac{(2k+2)!}{2^{2k}.2^2(k+1)!(k+1)!}$$

$$= \frac{(2k+2)(2k+1)(2k)!}{4.2^2(k+1)(k!)(k+1)(k!)}$$

$$= \frac{2(k+1)(2k+1)(2k)!}{4.(k+1)^2.2^{2k}.(k!)^2}$$

$$\le \frac{2(2k+1)}{4(k+1)} \cdot \frac{1}{\sqrt{3k+1}} \qquad \left[\text{Using equation } (1)\right]$$

$$\le \frac{(2k+1)}{2(k+1)} \cdot \frac{1}{\sqrt{3k+1}}$$

$$\le \frac{(2k+2)}{2(k+1)} \cdot \frac{1}{\sqrt{3k+3+1}}$$

$$\le \frac{1}{\sqrt{3k+4}} \qquad \left[\begin{array}{l}\text{Since, } 2k+1 < 2k+2 \\ 3k+1 \le 3k+4\end{array}\right]$$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q37

Let $P(n): 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}<2-\frac{1}{n}$ for all $n \geq 2$

For $n=2$

$1+\frac{1}{4}<2-\frac{1}{4}$

$=\frac{5}{4}<\frac{7}{4}$

$\Rightarrow \quad P(n)$ is true for $n=2$

Let $P(n)$ is true for $n=k$,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{k^2}<2-\frac{1}{k} \qquad ---(1)$$

Now, we have to show that,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{(k+1)}$$

Now,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$<2-\frac{1}{k}+\frac{1}{(k+1)^2} \qquad [\text{Using } (1)]$$

$$<2-\frac{k^2+2k+1-k}{k(k+1)^2}$$

$$<2-\frac{k^2+k+1}{k(k+1)^2}$$

$$<2-\frac{k^2+k}{k(k+1)^2}$$

$$<2-\frac{k(k+1)}{k(k+1)^2}$$

$$<2-\frac{1}{k+1}$$

$\Rightarrow \quad P(n)$ is true for $n=k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q38

Let $P(n): x^{2n-1} + y^{2n-1}$ is divisible by $(x+y)$

For $n = 1$

$x^{2(1)-1} + y^{2(1)-1}$

$= x + y$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$x^{2k-1} + y^{2k-1}$ is divisible by $(x+y)$

$x^{2k-1} + y^{2k-1} = (x+y)\lambda \qquad\qquad ---(1)$

We have to show that,

$x^{2k+1} + y^{2k+1} = \quad (x+y)\mu$

Now,

$x^{2k+1} + y^{2k+1}$

$= x^{2k-1}x^2 + y^{2k-1}.y^2$

$= \left[(x+y)\lambda - y^{2k-1}\right]x^2 + y^{2k-1}.y^2$

$= (x+y)\lambda x^2 - y^{2k-1}.x^2 + y^{2k-1}.y^2$

$= (x+y)\lambda x^2 - y^{2k-1}\left(x^2 - y^2\right)$

$= (x+y)\lambda x^2 - y^{2k-1}(x+y)(x-y)$

$= (x+y)\left[\lambda x^2 - y^{2k-1}(x-y)\right]$

$= (x+y)\mu$

$\Rightarrow \quad P(n)$ is true for $n = k+1$

$\Rightarrow \quad P(n)$ is true for all $n \in N$ by PMI

Q39

Let $P(n): \sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$

For $n = 1$

$$\sin x = \frac{\sin^2 x}{\sin x}$$

$$\sin x = \sin x$$

$\Rightarrow$ $P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\sin x + \sin 3x + \ldots + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x} \quad \text{---(i)}$$

We have to show that

$$\sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x = \frac{\sin^2 (k+1)x}{\sin x}$$

Now,

$$\{\sin x + \sin 3x + \ldots + \sin(2k-1)x\} + \sin(2k+1)x$$

$$= \frac{\sin^2 kx}{\sin x} + \frac{\sin(2k+1)x}{1}$$

Using equation (i),

$$= \frac{\sin^2 kx + \sin(2k+1)x \sin x}{\sin x}$$

$$= \frac{2\sin^2 kx + \cos[(2k+1)x - x] - \cos[2kx + x + x]}{2\sin x}$$

$$= \frac{2\sin^2 kx + \cos 2kx - \cos(2kx + 2x)}{2\sin x}$$

$$= \frac{1 - \cos 2kx + \cos 2kx - \cos 2x(k+1)}{2\sin x}$$

$$= \frac{1 - \cos 2x(k+1)}{2\sin x}$$

$$= \frac{2\sin^2 x(k+1)}{2\sin x}$$

$$= \frac{\sin^2 x(k+1)}{\sin x}$$

$\Rightarrow$ $P(n)$ is true for $n = k+1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI.

Q40

Let $P(n)$ be the statement given by

$$P(n): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots\cdots + \cos\left(\alpha+(n-1)\beta\right)$$

$$= \frac{\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}} \text{ for all } n \in N.$$

Step I:

$$P(1): \cos\alpha = \frac{\cos\left\{\alpha+\left(\frac{1-1}{2}\right)\beta\right\}\sin\left(\frac{1\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$\Rightarrow \cos\alpha = \frac{\cos\{\alpha+0\}\sin\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$\Rightarrow \cos\alpha = \cos\alpha$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots\cdots + \cos\left(\alpha+(m-1)\beta\right)$$

$$= \frac{\cos\left\{\alpha+\left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

We have to prove that $P(m+1)$ is true.

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots\cdots + \cos\left(\alpha+(m)\beta\right)$$

$$= \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots\cdots + \cos\left(\alpha+(m-1)\beta\right) + \cos\left(\alpha+(m)\beta\right)$$

$$= \frac{\cos\left\{\alpha+\left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}} + \cos\left(\alpha+(m)\beta\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots[\text{Using (i)}]$$

$$=\frac{\sin\frac{\beta}{2}\cos\ (\alpha+(m)\beta)+\cos\left\{\alpha+\left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$=\frac{\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m+1}{2}\right)\beta\right)-\sin\left(\alpha+\left(\frac{2m-1}{2}\right)\beta\right)\right]+\cos\left\{\alpha+\left(\frac{m-1}{2}\right)\beta\right\}\sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$=\frac{\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m+1}{2}\right)\beta\right)-\sin\left(\alpha+\left(\frac{2m-1}{2}\right)\beta\right)\right]+\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m-1}{2}\right)\beta\right)+\sin\left(-\alpha+\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$=\frac{\frac{1}{2}\left[\sin\left(\alpha+\left(\frac{2m+1}{2}\right)\beta\right)\right]+\frac{1}{2}\left[\sin\left(-\alpha+\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$=\frac{\frac{1}{2}\left[\sin\alpha\cos\left(\frac{2m+1}{2}\right)\beta+\cos\alpha\sin\left(\frac{2m+1}{2}\right)\beta+\sin\frac{\beta}{2}\cos\alpha-\cos\frac{\beta}{2}\sin\alpha\right]}{\sin\frac{\beta}{2}}$$

$$=\frac{\frac{1}{2}\left[\sin\alpha\left(\cos\left(\frac{2m+1}{2}\right)\beta-\cos\frac{\beta}{2}\right)+\cos\alpha\left(\sin\left(\frac{2m+1}{2}\right)\beta+\sin\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$=\frac{\frac{1}{2}\left[-2\sin\alpha\left(\left(\sin\left(\frac{m+1}{2}\right)\beta\right)\sin\frac{m\beta}{2}\right)+2\cos\alpha\left(\left(\sin\left(\frac{m+1}{2}\right)\beta\right)\cos\frac{m\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$=\frac{\sin\left(\frac{(m+1)\beta}{2}\right)\left[\cos\alpha\cos\frac{m\beta}{2}-\sin\alpha\sin\left(\frac{(m+1)\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

$$=\frac{\sin\left(\frac{(m+1)\beta}{2}\right)\cos\left(\alpha+\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q41

$$\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24},$$

Using induction we first show this is true for n=2:

$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}=\frac{14}{24}>\frac{13}{24}(True)$$

Now lets assume it is true for some n=k,

$$S_k=\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}>\frac{13}{24}$$

Finally we need to prove that this implies

it is also true for n=k+1:

$$S_{k+1}=\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{2k+2}$$

$$=\frac{-1}{k+1}+\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+...\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$=\frac{-1}{k+1}+S_k+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$=S_k+\frac{1}{2(2k+1)(k+1)}$$

$$>S_k$$

$$\therefore S_{k+1}>\frac{13}{24}$$

Q42

$$a_1 = \frac{1}{2}\left(a_0 + \frac{A}{a_0}\right), A_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right) \text{ and } a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$$

Let $\quad P(n) : \dfrac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\dfrac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$

For $n = 1$

$$\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$$

$$\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)$$

$\Rightarrow \quad P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right) \qquad \text{---(i)}$$

We have to show that

$$\frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^k}$$

$$\left(\frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}}\right)^{2^0}$$

$$= \left[\frac{\frac{1}{2}\left(a_k + \frac{A}{a_k}\right) - \sqrt{A}}{\frac{1}{2\left(a_k + \frac{A}{a_k}\right) + \sqrt{A}}}\right]^{2^0}$$

$$= \left[\frac{(a_k)^2 + A - 2ak\sqrt{A}}{(ak)^2 + A + 2a_k\sqrt{A}}\right]^{2^0}$$

$$= \frac{\left(a_k - \sqrt{A}\right)^2}{\left(a_k + \sqrt{A}\right)^2}$$

$$= \left[\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}}\right]^{2^1}$$

$$= \left[\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right]^{2^k}$$

$\Rightarrow \quad P(n)$ is true for $n = k + 1$

$\Rightarrow \quad P(n)$ is true for all $n \in NE$ by PMI

Q43

$P(n): 2^n \geq 3n$

It is given that $P(r)$ is true, so

$2^r \geq 3r$ $\qquad ---(1)$

Multiplying both the sides by 2,

$2^r.2 \geq 3r.2$

$2^{r+1} \geq 6r$

$2^{r+1} \geq 3r + 3r$

$2^{r+1} \geq 3 + 3r$ $\qquad [\text{Since } 3r \geq 3,\ 6r \geq 3 + 3r]$

$2^{r+1} \geq 3(r+1)$

So, $P(r+1)$ is true

But for $r = 1$

$2 \geq 3$

It is true, so

$P(n)$ is not true for all $n \in N$ by PMI

Q44

$S_n = 1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + \ldots\ldots$

Using induction we first show this is true for n=2,

we get $S_2 = 1^2 + 2\times 2^2 = 1+8 = 9$

From RHS, we have if n is even $S_n = \dfrac{n(n+1)^2}{2}$

$S_2 = \dfrac{2\times 9}{2} = 9$

Now using induction we first show this is true also

for n=3, we get $S_3 = 1+8+9 = 18$

From RHS, we have if n is odd $S_n = \dfrac{n^2(n+1)}{2}$

$S_3 = \dfrac{9\times 4}{2} = 18$

Lets assume above is true for n=k, we get

k is even, $S_k = 1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + \ldots\ldots + 2\times k^2 \;--1$

k is odd, $S_k = 1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + \ldots\ldots + k^2 \;----2$

Now lets prove for n=k+1

If k is even, k+1 is odd we get

$S_{k+1} = 1^2 + 2\times 2^2 + 3^2 + \ldots\ldots + 2\times k^2 + (k+1)^2 \;---3$

From above relation, we get

$S_k = 1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + \ldots\ldots + 2\times k^2 = \dfrac{k(k+1)^2}{2}$

Substitute this in 3, we get

$S_{k+1} = \dfrac{k(k+1)^2}{2} + (k+1)^2 = \dfrac{(k+1)^2(k+2)}{2}$

$= RHS$ (when '$k+1$'is odd)

Hence Proved

Q45

Let $P(n)$ be the statement given by

$P(n)$: The number of subsets of a set containing n distinct elements is 2^n for all $n \in N$.

Step I:

$P(1): 2^1 = 2$

For any set A contaiining 1 element, empty set and set A are two sets always subsets of A.

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

A set containing m distinct elements has 2^m subsets.........(i)

We have to prove that $P(m+1)$ is true.

Let the set A has $(m+1)$ elements.

$A = \{1,2,.......,m,m+1\}$

$A = \{1,2,.......,m\} \cup \{m+1\}$

Now using (i) we can say that $\{1,2,.......,m\}$ being m elemets has 2^m subsets.

For $\{m+1\}$, empty set and set itself $\{m+1\}$ are subsets.

So, $\{m+1\}$ has 2 subsets.

$\Rightarrow$ Set A has $2^m + 2$ subsets

$\Rightarrow$ Set A has 2^{m+1} subsets

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q46

Let P(n) be the statement given by

$P(n): a_n = 3 \times 7^{n-1}$ for all $n \in N$.

Step I:

$P(2): a_2 = 3 \times 7^{2-1} = 21$

Given that $a_k = 7\ a_{k-1}$ for all natural numbers $k \geq 2$

$a_2 = 7a_1 = 7 \times 3 = 21$

$\therefore$ P(2) is true.

Step II:

Let P(m) is true. Then,

$a_m = 3 \times 7^{m-1}$..........(i)

We have to prove that P(m + 1) is true.

$a_{m+1} = 7a_m$

$a_{m+1} = 7 \times a_m$

$a_{m+1} = 7^1 \times 3 \times 7^{m-1}$...........[from(i)]

$a_{m+1} = 3 \times 7^{m-1+1}$

$a_{m+1} = 3 \times 7^m$

$\Rightarrow$ P(m+1) is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q47

Let P(n) be the statement given by

$P(n): x_n = \frac{2}{n!}$ for all $n \in N$.

Step I:

$P(2): x_2 = \frac{2}{2!} = 1$

Given that $x_k = \frac{x_{k-1}}{n}$ for all natural numbers $k \geq 2$

$x_2 = \frac{x_1}{2} = \frac{2}{2} = 1$

$\therefore$ P(2) is true.

Step II:

Let P(m) is true. Then,

$x_m = \frac{2}{m!}$..........(i)

We have to prove that $P(m+1)$ is true.

$x_{m+1} = \frac{x_{m+1-1}}{m+1}$

$x_{m+1} = \frac{x_m}{m+1}$

$x_{m+1} = \frac{\frac{2}{m!}}{m+1}$............[from(i)]

$x_{m+1} = \frac{2}{m!(m+1)}$

$x_{m+1} = \frac{2}{(m+1)!}$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q48

Let $P(n)$ be the statement given by

$P(n): x_n = 5 + 4n$ for all $n \in N$.

Step I:

$P(1): x_1 = 5 + 4(1) = 5 + 4 = 9$

Given that $x_k = 4 + x_{k-1}$ for all natural numbers k

$x_1 = 4 + x_0 = 4 + 5 = 9$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$x_m = 5 + 4m$.........(i)

We have to prove that $P(m+1)$ is true.

$x_{m+1} = 4 + x_{m+1-1}$

$x_{m+1} = 4 + x_m$

$x_{m+1} = 4 + 5 + 4m$...........[from(i)]

$x_{m+1} = 5 + 4(m+1)$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q49

Let P(n) be the statement given by

$P(n): \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$.

Step I:

$P(2): \sqrt{2} = 1.4142$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{1.4142} = 1 + 0.7071 = 1.7071$$

$$\therefore \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$$

$\therefore$ P(2) is true.

Step II:

Let P(m) is true. Then,

$$\sqrt{m} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} \ldots\ldots\ldots(i)$$

We have to prove that P(m + 1) is true.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} > \sqrt{m} \ldots\ldots\ldots\ldots[\text{from}(i)]$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m} + \frac{1}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{\sqrt{m^2+m}+1}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{\sqrt{m^2}+1}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{m+1}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}$$

$\Rightarrow$ P(m+1) is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.

Q50

The distributive law from algebra states that for all real numbers c, a1 and a2, we have c (a1 + a2) = ca1 + ca2
Use this law and mathematical induction to prove that, for all natural numbers, n 2, if c (a1 + a2 + &. + an)

= ca1 + ca2 + ... + can

Let P(n) be the statement given by
$P(n): c(a_1 + a_2 + \ldots\ldots + a_n) = ca_1 + ca_2 + ca_3 + \ldots\ldots + ca_n$ for all natural numbers $n \geq 2$.

Step I:
$P(2): c(a_1 + a_2) = ca_1 + ca_2$

$\therefore$ P(2) is true.

Step II:
Let P(m) is true. Then,
$c(a_1 + a_2 + \ldots\ldots + a_m) = ca_1 + ca_2 + ca_3 + \ldots\ldots + ca_m \ldots\ldots\ldots(i)$

We have to prove that P(m + 1) is true.
$c(a_1 + a_2 + \ldots\ldots + a_m + a_{m+1}) = c[(a_1 + a_2 + \ldots\ldots + a_m) + a_{m+1}]$
$c(a_1 + a_2 + \ldots\ldots + a_m + a_{m+1}) = c(a_1 + a_2 + \ldots\ldots + a_m) + ca_{m+1}$
$c(a_1 + a_2 + \ldots\ldots + a_m + a_{m+1}) = ca_1 + ca_2 + ca_3 + \ldots\ldots + ca_m + ca_{m+1}$
$\ldots\ldots\ldots\ldots[\text{from}(i)]$

$\Rightarrow$ P(m+1) is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in N$.