Beyond Online Balanced Descent: An Optimal Algorithm for Smoothed Online Optimization

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Smoothed Online Convex Optimization

We consider Smoothed Online Convex Optimization (SOCO), a classic problem in the online learning community which has attracted recent attention due to a variety of successful applications in, e.g., data center capacity provisioning, demand response, speech animation, video streaming, network function virtualization, LQR control, and more. SOCO also has connections with other important problems like Online Body Chasing and Metrical Task System.

In SOCO, an online learner plays a series of rounds against an adaptive adversary. In the t-th round,

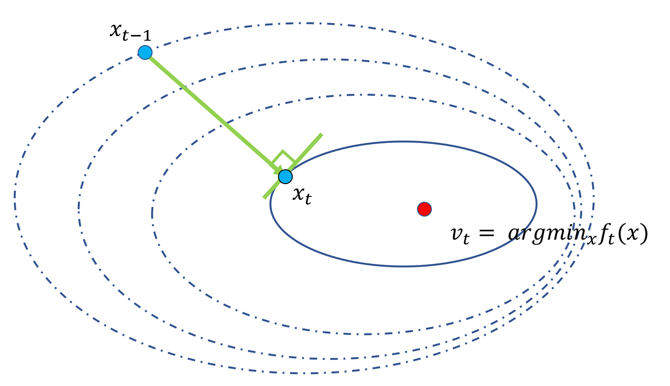
1. The adversary reveals an m-strongly-convex cost function $f_t : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ to the learner.
2. After observing f_t, the learner picks a point $x_t \in \mathbb{R}^d$.
3. The online learner pays the **hitting cost** $f_t(x_t)$ as well as a **switching cost** $\frac{1}{2}\|x_t - x_{t-1}\|_2^2$ which penalizes the learner for changing its decisions between rounds.

We focus on the setting where the cost functions f_t are m-strongly convex with respect to the ℓ_2 norm and the movement cost function is defined as $c(x_{t-1}, x_t) = \frac{1}{2}\|x_t - x_{t-1}\|_2^2$. The goal is to design an online algorithm with low **competitive ratio**, where the competitive ratio is defined as the ratio between the online learner's cost and the offline optimal cost. More precisely, we have

$$\texttt{Competitive Ratio} = \sup_{f_1,\dots,f_T} \frac{\sum_{t=1}^T f_t(x_t) + c(x_t, x_{t-1})}{\min_{x_1,\dots,x_T} \sum_{t=1}^T f_t(x_t) + c(x_t, x_{t-1})}.$$

Online Balanced Descent (OBD)

Online Balanced Descent (OBD) was the first algorithm to achieve a constant-competitive ratio beyond one-dimension; it achieves competitive ratio $3 + O(1/m)$. In every round, OBD projects the previously selected point x_{t-1} onto a level set K_t of the current cost function f_t. The set K_t is selected so that the switching cost incurred in the projection step is within a constant factor γ of the hitting cost associated with K_t. Here γ is a tuning parameter that trades off the two different costs.



We introduce an optimal online learning algorithm for online convex optimization with switching costs.

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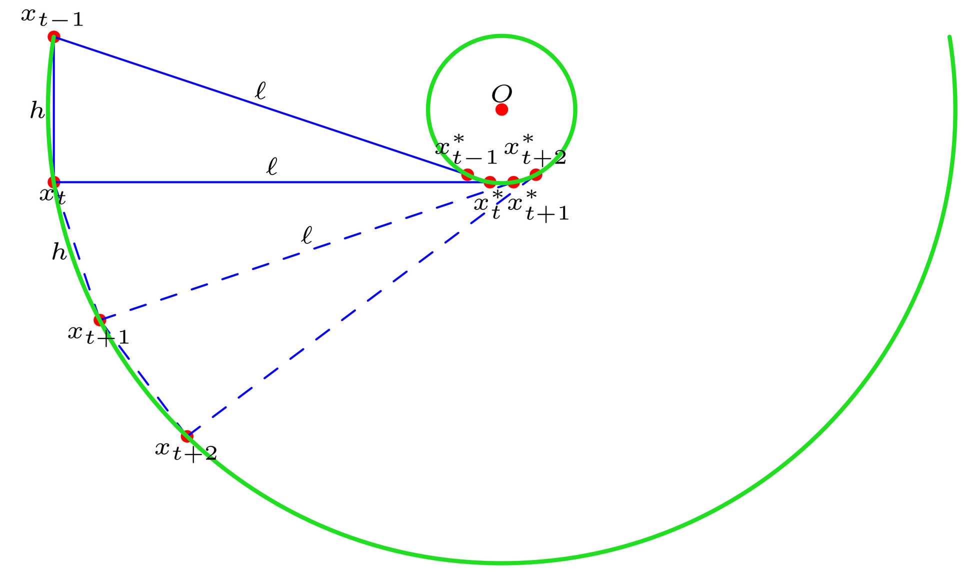
OBD is not optimal

Our first result shows a general lower bound on the competitive ratio of any SOCO algorithm and highlights that OBD cannot match the optimal bound. To prove this result we construct a counterexample so that the algorithm inevitably makes a large jump (the red dashed curve) at the last timestep, but the offline optimal can break the long journey into multiple small steps (the blue dashed curves) to avoid the large single jump.

Theorem 1. *On a SOCO instance with m-strongly convex costs and switching costs given by $c(x_t, x_{t-1}) = \frac{1}{2}\|x_t - x_{t-1}\|^2$, any online algorithm must have a competitive ratio of at least $\frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{m}}\right)$, which is $O(m^{-1/2})$ as $m \to 0^+$. In contrast, the competitive ratio of OBD is $\Omega(m^{-\frac{2}{3}})$ as $m \to 0^+$.*



An Optimal Algorithm: R-OBD

Our main result introduces a novel algorithm, R-OBD, which improves on the competitive ratio achieved by OBD and obtains the optimal, achievable competitive ratio. At round t, R-OBD selects the point

$$x_t \leftarrow \arg\min_x f_t(x) + \lambda_1 c(x, x_{t-1}) + \lambda_2 c(x, v_t),$$

where λ_1 and λ_2 are tunable parameters and $v_t = \arg\min_x f_t(x)$ is the minimizer of each cost function. While this may seem quite different from what OBD does, note that the first-order optimality condition implies that

$$\nabla f_t(x_t) + \lambda_1(x_t - x_{t-1}) + \lambda_2(x_t - v_t) = 0,$$

whereas the point picked by OBD satisfies

$$\nabla f_t(x_t) + \lambda_t(x_t - x_{t-1}) = 0$$

for some (possibly time-varying) choice of λ_t. By picking the parameters λ_1 and λ_2 appropriately, we can make the competitive ratio of R-OBD match the lower bound in Thm. 1, including constant factors.

Theorem 2. *There exists a choice of λ_1, λ_2 such that R-OBD achieves competitive ratio of at most $\frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{m}}\right)$.*