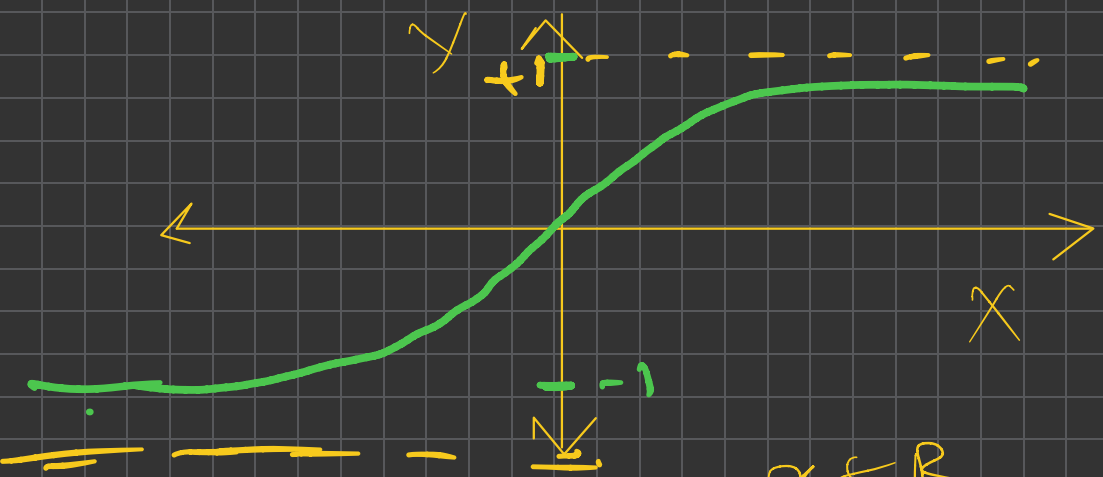
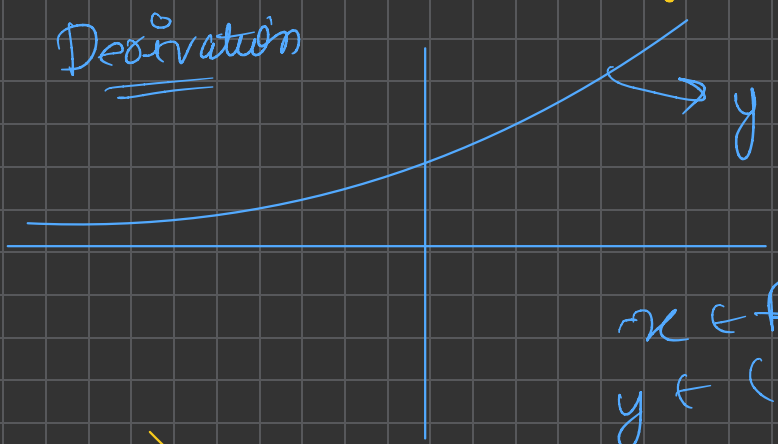
1) $\tanh = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$



$x \in R$

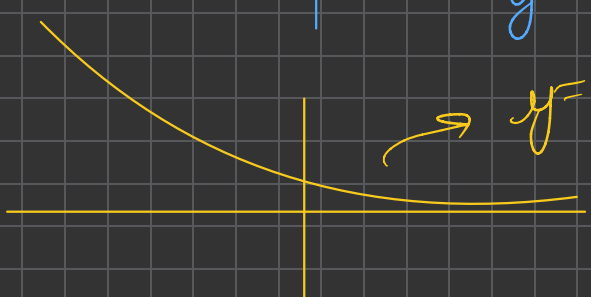
$y \in (-1, 1)$

Derivation



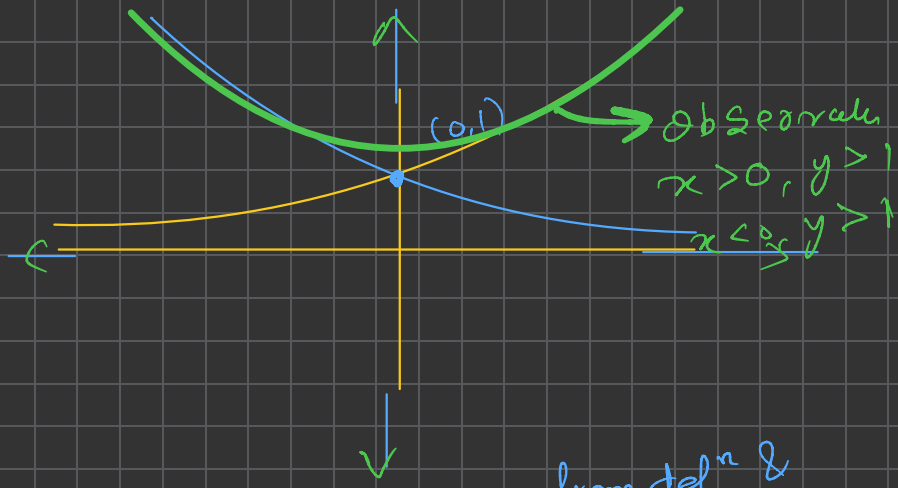
$x \in R$

$y \in (0, \infty)$



$x \in R$

$y \in (0, 1)$



$$f(x) = \frac{e^x + e^{-x}}{2}$$

→ from defⁿ & obs.

→ this new defⁿ is called Hyperbolic cosine

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \rightarrow \text{eq} \rightarrow (1)$$

Similarly

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \rightarrow \text{eq} \rightarrow 2$$

$\cosh(x)$, $x>0$, $x<0$, $|x| = |-x|$, same output y will be there

$\sinh(x)$, $x>0$, $x<0$, $|x| = |-x|$, Same output but diff. sign $|y|$ ✓

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\left(\frac{e^x - e^{-x}}{2}\right)}{\left(\frac{e^x + e^{-x}}{2}\right)}$$

$$\boxed{\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}} \rightarrow \text{Final formula}$$

Q what is h in tanh, sinh, cosh?

A h → hyperbola,

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

(Assuming $a = 1, b = 1$)

$$\boxed{x^2 - y^2 = 1} \rightarrow \text{eqn of hyperbola}$$

⇒

Q I want to find $\cosh^2 x - \sinh^2 x$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4}$$

→ eq (1)

$$\sinh^2(x) = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}$$

→ eq (2)

eq (1) − eq (2)

$$\cosh^2(x) - \sinh^2(x)$$

$$= \left(\frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4}\right) - \left(\frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - e^{2x} - e^{-2x} + 2e^x e^{-x}}{4}$$

$$= \frac{4e^x e^{-x}}{4}$$

$$= \frac{2e^x e^{-x} + 2e^x e^{-x}}{4} = \frac{4e^x e^{-x}}{4}$$

$$= e^x e^{-x} = e^0 = 1$$

conclusion

$$\cosh^2(x) - \sinh^2(x) = 1$$

we know that, $x^2 - y^2 = 1$ — Hyperbola analogs

considering a circle,

$$(x-a)^2 + (y-b)^2 = r^2 \quad (r=1,\ a=0,\ b=0)$$

$$x^2 + y^2 = 1$$

$$x = r\cos\theta, \quad y = r\sin\theta \quad (r=1)$$

$$\cos^2\theta + \sin^2\theta = 1$$

Circle	Hyperbola
$x^2 + y^2 = 1$	$x^2 - y^2 = 1$
$\cos^2\theta + \sin^2\theta = 1$	$\cosh^2(x) - \sinh^2(x) = 1$
$x = \cos\theta, \quad y = \sin\theta$	$x = \cosh(x)$, $y = \sinh(x)$

Note:- $\cos(x) \neq \cosh(x)$

$\sin(x) \neq \sinh(x)$

$\tan(x) \neq \tanh(x)$

2 circles → hyperbola

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$