ZX NOTES

Date: Y.

1 Working plan for Gautham July-September

Overall research question: *How we can use the representation of quantum computations as a MBQC pattern for optimization & compilation tasks?* Every work that fits into this question is useful. The research question splits into two main sub topics:

1. Which graph rewrites are useful on a MBQC pattern?
2. How can we translate MBQC patterns to hardware-adapted instructions?

MBQC patterns can be represented as graph-like diagrams in ZX-calculus.

1.1 Useful graph rewrites

There is no complete rule set for rewriting MBQC patterns so far. Yet, many rules of the complete rule set of standard ZX-diagrams have a similar representation for graph-like diagrams. The bialgebra rule seems to be covered by pivot (resp. local complementation), the copy rule by Z-insertion/Z-deletion, the fusion with again with pivot/ identity removal or neighbor unfusion in the other way.

$\Rightarrow$ Find a complete rule set for graph-like ZX-diagrams.

The most interesting rule for MBQC patterns seems to be local complementation: We can use it to change measurement labels and reduce the number of graph edges in a non trivial way. Optimizing the number of edges with local complementation in general is NP-hard even in the case that all local complementations commute. Yet, if would be interesting to know whether there are restricted graph types for which we can find a sequence of local complementations minimizing the number of edges in polynomial time. One example are graphs locally equivalent to trees. In such cases we can find a sequence of local complementations transforming the graph into a tree which then of course has the minimum number of edges possible. It would be interesting to see whether we can extend the algorithm somehow. Also we could use Pivot instead of local complementation for the same task (although it is restricted in its possibilities)

$\Rightarrow$ Study edge minimization with local complementation.

For an MBQC pattern with Pauli flow, each vertex can be extracted as a Pauli exponential on a quantum circuit, where the underlying graph of the pattern determines the exponential and the Pauli flow determines the order in which we can extract the vertices. We can represent this in a Pauli-dependency DAG. The goal here would be to examine how graph rewrites or changing measurement labels in the Pauli flow affect the exponentials. Yet, determining how the exponentials change is not easy, one can imagine that every operation pushes one or more Pauli strings through the PDDAG updating all Pauli exponentials where the strings anticommute and leaving others unchanged. Still these updates need to be formalized and understood more in order to find algorithms minimizing Pauli exponentials (reducing non-identities in the strings) or getting other desired forms like all Z-terms which are better realizable on neutral atom hardware.

$\Rightarrow$ Study graph and Pauli flow rewrites on PDDAG structure.

1. ZX Introduction

2. Measurements

2.1. Pauli Group.

Definition 1. ***Pauli strings.***

$$P_n = \{\bigotimes_{i=1}^{n} A_i | A_i \in P\} \tag{1}$$

where, P is the Pauli group,

$$P := \alpha\{I, X, Y, Z\}, \alpha \in \{\pm 1, \pm i\} \tag{2}$$

and I, X, Y, Z have matrix representation,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

Note that this is an *irreducible representation*. We can check that X and Z do not share any eigenvectors (and so any invariant 1dim subspaces).

$\mathbb{Q}_8$ is a subgroup of P. P is the smallest subgroup (of $U(2)$) generated by $\langle X, Y, Z\rangle$.

Quotienting P by the center yields the Klein four-group (TODO with other properties and lie algebra lie group).

Note that,

$$SU(2) \cap P_1 = \pm 1\{I\} \cup \pm i\{X, Y, Z\} \leq P_1 \tag{4}$$

This subgroup is isomorphic to $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ via the identification.

$$i \to iX, j \to iY, k \to iZ \tag{5}$$

2.2. Properties.

2.3. Measurements. In an MBQC we need to specify measurement operations for each non-output. We do this by assigning *measurement planes* for each qubit,

Any general measurement for a single qubit is specified by an axis on the Bloch sphere. Convention is to restrict these axes to a plane of the Bloch sphere – XY, YZ, or XZ. The axes selects two states – $|\eta\rangle$ and $|\eta'\rangle$ which are diametrically opposite on the sphere, then we form,

$$\Pi = |\eta\rangle\langle\eta| + |\eta'\rangle\langle\eta'| \tag{6}$$

and make a projective measurement.

We can write these states and axes explicitly. The Bloch sphere is parametrised as (θ, φ) – azimuthal and polar. We choose our axes by fixing one of the angles,

$$\theta = \pi/2 \ (XY), \varphi = 0 \ (YZ) \text{ or } \varphi = \pi/2 \ (XZ) \tag{7}$$

$$\begin{aligned} |+_{XY}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle) & |-_{XY}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - e^{i\alpha}|1\rangle) \\ |+_{XZ}\rangle &= \cos\frac{\alpha}{2}|0\rangle + \sin\frac{\alpha}{2}|1\rangle & |-_{XZ}\rangle &= \sin\frac{\alpha}{2}|0\rangle - \cos\frac{\alpha}{2}|1\rangle \\ |+_{YZ}\rangle &= \cos\frac{\alpha}{2}|0\rangle + i\sin\frac{\alpha}{2}|1\rangle & |-_{YZ}\rangle &= \sin\frac{\alpha}{2}|0\rangle - i\cos\frac{\alpha}{2}|1\rangle \end{aligned}$$

Note that any point (pure state) on the boundary of the Bloch sphere can be written as,

$$(\theta, \varphi) \mapsto \cos\frac{\varphi}{2}|0\rangle + e^{i\theta}\sin\frac{\varphi}{2}|1\rangle \tag{8}$$

The measurement axis coincide with X, Y or Z corresponds to $\alpha = a\pi$ with $a \in \{0, 1\}$ [1]

[1] more exactly $\alpha = \frac{2a}{\pi} \mod 4, a \in \{0, 1, 2, 3\}$ to pick out both axes in each plane.

After picking an axis, we construct our measurement as a projector,

$$\Pi_{\bullet\circ,\alpha} = |+_{\bullet\circ}(\alpha)\rangle\langle +_{\bullet\circ}(\alpha)| - |-_{\bullet\circ}(\alpha)\rangle\langle -_{\bullet\circ}(\alpha)| \tag{9}$$

It is useful to construct the measured Hermitian operators (with eigenbasis $|+\rangle, |-\rangle$) corresponding to each planar measurement.

$$\begin{aligned} XY &\mapsto \cos\alpha X + \sin\alpha Y =: M_{XY,\alpha} \\ XZ &\mapsto \sin\alpha X + \cos\alpha Z =: M_{XZ,\alpha} \\ YZ &\mapsto \sin\alpha Y + \cos\alpha Z =: M_{YZ,\alpha} \end{aligned}$$

Typically in MBQC, our "desired outcome" is the $+1$ eigenvalue collapse to the $|+\rangle$. This is usually denoted as outcome 0, and the undesired $|-\rangle$ collapse as outcome 1.

Depending on the plane of measurement, we can apply a pauli X, Y or Z to correct it. Z changes the relative phase, X swaps 0 and 1 and $ZX = Y$ does both.

$$Z|-_{XY}\rangle = |+_{XY}\rangle,\ Y|-_{XZ}\rangle = |+_{XZ}\rangle \text{ and } X|-_{YZ}\rangle = |+_{YZ}\rangle \tag{10}$$

3. Stabilisers

The idea is to specify a state (uniquely) via the generators of its stabiliser subgroup. "Errors" become changes to generators.

3.1. **Clifford group.**

Definition 2. *The **Clifford group** is the* normaliser *of the Pauli group P_n in $U(2^n)$,*

$$C_n := \{g \in SU(2^n) | g P_n g^{-1} = P_n\} \tag{11}$$

*A **Clifford gate** is an element of C_n.*

fact: the Clifford group on n-qubits is generated by Hadamard, Phase (i) and CNOT gates.

Theorem 3. ***(Gottesmann-Knill).*** *Any circuit involving only initial state $|0\rangle^{\otimes n}$, Clifford gates and Pauli measurements is (polynomial time) easily simulatable.*

So, we want to try to minimize the number of non-Clifford gates in our circuit.

fact: C_n is not a universal gate-set. The T gate cannot be finitely generated.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \tag{12}$$

Note that the Clifford group is not finite because if g normalises P_n then so does $e^{i\phi}g$. We can disregard global phases and just consider $C_n/U(1)$.

3.2. **Stabilisers and States.**

Definition 4. *The **subspace stabilised** by a subgroup $H \leq P_n$ is,*

$$V_H := \{|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} | h|\psi\rangle = |\psi\rangle \forall h \in H\} \tag{13}$$

Proposition 5. *For all $g \in U(2^n)$ and $H \leq P_n$, $V_{gHg^{-1}} = gV_H$.*

Note that if $-\mathbb{1} \in H$, then $-\mathbb{1}|\psi\rangle = |\psi\rangle \implies |\psi\rangle = 0$. This must be excluded for the stabiliser space to be non-trivial.

Proposition 6. *If $H \leq P_n$ and* $\dim(V_H) > 0$*, then H is abelian and $-\mathbb{1} \notin H$.*

Proof. $g_1, g_2 \in P_n$ either commute or anticommute. If they anticommute then $g_1g_2|\psi\rangle = -g_2g_1|\psi\rangle \implies |\psi\rangle = 0$ $\square$

Definition 7. *A set $S \leq H$ is* ***independent*** *if for all $g \in S$,*

$$\langle S \setminus \{g\}\rangle \neq \langle S\rangle \tag{14}$$

And finally, the theorem below allows us to specify a unique state by specifying n commuting Pauli strings. Let us denote V_S to be the subspace stabilised by a generating set $\langle S\rangle$.

Theorem 8. *If $S = \{g_1, \dots, g_l\}$ is indepdent, pairwise commutative such that $-\mathbb{1} \notin S$, then $\dim(V_S) = 2^{n-l}$.*

There is a surjective group homomorphism, $r : P_n \to \mathbb{F}_2^{2n}$ with $\ker(r) = \{\pm 1, \pm i\mathbb{1}\}$ via the map,

$$r(X_i) = e_i \text{ and } r(Z_i) = e_{i+n} \tag{15}$$

we can keep track of X_i, Y_i via this map as a $2 \times 2n$ matrix called the check matrix.

For example, the bell state ($|00\rangle + |11\rangle$) is the stabiliser state of,

$$\langle X_1X_2, Z_1Z_2\rangle \tag{16}$$

with associated check matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{17}$$

The stabiliser tableaux is an extension of the check matrix of the state.

Since we can specify states via its stabiliser (group), we can also track measurements. First, note that for any $g \in P_n$, the projectors P_+, P_- can be written as,

$$P_+ = \frac{1}{2}(I + g) \text{ and } P_- = \frac{1}{2}(I - g) \tag{18}$$

with probabiltiies,

$$p(+) = \frac{1}{2}(1 + \langle\psi|\tilde{g}|\psi\rangle) \text{ and } p(-) = \frac{1}{2}(1 - \langle\psi|\tilde{g}|\psi\rangle) \tag{19}$$

We want to measure some observable $\tilde{g} \in P_n$.

It is easy to check that any $g \in P_n$ either commutes or anticommutes with $\tilde{g}$. Consider a stabiliser V_S generated by $S = \langle g_1, \dots, g_n\rangle$. There are two cases,

(i) $\tilde{g}$ commutes with all stabilisers.

Then, for any g, $g(\tilde{g}|\psi\rangle) = \tilde{g}|\psi\rangle$. So, $\tilde{g}|\psi\rangle$ is a common $+1$ eigenvector of all stabilisers. Since V_S is one-dimensional, $\tilde{g}|\psi\rangle \propto |\psi\rangle$, but because it has eigenvalues ± 1, $\tilde{g}|\psi\rangle = \pm|\psi\rangle$. So, one of $\pm\tilde{g}$ is a stabiliser.

The outcome is deterministic $+$ or $-$ if $+\tilde{g} \in S$ or $-\tilde{g} \in S$. We do not need to update the stabilisers,

$$S = \langle g_1, \dots, g_n\rangle \tag{20}$$

(ii) $\tilde{g}$ anticommutes with some stabilisers.

wlog, we can assume that $\tilde{g}$ anticommutes with g_1 and commutes with all other g_i by picking g_1 such that $\{\tilde{g}, g_1\} = 0$ and for other i, $g_i \mapsto g_1g_i =: g_i'$,

$$gg_i' = g(g_1g_i) = -g_1(gg_i) = -g_1(-g_ig) = g_i'g \tag{21}$$

note that this is just a change of choice of generators and does not change S.

In this case, the measurement is *not* deterministic because,

$$p(+) = \frac{1}{2}(1 + \langle\psi|\tilde{g}|\psi\rangle) = \frac{1}{2}(1 + \langle\psi|\tilde{g}g_1|\psi\rangle) = \frac{1}{2}(1 - \langle\psi|\tilde{g}|\psi\rangle) = p(-) = \frac{1}{2} \tag{22}$$

In this case, depending on the measurement outcome we update the stabiliser (after measurement) to,

$$S = \langle \pm\tilde{g}, g_2, \dots, g_n\rangle \tag{23}$$

The GHZ state for example cannot be described by a stabiliser because $p(Z_1 = +) = \frac{1}{3}$.

Note that all of this assumes that we are making a *pauli measurement.* In an MBQC, before we make the X_u local measurement, we must update the tableaux to the above form – anticommuting with one (neighbour) vertex and commuting with all others as in 21.

This update for a X-measurement is the exact same as local complementation on the measured vertex (prove later).

4. MBQC

MBQC leverages entanglement (teleportation) instead of unitary gates to create a one-way model of quantum computation.

Definition 9. *A* ***measurement pattern*** *consists of an n-qubit register V with distinguished sets $I, O \subseteq V$ of inputs and outputs. Additionally there is a sequence of the following operations:*

i. Preperations – intialising all qubits $i \notin I$ in the $|+\rangle$ state.

ii. Entangling – applying a CZ_{ij} to qubit i as control and qubit j as target for (chosen) pairs $(i,j) \in V \times V$.

iii. Destructive measurements – which project qubits $i \notin O$ onto orthonormal basis $\{|+_{\lambda,\alpha}, |-_{\lambda}, \alpha\rangle\}$ as described in 2.3.

iv. Corrections – conditionally applying X or Z gates onto qubits $i \in V$.

The *graph* (state) for a measurement pattern is determined by the set of tuples $(i,j) \in V \times V$ such that we apply CZ_{ij}. Along with specifying measurement axes and angles for non-outputs $\bar{O} := V \implies$, we get a labelled open graph.

Definition 10. *A* ***labelled open graph*** *is a tuple $\Gamma = (G, I, O, \lambda)$ where $G = (V, E)$ is a graph, $I, O \subseteq V$ are (input and output) subsets of V and $\lambda : \bar{O} \to \{XY, YZ, XZ\} \times [0, 2\pi)$.*

Every measurjement yields an outcome 0 or 1. In total there are $2^{|\bar{O}|}$ possible outcomes – these are called *branches* of the measurement pattern.

After corrections we can sometimes ensure that every branch yields the same output. When this is possible, the measurement pattern is called **deterministic**. This is formalised via *flow.*

The **linear map** associated with a deterministic MBQC then looks like,

$$\underbrace{\left(\prod_{i\in\bar{O}} \langle +_{\lambda(i),\alpha(i)}|\right)}_{\text{(det) measurement}} \underbrace{\left(\prod_{i\sim j} CZ_{ij}\right)}_{\text{entangling}} \underbrace{\left(\prod_{i\in\bar{I}} |+\rangle_i\right)}_{\text{preperation}} \tag{24}$$

This acts on the input state $\bigotimes_{i\in I} |\psi_i\rangle$. The *resource state* used is,

$$\left(\prod_{i\sim j} CZ_{ij}\right)\left(\prod_{i\in\bar{I}} |+\rangle_i\right)|\psi\rangle_I \tag{25}$$

4.1. **Corrections.** Note that we cannot simply apply corrections by applying a Z or X gate on an incorrect outcome (typically a hardware/cost limitation on MBQC). Let us construct the stabiliser of the graph state.

First note that $|+\rangle_u = X_u|+\rangle_u$ and the identity,

$$CZ_{uv}X_u = X_uZ_vCZ_{uv} \tag{26}$$

Then, for any $w \in \bar{I}$,

$$\left(\prod_{u\sim v} CZ_{u,v}\right)\left(\prod_{u\in\bar{I}} |+\rangle_u\right) = \left(\prod_{u\sim v} CZ_{u,v}\right) X_w \left(\prod_{u\in\bar{I}} |+\rangle_u\right) \tag{27}$$

$$= \left(\prod_{v\in N_G(w)} Z_v\right) X_w \left(\prod_{u\sim v} CZ_{u,v}\right) \tag{28}$$

where we use the identity above when CZ involves vertex w, and otherwise it commutes with X_w. Also, the Z string commutes with X_w.

So, we see that,

$$\left[\left(\prod_{v\in N_G(w)} Z_v\right) X_w\right]\left(\prod_{u\sim v} |+\rangle_u\right) = \left(\prod_{u\sim v} |+\rangle_u\right) \tag{29}$$

For each vertex we get a stabiliser string of Z on the neighbours and X on the vertex.

Equivalently, we obtain the following operator equivalence that will allow us to perform corrections,

$$X_w \left(\prod_{u\sim v} |+\rangle_u\right) = \left(\prod_{v\in N_G(w)} Z_v\right)\left(\prod_{u\sim v} |+\rangle_u\right) \tag{30}$$

The above case is easy – to correct by applying a local X on vertex w can be achieved by applying a Z to all neighbours of w.

We can also take the product of stabilisers indexed (defined) by many vertices. To corrct by applying a local Z, we can take the stabiliser of vertices in in its neighbourhood.

Generally, we look for a stabiliser of the graph state,

$$S = Q_v \otimes P_{rest} \tag{31}$$

where Q_v is the local correction operator on vertex v. This is dependent (only) on the choice of plane and angle. The string of paulis P_{rest} is called an *extraction string.* Simple algebraic manipulation of a chosen stabiliser is the generator for corrections.

We want P_{rest} to be supported on the future (unmeasured) qubits. This is possible when we have a *flow* condition on the graph.

4.2. **Flow.** The simplest case is causal flow, we assume that all vertices are measured in the XY basis.

Definition 11. ***(causal flow).***
Given a labelled open graph $\Gamma = (G, I, O, \lambda)$ such that $\lambda(v) = XY$ for all $v \in \bar{O}$, a causal flow is a tuple $(f, \prec)$ where $f : \bar{O} \to \bar{I}$ and $\prec$ is strict partial order on V satisfying,

i. $v \sim f(v)$

ii. $v \prec f(v)$

iii. $\forall u \in N_G(f(v))$, $u = v$ *or* $v \prec u$

The partial ordering $\prec$ gives us the order in which to perform measurements.

Here, an error on any $v \in \bar{O}$ can be corrected via applying Z_v. Consider the stabiliser of vertex $f(v)$, rearranging we get,

$$\prod_{w \in N_G(f(v) \setminus \{v\})} Z_w X_{f(v)} = Z_v \tag{32}$$

Allowing measurements in all three planes and products of vertex stabilisers gives us generalised flow.

The following algebraic relation is very useful, let the stabiliser be specified by a set $g(v) \subseteq \bar{I}$,

$$\prod_{u \in g(v)} \left(\prod_{w \in N_G(u)} Z_w \right) X_u = \left(\prod_{u \in g(v)} \prod_{w \in N_G(u)} Z_w \right) \left(\prod_{u \in g(v)} X_u \right) \tag{33}$$

$$= \left(\prod_{u \in \mathrm{odd(g(v))}} Z_u \right) \left(\prod_{u \in g(v)} X_u \right) \tag{34}$$

$$= \left(\prod_{u \in \mathrm{odd}(g(v)) \setminus g(v)} Z_u \right) \left(\prod_{u \in g(v) \cap \mathrm{odd}(g(v))} Y_u \right) \left(\prod_{u \in g(v) \setminus \mathrm{odd}(g(v))} X_u \right) \tag{35}$$

where, in the second line we use that $Z_w^2 = I$, and that only Z on the odd neighbourhood of $g(v)$ survive.

We then write the expression as a disjoint product – either,

i. $u \in \mathrm{odd}(g(v))$ and $u \notin g(v)$ – apply Z

ii. $u \in g(v)$ and $u \in \mathrm{odd}(g(v))$ – apply Y

iii. $u \in g(v)$ and $u \notin \mathrm{odd}(g(v))$ – apply X

Finally, based on whether we require a X_v, Y_v or Z_v correction, we choose $g(v)$ appropriately such that v belongs to one of the corresponding sets. We search for subsets $g(v)$ and the ordering $\prec$ simultaneously, not separately.

Definition 12. ***(generalised flow).***
Given a labelled open graph $\Gamma = (G, I, O, \lambda)$ such that $\lambda(v) \in \{XY, XZ, YZ\}$ for all $v \in \bar{O}$, a generalised flow *or* glow *for Γ is a tuple $(g, \prec)$ where $g : \bar{O} \to \mathcal{P}(\bar{I})$ and $\prec$ is a strict partial order over V satisfying for all $v \in \bar{O}$,*

i. for all $u \in g(v)$, $v \neq u \implies v \prec u$

ii. for all $u \in \mathrm{odd}(g(v))$, $v \neq u \implies v \prec u$

iii. $\lambda(v) = XY \implies v \in \mathrm{odd}(g(v)) \setminus g(v)$

iv. $\lambda(v) = XZ \implies v \in g(v) \cap \mathrm{odd}(g(v))$

v. $\lambda(v) = YZ \implies v \in g(v) \setminus \mathrm{odd}(g(v))$

Conditions i, ii ensure that correction sets are in the *future*, and iii - v follow from the stabiliser algebra.

We can generalise this by noting that if some measurements are pauli, $\lambda(u) \in \{X, Y, Z\}$, then u can be in the correcting set of some v in the future ($u \prec v$) as long as correcting v induces the same pauli $\lambda(u)$ on u. This is described as a *free correction in the past.* Note that the $\pm$ sign must still be tracked, but doesn't require the application of any operator.

Definition 13. ***(Pauli flow).***
Given a labelled open graph $\Gamma = (G, I, O, \lambda)$, a Pauli flow *for Γ is a tuple $(p, \prec)$ where $p : \bar{O} \to \mathcal{P}(\bar{I})$ and $\prec$ is a strict partial order over V satisfying for all $v \in \bar{O}$,*

$[\prec .X]$ *for all $u \in p(v)$, $v \neq u$ and $\lambda(u) \notin \{X, Y\} \implies v \prec u$.*

$[\prec .Z]$ *for all $u \in \mathrm{odd}(p(v))$, $v \neq u$ and $\lambda(u) \notin \{Y, Z\} \implies v \prec u$.*

$[\prec .Y]$ *for all $v \not\prec u$, $v \neq u$ and $\lambda(u) = Y \implies u \notin p(v) \triangle \mathrm{odd}(p(v))$*

$[\lambda.XY]$ $\lambda(u) = XY \implies v \in \mathrm{odd}(p(v))$ *and* $v \notin p(v)$

$[\lambda.XZ]$ $\lambda(u) = XZ \implies v \in p(v)$ *and* $v \in \mathrm{odd}(p(v))$

$[\lambda.YZ]$ $\lambda(u) = YZ \implies v \in p(v)$ *and* $v \notin \mathrm{odd}(p(v))$

$[\lambda.X]$ $\lambda(u) = X \implies v \in \mathrm{odd}(p(v))$

$[\lambda.Z]$ $\lambda(u) = Z \implies v \in p(v)$

$[\lambda.Y]$ $\lambda(u) = Y \implies v \in p(v) \triangle \mathrm{odd}(p(v))$

where, $\triangle$ denotes the symmetric difference of sets.

The first three conditions are on the ordering $\prec$ similar to gflow, with the additional exceptional case for planar measurements in the past. The other six conditions are identical to gflow, with the last three explicitly written for the planar case.

We can describe the stabiliser algebra for the correcting set $p(v)$ as,

$$S(p(v)) = \prod_{w \in p(v)} X_w \prod_{w \in \mathrm{odd}(p(v))} Z_w \tag{36}$$

The local operator at some vertex $u \in p(v)$ is,

$$P_u = X^{1_{u \in p(v)}} Z^{1_{u \in \mathrm{odd}(p(v))}} \tag{37}$$

where, $1_{x \in A}$ denotes the characteristicsfunction – 1 if $x \in A$ and 0 if $x \notin A$.

The third axiom $[\prec .Y]$ describes the $(0, 0)$ corresponding to a local I and $(1, 1)$ corresponding to a local Y case. Note that writing $u \notin p(v) \cap \mathrm{odd}(p(v))$ would not allow the trivial identity case.

In Pauli flow, past vertices with pauli measurements are allowed to be in future correcting sets as long as the induced local operator commutes.

Focussed flow requires that this is *necessarily* the case for every correction regardless of ordering. That is, the induced $P_{v \to u} \in \{\mathrm{I}, \lambda(u)\}$. Here,

$$P_{v \to u} := X^a Z^b, \quad a = 1_{u \in p(v)}, \; b = 1_{u \in \mathrm{odd}(p(v))} \tag{38}$$

whether $v \prec u$ or $u \prec v$.

Definition 14. ***(focussed pauli flow).***
a pauli flow $(p, \prec)$ is focussed if for all $u, v \in \bar{O}$ and $u \neq v$,

$$P_{v \to u} \in \{I, \lambda(u)\}$$

There is an amibiguity here with treating non-planar measurements, so we identify measurement planes with the pauli at angle 0.

A bit more generally,

Definition 15. ***(focussed sets)*** *Given a labelled open graph Γ, a set $\tilde{p} \subseteq \bar{I}$ is focussed over $S \subseteq \bar{O}$ if,*

$[FX]$ *for all $v \in S \cap \tilde{p}$, $\lambda(v) \in \{XY, X, Y\}$*

$[FZ]$ *for all $v \in S \cap \text{odd}(\tilde{p})$, $\lambda(v) \in \{XZ, YZ, Y, Z\}$*

$[FY]$ *for all $v \in S$, $\lambda(v) = Y \implies (v \in \tilde{p} \iff v \in \text{odd}(\tilde{p}))$*

$\tilde{p}$ is a focussed set *for Γ if it is focussed over $\bar{O}$.*

*A pauli flow $(p, \prec)$ is a **focussed pauli flow** if for all $v \in \bar{O}$, $p(v)$ is focussed over $\bar{O} \setminus \{v\}$. We can check that in this case the definition of a focussed set coincides with the condition in definition 14.*

5. PDDAG AND REWRITE RULES

5.1. **Extraction strings.** Recall 31, we want to use focussed pauli flow, write every measurement as a rotation + pauli, then finally push the rotations to just the outputs. This requires identifying a stabiliser $S_v = Q_v \otimes P_{rest}$ for each vertex where Q_v is the local correction on vertex v, so orthogonal to $\lambda(v)$.

To be precise, $P_{rest} \equiv P_{abs} \otimes P_{outs}$ can consist of any operations on the outputs, and also local paulis that are absorbed by other vertices [2] 15 – the fact that we can always find such a stabiliser S_v is guaranteed by focussed pauli flow. The following rotation lemma computes the 'extraction strings' on the outputs.

Lemma 16. ***(rotation lemma),***
Let Q, S be commuting pauli strings, and let S be a stabiliser of $|G\rangle$ [3]. Then,

$$e^{-i\frac{\theta}{2}Q}|G\rangle = e^{-i\frac{\theta}{2}QS}|G\rangle$$

To compute the extraction strings, we use the rotation lemma and set $Q = Q_v$ and $S = S_v$ – then, in the product QS we are left with $\mathbb{1} \otimes P_{rest}$.

The only problem that remains is that if $u \in p(v), u \succ v$, and $\lambda(u)$ is planar, then $P_{rest}^{v \to u}$ might not be absorbed by $\lambda(u)$. To resolve this, we process vertices in reverse order of flow (i.e, increasing measurement depth order) and at each step convert the measurement labels into paulis while pushing the rotations to the output.

5.2. **PDDAG.**

Definition 17. ***(extraction strings)*** *Let (Γ, α) be a measurement pattern with focussed pauli flow $(p, \prec)$. Let $v \in \bar{O}$. Then, a pauli string P is a Q-extraction string ($Q \in \{X, Y, Z\}$) for v if $Q_v P_O$ is a stabiliser of,*

$$\left(\prod_{\substack{u \in \bar{O} \\ u \succ v \\ \lambda(u) \neq \{X,Y,Z\}}} \langle +_{\lambda(u),0}|\right)\left(\prod_{\substack{u \in \bar{O}\setminus\{v\} \\ \lambda(u) \in \{X,Y,Z\}}} \langle +_{\lambda(u),0}|\right)\left(\prod_{u \sim w} CZ_{u,w}\right)\left(\prod_{u \in \bar{I}} |+\rangle_u\right)$$

The first projector (on the right) are for pauli measurements, which can be performed whenever (not part of the PDDAG), and the second projector is for all vertices that have already been extracted and written to the outputs.

[2]pauli flow will only guarantee that paulis are absorbed by past vertices, not future

[3]or more generally, of any linear map C

6. Rewrites for Circuit Simplification

Let Γ be a measurement pattern with focussed pauli flow $(p, \prec)$ and $v \in \bar{O}$. Define the following set,

$$A(v) := \{p(v) \cup \mathrm{odd}(p(v))\} \tag{39}$$

Then,

Definition 18. *The* ***pauli-weight*** *of vertex* $v \in \bar{O}$ *is given by,*

$$\omega(v) := |A(v) \cap O|$$

The ***pauli-weight*** *of the pattern is then,*

$$\omega(\Gamma) := \sum_{\substack{v \in \bar{O} \\ \lambda(v) \notin \{X,Y,Z\}}} \omega(v)$$

Note that when local complementation is applied to the graph, we must update the flow,

$$(p, \prec) \xrightarrow{\text{loc compl.}} (p', \prec) \tag{40}$$

Then, our problem reduces to applying local complementation such that,

$$\underbrace{\sum_{v \in \bar{O}} |\left(p'(v) \cap \mathrm{odd}(p'(v))\right) \cap O|}_{\omega(\Gamma')} \leq \underbrace{\sum_{v \in \bar{O}} |\left(p(v)\right) \cap \mathrm{odd}(p(v)) \cap O|}_{\omega(\Gamma)} \tag{41}$$

Concretely, local complementation updates all correction sets (and thusly odd neighbourhoods) according to *Lemma D.14* from Simmons' paper. For all $u \in \bar{O}$, local complementation about a vertex $v \in \bar{O}$ yields,

$$p'(u) := \begin{cases} p(u) \triangle \{v\} & v \in \mathrm{odd}(p(v)) \\ p(u) & v \notin \mathrm{odd}(p(v)) \end{cases} \tag{42}$$

Note that this is the new pauli flow (not focussed!). The focussed condition is a bit more complicated to account for various cases [*Lemma D.15*].

6.1. **Matrix formalism.** An idea is to compute the sets of interest with the following matrix ingredients, and see how local complementation changes it.

1. Adjacency matrix A
2. Matrix of correction sets P, where for a fixed vertex v, $p(v)$ is the indicator column vector $\in \{0,1\}^V$
3. The product AP (modulo 2) will yield the matrix of odd neighbourhoods.

Then finally,

i. Define $W := P \vee AP$ (entrywise OR sum)

ii. Extract the rows corresponding to the outputs O

iii. Delete the columns corresponding to pauli vertices and obtain the reduced W_ω

iv. $\omega(\Gamma) := \sum_{i,j} W_{\omega ij}$

We can also compute the new correction sets after local complementation via $P' = P \oplus AP$ (modulo 2) [TODO and CHECK]

7. [Incorrect (ignore)]

Let Γ be a measurement pattern with focussed pauli flow $(p, \prec)$ and $v \in \bar{O}$ such that $\lambda(v) \notin \{X, Y, Z\}$ (appears in the PDDAG). Define the following sets of vertices,

$$A(v) := \{p(v) \cup \mathrm{odd}(p(v)) \setminus \{v\}\}$$

and,

$$B(v) := \{u \in A(v) | u \notin O \text{ and } u \succ v \text{ or } \lambda(u) \in \{X, Y, Z\}$$

Definition 19. *The **pauli-weight** of vertex v is given by,*

$$\omega(v) := |A(v)| - |B(v)|$$

*The **pauli-weight** of the pattern is then,*

$$\omega(\Gamma) := \sum_{\substack{v \in V \\ \lambda(v) \notin \{X,Y,Z\}}} \omega(v)$$

This follows from the extraction procedure. First, we obtain a pauli for each vertex in $A(v)$, then both the future vertices except outputs, and pauli vertices in $A(v)$ are absorbed. Also, the pauli for vertex v itself is absorbed by its rotation. Note that $B(v) \subseteq A(v)$ [probably a strict subset but why?].

It also follows that $0 \leq \omega(v) \leq |O|$.

Another useful way to calculate weight could be as,

$$\omega(\Gamma) = \sum_{o_i \in O} c(o_i) \tag{43}$$

where, $c(o_i)$ counts the number of vertices that contribute a pauli to output o_i.

7.1. **matrix representation.** Let V be the set of vertices of a measurement pattern Γ' and $V' \subset V$ be the set of non-output, non-pauli vertices. We can construct a $|V'| \times |V'|$ matrix M as follows,

$$M_{v,u} = \begin{cases} 1 & u \in A(v) \text{ and } u \notin B(v) \\ 0 & u \in A(v) \text{ and } u \in B(v) \end{cases} \tag{44}$$

After choosing (any) ordering of V'. So, $\sum_{u \in V'} M_{v,u} = \omega(v)$ and $\sum_{u,v \in V'} M_{v,u} = \omega(\Gamma)$.

THIS IS WRONG!!! but might have stumbled on a necessary condition $M = 0$ for focussed pauli flow...wonder what would make it sufficient.

$$\text{I think: pf} + M = 0 \iff \text{fpf}$$

[*Maybe a more revealing/sensible ordering of V' based on $\prec$?*]

Then, we can look at how local complementation changes M.

Example – C.13 from Simmons' paper. Then, corresponding to the vertex ordering (basis) $\{i, a, b, c\}$

8. References

(1) Pauli and Stabiliser

(2) MBQC –

(3) PDDAG –