Handout #13

P.C Yeh

Radix-2 FFT Algorithms

I. Decimation-in-time:

X(k) is the N-DFT of x(n) $\Rightarrow X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} = \sum_{n\ \mathrm{even}} x(n)W_N^{kn} + \sum_{n\ \mathrm{odd}} x(n)W_N^{kn}$

$$= \sum_{m=0}^{(N/2)-1} x(2m)W_N^{k2m} + \sum_{m=0}^{(N/2)-1} x(2m+1)W_N^{k(2m+1)}$$

$$= \sum_{m=0}^{(N/2)-1} x(2m)W_{N/2}^{km} + W_N^k \sum_{m=0}^{(N/2)-1} x(2m+1)W_{N/2}^{km}$$

$$= F_1(k) + W_N^k F_2(k),$$

where $W_N^k = e^{-j\frac{2k\pi}{N}} \Rightarrow W_N^{2km} = e^{-j\frac{4km\pi}{N}} = e^{-j\frac{2km\pi}{N/2}} = W_{N/2}^{km}$,

$F_1(k)$ is the N/2-DFT of {x(0), x(2), …, x(N-2)},

$F_2(k)$ is the N/2-DFT of {x(1), x(3), …, x(N-1)}.

When k=0, 1, …, N/2-1

1) $X(k) = \sum_{m=0}^{(N/2)-1} x(2m)W_{N/2}^{km} + W_N^k \sum_{m=0}^{(N/2)-1} x(2m+1)W_{N/2}^{km} = F_1(k) + W_N^k F_2(k)$.

2) $X(N/2+k) = \sum_{m=0}^{(N/2)-1} x(2m)W_{N/2}^{(N/2+k)m} + W_N^{(N/2+k)} \sum_{m=0}^{(N/2)-1} x(2m+1)W_{N/2}^{(N/2+k)m}$

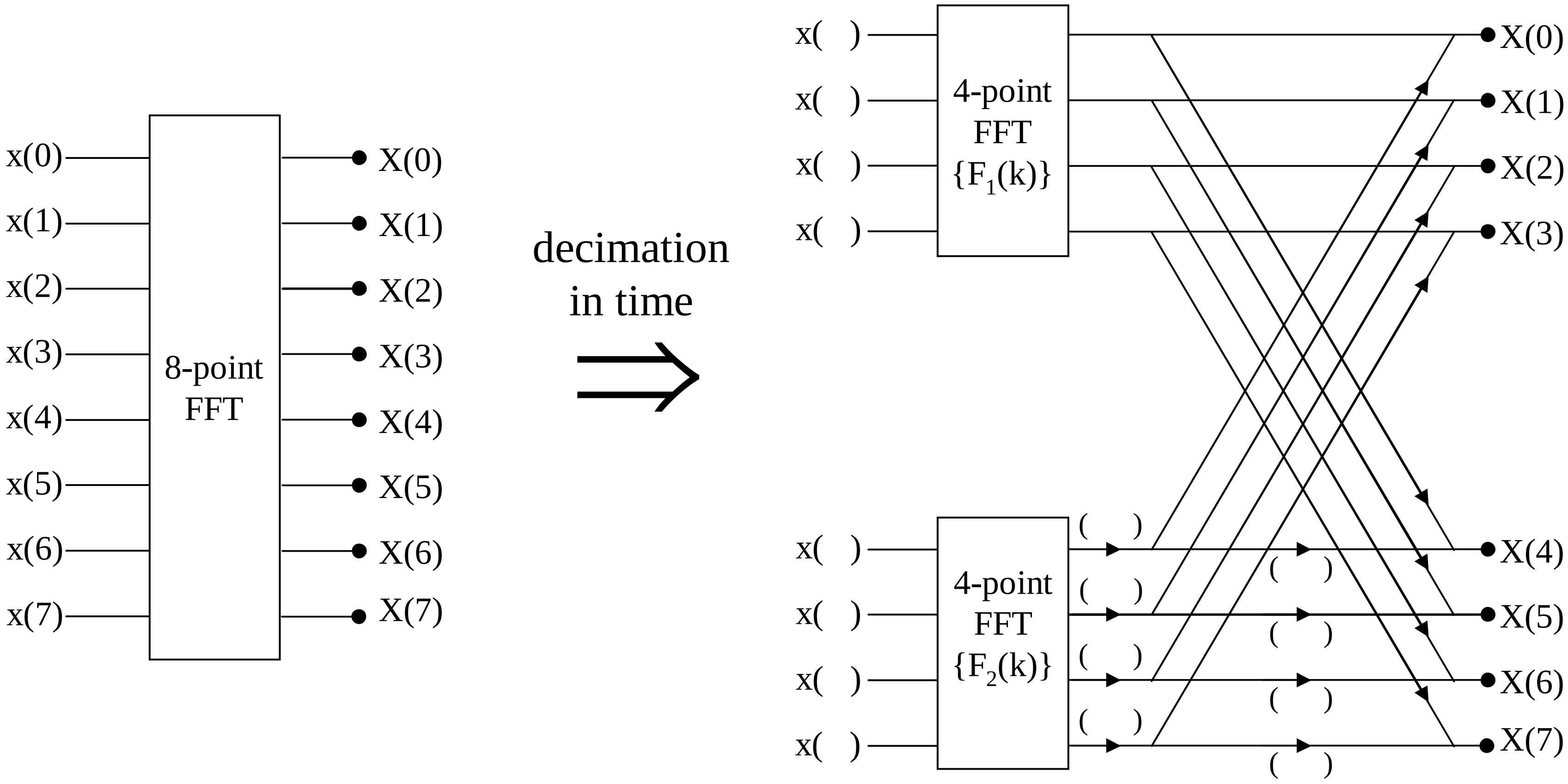
$$= \sum_{m=0}^{(N/2)-1} x(2m)W_{N/2}^{km} - W_N^k \sum_{m=0}^{(N/2)-1} x(2m+1)W_{N/2}^{km}$$

$$= F_1(k) - W_N^k F_2(k)$$

Note: $W_N^{(N/2+k)} = e^{-j\frac{2(N/2+k)\pi}{N}} = e^{-j\pi} e^{-j\frac{2k\pi}{N}} = -W_N^k$

and $W_{N/2}^{(N/2+k)m} = e^{-j\frac{2(N/2+k)m\pi}{N/2}} = e^{-j2m\pi} e^{-j\frac{2km\pi}{N/2}} = W_{N/2}^{km}$.

■ Ex: 8-DFT of x(n) = {x(0), x(1), …, x(7)}



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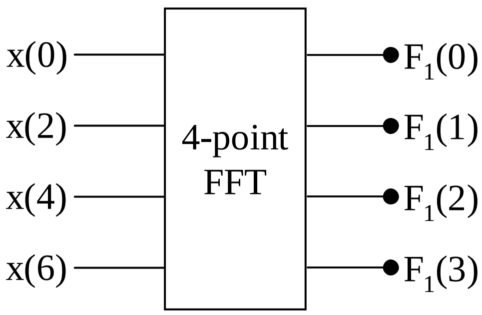
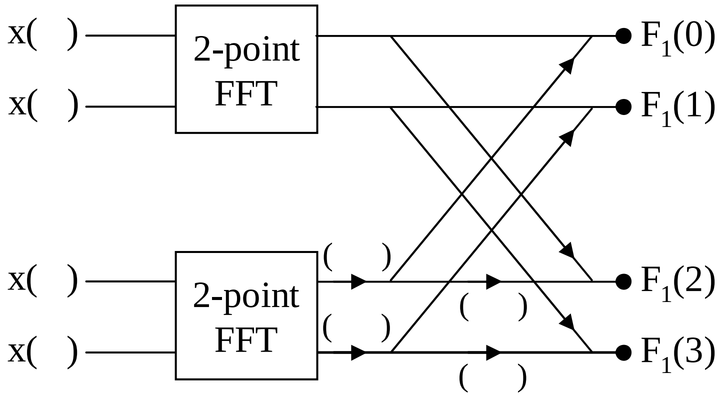
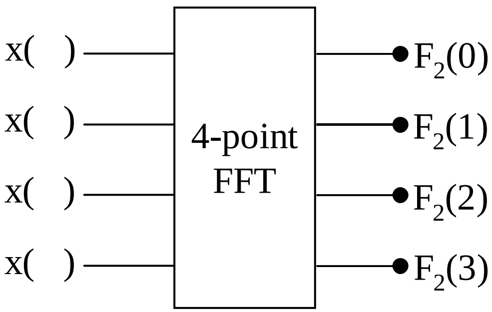


decimation in time

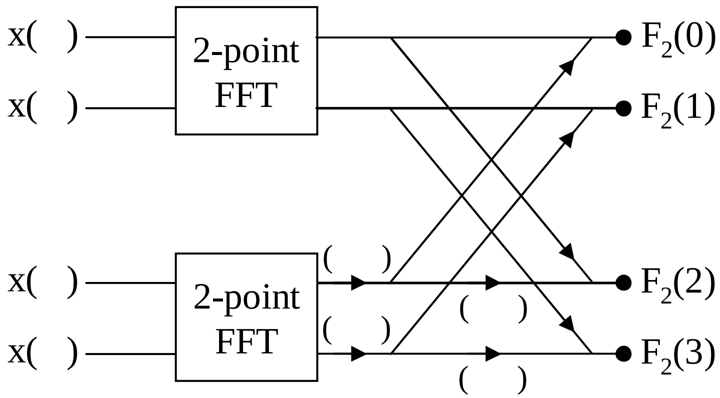
$\Longrightarrow$

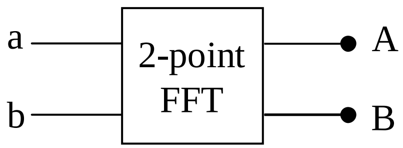




decimation in time

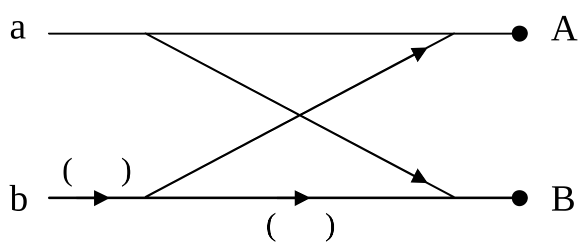
$\Longrightarrow$



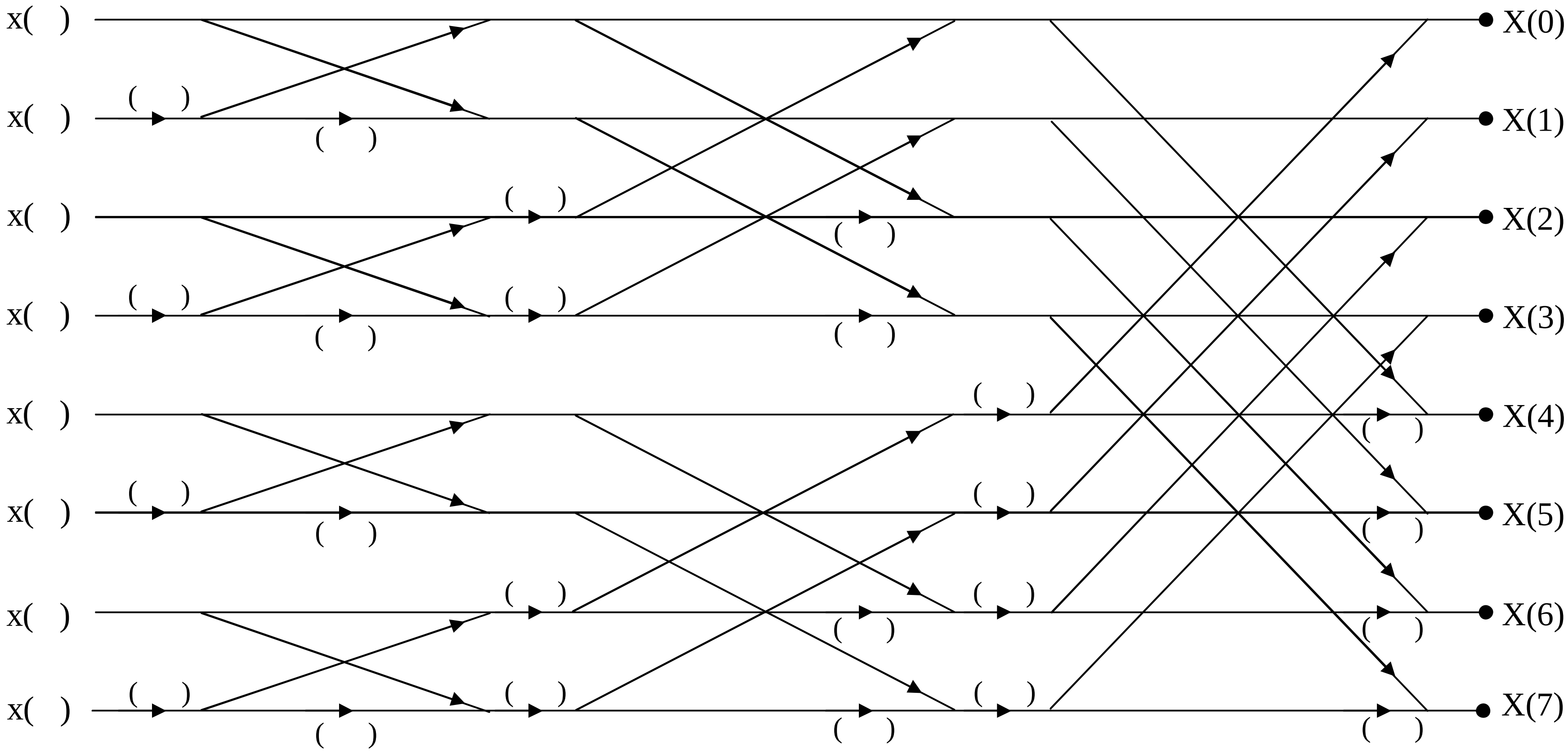


decimation in time

$\Longrightarrow$



Hence we have the final form of 8 point FFT (decimation-in-time):



II. Decimation-in-frequency:

X(k) is the N-DFT of x(n)

$$\Rightarrow X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=0}^{(N/2)-1} x(n+N/2) W_N^{k(n+N/2)}$$

$$= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{(N/2)-1} x(n+N/2) W_N^{kn}$$

$$= \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^k x(n+N/2) \right] W_N^{kn} .$$

Note: $W_N^{kN/2} = e^{-j\frac{2k\pi N/2}{N}} = e^{-jk\pi} = (-1)^k$.

For k=0, 1, …, N/2-1

1) $X(2k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^{2k} x(n+N/2) \right] W_N^{2kn} = \sum_{n=0}^{(N/2)-1} \left[x(n) + x(n+N/2) \right] W_{N/2}^{kn}$,

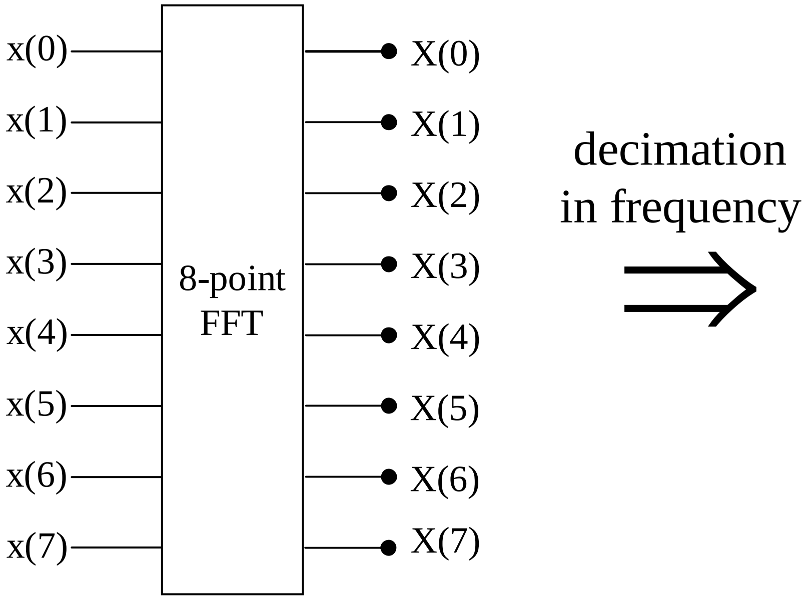
i.e. X(2k) is the N/2-DFT of $g_1(n)$ = x(n)+x(n+N/2), n=0, 1, …, (N/2)-1.

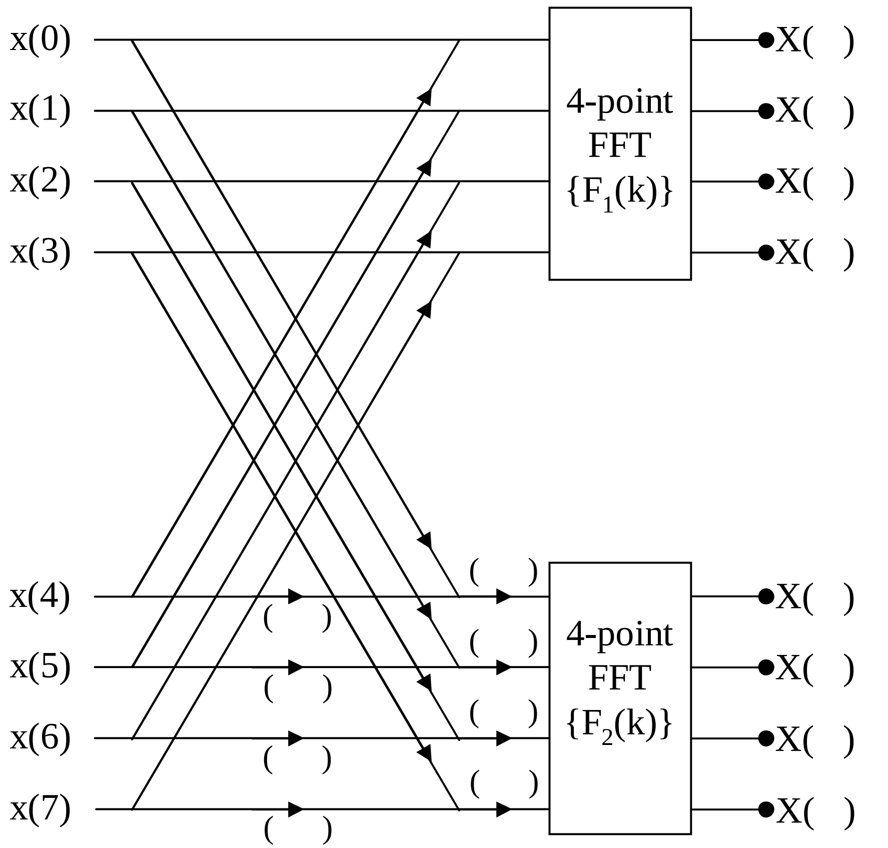
2) $X(2k+1) = \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^{2k+1} x(n+N/2) \right] W_N^{(2k+1)n} = \sum_{n=0}^{(N/2)-1} \left\{ \left[x(n) - x(n+N/2) \right] W_N^{n} \right\} W_{N/2}^{kn}$,

i.e. X(2k+1) is the N/2-DFT of $g_2(n)$ = { x(n) - x(n+N/2) } W_N^{kn} , n=0, 1, …, (N/2)-1.

Note: $W_N^{2kn} = e^{-j\frac{2(2kn)\pi}{N}} = e^{-j\frac{2kn\pi}{N/2}} = W_{N/2}^{kn}$ and $W_N^{(2k+1)n} = e^{-j\frac{2(2k+1)n\pi}{N}} = e^{-j\frac{2(2kn)\pi}{N}} e^{-j\frac{2n\pi}{N}} = W_{N/2}^{kn} W_N^{n}$.

■ Ex: 8-DFT of x(n) = {x(0), x(1), …, x(7)}





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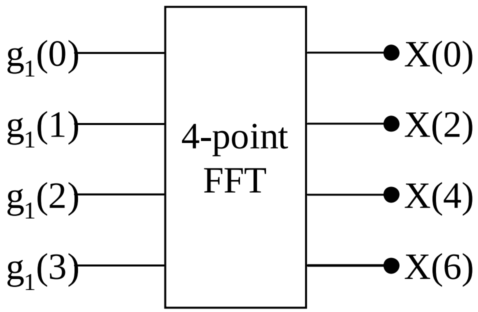
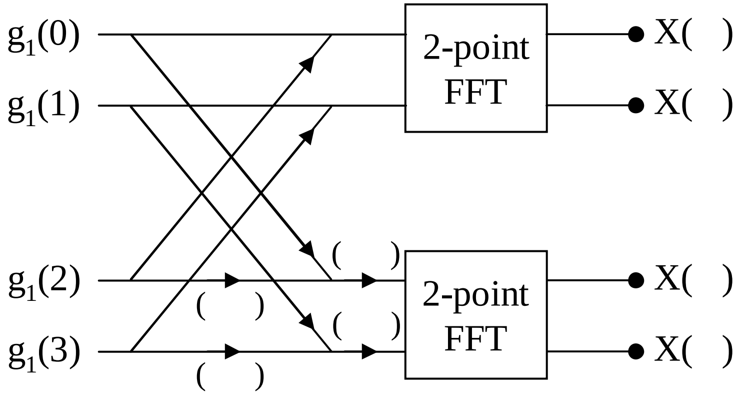
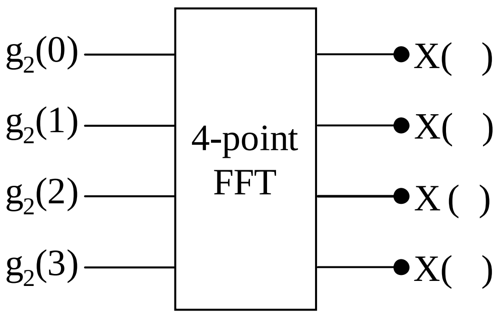
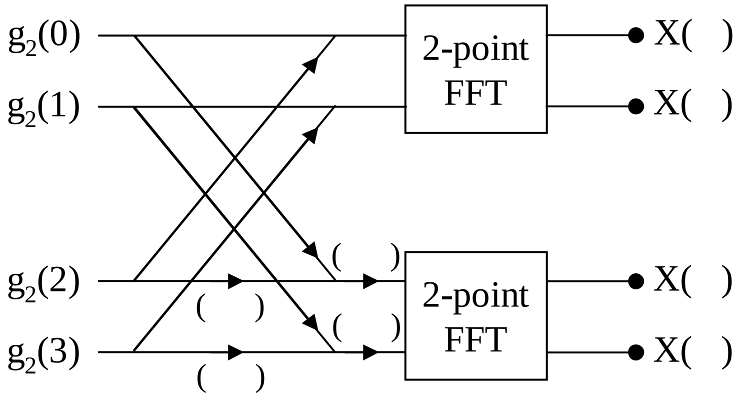


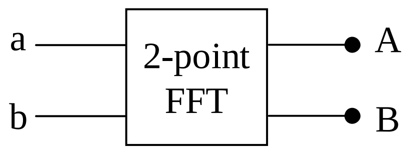
decimation
in frequency
$\Rightarrow$



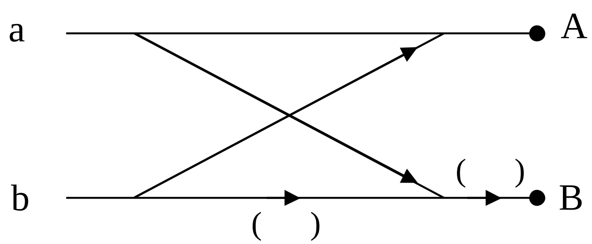


decimation
in frequency
$\Rightarrow$





decimation
in frequency
$\Rightarrow$



Hence we have the final form of 8 point FFT (decimation-in-frequency):