CORRECTION OF TIME OF FLIGHT MOMENTUM DUE TO ENERGY LOSS

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This note idetifies a common method for correcting for energy loss due to time-of-flight measurements, regardless of whether the time of flight is measured between TOF0 and TOF1 or between TOF1 and TOF2. This method will determine the momentum at the absorber, based on the TOF01 or TOF12 measurements. Throughout the analysis, we will assume straight tracks between measurement points.

Assume the time of flight between two points on a curve, s_1 and s_2, is given by;

$$\Delta t = \int_{s_1}^{s_2} \frac{ds}{c\beta} \tag{1}$$

where

$$\beta = \frac{pc}{E} = \frac{1}{\sqrt{1 + \frac{m^2c^2}{p^2}}}. \tag{2}$$

If there is no energy loss, the energy E (and the momentum p) is constant and

$$\Delta t = \frac{\Delta s}{c\beta} = \frac{\Delta s}{c}\sqrt{1 + \frac{m^2c^2}{p^2}} \tag{3}$$

(where $\Delta s = s_2 - s_1$) and with minor algebra

$$p = \frac{mc}{\sqrt{\frac{c^2\Delta t^2}{\Delta s^2} - 1}}. \tag{4}$$

It is a little different with energy loss. Suppose that we have an energy at some point s_a along the trajectory (for example, at the centre of the absorber), so that the energy (to first order) is given by

$$E = E_a - \frac{dE}{ds}(s - s_a). \tag{5}$$

If we assume that $\frac{dE}{ds}$ is a positive quantity then this formula agrees with our expectation. If $s > s_a$, then $E < E_a$ and if $s < s_a$, then $E > E_a$.

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Then the time of flight between s_1 and s_2 becomes

$$\begin{aligned}\Delta t &= \int_{s_1}^{s_a} \frac{ds}{c\beta} + \int_{s_a}^{s_2} \frac{ds}{c\beta} \\ c\Delta t &= \int_{s_1}^{s_a} ds \frac{E_a - \frac{dE}{ds}(s-s_a)}{\sqrt{(E_a - \frac{dE}{ds}(s-s_a))^2 - m^2c^4}} + \int_{s_a}^{s_2} ds \frac{E_a - \frac{dE}{ds}(s-s_a)}{\sqrt{(E_a - \frac{dE}{ds}(s-s_a))^2 - m^2c^4}}\end{aligned}$$

We can use the integral

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2} + C$$

to obtain

$$c\Delta t = \frac{1}{\frac{dE}{ds}}\left(p_a c - \sqrt{(E_a - \frac{dE}{ds}(s_2-s_a))^2 - m^2c^4} + \sqrt{(E_a - \frac{dE}{ds}(s_1-s_a))^2 - m^2c^4} - p_a c\right) \tag{6}$$

$$c\Delta t\frac{dE}{ds} = p_a c\sqrt{1 - \frac{2E_a}{p_a^2c^2}\frac{dE}{ds}(s_1-s_a) + \frac{1}{p_a^2c^2}\left(\frac{dE}{ds}(s_1-s_a)\right)^2} - p_a c\sqrt{1 - \frac{2E_a}{p_a^2c^2}\frac{dE}{ds}(s_2-s_a) + \frac{1}{p_a^2c^2}\left(\frac{dE}{ds}(s_2-s_a)\right)^2} \tag{7}$$

(8)

In the case in which $\left|\frac{dE}{ds}(s-s_a)\right| << p_a c$ we can perform the Taylor expansion

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

and we keep terms up to second order in $\left(\frac{\frac{dE}{ds}(s-s_a)}{p_a c}\right)^2$. In this case

$$\begin{aligned}c\Delta t\frac{dE}{ds} = p_a c\Bigg[&\frac{1}{2}\left(\frac{dE}{ds}\right)^2\frac{(s_1-s_a)^2}{p_a^2c^2} - \frac{1}{2}\left(\frac{dE}{ds}\right)^2\frac{(s_2-s_a)^2}{p_a^2c^2} \\ &-\frac{E_a}{p_a^2c^2}\left(\frac{dE}{ds}\right)(s_1-s_a) + \frac{E_a}{p_a^2c^2}\left(\frac{dE}{ds}\right)(s_2-s_a) \\ &\frac{E_a^2}{2p_a^4c^4}\left(\frac{dE}{ds}\right)^2(s_1-s_a)^2 + \frac{E_a^2}{2p_a^4c^4}\left(\frac{dE}{ds}\right)^2(s_2-s_a)^2\Bigg].\end{aligned}$$

Therefore:

$$c\Delta t\frac{dE}{ds} = \frac{1}{2p_a c}\left(\frac{dE}{ds}\right)^2\left[(s_1-s_a)^2-(s_2-s_a)^2\right]+\frac{E_a}{p_a c}\left(\frac{dE}{ds}\right)(s_2-s_1) - \frac{E_a^2}{2p_a^3c^3}\left(\frac{dE}{ds}\right)^2\left[(s_1-s_a)^2-(s_2-s_a)^2\right].$$

For convenience, we define

$$\Delta s_1 = s_1 - s_a \tag{9}$$

$$\Delta s_2 = s_2 - s_a \tag{10}$$

$$\Delta s = s_2 - s_1. \tag{11}$$

Therefore, we obtain the expression:

$$\frac{c\Delta t}{\Delta s} = \frac{1}{p_a c}\left(\frac{dE}{ds}\right)\frac{\Delta s_1^2-\Delta s_2^2}{\Delta s}\left(1-\frac{E_a^2}{p_a^2c^2}\right)+\frac{E_a}{p_a c}. \tag{12}$$

We also use the following expression:

$$\frac{\Delta s_1^2-\Delta s_2^2}{\Delta s} = \Delta s_1+\Delta s_2, \tag{13}$$

to obtain

$$\frac{c\Delta t}{\Delta s} = \frac{1}{p_a c}\left(\frac{dE}{ds}\right)(\Delta s_1+\Delta s_2)\left(1-\frac{E_a^2}{p_a^2c^2}\right)+\frac{E_a}{p_a c}. \tag{14}$$

To get this into a useful expression that can be solved for the momentum at the absorber, we use

$$\frac{E_a}{p_a c} = \sqrt{1+\frac{m^2c^2}{p_a^2}}.$$

Therefore:

$$\left[\frac{c\Delta t}{\Delta s}+\frac{m^2c^4}{2p_a^3c^3}\left(\frac{dE}{ds}\right)(\Delta s_1+\Delta s_2)\right]^2 = 1+\frac{m^2c^2}{p_a^2}. \tag{15}$$

If we define a new dimensionless variable $x = \frac{mc}{p_a}$, then we obtain a sixth order polynomial equation:

(16)

$$x^6\frac{1}{4m^2c^4}\left(\frac{dE}{ds}\right)^2(\Delta s_1+\Delta s_2)^2+x^3\frac{c\Delta t}{\Delta s}\frac{1}{mc^2}\left(\frac{dE}{ds}\right)(\Delta s_1+\Delta s_2)-x^2+\left(\frac{c^2\Delta t^2}{\Delta s^2}-1\right)=0.$$

This formula was tested by comparing the naive calculation (equation 4) between TOF1 and TOF2 to the more accurate calculation (equation 16). The position of TOF0 is at $z_0 = 5285$ mm, of TOF1 is at $z_1 = 12929$ mm, of TOF2 is at $z_2 = 21152$ mm and of the absorber is at $z_a = 16952$ mm. Therefore, the distance between TOF1 and TOF2 is $\Delta s = 8223$ mm, and the two distances to the absorber are $\Delta s_1 = -4023$ mm and $\Delta s_2 = 4200$ mm. Therefore, $\Delta s_1 + \Delta s_2 = 177$ mm. Figure 1 shows the comparison of the naive calculation (equation 4, Left) between TOF1 and TOF2 to the more accurate calculation (equation 16, Right). The residuals of both calculations are shown in Figure 2. The bias from the naive calculation is 3.77 MeV/c (with a resolution of 4.85 MeV/c) and the bias from the more accurate calculation is 3.50 MeV/c (with a resolution of 4.86 MeV/c).

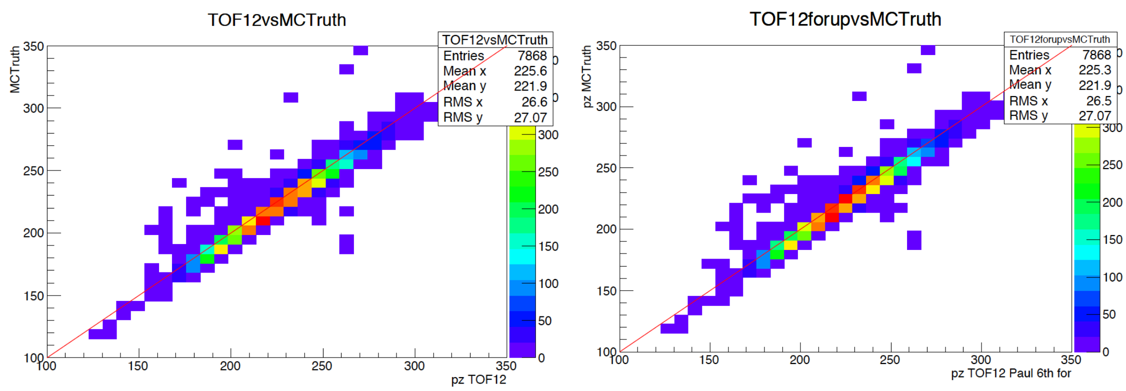

FIGURE 1. Left: Naive calculation of momentum at the absorber from TOF12 (equation 4) compared to Monte Carlo truth. Right: Calculation of momentum at the absorber from TOF12 using 6th order polynomial (equation 16).

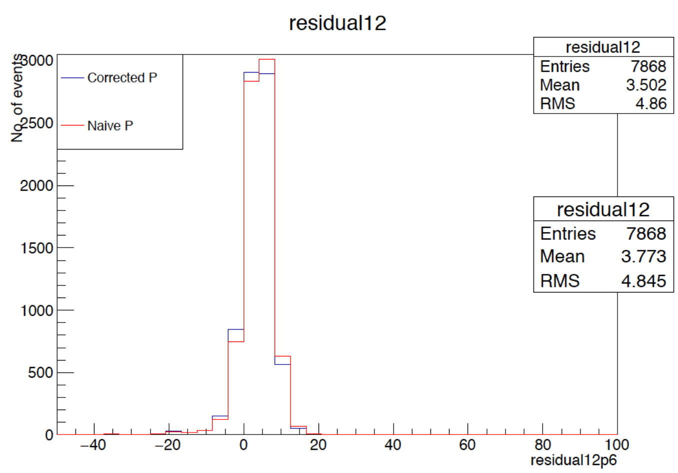

FIGURE 2. Residuals of the naive calculation of the momentum at the absorber and the 6th order polynomial, using TOF12, minus the Monte Carlo momentum at the absorber.

If TOF2 is not present then we need a calculation for the momentum from the TOF0 to TOF1 distance. This calculation is similar to the TOF12 calculation. The naive calculation is:

$$p = \frac{mc}{\sqrt{\frac{c^2\Delta t^2}{\Delta s^2} - 1}}, \tag{17}$$

with $\Delta s^2 = s_1 - s_0$. The calculation that corrects for energy loss, taking into account the extrapolation to the absorber, assuming a constant energy loss from TOF0 to the absorber and from TOF1 to the absorber is:

$$x^6 \frac{1}{4m^2c^4}\left(\frac{dE}{ds}\right)^2 (\Delta s_0 + \Delta s_1)^2 + x^3 \frac{c\Delta t}{\Delta s}\frac{1}{mc^2}\left(\frac{dE}{ds}\right)(\Delta s_0 + \Delta s_1) - x^2 + \left(\frac{c^2\Delta t^2}{\Delta s^2} - 1\right) = 0, \tag{18}$$

in which $\Delta s_0 = s_0 - s_a = -11667$ mm. Therefore, $\Delta s_0 + \Delta s_1 = -15690$ mm. The naive calculation overrestimates the momentum at the absorber because the momentum is larger between TOF0 and TOF1 than at the absorber (figure 3, Left). The sixth order polynomial caclulation corrects for the energy loss to have better agreement with the momentum at the absorber, but over-corrects, since the assumption of a continuous energy loss, rather than an instantaneous energy loss at the absorber, makes the momentum smaller (figure 3, Right). The residuals of both calculations are shown in Figure 4. The bias from the naive calculation is 28.64 MeV/c (with a resolution of 9.87 MeV/c) and the bias from the more accurate calculation is -4.03 MeV/c (with a resolution of 10.25 MeV/c).

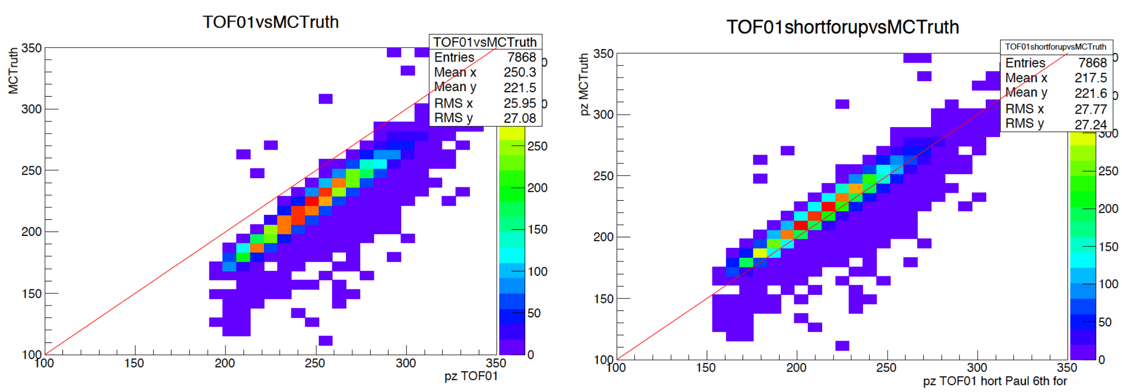

FIGURE 3. Left: Naive calculation of momentum at the absorber from TOF01 (equation 17) compared to Monte Carlo truth. Right: Calculation of momentum at the absorber from TOF01 using 6th order polynomial (equation 18).

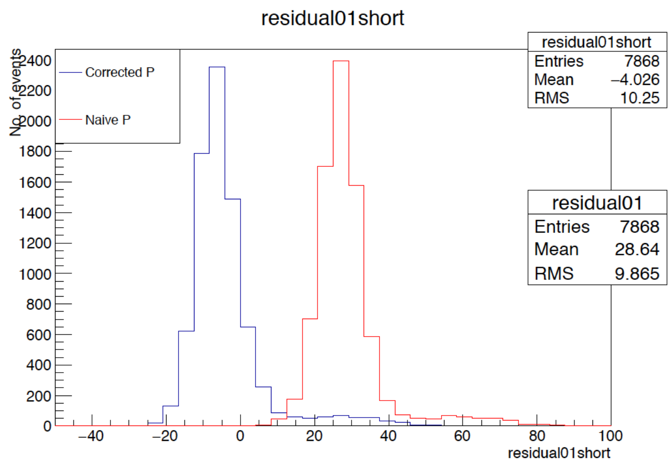

FIGURE 4. Residuals of the naive calculation of the momentum at the absorber and the 6th order polynomial, using TOF01, minus the Monte Carlo momentum at the absorber.