Solving Matrix Completion as Noisy Matrix Sensing

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Abstract

Matrix completion, a crucial sub-problem of non-convex matrix sensing, is integral to numerous machine learning applications such as recommender systems. Traditionally, matrix completion suffers from limited theoretical recovery guarantees, primarily due to its dependency on parameters like incoherence. This paper introduces an innovative method that resolves matrix completion challenges by transforming them into noisy matrix sensing problems, which in turn warrants the use of over-parametrization to achieve global guarantees. This approach circumvents traditional limitations by dropping requirements on observation patterns and incoherence. This novel strategy broadens the theoretical framework via the introduction of ϵ-MC problems and paves way for more effective handling of complex optimization tasks in real-world scenarios with incomplete data.

1 Introduction

Matrix completion (MC) and matrix sensing (MS) are pivotal in the fields of machine learning and signal processing, tasked with the reconstruction of a low-rank matrix from partial observations or linear measurements. These problems have broad applications ranging from collaborative filtering in recommendation systems (Koren et al., 2009), motion detection (Fattahi and Sojoudi, 2020), power system state estimation (Zhang et al., 2017; Jin et al., 2019) to image recovery (Gu et al., 2014) and biomedical imaging (Lustig et al., 2008). More importantly, since this framework can represent arbitrary polynomial optimization problems (Molybog et al., 2020) and is identical to the training of two-layer quadratic neural networks (Li et al., 2018), it has much greater theoretical implications in the machine learning community outside of its direct applications.

Matrix sensing generally involves recovering a matrix from a set of linear measurements, formulated as:

$$\min_{X \in \mathbb{R}^{n \times r_{\text{search}}}} \frac{1}{2} \| \|\mathcal{A}(XX^T) - b\|_2^2 := f(XX^T) := h(X) \quad \text{(MS)} \tag{1}$$

Here, $\mathcal{A}$ acts on the rank-r matrix product XX^T and compares it to a vector of observations $b = (M^*)$, with M^* being the rank-r ground truth matrix of interest. $\mathcal{A}(\cdot)$ is comprised of m symmetric sensing matrices, and $\mathcal{A}(M) = [\langle A_1, M \rangle, \dots, \langle A_m, M \rangle]^T$. Without explicit explanation, we set $r_{\text{search}} = r$. The matrix completion challenge, a special case of matrix sensing, is given by:

$$\min_{X \in \mathbb{R}^{n \times r}} \frac{1}{2} \|\mathcal{A}_\Omega(XX^T - M^*)\|_2^2 \quad \text{(MC)} \tag{2}$$

where $\Omega \subseteq [n] \times [n]$ represents the observed entries of an $n \times n$ matrix. We use the notation N_Ω to denote the matrix where

$$(N_\Omega)_{i,j} = N_{i,j} \cdot \mathbf{1}_{(i,j) \in \Omega} \tag{3}$$

for any arbitrary $N \in \mathbb{R}^{n \times n}$. $\mathcal{A}_\Omega(\cdot)$ is used to specifically denote the sensing operator of the matrix completion problem, where $\mathcal{A}_\Omega(M) = \text{vec}(M_\Omega)$.

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Matrix completion is distinguished by its reliance on the sample rate and the matrix's incoherence parameters. These parameters dictate the spread of matrix information across its entries and singular vectors (Candès and Recht, 2009; Candès and Tao, 2010). This dependency complicates matrix completion compared to matrix sensing, where challenges are often more tractable due to properties like Restricted Strong Convexity (RSC) and Smoothness (RSS), or the Restricted Isometry Property (RIP) (Recht et al., 2010).

Definition 1 (Restricted Strong Smoothness (RSS) and Restricted Strong Convexity (RSC))**.** The linear operator $\mathcal{A} : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^m$ satisfies the (L_s, r)-RSS property and the (α_s, r)-RSC property if

$$f(M) - f(N) \leq \langle M - N, \nabla f(N)\rangle + \frac{L_s}{2}\|M - N\|_F^2$$
$$f(M) - f(N) \geq \langle M - N, \nabla f(N)\rangle + \frac{\alpha_s}{2}\|M - N\|_F^2$$

are satisfied, respectively for all $M, N \in \mathbb{R}^n$ with $\operatorname{rank}(M), \operatorname{rank}(N) \leq r$. Note that RSS and RSC provide a more expressible way to represent the RIP property, with $\delta_r = (L_s - \alpha_s)/(L_s + \alpha_s)$.

Definition 2 (μ_0-incoherence)**.** (Ge et al., 2017) Given a rank-r matrix $M \in \mathbb{R}^{n_1\times n_2}$, we say it is μ_0-incoherent if its truncated SVD decomposition $U\Sigma V^\top$ satisfies

$$\|e_i^\top U\|_2 \leq \sqrt{\mu_0 r/n_1}, \quad \|e_j^\top V\|_2 \leq \sqrt{\mu_0 r/n_2}, \quad \forall i, j \in [n_1], [n_2]$$

where e_i is the i-th standard basis of $\mathbb{R}^{n_1}$ and e_j is the j-th standard basis of $\mathbb{R}^{n_2}$.

Since μ_0 is hard to gauge prior to solving this problem, applying matrix completion guarantees can be more challenging than its highly-related matrix sensing problem. Thus, this paper introduces novel methodologies to solve matrix completion, thereby broadening its theoretical accessibility and enhancing its practical applicability. The proposed approaches aim to reduce the reliance on complex parameters like incoherence, making these powerful techniques more accessible to a wider range of real-world applications.

1.1 Related Works

There are already a plethora of works devoted to the solving of matrix sensing and matrix completion problems. In particular, we give a brief review of the works that focus on giving recovery guarantees of low-rank matrix recovery problems.

Matrix Sensing

It has long been known since the landmark papers from Recht et al. (2010); Candès and Tao (2010) that the RIP constants (see Definition 1) play a central role in determining whether this non-convex problem could be solved to optimality with guarantees. It is widely understood that $\delta_{2r} = 1/2$ is a sharp threshold for the factorized Burer-Monteiro (BM) formulation (1) (Zhang et al., 2021; Ma et al., 2022), and a sufficient bound for SDP relaxation (Cai and Zhang, 2013).

Recent studies have highlighted over-parametrization as a crucial strategy in matrix sensing when RIP constants are suboptimal (i.e., $\delta_{2r} \geq 1/1$). Research by Zhang (2021, 2022) examined cases where the search rank r_{search} exceeds the true rank r, thus increasing the problem's parametrization. Zhang (2022) demonstrated that for $r_{\text{search}} > r[(1+\delta_n)/(1-\delta_n) - 1]^2/4$ and $r^* \leq r < n$, each solution $\hat{X}$ satisfies $\hat{X}\hat{X}^\top = M^*$. Similarly, Ma and Fattahi (2022) established analogous results under RIP-type conditions for the ℓ_1 loss. Setting $r_{\text{search}} = n$, the most extreme case, equates the problem's parametrization to that of SDP relaxation. Here, Yalcin et al. (2023) showed that the RIP threshold for exact recovery using SDP can approach 1 when M^* has a high true rank, thus underscoring the efficacy of over-parametrization. Nevertheless, the practical applicability of these conditions is limited, leading Ma et al. (2023) to explore tensor-based optimizations inspired by Sums-of-Squares to navigate non-convex landscapes in high δ_{2r} scenarios. Despite its utility in resolving spurious solutions, this tensor approach's applicability to matrix completion remains constrained by the need for a valid RIP constant.

Matrix Completion

The foundational work by Candès and Recht (2009) established that exact matrix recovery is possible from few entries, requiring a sample size of $\mu_0 n^{1.2} r \log(n)$ for $n \times n$ matrices of rank r with incoherence parameter μ_0. Enhancements in recovery guarantees and computational efficiencies followed, including spectral-gradient descent algorithms by Keshavan et al. (2010) and deeper insights into incoherence by Candès and Tao (2010). Studies by Recht (2011) and Gross (2011) expanded on these by demonstrating successful recovery without uniform random sampling, while Ding and Chen (2020) refined sampling orders further to $\mu_0 r \log(\mu_0 r) n \log(n)$.

Research into Burer-Monteiro factorization has explored non-convex optimization strategies for matrix completion, including greedy algorithms (Lee and Bresler, 2010; Wang et al., 2014), alternating minimization (Haldar and Hernando, 2009; Tanner and Wei, 2016; Wen et al., 2012), and Riemannian optimization (Mishra et al., 2014; Dai and Milenkovic, 2010)—reviewed comprehensively in Nguyen et al. (2019). Although lacking explicit recovery guarantees, these methods demonstrate empirical effectiveness, with some, like ADMiRA (Lee and Bresler, 2010), dependent on RIP conditions not generally applicable in matrix completion.

Recent developments (Ge et al., 2016, 2017; Du et al., 2017) have provided robust recovery guarantees for matrix completion using gradient descent and variants. These studies confirm that absent spurious solutions if each entry is observed with a probability $p \geq \text{poly}(\kappa, r, \mu_0, \log n)/n$, ensuring the success of BM in polynomial time with saddle-escaping algorithms, reflecting SDP literature findings. This raises the question of applying matrix sensing's RIP-based literature to matrix completion, an area that remains largely unexplored despite initial efforts like Zhang et al. (2023) to bridge this theoretical gap.

1.2 Our Approach and Main Contributions

In an effort to bridge the theoretical gaps identified in matrix completion (MC) problems, our research introduces a novel framework designed to imbue these problems with Restricted Isometry Property (RIP) characteristics, albeit with a trade-off in solution precision. The core of our methodology involves strategically perturbing the sensing matrices to make their nullspace to be trivial and employing over-parametrization in problem solving. Here are the principal steps of our approach:

1. **Perturbation for RIP Compliance:** We modify the original MC problem to enforce a valid RIP constant. This is achieved by introducing controlled perturbations to the sensing matrices, transforming a MC scenario into a manageable noisy MS problem. This step is crucial for aligning the MC problem with the more favorable theoretical properties of MS.
2. **Verification of Solution Fidelity:** We establish that the global solution to the perturbed problem will be close to the ground truth M^* with high probability under mild assumptions.
3. **Adaptation of Lifted Tensor Framework:** We extend the lifted tensor framework, originally discussed in Ma et al. (2024), to our perturbed MS problem. This is because we need the power of over-parametrization to handle the perturbed MS problem with very high RIP constants.

While the details of these steps might appear counter-intuitive at first glance, we will provide a thorough exposition in subsequent sections to clarify our methods and findings. Moreover, this strategy leads to two significant contributions to the field of low-rank matrix recovery:

- Proposes a framework to perform matrix completion with global guarantees without the need for the ground truth to obey incoherence conditions or for the observed entries to have certain structures, enabling a much wider range of MC problems to be solved with guarantees.
- We validate that the lifted tensor framework (Ma et al., 2023, 2024) remains effective in scenarios with noise corruption, thereby expanding its utility and robustness.

While this work specifically employs the lifted tensor framework to address the perturbed MC problem, our formulation is designed to be versatile, accommodating various solution methodologies. According to our theoretical guarantees, any method that resolves the problem with some degree of certainty can achieve a solution that closely approximates the true matrix M^*. This underscores the robustness and adaptability of our reformulation, highlighting its potential to effectively handle a wide range of scenarios in matrix recovery.

2 Notation

This document utilizes several standard notations. Scalar values such as $\sigma_i(M)$ and $\lambda_i(M)$ represent the i-th largest singular value and eigenvalue of matrix M, respectively. The Euclidean norm of a vector v is denoted as $\|v\|$, while $\|M\|_F$ and $\|M\|_2$ are used for the Frobenius and induced l_2 norms of a matrix M. Vectorization of matrix M is performed by $\mathrm{vec}(M)$, which stacks the columns of M into a vector. Conversely, $\mathrm{mat}(v)$ transforms a vector $v \in \mathbb{R}^{n^2}$ into a square matrix M. The set of integers from 1 to n is expressed as $[n]$. For notation in operations, $\circ l$ indicates a repeated Cartesian product, $\oslash$ refers to the Kronecker product, and $\otimes$ signifies the tensor outer product. If these notations come with subscripts, they denote the dimension along which the operation is performed. Finally, if $S \in [n] \times [m]$ represents a subset of indices of a $n \times m$ matrix, then N_S refers to the sub-matrix of $N \in \mathbb{R}^{n\times m}$ relevant to S as per (3), and $\|N\|_{S,F}$ denotes the Frobenius norm of N_S.

3 The Perturbed Matrix Completion Formulation

As explained above, most literature regarding the recovery guarantees of matrix sensing problems require some valid RIP constant. However, the attainment of such a constant automatically implies a trivial nullspace, meaning that $\mathcal{A}$ only maps a zero matrix to a zero vector, which is impossible for matrix completion problems. To demonstrate why this is, let's consider a 2×2 matrix recovery problem, and say we observed three entries of some $M \in \mathbb{R}^{2\times 2}$ except for the lower-right entry. This will correspond to the case where

$$\mathcal{A}_\Omega(M) = \mathrm{vec}(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} \odot M) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix} \mathrm{vec}(M) := T_\Omega \,\mathrm{vec}(M) \in \mathbb{R}^4 \tag{4}$$

For this $\mathcal{A}_\Omega$ to exhibit any RIP constant $\delta < 1$, it is required that $\|\mathcal{A}_\Omega(M)\|_2^2 \geq (1-\delta)\|M\|_F^2$, meaning that the T_Ω matrix above cannot output 0 unless M is a zero matrix. Nevertheless, for this specific example, even if we observed three out of four entries, we can simply set $\mathrm{vec}(M) = [0, 0, 0, 1]^\top$ to make $\mathcal{A}_\Omega(M) = 0$, violating the RIP condition. This is a simple example showing us that RIP condition will not hold for matrix completion problems unless all entries are observed, regardless of its size. Therefore, it begs the question of *whether we could find a simple way to let MC problems exhibit RIP property without changing its solution?*

Despite this limitation, a surprisingly simple solution exists to impart the RIP property to matrix completion problems. The primary issue is the zero entries in the diagonal of T_Ω, contributing to a non-trivial nullspace. By perturbing these zero entries slightly with a small number $\epsilon \in (0, 1]$, we can eliminate the nullspace. Revisiting (4), consider a perturbed sensing operator $\mathcal{A}_{\Omega,\epsilon}$:

$$\mathcal{A}_{\Omega,\epsilon}(M) = \mathrm{vec}\left(\begin{bmatrix}1 & 1\\1 & \epsilon\end{bmatrix} \odot M\right) = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & \epsilon\end{bmatrix} \mathrm{vec}(M) := T_{\Omega,\epsilon} \,\mathrm{vec}(M) \tag{5}$$

in which $T_{\Omega,\epsilon}$ has a trivial nullspace as promised. However, different operators lead to different observations. For example, when considering the case of (4), using $\mathcal{A}_\Omega$ and $\mathcal{A}_{\Omega,\epsilon}$ on the same matrix results in:

$$\mathcal{A}_\Omega(M) = \begin{bmatrix}M_{1,1}\\M_{1,2}\\M_{2,1}\\0\end{bmatrix} \longrightarrow \mathcal{A}_{\Omega,\epsilon}(M) = \begin{bmatrix}M_{1,1}\\M_{1,2}\\M_{2,1}\\\epsilon M_{2,2}\end{bmatrix}$$

Therefore, given a ground truth matrix M^*, the normal matrix completion problem will give you an observation

$$b = \mathcal{A}_\Omega(M^*) = \mathcal{A}_{\Omega,\epsilon}(M^*) + w_\epsilon, \quad w_\epsilon = \begin{bmatrix}0 & 0 & 0 & -\epsilon M^*_{2,2}\end{bmatrix}^\top \tag{6}$$

where $w_\epsilon \in \mathbb{R}^{n^2}$ can be considered a noise term. With this idea in place, we formally introduce our perturbed MC problem to solve:

$$\min_{X\in\mathbb{R}^{n\times r_{\text{search}}}} \|\mathcal{A}_{\Omega,\epsilon}(XX^T) - b\|_2^2 := f_{w_\epsilon}(XX^\top) := h_{w_\epsilon}(X) \quad (\epsilon\text{-MC}) \tag{7}$$

Essentially, we're transforming a noiseless matrix completion problem into a noisy matrix sensing problem with operator $\mathcal{A}_{\Omega,\epsilon}$ and deterministic noise w_ϵ. This approach, notably, ensures the attainment of valid RSS/RSC parameters, equivalent to RIP constants.

Lemma 1. *Given an arbitrary matrix completion problem with sensing operator $\mathcal{A}_\Omega$, if this operator is perturbed to produce $\mathcal{A}_{\Omega,\epsilon}$ according to* (5) *with a scalar $\epsilon \in (0, 1]$, then the ϵ-MC problem will exhibit $(1, n)$-RSS property and the (ϵ^2, n)-RSC property.*

The proof is straightforward and thus omitted for brevity. With that said, another major challenge that the perturbed formulation of ϵ-MC problems brings is that the global solution of ϵ-MC might not be M^* anymore. This can be easily seen since $\|\mathcal{A}_{\Omega,\epsilon}(M^*) - b\|_2^2 \neq 0$. As a result, even if we were able to find the global optima of (7), we may still fail our mission to find M^*. Therefore, one thing that we hope to investigate is under what conditions can the ground truth be preserved. Inspired by Ma and Fattahi (2023), we hope to link it with the number of corrupted observations. If we adopt the standard assumption that each entry of the matrix is independently observed with probability p, then we could generalize this observation by linking it to p. As our next step, we show that the global solution of (7), denoted as $M^\dagger$, will be very close to M^* with high probability, and we can further achieve a tradeoff tradeoff between sample rate p and geometric uniformity captured by ϵ.

We will briefly go over the high-level ideas in this derivation and present our formal theorem in the end. Since we assumed that $M^\dagger$ is the global optimum of (7), then by definition it gives that $f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*) \leq 0$. If we partition the set Ω into $\bar{S}$, the observed, noiseless entries, and S, the unobserved, perturbed entries, then we could decompose $f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*)$ further.

$$\begin{aligned}0 \geq f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*) =& \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 + \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*) - w_\epsilon\|_{S,2}^2 - \frac{1}{2}\|w_\epsilon\|_{S,2}^2 \\ =& \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 + \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger)\|_{S,2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,F}^2 \\ \geq& \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,F}^2\end{aligned} \tag{8}$$

where $\|\cdot\|_{S,2}$ denotes the $l2$/Frobenius norm of the sub-vector with entries in an arbitrary set S. Then if we add $\frac{1}{2}\|M^\dagger - M^*\|_{S,2}^2$ to both sides of (8), it is easy to show

$$\frac{1}{2}\|M^\dagger - M^*\|_F^2 \leq \frac{1}{2}\left(\|M^\dagger - M^*\|_{S,F}^2 + \epsilon^2\|M^*\|_{S,F}^2\right) \tag{9}$$

Here we look into the right hand side terms of (9) a bit more carefully, and realize that both $\|M^\dagger - M^*\|_{S,F}^2$ and $\|M^*\|_{S,F}^2$ are random variables with their sizes dependent on the sampling rate. Since $\|\cdot\|_S^2$ only denotes the size of the sub-matrix that are not observed (therefore perturbed by ϵ), if our sample rate p is small, this norm would also be small in expectation. Combining this intuition with concentration inequality to control deviation, gives this next theorem which serves as our main result showing why the ϵ-MC problem (7) can serve as a meaningful surrogate to the original MC problem.

Theorem 1. *Assume that $M^\dagger \in \mathbb{R}^{n\times n}$ is a symmetric, rank-r matrix that is a global optimum of* (7) *with an $\epsilon \in (0, 1]$. Assume that each entry of the original MC problem is independently observed with probability p, then for any $\eta \leq p \in \mathbb{R}$,*

$$\|M^\dagger - M^*\|_F^2 \leq \frac{1-p+\eta}{p-\eta}\epsilon^2\|M^*\|_F^2 \tag{10}$$

holds with probability at least $1 - \exp\left(-2\eta^2\|d\|_1^2/\|d\|_2^2\right)$, where $d \in \mathbb{R}^{n^2}$ is defined as

$$d := \mathrm{vec}(M^\dagger - M^*) \odot \mathrm{vec}(M^\dagger - M^*) + \epsilon^2\,\mathrm{vec}(M^*) \odot \mathrm{vec}(M^*) \tag{11}$$

We begin by noting that for any vector d, elementary inequalities ensure that $1 \leq \|d\|_1^2/\|d\|_2^2 \leq n^2$. This ratio increases as the values of d become more evenly distributed. In proving our theorem, we employed Hoeffding's inequality to achieve clear and interpretable results. While other concentration inequalities like Bennett's inequality can also be applied to independent, bounded variables, they do not always provide a tighter bound and would complicate the expression, hence they are not included in this work. We recognize the potential for employing more advanced statistical tools to refine these bounds. Readers interested in exploring this further can find the proof of the theorem in the Appendix.

4 Lifted Tensor Framework with Noise

Now that we are able to reformulate the original MC problem (2) into the new ϵ-MC problem (7), it presents us with a new challenge. Although now (7) admits valid RSC/RSS constants, this is nevertheless still a difficult matrix sensing problem to solve due to its small RSC constant (or large RIP constant). Thus, it is important that we apply an over-parametrized framework to deal with it in order to compensate for the poor geometric uniformity.

To this end, we employ the lifted tensor framework proposed in Ma et al. (2023), since it has the ability to deal with really small RSC constants, like those we have in ϵ-MC. However, in their original work, measurements were assumed to be clean, and this is incompatible with our framework since we hope to deal with noisy MS problems. Thus, we generalize the original results in Ma et al. (2023), and also in its subsequent work Ma et al. (2024) to demonstrate how the inclusion of noise could affect guarantees in when using a higher-order tensor parametrization.

First of all, we present the lifted tensor problem when our observations are corrupted by some random noise $\tilde{w} \in \mathbb{R}^m$,

$$\min_{\mathbf{w} \in \mathbb{R}^{nr \circ l}} \quad \|\langle \mathbf{A}^{\otimes l}, \langle \mathbf{P}(\mathbf{w}), \mathbf{P}(\mathbf{w}) \rangle_{2*[l]} \rangle - \tilde{b}^{\otimes l}\|_F^2, \quad \tilde{b} = \mathcal{A}(M^*) + \tilde{w} \tag{12}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n \times n}$ is a three-way tensor which can be seen as a concatenation of all sensing matrices $\{A_i\}_{i=1}^m$, and $\mathbf{w} \in \mathbb{R}^{nr \circ l}$ is the tensor decision variable used to increase the parametrization of $X \in \mathbb{R}^{n \times r}$. Here, $^{\otimes l}$ simply denotes l times of repeated tensor outer product, and $\mathbf{P}$ is just another constant permutation tensor used for correct multiplication. The gist of this paper is not on the tensor formulation, thus many details are deferred to Appendix A, and interested readers can learn more about general tensor knowledge and problem details there. For convenience's sake, we define $f^l(\cdot) : \mathbb{R}^{n \circ 2l} \mapsto \mathbb{R}$ and $h^l(\cdot) : \mathbb{R}^{[n \times r] \circ l} \mapsto \mathbb{R}$ as $f^l(\mathbf{M}) := \|\langle \mathbf{A}^{\otimes l}, \mathbf{M} \rangle - \tilde{b}^{\otimes l}\|_F^2$ and $h^l(\mathbf{w}) = f^l(\langle \mathbf{w}, \mathbf{w} \rangle_{2*[l]})$, with $\nabla f^l(\cdot) = \nabla_{\mathbf{M}} f^l(\cdot)$ and $\nabla h^l(\cdot) = \nabla_{\mathbf{w}} h^l(\cdot)$.

In the original works, it was proven that the lifted formulation (12) is able to convert spurious solutions in (1) to strict saddles via its drastic over-parametrization if this spurious solution is somehow far away from the ground truth M^*. However, the first thing to note here is that for any corrupted MS problem (its observation b is not clean and affected by noise), its global solution might not correspond to M^* anymore, which is the same challenge that we faced in the ϵ-MC formulation. This means that spurious solutions have to be even more distant to M^* for it be converted into strict saddles, depending on the intensity of noise. The result is summarized in the following theorem:

Theorem 2. *Consider an arbitrary second-order point $\hat{X} \in \mathbb{R}^{n \times r}$ of the factorized matrix sensing objective in the form of* (1) *where its observations b could be potentially corrupted by some random noise $\tilde{w} \in \mathbb{R}^m$ (i.e. $b = \tilde{b}$). Assuming that the linear operator $\mathcal{A}(\cdot)$ in* (1) *satisfies the RSC and RSS conditions with constants α_s, L_s respectively. Then $\hat{\mathbf{w}} = \mathrm{vec}(\hat{X})^{\otimes l}$ is a strict saddle of* (12) *with a rank-1 symmetric escape direction if*

$$\|M^* - \hat{X}\hat{X}^\top\|_F^2 \geq \frac{L_s}{\alpha_s} \lambda_r(\hat{X}\hat{X}^\top) \operatorname{tr}(M^*) + \frac{\|\tilde{w}\|_2^2}{\alpha_s} \tag{13}$$

with an odd l satisfying

$$l > \frac{1}{1 - \log_2(2\beta)}, \quad \beta := \frac{L_s \operatorname{tr}(M^*) \lambda_r(\hat{X}\hat{X}^\top)}{\alpha_s \|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}. \tag{14}$$

The proof of this theorem is located in Appendix B. The theorem highlights how the conversion radius from spurious solutions to strict saddles is influenced by the norm of the noise $\tilde{w}$. Setting $\tilde{w} = 0$ allows this theorem to coincide with Theorem 4 from Ma et al. (2024). More importantly, it is crucial for the critical point $\hat{\mathbf{w}}$ in (12) to be a rank-1 tensor to possess a negative escape direction. For a detailed definition of tensor rank, please see Appendix A. According to Ma et al. (2024), employing a gradient descent (GD) algorithm with sufficiently small initialization ensures that the search is conducted over approximately rank-1 tensors throughout the GD trajectory. This work further establishes that this characteristic remains unchanged when b is substituted with $\tilde{b}$, indicating that the observations are impacted by noise. This finding is substantiated by the following result:

Theorem 3. *Consider a finite-horizon gradient descent trajectory $\{\mathbf{w}_t\}_{t\in[T]}$ of (12) with $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta\nabla h^l(\mathbf{w}_t)$ starting from the initialization $\mathbf{w}_0 = \xi x_0^{\otimes l}$ with $\xi \in \mathbb{R}$ denoting the scale of the initialization, η representing the step-size and $x_0 \in \mathbb{R}^{nr}$ being an arbitrary vector with $\|x_0\|_2^2 = 1$. Then there exists $t(\kappa, l) \geq 1$ and $\kappa < 1$ such that*

$$\frac{\lambda_2^v(\mathbf{w}_t)}{\lambda_1^v(\mathbf{w}_t)} \leq \kappa, \qquad \forall t \in [t(\kappa, l), t_T] \tag{15}$$

if the initialization scale ξ is sufficiently small, where $t(\kappa, l)$ is expressed as

$$t(\kappa, l) = \left\lceil \ln\left(\frac{\|x_0\|_2^l}{\kappa|v_1^\top x_0|^l}\right) \ln\left(\frac{1+\eta\sigma_1^l(U)}{1+\eta\sigma_2^l(U)}\right)^{-1} \right\rceil \tag{16}$$

where $\sigma_1(U)$ and $\sigma_2(U)$ denote the first and second singular values of U and v_1, v_2 are the associated singular vectors, with

$$U = \langle \mathbf{A}_r, \tilde{b}\rangle_1 \in \mathbb{R}^{nr\times nr}, \quad \mathbf{A}_r := I_r \oslash_{2,3} \mathbf{A} \tag{17}$$

The proof of this theorem can again be found in Appendix B. If x_0 is initialized according to Lemma 14 in Ma et al. (2024), then we can also show that $\mathbf{w}_t$ will be κ-rank-1 as soon as $t \asymp \ln(1/\kappa) \ln\left((1+\eta\sigma_1^l(U))/(1+\eta\sigma_2^l(U))\right)^{-1}$, if ξ is chosen as a function of U, r, n, L_s, without the need for it to be arbitrarily small. However, since such results are not the main focal point of this work, we will not elaborate here for the sake of succinctness. The main takeaway is that by incorporating the noise $\tilde{w}$ into our objective (12), the ability of gradient descent algorithms to induce implicit bias is untouched. It is also worth noting that the results presented in this subsection applies to all tensor problems in the form of (12), which are lifted from general noisy matrix sensing problems, and not specific to our ϵ-MC problem.

5 Main Results

Our goal is to achieve a globally optimal solution for the ϵ-MC problem because it closely represents the M^* solution. However, this becomes challenging due to the α_s constant in equation (7), which depends on the small value of ϵ. To address this, rather than solving the problem using its basic matrix (BM) factorized form (as shown in equation (1)), which lacks global optimization guarantees, we apply more complex techniques with over-parametrization. We previously demonstrated that the lifted tensor framework (12), independent of the specific ϵ-MC problem, effectively handles noise in the observed data (when b becomes $\tilde{b}$), with the quality of the guarantee degrading as the magnitude of corruption increases.

By integrating these methodologies, we demonstrate a robust way to approximately solve the generic MC problem (as formulated in equation (2)) while still providing reliable global solutions, as elaborated in our main theorem below

Theorem 4. *Consider the matrix completion problem of completing a $n \times n$, rank-r matrix M^*, where $\Omega \subseteq [n] \times [n]$ denotes the set of observed entries and $\bar{\Omega}$ denotes the unobserved entries. Introduce a perturbation $\epsilon \in (0, 1]$ to formulate an ϵ-MC problem as per (7). Applying the tensor framework described in (12) to this ϵ-MC problem yields the following results:*

For any rank-1 critical point $\hat{\mathbf{w}} = \mathrm{vec}(\hat{X})^{\otimes l}$ of (12), if it is a second-order point (local minima), this implies that

$$\|M^* - \hat{X}\hat{X}^\top\|_F < \frac{1}{\epsilon}\lambda_r(\hat{X})\sqrt{\mathrm{tr}(M^*)} + e_1 \tag{18}$$

holds with probability at least q, under the condition that l is odd and meets the requirement:

$$l > \frac{1}{1-\log_2(2\beta)}, \quad \beta := \frac{\mathrm{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\epsilon^2\left(\|M^* - \hat{X}\hat{X}^\top\|_F^2 - e_2\right)}. \tag{19}$$

For all instances of the MC problem, the following hold:

$$e_1 = \|M^*\|_{\bar{\Omega},F}, \quad e_2 = \|M^*\|_{\bar{\Omega},F}^2, \quad q = 1 \tag{20}$$

Alternatively, if all entries are observed independently with probability p*, the expressions modify to:*

$$e_1 = \sqrt{\frac{1-p+\eta}{p-\eta}}\epsilon\|M^*\|_F, \quad q = 1 - \exp\left(-2\eta^2\|d\|_1^2/\|d\|_2^2\right), \quad e_2 = 0 \tag{21}$$

where d *is defined as per* (11).

Our main theorem builds directly on the results of Theorem 2, applying specific parameters ($L_s = 1$, $\alpha_s = \epsilon^2$) along with the definition of w_ϵ from equation (6). This leads to a deterministic outcome where the probability q equals 1. However, there's a critical aspect to consider: the transition from a spurious solution is contingent upon the condition described in (18). A significant challenge arises if $\|M^*\|^2_{\Omega,F}$ is large, potentially rendering this condition overly lenient. Here, the utility of Theorem 1 becomes evident. Under its probabilistic framework, we apply a triangle inequality to reduce the bound e_1 to $\sqrt{\frac{1-p+\eta}{p-\eta}}\epsilon\|M^*\|_F$. This adjustment is particularly valuable, as the presence of ϵ and p can significantly diminish the error term, effectively countering the inaccuracies introduced by our approximation method. The proof to this theorem can be found in Appendix B.

This theorem presents a new approach to matrix completion that is not reliant on the incoherence parameter or strict observation modes, diverging from established results like those in Ge et al. (2017); Candes and Plan (2010), which require a high sample rate on the magnitude of $\mathcal{O}(\mu_0 n^{1.2} r \log(n))$ with unknown constant scale. Our method offers a flexible tradeoff between observation probability and solution accuracy, effectively managing a gradual degradation in assurance. Furthermore, with the introduction of ϵ, it enables us to actively trade-off solution accuracy with computational complexity. This adjustment is particularly effective in noisy scenarios where some degree of inaccuracy is unavoidable, making our approach both practical and justifiable for real-world applications.

The results of Theorem 4 only apply to rank-1 critical points. To adhere to this, we can start our gradient descent algorithm at a small scale, leveraging Theorem 3 to maintain the rank constraint. Theorem 5 in Appendix A summarizes the results, and it is not presented here as its complex details could detract from the central narrative of Theorem 4.

6 Numerical Experiments

In this section, we numerically demonstrate the effectiveness of our method[1] against traditional BM based and semi-definite relaxation (SDP) methods.

We first hope to visualize the probabilistic guarantees made in Theorem 1, since it could be rather abstract to understand in its rigorous form. In Figure 1, we plot the probability lower-bound for $\|M^\dagger - M^*\|_F^2$ to be smaller than a given value with respect to different ϵ values. For instance, say in our plot the x-value is a, y-value is b, and the contour value (represented by color) is c, then this means that the normed difference between $M^\dagger$ and M^* will be smaller than a with probability c given that $\epsilon = b$. We plot this graph for p values of $0.3, 0.4$ and 0.5. As one can observe, as p increases, the normed difference will be much smaller with respect to all ϵ values, and even when $p = 0.3$, which is rather low, $M^\dagger$ will still be close to M^* with high probability given a small ϵ. Note here $M^\dagger$ is chosen as a random rank-r matrix close to M^* for good visualization since it is intractable to actually compute (hence the probabilistic guarantee).

To further the investigation, we introduce a benchmark matrix completion problem described in Yalçın et al. (2022), known to be difficult:

$$\Omega = \{(i,i), (i,2k), (2k,i) | \ \forall i \in [n], k \in [\lfloor n/2 \rfloor]\}, \tag{22}$$

M^* is also chosen identically to Example 1 from Yalçın et al. (2022) to ensure consistency. The study applies the BM factorized formulation (2), and our approach to address (22). Our approach employs the lifted problem (12) with $l = 3$, culminating in a tensor $\mathbf{w}_T$ after T iterations of gradient descent. A tensor PCA algorithm (Ma et al., 2024) extracts the principal component $X_T \in \mathbb{R}^{n \times r}$, approximating $\mathbf{w}_T$ as $\text{vec}(X_T)^{\otimes l}$. X_T then serves as the solution to the original problem (2). A successful instance of gradient descent is defined by $\|X_T X_T^\top - M^*\|_F \leq 0.05$. Preferring the success rate metric over average reconstruction error minimizes the impact of outliers and reduces

[1]'https://github.com/anonpapersbm/mc_noisy_ms',run on M1 Max Macbook Pro

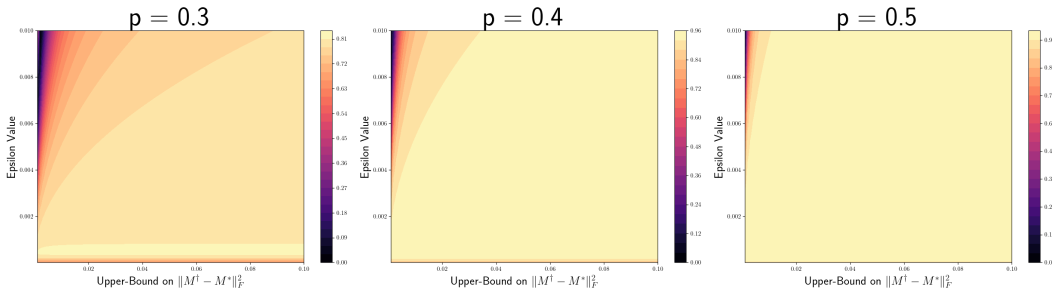
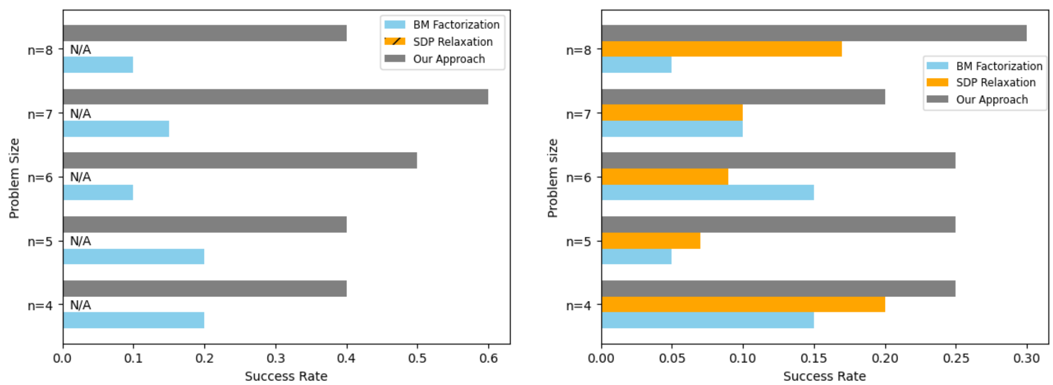


Figure 1: Probability Lower-Bound for Theorem 1.

variance, providing a more reliable measure of efficacy. This approach is also tested against a standard model where each entry of M^* is observed with a probability $p = 0.15$, since this was the main focus of classic matrix completion literature. Results are documented in Figure 2.



(a) Problem Instance (22)

(b) Problem Instance with Independent Entires

Figure 2: Success Rates for Our Approach compared to Standard Approaches.

Figure 2 clearly demonstrates the superior success rate of the proposed method compared to both the BM formulation and SDP relaxation (Candès and Tao, 2010) across different settings and problem sizes n. It is noteworthy that the SDP relaxation, being a convex problem, is executed only once per scenario, as it reliably converges to a global solution. However, for the specific instance (22), the SDP approach will be invalid since there are other SDP matrices N such that $N_\Omega = M^*_\Omega$. For the independent observation model, the variability of Ω necessitates running 10 distinct problem instances for each size n, with each instance undergoing 20 trials to estimate the success rate. This testing approach showcases the higher success rate of the proposed method. Additionally, the convex relaxation typically surpasses the BM formulation, as indicated in Figure 2b. Experimental conditions included a $\epsilon = 5 * 10^{-5}$, a learning rate of 2e-2, an initialization scale of $\xi = 10^{-4}$ for (12), and the utilization of the Adam optimizer (Kingma and Ba, 2014) for all experiments except those involving the semi-definite problem, where the open-source SCS solver was employed.

7 Conclusion

The methodologies unveiled in this study signify a potential paradigm shift within the realms of machine learning and optimization. By challenging the conventional strategy of minimizing noise to solve complex problems, our research introduces a controlled noise mechanism that not only elevates theoretical promises but also enables a strategic management of trade-offs in problem-solving. The developed ϵ-MC framework enhances practical application by allowing the integration of matrix sensing techniques, providing a flexible framework that could benefit general matrix completion problems.

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A Additional Details for Noisy Lifted Framework

A.1 Additional Definitions

Definition 3 (Tensor)**.** As a generalization of the way vectors are used to parametrize finite-dimensional vector spaces, we use *arrays* to parametrize tensors generated from product of finite-dimensional vector spaces, as per Comon et al. (2008). In particular, we define an l-way array as such:

$$\mathbf{a} = \{a_{i_1 i_2 \dots i_l} | 1 \le i_k \le n_k, 1 \le k \le l\} \in \mathbb{R}^{n_1 \times \cdots \times n_l}$$

Note that in this paper tensors and arrays can be regarded as synonymous since there exists an isomorphism between them. Moreover, if $n_1 = \cdots = n_l$, then we call this tensor(array) an l-order(way), n-dimensional tensor. For the convenience of tensor representation, we use the notation $\mathbb{R}^{n \circ l}$ with $n \circ l \coloneqq n \times \cdots \times n$. In this work, tensors are denoted with bold variables, and other fonts are reserved for matrices, vectors, and scalars unless specified otherwise.

Definition 4 (Symmetric Tensor)**.** Similar to the definition of symmetric matrices, for an order-l tensor $\mathbf{a}$ with the same dimensions (i.e., $n_1 = \cdots = n_l$), also called a cubic tensor, it is said that the tensor is symmetric if its entries are invariance under any permutation of their indices:

$$a_{i_{\sigma(1)} \cdots i_{\sigma(l)}} = a_{i_1 \cdots i_l} \quad \forall \sigma, \quad i_1, \dots, i_l \in \{1, \dots, n\}$$

where $\sigma \in \mathcal{G}_l$ denotes a specific permutation and $\mathcal{G}_l$ is the symmetric group of permutations on $\{1, \dots, l\}$. We denote the set of symmetric tensors as $\mathrm{S}^l(\mathbb{R}^n)$.

Definition 5 (Rank of Tensors)**.** The rank of a cubic tensor $\mathbf{a} \in \mathbb{R}^{n \circ l}$ is defined as

$$\operatorname{rank}(\mathbf{a}) = \min\{r | \mathbf{a} = \sum_{i=1}^{r} u_i \otimes v_i \otimes \cdots \otimes w_i\}$$

for some vector $u_i, \dots, w_i \in \mathbb{R}^n$. Furthermore, according to Kolda (2015), if $\mathbf{a}$ is a symmetric tensor, then it can be decomposed as:

$$\mathbf{a} = \sum_{i=1}^{r} \lambda_i u_i \otimes \cdots \otimes u_i \coloneqq \sum_{i=1}^{r} \lambda_i u_i^{\otimes l}$$

and the rank is conveniently defined as the number of nonzero λ_i's, which is very similar to the rank of symmetric matrices indeed. The most important concept in our paper is rank-1 tensors, and for any tensor $\mathbf{a}$, a necessary and sufficient condition for it to be rank-1 is that

$$\mathbf{a} = u^{\otimes l}$$

for some $u \in \mathbb{R}^n$.

Definition 6 (Tensor Multiplication)**.** Outer product is an operation carried out on a pair of tensors, denoted as $\otimes$. The outer product of 2 tensors $\mathbf{a}$ and $\mathbf{b}$, respectively of orders l and p, is a tensor of order $l + p$, denoted as $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$ such that:

$$c_{i_1 \dots i_l j_1 \dots j_p} = a_{i_1 \dots i_l} b_{j_1 \dots j_p}$$

When the 2 tensors are of the same dimension, this product is such that $\otimes : \mathbb{R}^{n \circ l} \times \mathbb{R}^{n \circ p} \mapsto \mathbb{R}^{n \circ (l+p)}$. Henceforth, we use the shorthand notation

$$\underbrace{a \otimes \cdots \otimes a}_{l \text{ times}} \coloneqq a^{\otimes l}$$

We also define an inner product of two tensors. The mode-q inner product between the 2 aforementioned tensors having the same q-th dimension is denoted as $\langle \mathbf{a}, \mathbf{b} \rangle_q$. Without loss of generality, assume that $q = 1$ and

$$[\langle \mathbf{a}, \mathbf{b} \rangle_q]_{i_2 \dots i_l j_2 \dots j_p} = \sum_{\alpha=1}^{n_q} a_{\alpha i_2 \dots i_l} b_{\alpha j_2 \dots j_p}$$

Note that when we write $\langle \cdot, \cdot \rangle_q$, we count the q-th dimension of the first entry. Indeed, this definition of inner product can also be trivially extended to multi-mode inner products by just summing over all modes, denoted as $\langle \mathbf{a}, \mathbf{b} \rangle_{q, \dots, s}$.

Lemma 2 (Section 10.2 Petersen et al. (2008))**.** *For four arbitrary matrices A, B, C, D of compatible dimensions, it holds that*

$$\langle A \otimes B, C \otimes D \rangle_{2,4} = AC \otimes BD \tag{23}$$

A.2 Formulation Details

As noted in our main formulation (12), the decision variable $\mathbf{w}$ is a tensor of dimension $nr \times \cdots \times nr$, since it serves as a repeated outer product of $\text{vec}(X)$ with $X \in \mathbb{R}^{n\times r}$ being our original decision variable in (1) (here we assume $r_{\text{search}} = r$). The permutation $\mathbf{P}$ is needed in order to convert $\mathbf{w} \in \mathbb{R}^{nr \circ l}$ back to $\mathbb{R}^{[n\times r]\circ l}$ in order to do meaningful inner products. $\mathbf{P} \in \mathbb{R}^{n\times r\times nr}$ is defined as

$$\langle \mathbf{P}, \text{vec}(X)\rangle_3 = X \quad \forall X \in \mathbb{R}^{n\times r}, n, r \in \mathbb{Z}^+$$

Such $\mathbf{P}$ can be easily constructed via filling appropriate scalar "1"s in the tensor. Via Lemma A.1, we also know that

$$\langle \mathbf{P}^{\otimes l}, \text{vec}(X)^{\otimes l}\rangle_{3*[l]} = (\langle \mathbf{P}, \text{vec}(X)\rangle_3)^{\otimes l} = X^{\otimes l} \tag{24}$$

Notationally, we abbreviate $\langle \mathbf{P}^{\otimes l}, \mathbf{w}\rangle_{3*[l]}$ as $\mathbf{P}(\mathbf{w})$ for enhanced readability for an arbitrary tensor $\mathbf{w}$ with dimension greater or equal to 2.

Since we also make extensive use of first and second order critical points of (12), we present them here for accessibility:

Lemma 3. *The tensor $\hat{\mathbf{w}} \in \mathbb{R}^{nr \circ l}$ is an SOP of (12) if and only if*

$$\langle \nabla f^l(\langle \mathbf{P}(\hat{\mathbf{w}}), \mathbf{P}(\hat{\mathbf{w}})\rangle_{2*[l]}), \mathbf{P}(\hat{\mathbf{w}})\rangle_{2*[l]} = 0, \tag{25a}$$

$$\begin{aligned} &2\langle \nabla f^l(\langle \mathbf{P}(\hat{\mathbf{w}}), \mathbf{P}(\hat{\mathbf{w}})\rangle_{2*[l]}), \langle \mathbf{P}(\Delta), \mathbf{P}(\Delta)\rangle_{2*[l]} + \\ &\|\langle \mathbf{A}^{\otimes l}, \langle \mathbf{P}(\hat{\mathbf{w}}), \mathbf{P}(\Delta)\rangle_{2*[l]} + \langle \mathbf{P}(\Delta), \mathbf{P}(\hat{\mathbf{w}})\rangle_{2*[l]}\rangle\|_F^2 \geq 0 \quad \forall \Delta \in \mathbb{R}^{nr \circ l} \end{aligned} \tag{25b}$$

with (25b) being a necessary and sufficient condition for $\hat{\mathbf{w}}$ to be a FOP and $\nabla f_w^l(\mathbf{M})$ is defined as

$$\nabla f_w^l(\mathbf{M}) = \langle (\mathbf{A}^{\otimes l})^* \mathbf{A}^{\otimes l}, \mathbf{M}\rangle - [\langle \mathbf{A}^*\mathbf{A}, M^*\rangle + \langle \mathbf{A}^*, \tilde{w}\rangle]^{\otimes l} \tag{26}$$

The proof to this lemma is highly technical and can be obtained by slightly changing the proof to Lemma 7 in Ma et al. (2024) by changing $b = \mathcal{A}(M^*)$ to $\tilde{b} = \mathcal{A}(M^*) + \tilde{w}$ defined above.

Next, we present a technical extension of Theorem 4 and Theorem 3, showing how gradient descent initialized with small scale can help ensure that second-order points of lifted version of (7) remain very close to M^* along the optimization trajectory

Theorem 5. *Consider a generic matrix completion problem under the same premise as given in Theorem 4. Assume that the symmetric tensor $\hat{\mathbf{w}} \in \mathbb{R}^{nr \circ l}$ is a second-order point (local minima) of (12) that is κ-rank-1 with $\kappa \leq \mathcal{O}(1/\|M^*\|_F^2)$. This can be achieved by initializing the vanilla gradient algorithm at $\mathbf{w}_0 = \xi x_0^{\otimes l}$ with a sufficiently small $\xi > 0 \in \mathbb{R}$. Then after iterations $t(\kappa, l)$ given in (16), Theorem 3 ensures that all tensors along the trajectory will become κ-rank-1.*

If $\hat{\mathbf{w}}$'s major spectral decomposition is given as $\hat{\mathbf{w}} = \lambda_S \hat{x}^{\otimes l} + \hat{\mathbf{w}}^\dagger$ with $\hat{x} \in \mathbb{R}^{nr}$ being a FOP of (7) (ensured by Proposition 2 in Ma et al. (2024)), we know that

$$\|M^* - \hat{X}\hat{X}^\top\|_F < \frac{1}{\epsilon}\lambda_r(\hat{X})\sqrt{\text{tr}(M^*)} + \mathcal{O}(\sqrt{r}\kappa^{1/2l}) + e_1 \tag{27}$$

holds with probability at least q, under the condition that l is odd and meets the requirement:

$$l > \frac{1}{1 - \log_2(2\beta)}, \quad \beta := \frac{\text{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\epsilon^2\left(\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \mathcal{O}(r\kappa^{1/l}) - e_2\right)}. \tag{28}$$

where e_1, e_2, and q are identical to those given in Theorem 4 depending on different MC instances.

The proof of this Theorem is omitted because it directly follows from Theorem 4, Theorem 3, and Theorem 2 in Ma et al. (2024).

B Missing Proofs

Proof to Theorem 1. To begin with, we reiterate our elementary results which follows from the definition of $M^\dagger$ and that of (7):

$$
\begin{aligned}
&0 \geq f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*) \\
=&\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 + \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*) - w_\epsilon\|_{S,2}^2 - \frac{1}{2}\|w_\epsilon\|_{S,2}^2 \\
=&\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 + \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger)\|_{S,2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,2}^2 \\
\geq&\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,2}^2
\end{aligned}
$$

This follows from the simple observation that

$$(w_\epsilon)_i = -\epsilon\,\mathrm{vec}(M^*)_i \quad \forall\ i \in [n^2]$$

Then moving $\frac{1}{2}\epsilon^2\|M^*\|_{S,2}^2$ to the left hand side, and adding $\frac{1}{2}\|M^\dagger - M^*\|_{S,2}^2$ to both sides gives

$$\|M^\dagger - M^*\|_F^2 \leq \|M^\dagger - M^*\|_S^2 + \epsilon^2\|M^*\|_{S,2}^2 \tag{29}$$

If we define a new vector $d \in \mathbb{R}^{n^2}$ in which

$$d \coloneqq \mathrm{vec}(M^\dagger - M^*) \odot \mathrm{vec}(M^\dagger - M^*) + \epsilon^2\,\mathrm{vec}(M^*) \odot \mathrm{vec}(M^*)$$

then we know that

$$d_i = \left(\mathrm{vec}(M^\dagger)_i - \mathrm{vec}(M^*)_i\right)^2 + \epsilon^2\,\mathrm{vec}(M^*)_i^2 \geq 0 \quad \forall\ i \in [n^2]$$

So if we further define a series of random variables $\{r_1, r_2, \ldots, r_{n^2}\}$ with

$$r_i = \begin{cases} 0 & \text{with probability } p \\ d_i & \text{with probability } 1-p \end{cases} \tag{30}$$

Then we know that

$$\|M^\dagger - M^*\|_S^2 + \epsilon^2\|M^*\|_{S,2}^2 = \sum_{i=1}^{n^2} r_i \coloneqq R \tag{31}$$

because for any matrix $M \in \mathbb{R}^{n_1 \times n_2}$, we have

$$\|M\|_S^2 = \sum_i^{n_1 n_2} m_i^2, \quad m_i = \begin{cases} 0 & \text{with probability } p \\ \mathrm{vec}(M)_i & \text{with probability } 1-p \end{cases}$$

Then we simply acknowledge that $0 \leq r_i \leq d_i$ almost surely, which sets up the premise to use Hoeffding's inequality (Hoeffding, 1994). This concentration inequality gives that

$$\mathbb{P}\left(R \leq \mathbb{E}[R] + t\right) \geq 1 - \exp\left(\frac{-2t^2}{\sum_i^{n^2}(d_i - 0)^2}\right) = 1 - \exp\left(\frac{-2t^2}{\|d\|_2^2}\right) \tag{32}$$

First of all, we could easily derive that

$$
\begin{aligned}
\mathbb{E}[R] &= \sum_i^{n^2}(1-p)\left[\left(\mathrm{vec}(M^\dagger)_i - \mathrm{vec}(M^*)_i\right)^2 + \epsilon^2\,\mathrm{vec}(M^*)_i^2\right] \\
&= (1-p)\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right)
\end{aligned} \tag{33}
$$

Therefore combining (29), (32) and (33) we have

$$\mathbb{P}\left(\|M^\dagger - M^*\|_F^2 \leq (1-p)\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right) + t\right) \geq 1 - \exp\left(\frac{-2t^2}{\|d\|_2^2}\right) \tag{34}$$

Then we can choose

$$t = \eta\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right) = \eta\|d\|_1$$

for some constant $\eta \leq p$. This will then transform (34) into

$$\begin{aligned} &\mathbb{P}\left(\|M^\dagger - M^*\|_F^2 \leq (1-p+\eta)\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right)\right) \geq 1 - \exp\left(\frac{-2t^2}{\|d\|_2^2}\right) \\ \implies &\mathbb{P}\left((p-\eta)\|M^\dagger - M^*\|_F^2 \leq (1-p+\eta)\epsilon^2\|M^*\|_F^2\right) \geq 1 - \exp\left(\frac{-2\eta^2\|d\|_1^2}{\|d\|_2^2}\right) \\ \implies &\mathbb{P}\left(\|M^\dagger - M^*\|_F^2 \leq \frac{1-p+\eta}{p-\eta}\epsilon^2\|M^*\|_F^2\right) \geq 1 - \exp\left(\frac{-2\eta^2\|d\|_1^2}{\|d\|_2^2}\right) \end{aligned} \tag{35}$$

which proves our desired result directly. □

Proof of Theorem 2. First of all, we hope to decompose the Hessian of (1) at a second order point $\hat{X} \in \mathbb{R}^{n\times r}$. Classic matrix sensing literatures like Ha et al. (2020); Zhang et al. (2021); Li et al. (2019) give that the second-order critical condition of (1) are given as

$$\nabla f(\hat{X}\hat{X}^\top)\hat{X} = 0, \tag{36}$$

$$2\langle \nabla f(\hat{X}\hat{X}^\top), UU^\top\rangle + [\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top + U\hat{X}^\top, \hat{X}U^\top + U\hat{X}^\top) \geq 0 \quad \forall U \in \mathbb{R}^{n\times r} \tag{37}$$

with (36) being the first order critical condition. Moreover, since the sensing matrices $\{A_i\}_{i\in[m]}$ can be assumed be to symmetric without loss of generality (Zhang et al., 2021), we have that

$$[\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top + U\hat{X}^\top, \hat{X}U^\top + U\hat{X}^\top) = 4[\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top, \hat{X}U^\top).$$

We then could decompose LHS of (37) as $2C_1 + 4C_2$ where

$$C_1 \coloneqq \langle \nabla f(\hat{X}\hat{X}^\top), UU^\top\rangle, \quad C_2 \coloneqq [\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top, \hat{X}U^\top)$$

Given the assumption that (1) obeys some RSS condition, it is possible to upper-bound C_2 by observing

$$[\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top + U\hat{X}^\top, \hat{X}U^\top + U\hat{X}^\top) \leq L_s\|\hat{X}U^\top + U\hat{X}^\top\|_F^2$$

Therefore, if want to somehow create an negative escape direction for $\hat{X}$, it is important that we find a U such that C_1 is negative and large in magnitude, and then amplify this term via tensor parametrization. To do so, we first do a more in-depth analysis of $\nabla f(\hat{X}\hat{X}^\top)$. As mentioned above, since $\nabla f(\cdot)$ can be assumed to be symmetric, one can select $u \in \mathbb{R}^n$ such that $u^\top \nabla f(\hat{x}\hat{x}^\top)u = \lambda_{\min}(\nabla f(\hat{x}\hat{x}^\top))$. Then via the definition of RSC we have

$$f(M^*) \geq f(\hat{X}\hat{X}^\top) + \langle \nabla f(\hat{X}\hat{X}^\top), M^* - \hat{X}\hat{X}^\top\rangle + \frac{\alpha_s}{2}\|\hat{X}\hat{X}^\top - M^*\|_F^2. \tag{38}$$

With $\hat{X}$ being a first-order point, according to (36)

$$\nabla f(\hat{X}\hat{X}^\top)\hat{X} = 0 \implies \langle \nabla f(\hat{X}\hat{X}^\top), \hat{X}\hat{X}^\top\rangle = 0$$

Therefore, if in (1) our b is corrupted as $\mathcal{A}(M^*) + \tilde{w}$, then plugging it back into (38) gives

$$\begin{aligned} \langle \nabla f(\hat{X}\hat{X}^\top), M^*\rangle &\leq -\frac{\alpha_s}{2}\|\hat{x}\hat{x}^\top - M^*\|_F^2 + f(M^*) - f(XX^\top) \\ &\leq -\frac{\alpha_s}{2}\|\hat{x}\hat{x}^\top - M^*\|_F^2 + f(M^*) \\ &= -\frac{\alpha_s}{2}\|\hat{x}\hat{x}^\top - M^*\|_F^2 + \frac{\|\tilde{w}\|_2^2}{2} \end{aligned} \tag{39}$$

where the second inequality follows from the fact that $f(\cdot) \geq 0$ in its entire domain and the last inequality follows from $f(M^*) = 1/2\|\mathcal{A}(M^*) - \mathcal{A}(M^*) - \tilde{w}\|_2^2 = \|\tilde{w}\|_2^2/2$. Furthermore, since both $\nabla f(\hat{X}\hat{X}^\top)$ and M^* are assumed to be positive semidefinite,

$$\langle \nabla f(\hat{X}\hat{X}^\top), M^*\rangle \geq \lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top))\operatorname{tr}(M^*)$$

which implies that

$$\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)) \leq \frac{-\alpha_s\|\hat{X}\hat{X}^\top - M^*\|_F^2 + \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)} \tag{40}$$

Furthermore, (13) gives us

$$\|\hat{X}\hat{X}^\top - M^*\|_F^2 \geq \|\tilde{w}\|_2^2/\alpha_s$$

since $\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) \geq 0$ by definition. This means that

$$\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)) \leq \frac{-\alpha_s\|\hat{X}\hat{X}^\top - M^*\|_F^2 + \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)} \leq 0 \tag{41}$$

Thus, with this result equipped, we can further find a U that makes C_1 small. In the most convenient manner, we first consider the eigenvector $u \in \mathbb{R}^n$ of $\nabla f(\hat{X}\hat{X}^\top)$ associated with $\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top))$. Additionally we consider $q \in \mathbb{R}^r$ to be the r-th singular value of $\hat{X}$, with

$$\|\hat{X}q\|_2 = \sigma_r(\hat{X}), \qquad \|q\|_2 = 1$$

Then choosing $U \in \mathbb{R}^{n\times r} = uq^\top$ leads to

$$C_1 = \langle \nabla f(\hat{X}\hat{X}^\top), UU^\top\rangle = \langle \nabla f(\hat{X}\hat{X}^\top), uu^\top\rangle = -G$$

where $G \coloneqq -\lambda_{\text{min}}(\nabla f(\hat{X}\hat{X}^\top)) \geq 0$. By recalling $\hat{X}^\top u = 0$ according to the first-order condition (36), we can further bound C_2 with this chocie of U as

$$\begin{aligned}
L_s\|\hat{X}U^\top + U\hat{X}^\top\|_F^2 &= L_s\|u(\hat{X}q)^\top + (\hat{X}q)u^\top\|_F^2 \\
&= 2L_s\|\hat{X}q\|_F^2 + 2L_s(q^\top(\hat{X}^\top u))^2 \\
&= 2L_s\lambda_r(\hat{X}\hat{X}^\top),
\end{aligned}$$

leading to

$$C_2 \leq \frac{1}{2}L_s\lambda_r(\hat{X}\hat{X}^\top)$$

Now, if we choose $\Delta = \operatorname{vec}(U)^{\otimes l}$ for the aforementioned $U \in \mathbb{R}^{n\times r}$, the LHS of (25b) can be expressed as:

$$\begin{aligned}
&2(\langle \mathbf{A}, \hat{X}\hat{X}^\top\rangle_{2,3}^\top \langle \mathbf{A}, uu^\top\rangle_{2,3})^l - 2\left((\langle \mathbf{A}, M^*\rangle_{2,3} + \tilde{w})^\top \langle \mathbf{A}, uu^\top\rangle_{2,3}\right)^l + 4(\|\langle \mathbf{A}, \hat{X}U^\top\rangle_{2,3}\|_2^2)^l \\
\leq& 2(\lambda_{\text{min}}(\nabla f(\hat{X}\hat{X}^\top)))^l + 4C_2^l \\
=& 2C_1^l + 4C_2^l
\end{aligned} \tag{42}$$

where the inequality follows from:

$$a^n - b^n \leq (a-b)^n, \quad \forall b \geq a \geq 0$$

Here, since $a - b = C_1 \leq 0$, the above inequality can be used. As a result,

$$\text{LHS of (25b)} \leq \underbrace{-2G^l}_{\text{Part 1}} + \underbrace{\frac{2}{2^{l-1}}L_s^l\lambda_r(\hat{X}\hat{X}^\top)^l}_{\text{Part 2}}$$

We know since $G \geq 0$, Part 1 is always negative assuming l is odd, and Part 2 is always positive. Therefore, it suffices to find an order l such that

$$G^l > (1/2^{l-1})L_s^l\lambda_r(\hat{X}\hat{X}^\top)^l \tag{43}$$

Conveniently, (41) says that

$$G \geq \frac{\alpha_s\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)}, \tag{44}$$

which can be used to derive sufficient condition for (43). Therefore, if

$$\left(\frac{\alpha_s\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)}\right)^l > (1/2^{l-1})L_s^l\lambda_r(\hat{X}\hat{X}^\top)^l,$$

we can conclude that (43) holds, which implies that the LHS of (25b) is negative, directly proving that $\hat{X}^{\otimes l}$ is not an SOP anymore. Elementary manipulations of the above equation give that a sufficient condition is

$$\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2/\alpha_s > 2^{1/l}\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) \tag{45}$$

We now consider (13), which means that

$$\lambda_r(\hat{X}\hat{X}^\top) \leq \frac{\alpha_s\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{L_s\operatorname{tr}(M^*)} \tag{46}$$

Subsequently, define a constant γ such that

$$L_s\lambda_r(\hat{X}\hat{X}^\top) = \gamma\left[\frac{\alpha_s\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)}\right]$$

Then, (44) and (46) together imply that $1 \leq \gamma < 2$. Using this simplified notation, our sufficient condition (45) becomes

$$1 > \frac{\gamma}{2^{(l-1)/l}} \tag{47}$$

Given $1 \leq \gamma < 2$, there always exists a large enough l such that (47) holds, which proves that LHS of (25b) is negative, proving that $\operatorname{vec}(\hat{X})^{\otimes l}$ is a strict saddle, concluding the proof.

To derive a sufficient l, we simply acknowledge

$$\gamma = \frac{2L_s\operatorname{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\alpha_s\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2} := 2\beta$$

and that $\beta \leq 1$ due to assumption (13). Therefore, for (47) to hold true, it is enough to have

$$2^{(l-1)/l} > 2\beta \implies \frac{l-1}{l} > \log_2(2\beta) \implies l > \frac{1}{1-\log_2(2\beta)}$$

$\square$

Proof of Theorem 3. First of all, we hope to decompose the GD trajectory of (12) $\{\mathbf{w}_t\}_{t=0}^T$ as follows:

$$\mathbf{w}_{t+1} = \langle \mathbf{Z}_t, \mathbf{w}_0\rangle - \mathbf{E}_t := \tilde{\mathbf{w}}_t - \mathbf{E}_t \tag{48}$$

where

$$\begin{aligned}
\mathbf{Z}_t &:= (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)^t, \quad \mathbf{A}_r = I_r \oslash_{2,3} A \\
\mathbf{E}_t &:= \sum_{i=1}^t (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)^{t-i}\hat{\mathbf{E}}_i \\
\hat{\mathbf{E}}_i &:= \eta\langle\langle(\mathbf{A}_r^l)^*\mathbf{A}^l, \langle \mathbf{P}(\mathbf{w}_{i-1}), \mathbf{P}(\mathbf{w}_{i-1})\rangle_{2*[l]}\rangle, \mathbf{w}_{i-1}\rangle_{2*[l]} \\
(\mathbf{A}_r^l)^*\mathbf{A}^l &:= \langle(\mathbf{A}_r)^{\otimes l}, \mathbf{A}^{\otimes l}\rangle_{3,6,\ldots,3l} \in \mathbb{R}^{[nr\times nr\times n\times n]\circ l}
\end{aligned}$$

This can be proved via induction where

$$\begin{aligned}
\mathbf{w}_1 &= \left(\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} - (\mathbf{A}^{\otimes l})^*\langle \mathbf{P}(\mathbf{w}_0), \mathbf{P}(\mathbf{w}_0)\rangle\right)\mathbf{w}_0 \\
&= (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)\mathbf{w}_0 - \eta\langle\langle(\mathbf{A}_r^l)^*\mathbf{A}^l, \langle \mathbf{P}(\mathbf{w}_0), \mathbf{P}(\mathbf{w}_0)\rangle\rangle\mathbf{w}_0 \\
&= \langle \mathbf{Z}_1, \mathbf{w}_0\rangle - \mathbf{E}_1
\end{aligned}$$

This serves as our base case, and the induction step can be proven as

$$\begin{aligned}
\mathbf{w}_{t+1} &= \left(\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} - (\mathbf{A}^{\otimes l})^* \langle \mathbf{P}(\mathbf{w}_t), \mathbf{P}(\mathbf{w}_t)\rangle\right) \mathbf{w}_t \\
&= (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)\mathbf{w}_t - \eta\langle (\mathbf{A}_r^{l})^* \mathbf{A}^{l}, \langle \mathbf{P}(\mathbf{w}_t), \mathbf{P}(\mathbf{w}_t)\rangle\rangle \mathbf{w}_t \\
&= (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)\mathbf{w}_t - \hat{\mathbf{E}}_{t+1} \\
&= (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)\left(\tilde{\mathbf{w}}_t - \sum_{i=1}^{t} (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)^{t-i} \hat{\mathbf{E}}_i\right) - \hat{\mathbf{E}}_{t+1} \\
&= \tilde{\mathbf{w}}_{t+1} - \sum_{i=1}^{t} (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)^{t+1-i} \hat{\mathbf{E}}_i - \hat{\mathbf{E}}_{t+1} \\
&= \tilde{\mathbf{w}}_{t+1} - \sum_{i=1}^{t+1} (\mathcal{I} + \eta\langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l}\rangle)^{t+1-i} \hat{\mathbf{E}}_i \\
&= \tilde{\mathbf{w}}_{t+1} - \mathbf{E}_t
\end{aligned}$$

Therefore, we can then use a version of Lemma 13 and Lemma 2 in Ma et al. (2024) with $U = \langle \mathbf{A}_r^* \mathbf{A}, M^* \rangle$ replaced with $U = \langle \mathbf{A}_r, \tilde{b} \rangle_1$ to prove this theorem, following the steps in proof to Theorem 1 in Ma et al. (2024). □

Proof to Theorem 4. For the general matrix completion case, as promised by Lemma 1, the theorem is proved by substituting $L_s = 1$, $\alpha_s = \epsilon^2$ into Theorem 4, and since this is a deterministic result, it happens with probability 1, meaning that for any second-order point $\hat{\mathbf{w}} = \hat{X}^{\otimes l}$ of (12), it satisfies that

$$\begin{aligned}
\|M^* - \hat{X}\hat{X}^\top\|_F &< \sqrt{\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) + \frac{\epsilon^2 \|M^*\|_{\bar{\Omega},F}^2}{\epsilon^2}} \\
&= \sqrt{\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) + \|M^*\|_{\bar{\Omega},F}^2} \\
&\le \sqrt{\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*)} + \|M^*\|_{\bar{\Omega},F} \\
&= \frac{1}{\epsilon}\lambda_r(\hat{X})\sqrt{\operatorname{tr}(M^*)} + \|M^*\|_{\bar{\Omega},F}
\end{aligned} \tag{49}$$

where l has to obey equation (14) as stated in Theorem 2.

In the case of each entry of M^* being observed independently with probability p, we first apply Theorem 4 with $\tilde{w} = 0$ to (7), meaning that we first assume that no noise exists in b. This is the case where we are actually trying to recover the global solution of (7), denoted as $M^\dagger$. This means that for any rank-1 critical point $\hat{\mathbf{w}} = \hat{X}^{\otimes l}$ of (12), it is a second-order point only if

$$\|\hat{X}\hat{X}^\top - M^\dagger\|_F^2 < \frac{1}{\epsilon^2}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) \tag{50}$$

holds, when l is odd and satisfies

$$l > \frac{1}{1 - \log_2(2\beta)}, \quad \beta := \frac{L_s \operatorname{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\epsilon^2 \|M^* - \hat{X}\hat{X}^\top\|_F^2}. \tag{51}$$

The above statement holds deterministically. However, Theorem 1 also tells us that

$$\|M^\dagger - M^*\|_F^2 \le \frac{1 - p + \eta}{p - \eta}\epsilon^2 \|M^*\|_F^2$$

with high probability, so then by a triangle inequality we have that the conversion criterion above transforms to

$$\begin{aligned}
\|\hat{X}\hat{X}^\top - M^*\|_F &\le \|\hat{X}\hat{X}^\top - M^\dagger\|_F + \|M^\dagger - M^*\|_F \\
&< \frac{\lambda_r(\hat{X})\sqrt{\operatorname{tr}(M^*)}}{\epsilon} + \sqrt{\frac{1 - p + \eta}{p - \eta}}\epsilon \|M^*\|_F
\end{aligned} \tag{52}$$

with the same probability stated in Theorem 1, thereby concluding the proof. □