Aufgabe: Berechne die die red. Gröbnerbasis von $\langle f_1, f_2 \rangle$ mit $f_1 = x^2+2y^2-4$, $f_2 = xy-1$ bzgl. der Lex-Ordnung.

$$S(f_1, f_2) = yf_1 - xf_2 = (x^2y+2y^3-4y) - (x^2y - \cancel{x})$$
$$= x + 2y^3 - 4y$$

$$\mathrm{Rem}(S(f_1,f_2); f_1, f_2) = x+2y^3-4y =: f_3$$

$$S(f_1, f_3) = 1\cdot f_1 - xf_3 = (x^2+2y^2-4) - (x^2+2xy^3-4xy)$$
$$- 2xy^3 + 4xy + 2y^2 - 4$$

$$\mathrm{Rem}(S(f_1,f_3); f_1, f_2, f_3) = 0$$

$-2xy^3 + 4xy + 2y^2 - 4$	$+2y^2 f_2 = +2xy^3 - 2y^2$
$4xy - 4$	$-4 f_2 = -4xy + 4$
0	

$$S(f_2, f_3) = 1\cdot f_2 - y\cdot f_3 = (xy-1) + (-xy - 2y^4 + 4y^2)$$
$$= -2y^4 + 4y^2 - 1$$

$$\mathrm{Rem}(S(f_2,f_3); f_1, f_2, f_3) = -2y^4 + 4y^2 - 1 =: f_4$$

$f_1 = x^2 + 2y^2 - 4,$

$f_2 = xy - 1$

$\mathrm{Rem}(S(f_1, f_2); f_1, f_2) = x + 2y^3 - 4y =: f_3$

$\mathrm{Rem}(S(f_1, f_3); f_1, f_2, f_3) = 0$

$\mathrm{Rem}(S(f_2, f_3); f_1, f_2, f_3) = -2y^4 + 4y^2 - 1 =: f_4$

$$S(f_1, f_4) = y^4 f_1 + \tfrac{1}{2} x^2 f_4 = (x^2y^4 + 2y^6 - 4y^4) + (-x^2y^4 + 2x^2y^2 - \tfrac{1}{2}x^2)$$
$$= 2x^2y^2 - \tfrac{1}{2}x^2 + 2y^6 - 4y^4$$

$\mathrm{Rem}(S(f_1, f_4); f_1, f_2, f_3, f_4) = 0$

$2x^2y^2 - \frac{1}{2}x^2 + 2y^6 - 4y^4$	$-2y^2 f_1 = -2x^2y^2 - 4y^4 + 8y^2$
$-\frac{1}{2}x^2 + 2y^6 - 8y^4 + 8y^2$	$+\frac{1}{2} f_1 = \frac{1}{2}x^2 + y^2 - 2$
$2y^6 - 8y^4 + 9y^2 - 2$	$+y^2 f_4 = -2y^6 + 4y^4 - y^2$
$-4y^4 + 8y^2 - 2$	$-2f_4 = 4y^4 - 8y^2 + 2$
0	

$f_1 = x^2 + 2y^2 - 4,$

$f_2 = xy - 1$

$\mathrm{Rem}(S(f_1, f_2); f_1, f_2) = x + 2y^3 - 4y =: f_3$

$\mathrm{Rem}(S(f_1, f_3); f_1, f_2, f_3) = 0$

$\mathrm{Rem}(S(f_2, f_3); f_1, f_2, f_3) = -2y^4 + 4y^2 - 1 =: f_4$

$\mathrm{Rem}(S(f_1, f_4); f_1, f_2, f_3, f_4) = 0$

$S(f_1, f_4) = y^3 f_2 + \frac{1}{2} x f_4 = (xy^4 - y^3) + (-xy^4 + 2xy^2 - \frac{1}{2}x)$

$= 2xy^2 - \frac{1}{2}x - y^3$

$\mathrm{Rem}(S(f_2, f_4); f_1, f_2, f_3, f_4) = 0$

$2xy^2 - \frac{1}{2}x - y^3$	$-2y f_2 = -2xy^2 + 2y$
$-\frac{1}{2}x - y^3 + 2y$	$\frac{1}{2} f_3 = \frac{1}{2}x + y^3 - 2y$
0	

$f_1 = x^2 + 2y^2 - 4,$

$f_2 = xy - 1$

$\mathrm{Rem}(S(f_1, f_2); f_1, f_2) = x + 2y^3 - 4y =: f_3$

$\mathrm{Rem}(S(f_1, f_3); f_1, f_2, f_3) = 0$

$\mathrm{Rem}(S(f_2, f_3); f_1, f_2, f_3) = -2y^4 + 4y^2 - 1 =: f_4$

$\mathrm{Rem}(S(f_1, f_4); f_1, f_2, f_3, f_4) = 0$

$\mathrm{Rem}(S(f_2, f_4); f_1, f_2, f_3, f_4) = 0$

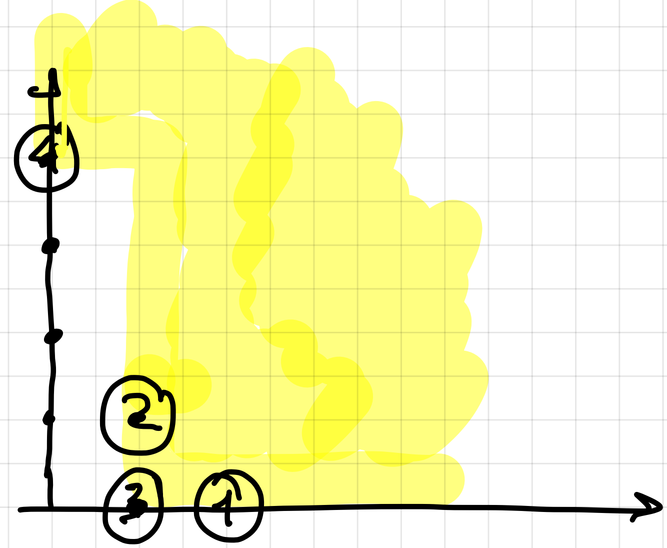
$S(f_3, f_4) = y^4 f_3 + \frac{1}{2} x f_4 = (xy^4 + 2y^7 - 4y^5) + (-xy^4 + 2xy^2 - \frac{1}{2}x)$

$= 2xy^2 - \frac{1}{2}x + 2y^7 - 4y^5$

$\mathrm{Rem}(S(f_3, f_4); f_1, f_2, f_3, f_4) = 0$

$$\boxed{\begin{array}{l|l} 2xy^2 - \frac{1}{2}x + 2y^7 - 4y^5 & -2y f_2 = -2xy^2 + 2y \\ -\frac{1}{2}x + 2y^7 - 4y^5 + 2y & \frac{1}{2} f_3 = \frac{1}{2}x + y^3 - 2y \\ 2y^7 - 4y^5 + y^3 & + y^3 f_4 = -2y^7 + 4y^5 - y^3 \\ 0 & \end{array}}$$

$$\left.\begin{aligned} f_1 &= x^2+2y^2-4, \\ f_2 &= xy-1 \\ f_3 &= x+2y^3-4y \\ f_4 &= -2y^4+4y^2-1 \end{aligned}\right\} \text{Gröbnerbas.}$$



$$\left\{\begin{aligned} &x+2y^3-4y \\ &y^4+2y^2-\tfrac{1}{2} \end{aligned}\right.$$

Minimale Gröbnerbasis
und eine reduzierte Gröbnerbasis.

$$f_1 = x^2 + y^2 - 1$$

$$f_2 = xy - 1$$

$$f_3 = x + 2y^3 - 2y = \mathrm{Rem}(S(f_1, f_2); f_1, f_2)$$

$$f_4 = -y^2 - 3 = \mathrm{Rem}(S(f_1, f_3); f_1, f_2, f_3)$$

$-2xy^3 - 2xy + y^2 - 1$	$2y^2 f_2 = 2xy^3 - 2y^2$
$-2xy - y^2 - 1$	$2f_2 = 2xy - 2$
$-y^2 - 3$	ues $\longrightarrow$ Rest