How do we invert a linear map given by a matrix $A \in F^{n \times n}$? The map takes $x \in F^n$ as the input and produces $Ax = y$ as the output. So, to invert the map we need to find the "way back" - from y to x. For a given y, $Ax = y$ is a system of linear equations in the unknowns $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

So, to invert a linear map, we need to learn how to solve linear systems.

Let's consider an arbitrary linear system with m equations and n unknowns $x_1, \dots, x_n$:

$$\text{(LS)} \qquad \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, \dots, m)$$

where $a_{ij}, b_i \in F$ are given coefficients of the system.

In the matrix form the system is written as

$$Ax = b$$

where $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \in F^{m\times n}$, $b \in (b_i)_{i=1,\dots,m} \in F^m$

and $x = (x_j)_{j=1,\dots,n} \in F^n$ is unknown.

Bsp

$$\begin{cases} x_1 + 2x_2 = 4 & (1) \\ 3x_1 + 4x_2 = -1 & (2) \end{cases} \quad -3\times$$

system over $\mathbb{Q}$

$$\begin{cases} x_1 + 2x_2 = 4 & (1') \\ \boxed{-2x_2 = -13} & (2') \end{cases}$$

$$\left.\begin{array}{l} x_1 + 2x_2 = 4 \Rightarrow -3x_1 - 6x_2 = -12 \\ 3x_1 + 4x_2 = -1 \Rightarrow 3x_1 + 4x_2 = -1 \end{array}\right] \Rightarrow -2x_2 = -13$$

If (1') and (2') are true, then (1) is also true,
Because (1) is just the same as (1').

What equation do we get by adding 3x (1') to (2')?

$$\underbrace{3(x_1 + 2x_2)}_{\text{from } (1')} - \underbrace{2x_2}_{\text{from } (2')} = \underbrace{3\cdot 4}_{\text{from } (1')} - \underbrace{13}_{\text{from } (2')}$$

$3x_1 + 4x_2 = -1$, which is equation (2)

$$\begin{cases} x_1 + 2x_2 = 4 & (1') \\ -2x_2 = -13 & (2') \end{cases} \quad +1\cdot$$

$\Updownarrow$

$$\begin{cases} x_1 = -9 \\ -2x_2 = -13 \end{cases} \iff \begin{cases} x_1 = -9 \\ x_2 = \frac{13}{2} \end{cases}$$

Observations: it's nice to use transformations of equations as above.

If a variable x_k occurs in exactly one equation, we are happy: it does not interfere with any other equations and we get its value from the equation, in which it occurs.

Def. We say that a variable x_k occurs in the equation
$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{if} \quad a_{ik} \neq 0.$$

Def. We introduce the following operations on the equations:

- changing the order of two equations:
$$(s) \longleftrightarrow (t)$$
(swapping s-th and t-th equations)

- multiplying an equation with a non-zero number: $(s) := \gamma \cdot (s)$ for $\gamma \in F \setminus \{0\}$.

- Adding a multiple of one equation to another one:

$$(s) := (s) + \gamma \cdot (t)$$

to the s-th equation we add γ times t-th equation $(s \neq t)$.

We call the above transformations of linear systems.

Proposition 30 Application of elementary transformations to a linear system does not change the solution set of a linear system. (That is, by carrying out elementary transformation one arrives at an equivalent system.)

Let's develop a method that solves (LS).

We introduce J, the set of indices j of the variables that are contained in exactly one equation and the set I of the indices i of the equations that contain such variables.

We have $|I| = |J|$.

We want to make this property an invariant and we want to make I and J larger in each iteration.

$$\begin{cases} x_1 \quad + \quad 2x_3 - x_4 = 5 \\ \quad x_2 \quad - x_3 - x_4 = 4 \\ \quad\quad x_3 + x_4 = 10 \end{cases}$$

$J = \{1, 2\}$

$I = \{1, 2\}$

Algorithm. (Gaussian elimination)

$I := \emptyset$

$J := \emptyset$

while there exists a_{ij} with $a_{ij} \neq 0$ and $i \notin I$, $j \notin J$, eliminate x_j from ~~all~~ the equations but the i-th one, add i to I and j to J.
(eliminate using elementary transformations).

Example

$$\begin{cases} x_1 + x_2 + x_3 = 5 & (1) \\ x_1 - x_2 + x_3 = 1 & (2) \\ x_1 + 2x_2 - x_3 = 0 & (3) \end{cases}$$

$I = \emptyset$

$J = \emptyset$

$$\begin{cases} x_1 + x_2 + x_3 = 5 & (1) \\ \quad -2x_2 = -4 & (2) \\ \quad x_2 - 2x_3 = -5 & (3) \end{cases}$$

$I = \{1\}$

$J = \{1\}$

$$\begin{cases} x_1 \qquad + 3x_3 = +10 & (1) \\ \qquad -4x_3 = -14 & (2) \\ x_2 - 2x_3 = -5 & (3) \end{cases}$$

$I = \{1, 3\}$

$J = \{1, 2\}$

$$\begin{cases} x_1 = +10 - 3 \cdot 3{,}5 \\ x_3 = 3{,}5 \\ x_2 = -5 + 2 \cdot 3{,}5 \end{cases}$$

$I = \{1, 2, 3\}$

$J = \{1, 2, 3\}$

$$\begin{cases} x_1 = -0{,}5 \\ x_2 = 2 \\ x_3 = 3{,}5 \end{cases}$$

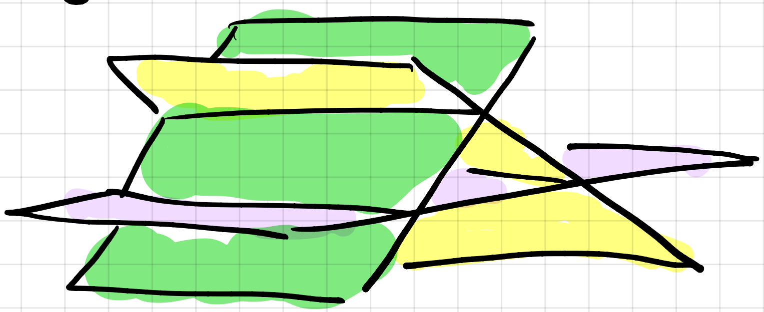
Example

$$\begin{cases} x_1 + x_2 + x_3 = 5 & (1) \\ x_1 - x_2 + x_3 = 1 & (2) \\ 3x_1 + x_2 + 3x_3 = 0 & (3) \end{cases}$$

$\Downarrow$ Gaussian Elimination

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \\ 0 = 1 \end{cases}$$

The system has no solutions.



Example To invert a map, given by $A \in F^{n \times n}$, that sends $x \in F^n$ to $Ax = y \in F^n$ we need to solve the system $Ax = y$ in the unknown y for an arbitrary choice of y.

That means, on the left-hand-side we have unknowns $x_1, \dots, x_n$ and on the right-hand-side we also have unknowns: $y_1, \dots, y_n$, and we solve the system $Ax = y$ in the unknowns $x_1, \dots, x_n$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathbb{Q}^{2 \times 2}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{cases} 1 \cdot x_1 + 2 \cdot x_2 = y_1 \\ 3 \cdot x_1 + 4 \cdot x_2 = y_2 \end{cases}$$

$$\begin{cases} 1 \cdot x_1 + 2 \cdot x_2 = 1 \cdot y_1 + 0 \cdot y_2 \\ 3 \cdot x_1 + 4 \cdot x_2 = 0 \cdot y_1 + 1 \cdot y_2 \end{cases}$$

One usually represents these equations with a matrix:

$$\begin{array}{r c} & \begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} \text{1st equation} \rightarrow \\ \text{2nd equation} \rightarrow \end{array} & \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \end{array}$$

By transforming this system with the Gaussian-elimination, we arrive at an equivalent system

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \quad -3\times$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad +1\times$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad \times\left(-\tfrac{1}{2}\right)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_2 = y_1 \\ 3x_1 + 4x_2 = y_2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x_1 = -2y_1 + y_2 \\ x_2 = \frac{3}{2} y_1 - \frac{1}{2} y_2 \end{cases}$$

So, the inverse of the linear map

$T: \mathbb{Q}^2 \to \mathbb{Q}^2$ given by

$T(x_1, x_2) = (x_1 + 2x_2, 3x_1 + 4x_2)$ is the map

$T^{-1}: \mathbb{Q}^2 \to \mathbb{Q}^2$ given by

$$T^{-1}(y_1, y_2) = \left(-2y_1 + y_2, \ \frac{3}{2}y_1 - \frac{1}{2}y_2\right).$$

Example Consider the linear map

$T: \mathbb{Q}^3 \to \mathbb{Q}^3$ given by

$$T(x) = A \cdot x \quad \text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & -1 & 8 \end{bmatrix}$$

$T(x) = y$, where

$$\begin{cases} x_1 + x_2 + x_3 = y_1 \\ x_1 - x_2 + 2x_3 = y_2 \\ 5x_1 - x_2 + 8x_3 = y_3 \end{cases}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 5 & -1 & 8 & 0 & 0 & 1 \end{array}\right]$$

Gaussian elimination

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -6 & 3 & -5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right]$$

$$\begin{cases} x_1 + 3x_2 = 2y_1 - y_2 \\ -2x_2 + x_3 = -y_1 + y_2 \\ 0 = -2y_1 - 3y_2 + y_3 \end{cases}$$

$$\boxed{\text{There is no any } x \in \mathbb{Q}^3 \text{ with } T(x) = y \text{ for } y = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}}$$

$\Uparrow$

$\boxed{\text{false}}$

$$T(x) = y \iff$$

$$\begin{cases} x_1 + 3x_2 = 2 \cdot 1 - 1 \\ -2x_2 + x_3 = -1 + 1 \\ 0 = -2 \cdot 1 - 3 \cdot 1 + 5 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x_1 + 3x_2 = 1 \\ -2x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x_1 = 4 \\ x_2 = -1 \\ x_3 = -2 \end{cases}$$

$$\Updownarrow$$

$$x = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

For $y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, there is not any $x \in \mathbb{Q}^3$ such that $T(x) = y$.

Because

$$T(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \iff$$

$$\begin{cases} x_1 + 3x_2 = 2 \cdot 0 - 0 \\ -2x_2 + x_3 = -0 + 0 \\ 0 = -2 \cdot 0 - 3 \cdot 0 + 1 \end{cases}$$

$$0 = 1$$

The map is also not injective!

Let's consider all $x \in \mathbb{Q}^3$ with $T(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

By Gaussian elimination:

$$T(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} x_1 + 3x_2 = 0 \\ -2x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 = -3x_2 \\ x_3 = 2x_2 \\ x_2 \in \mathbb{R} \end{cases}$$

So, T is not injective, because (for example)

$$T(0,0,0) = T(-3,1,2) = (0,0,0)$$

So, T is neither injective nor surjective.

Theorem 31 For a field F, every linear map $T: F^n \to F^n$ $(n \in \mathbb{N})$ is either bijective or neither injective nor surjective.

Proof is omitted, but see the above two examples.