Example

$$(1+t+2t^2)\cdot(3-t) =$$

$$3+3t+6t^2-t-t^2-2t^3 =$$

$$3+2t+5t^2-2t^3$$

$$(1+t+2t^2)+(3-t) = 4+2t^2$$

Remarks

- t is a formal variable, which means that equality of the polynomials is <u>defined</u> by the comparison of the coefficients

$$\sum_i a_i t^i = \sum_i b_i t^i \text{ in } K[t] \text{ means,}$$

by definition that $a_i = b_i$ for every $i \in \mathbb{N}_0$.

- Polynomials can be evaluated by substituting a value for t.

$$f = 1 + t + 2t^2 \in \mathbb{Q}[t] \implies$$

$$f(\sqrt{2}) = 1 + \sqrt{2} + 2(\sqrt{2})^2 = 5 + \sqrt{2}$$

- K is a subset of $K[t]$, because t^0 is defined as 1.

Proposition 20 For a field K, the structure $(K[t], +, \cdot)$ is a unitary commutative ring without zero divisors. Furthermore, the set of invertible elements of $K[t]$ is exactly $K \setminus \{0\}$.

We omit the proof.

Def Let K be a field. Then a polynomial $f \in K[t]$ is called <u>irreducible</u> if $\deg(f) \geq 1$ and for all $g, h \in K[t]$ such that $f = g \cdot h$, one has $g \in K \setminus \{0\}$ or $h \in K \setminus \{0\}$.

Furthermore, f is called <u>reducible</u> if $f = g \cdot h$ for some $g, h \in K[t]$ with $\deg(g) \geq 1$ and $\deg(h) \geq 1$.

Examples

- $f = t^2 + 3t + 2 \in \mathbb{Q}[t]$ $\qquad (t+2)(t+1) = t^2 + 3t + 2$
  $\Rightarrow f$ reducible

- $t+1 \in \mathbb{Q}[t]$ is irreducible

- $t^2 - 2 \in \mathbb{Q}[t]$ irreducible (why?)

- $t^2 - 2 \in \mathbb{R}[t]$ is reducible $t^2-2 = \underbrace{(t-\sqrt{2})}_{\in \mathbb{R}[t]} \cdot \underbrace{(t+\sqrt{2})}_{\in \mathbb{R}[t]}$

Def Let K be a field. In the ring $K[t]$ of polynomials in the variable t, a polynomial $f = \sum_{i=0}^{d} a_i t^i$ with $d \in \mathbb{N}_0$ & $a_i \in K$, $a_d \neq 0$ is called <u>monic</u> if $a_d = 1$.

Theorem 21 Let K be a field and $f \in K[t] \setminus \{0\}$.
Then f admits a factorization

$$f = c \cdot g_1 \cdot \ldots \cdot g_m,$$

where $c \in K \setminus \{0\}$, $m \in \mathbb{N}_0$ and $g_1, \ldots, g_m \in K[t]$ are monic irreducible polynomials in the ring $K[t]$. Furthermore, this factorization is unique up to the reindexing the polynomials $g_1, \ldots, g_m$.

We leave the proof out.

Remark The assumption of being monic is needed for uniqueness.

Example • $f = 2t^4 - 4t^2 + 2 \in \mathbb{Q}[t]$

$$= 2 \cdot (t^4 - 2t^2 + 1)$$
$$= 2 \cdot (t^2 - 1)^2$$
$$= 2 \cdot (t^2 - 1) \cdot (t^2 - 1)$$
$$= 2 \cdot (t-1) \cdot (t+1) \cdot (t-1) \cdot (t+1)$$
$$= c \cdot g_1 \cdot g_2 \cdot g_3 \cdot g_4$$

where $c = 2$

$$\left.\begin{aligned} g_1 &= t-1 \\ g_2 &= t+1 \\ g_3 &= t-1 \\ g_4 &= t+1 \end{aligned}\right\} \text{irreducible and monic}$$

- $f = 10t^4 - 10 \in \mathbb{Q}[t]$

$$= 10 \cdot (t^4 - 1)$$
$$= 10 \cdot (t^2 - 1) \cdot (t^2 + 1)$$
$$= 10 \cdot (t - 1) \cdot (t + 1) \cdot (t^2 + 1)$$

$$f = c \cdot g_1 \cdot g_2 \cdot g_3$$

$$c = 10$$
$$\left.\begin{array}{l} g_1 = t - 1 \\ g_2 = t + 1 \\ g_3 = t^2 + 1 \end{array}\right\} \text{ irreducible in } \mathbb{Q}[t] \text{ and monic}$$

- $f = t^4 - 1 \in \mathbb{C}[t]$

$$f = (t^2 - 1) \cdot (t^2 + 1) = (t - 1) \cdot (t + 1)(t^2 + 1)$$
$$= (t - 1) \cdot (t + 1) \cdot (t + i) \cdot (t - i),$$

where $i = \sqrt{-1}$.

Proposition 22 Let K be a field, $f \in K[t]$ and $g \in K[t] \setminus \{0\}$. Then f admits a representation

$$f = q \cdot g + r$$

where $q, r \in K[t]$ and $\deg r < \deg g$.

<u>Proof</u>: Exercise.

Remark q and r can be found by an algorithm analogous to the long-division algorithm over integers.

This is a long division algorithm in $K[t]$.

Example In $\mathbb{Q}[t]$ we carry out the division of $f = t^3 - t^2 + 2t - 3$ by $g = t^2 + t + 1$

$$
\begin{array}{l}
(t^3 - t^2 + 2t - 3) : (t^2 + t + 1) = t - 2 \\
\underline{\;t^3 + t^2 + t\qquad\qquad} \\
\qquad -2t^2 + t - 3 \\
\qquad \underline{-2t^2 - 2t - 2} \\
\qquad\qquad\quad 3t - 1
\end{array}
$$

We have

$$
\underbrace{t^3 - t^2 + 2t - 3}_{f} = \underbrace{(t-2)}_{\substack{\text{The quotient} \\ \text{of the long division} \\ \text{of } f \text{ by } g}} \cdot \underbrace{(t^2+t+1)}_{g} + \underbrace{(3t-1)}_{\substack{\text{The remainder} \\ \text{of the} \\ \text{long division} \\ \text{of } f \text{ by } g.}}
$$

Example Let's do a long division of

$$f = t^2+t+1 \in \mathbb{Q}[t] \qquad \text{by } g = t-2 \,?$$

$$f(2) = 2^2+2+1 = 7$$

$$
\begin{array}{l}
(t^2+t+1) : (t-2) = t+3 \\
\underline{t^2-2t} \\
\quad 3t+1 \\
\quad \underline{3t-6} \\
\qquad 7
\end{array}
$$

$$t^2+t+1 = (t+3)\cdot(t-2)+7.$$

Remark We introduce the divisibility and the greatest common divisor in the ring $K[t]$, where K is a field, in the same way as we did it for integers.

An $f \in K[t]$ is said to be divisible by $g \in K[t] \setminus \{0\}$ if $f = g \cdot h$ for some $h \in K[t]$.

For polynomials $f_1, f_2 \in K[t]$, the greatest common divisor of f_1 and f_2 is a polynomial of the highest degree that divides both f_1 and f_2. This polynomial, which we denote by $\gcd(f_1, f_2)$, is defined uniquely up to a multiple in $K \setminus \{0\}$. In the exceptional case $f_1 = f_2 = 0$, we define $\gcd(f_1, f_2)$ as 0.

Proposition 23 Let K be a field, $a \in K$ and $f \in K[t]$. Then f is divisible by $t-a$ if and only if $f(a) = 0$.

Proof: By Proposition 22 applied to $g = t-a$, f admits a representation

$$f = q \cdot g + r \qquad (*)$$

where $q, r \in K[t]$ and $\deg r < \deg g$.

Since $\deg g = \deg(t-a) = 1$, we have $\deg r < \deg g = 1$. Since $\deg r < 1$, we have $r \in K$.

Evaluation of $(*)$ at $t = a$ gives:

$$f(a) = q(a) \cdot g(a) + \underset{\underset{K}{\cap}}{r} = q(a) \cdot (a-a) + r$$

$$= q(a) \cdot 0 + r = 0 + r = r.$$

$\Rightarrow f(a) = r.$

Hence, if $f(a)=0$, then $r=0$ and $(*)$ amounts to $f=q\cdot g$, which shows that f is divisible by $g=t-a$. Conversely, if f is divisible by $t-a$, then

$$f = q\cdot(t-a) \qquad (**)$$

for some $q\in K[t]$. So, evaluation of $(**)$ at $t=a$ gives $f(a)=q(a)\cdot(a-a)=$

$$=q(a)\cdot 0 = 0. \qquad \square$$