$K \subseteq \mathbb{R}^n$ convex body

$K^\circ = \{ x \in \mathrm{span}(K) \mid x^T y \leq 1, \ \forall y \in K \}$

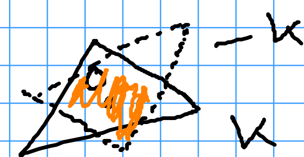
basic properties $\qquad 0 \in \mathrm{int}(K)$, $0 \in \mathrm{int}(L)$ $\qquad$ orth. proj. of K onto F

(i) $(K^\circ)^\circ = K$

(ii) $\forall$ subspace F holds $\pi_F(K)^\circ = K^\circ \cap F$

(iii) $(K \cap L)^\circ = \mathrm{conv}(K^\circ \cup L^\circ)$

barycenter / centroid of K



$$b(K) = \frac{1}{\mathrm{vol}(K)} \int_K x \, dx$$

Milman-Pajor:

$$b(K) = 0 \Rightarrow \mathrm{vol}(K) \leq 2^n \cdot \mathrm{vol}(K \cap (-K))$$

Thm.: (RR, Prop. 19, v1)

$K \subseteq \mathbb{R}^n$, $\boxed{b(K^\circ) = 0}$, F d-subspace

$$\Rightarrow \mathrm{vol}_d(\pi_F(K))^{\frac{1}{d}} \lesssim \left(\frac{n}{d}\right)^3 \mathrm{vol}_d(\pi_F(K \cap (-K)))^{\frac{1}{d}}$$

Thm: (Vritsiou '23)

$K \subseteq \mathbb{R}^n$, $\boxed{b(K) = 0}$, F d-subspace

$$\Rightarrow \text{—''—} \lesssim \left(\frac{n}{d}\right)^5 \log\left(\frac{en}{d}\right)^2 \text{—''—}$$

$$\mu_{KL}(K,\Lambda) \longleftrightarrow \mu(K,L)$$

μ_{KL} attained ?

$$= \max_{F} \left(\frac{\det(\pi_F(\Lambda))}{\mathrm{vol}((\pi_F(K))} \right)^{\frac{1}{\dim F}}$$

<ins>proof sketch of RR:</ins>

$0 \in \mathrm{int}(K)$

- look at polar

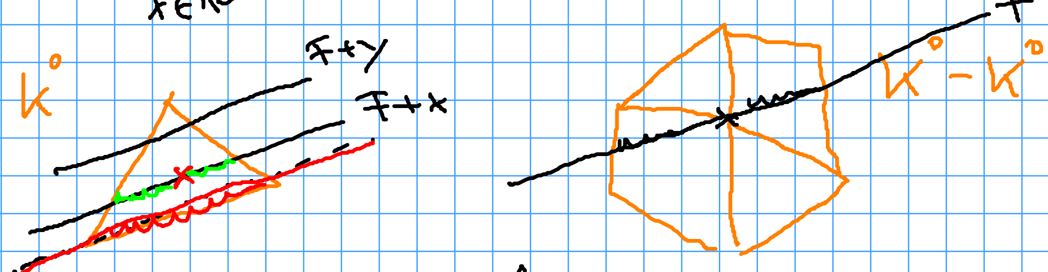
$$\mathrm{vol}_d(\pi_F(K\cap -K)^\circ)^{\frac{1}{d}} \leq \mathrm{vol}_d((K^\circ - K^\circ)\cap F)^{\frac{1}{d}}$$

$$\left[(K\cap -K)^\circ \cap F = \mathrm{conv}(K^\circ \cup (-K)^\circ)\cap F \right]$$

$$\lesssim \frac{n}{d} \cdot \max_{x\in\mathbb{R}^n} \mathrm{vol}_d(K^\circ \cap (F+x))^{\frac{1}{d}}$$

[Rudelson '98]

(*)



$$\leq \left(\frac{n}{d}\right)^2 \mathrm{vol}_d(K^\circ \cap F)^{\frac{1}{d}}$$

[Fradelizi '97]

$b(K^\circ)=0$

- back to primal

$\nu_d = \mathrm{vol}(B_2^d)$

$$\mathrm{vol}_d(\pi_F(K\cap -K))^{\frac{1}{d}} \gtrsim \frac{\nu_d^{2/d}}{\mathrm{vol}_d(\pi_F(K\cap -K)^\circ)^{\frac{1}{d}}}$$

reverse [Blaschke-Santaló]

$$\left[L=-L \Rightarrow c^n \nu_n^2 \leq \mathrm{vol}(L)\,\mathrm{vol}(L^\circ) \leq \mathrm{vol}(B_2^n)\,\mathrm{vol}(B_2^n) \right]$$

$$\overset{(*)}{\gtrsim} \left(\frac{d}{n}\right)^2 \frac{\omega_d^{2/d}}{\mathrm{vol}_d(K^\circ \cap F)^{\frac{1}{d}}}$$

$$\gtrsim \left(\frac{d}{n}\right)^3 \mathrm{vol}_d\Big(\underbrace{(K^\circ \cap F)^\circ}_{=\pi_F((K^\circ)^\circ)=\pi_F(K)}\Big)^{\frac{1}{d}} \qquad \text{[Lemma I, } b(K^\circ)=0\text{]}$$

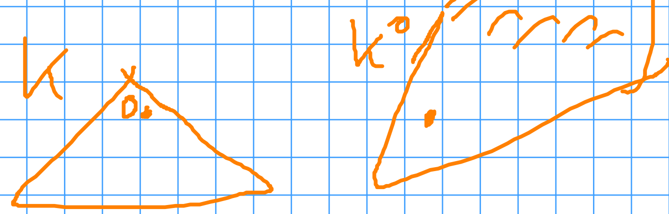
$$= \left(\frac{d}{n}\right)^3 \mathrm{vol}_d(\pi_F(K))^{\frac{1}{d}}.$$

$\blacksquare$

Lemma I: (RR, vA, Lemma 50)

$K \subseteq \mathbb{R}^n$, $b(K)=0$, F d-subspace

$$\Rightarrow \mathrm{vol}_d(K\cap F)\,\mathrm{vol}_d((K\cap F)^\circ) \le \left(\frac{n+1}{d}\right)^d \omega_d^2.$$



$d=n$: $\left(\frac{n+1}{n}\right)^n \le e$

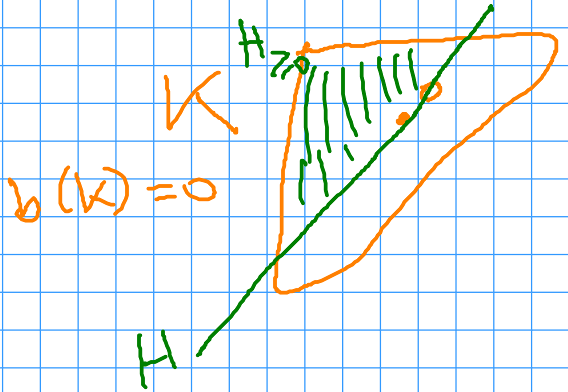
two main ingredients

(A) **Thm.:** (Myroshnychenko, Stephen, Zhang '18)

K, $b(K)=0$, F, H hyperplane through 0

$$\Rightarrow \mathrm{vol}_d(K\cap F\cap H_\geq) \ge \left(\frac{d}{n+1}\right)^d \mathrm{vol}_d(K\cap F)$$

extension of Grünbaum's inequality ($d=n$)



$\mathrm{vol}(K \cap H_{\geq}) \geq \frac{1}{e}\,\mathrm{vol}(K)$

(B) (Lemma 49, RR, vN)

$P \subseteq \mathbb{R}^n$ convex body with

(#) $\quad \mathrm{vol}(P \cap H_{\geq}) \geq \delta \cdot \mathrm{vol}(P)$ for some $0 < \delta \leq \frac{1}{2}$

and for all hyperplanes H through 0.

$$\Rightarrow \mathrm{vol}(P)\,\mathrm{vol}(P^{\circ}) \leq \frac{1}{\delta}\,\nu_n^2 .$$

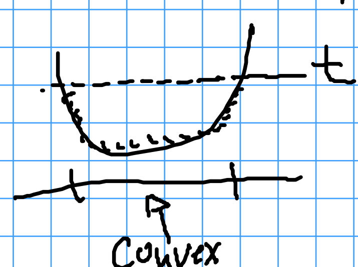
proof idea for (B):

function $x \mapsto \mathrm{vol}((P-x)^{\circ})$ for $x \in P$

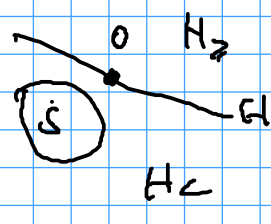
Meyer-Werner '98 $\Rightarrow$ convex fct.

$t \geq 1 \rightsquigarrow S(P,t) := \{x \in P \mid \mathrm{vol}(P)\,\mathrm{vol}((P-x)^{\circ}) \leq t \cdot \nu_n^2\}$

sublevel set $\Rightarrow$ is convex



goal: $0 \in S(P, \frac{1}{\delta})$



assume $0 \notin S(P, \frac{1}{\delta})$

$\Rightarrow \exists$ separating hyperplane H with $S(P, \frac{1}{\delta}) \subseteq H_{<}$

Meyer-Werner '98 using (#)

$\Rightarrow \exists x^* \in H \cap \mathrm{int}(P)$ such that

$$x^* \in S\left(P, \underbrace{\frac{1}{4\delta(1-\delta)}}_{\le \frac{1}{\delta} \text{ because } \delta \le \frac{1}{2}}\right) \subseteq S\left(P, \frac{1}{\delta}\right)$$

$$\Downarrow \quad x^* \in P \cap H, \; S\left(P, \frac{1}{\delta}\right) \subseteq H_< \qquad \blacksquare$$