TODAY: SECTION 3 IN "REGEV + STEPHENS-DAVIDOWITZ"
(REVERSE MINKOWSKI THEOREM)

LAST TIME: $\mathcal{V}(\Lambda) = \{x : \mathrm{dist}(x,\Lambda) = \|x\|\}$
$= \{x : \langle x, v\rangle \le \frac{\|v\|^2}{2}, \ \forall v \in \Lambda \setminus 0\}$

- ▷ $\mathcal{V}(\Lambda)$ IS A FUNDAMENTAL BODY OF Λ.
- ▷ $A\mathcal{V}(\Lambda)$ IS — " — $A\Lambda$.

THM: $f \in \mathbb{R}_{\ge 0} \to \mathbb{R}$ CONT' DIFFERENTIABLE, $\Lambda \subseteq \mathbb{R}^n$ LATTICE

$$g: \quad GL_n(\mathbb{R}) \longrightarrow \mathbb{R} \qquad A \longmapsto \frac{1}{\det A} \int_{\mathcal{V}(A\Lambda)} f(\|x\|^2)\,dx$$
$$h: \qquad\qquad\qquad\qquad\qquad A \longmapsto \frac{1}{\det A} \int_{A\mathcal{V}(\Lambda)} f(\|x\|^2)\,dx$$

$\Rightarrow h$ & g ARE DIFFERENTIABLE AT $A=I$ AND

$$\nabla h(I) = \nabla g(I) = 2\int_{\mathcal{V}(\Lambda)} f'(\|x\|^2)\, x x^T \,dx$$

OBSERVE: FOR h THIS FOLLOWS FROM THE CHAIN RULE

LEMMA (CLAIM 3.3): $\forall \Lambda \subseteq \mathbb{R}^n \ \exists \nu > 1 \ \forall M \in \mathbb{R}^{n,n}$. $\|M\| < \nu^{-1}$

(C1) $(1 - \nu\|M\|)\mathcal{V}(\Lambda) \subseteq A\mathcal{V}(\Lambda) \subseteq (1+\nu\|M\|)\mathcal{V}(\Lambda)$

$A = I + M$

(C2) — " — $\subseteq \mathcal{V}(A\Lambda) \subseteq$ — " — $\mathcal{V}(\Lambda)$

PROOF: WRITE $B^n \triangleq$ EUCLIDEAN UNIT BALL

$\mu = \mu(B^n ; \Lambda) \leftarrow$ CIRCUMRADIUS OF $\mathcal{V}(\Lambda)$

$\lambda_1 = \underbrace{\lambda_1(B^n ; \Lambda)}_{\text{SHORTEST VECTOR OF } \Lambda \setminus 0.} \leftarrow \frac{1}{2}$INRADIUS OF $\mathcal{V}(\Lambda)$

CHOOSE $\nu = \overset{c>1}{c} \cdot \mu/\lambda_1 > 1$

$\underline{L2}$: $x \in \mathcal{V}(\Lambda)$, $y \in \Lambda \setminus 0$

$$\frac{\langle Ay, x\rangle}{\|Ay\|^2} \overset{A = I + M}{\le} \frac{\langle y, x\rangle + \langle My, x\rangle}{(1-\|M\|)^2\,\|y\|^2}$$

$\|A\| \ge |\,\|I\| - \|M\|\,| = 1 - \|M\|$

$$\le \frac{\langle y, x\rangle}{(1-\|M\|)^2\|y\|^2} + \frac{\|M\|\cdot\|x\|\cdot\cancel{\|y\|}}{(1-\|M\|)^2\|y\|^{2}}$$

$$\le \quad \text{— " —} \quad + \frac{\|M\|\cdot\|x\|}{(1-\|M\|)^2\lambda_1}$$

$$\underset{x\in\mathcal{V}(\Lambda)}{\le} \frac{1}{2}\,\frac{1}{(1-\|M\|)^2} + \|M\| \cdot \frac{\mu}{\lambda_1\,(1-\|M\|)^2}$$

$$\le \frac{1}{2}\left(1 + c\,\frac{\mu}{\lambda}\,\|M\|\right)$$

$$\Rightarrow \quad \langle Ay, x\rangle \le \frac{\|Ay\|^2}{2}\left(1 + c\,\frac{\mu}{\lambda}\,\|M\|\right) \qquad \square$$

PROOF OF THE THM: SUFFICES TO SHOW

INDEPENDENT OF M

(*) $$|\det(I+M)|\cdot|h(I+M) - g(I+M)| \le C^* \cdot \|M\|^2, \quad \forall \|M\| \le \frac{1}{nC}$$

$$\Rightarrow \quad \frac{|h(I+M) - g(I+M)|}{\|M\|} \longrightarrow 0$$

HAVE
$$|\det A|\, g(A) = \int_{\mathcal{V}(A\Lambda)} f(\|x\|^2)\,dx \overset{\text{DEF' } \mathcal{V}(A\Lambda)}{=} \int_{\mathcal{V}(A\Lambda)} f(\mathrm{dist}(x, A\Lambda)^2)\,dx$$

$$= \int_{A\mathcal{V}(\Lambda)} f(\mathrm{dist}(x, A\Lambda)^2)\,dx$$

$x \mapsto \mathrm{dist}(x, A\Lambda)$ IS $A\Lambda$-PERIODIC & $A\mathcal{V}(\Lambda)$, $\mathcal{V}(A\Lambda)$ ARE 'FUND' BODIES.

WRITE $\hat{\mathcal{V}} = A\mathcal{V}(\Lambda) \setminus \mathcal{V}(A\Lambda)$

$$\text{LHS OF (*)} = \int_{A\mathcal{V}(\Lambda)} \overbrace{f(\|x\|^2)}^{h} - \overbrace{f(\mathrm{dist}(x, A\Lambda)^2)}^{g}\,dx$$

$$= \int_{\hat{\mathcal{V}}} \quad \text{— " —}$$

$$\le \mathrm{vol}\,\hat{\mathcal{V}} \cdot \max_{x \in \hat{\mathcal{V}}} f(\|x\|^2) - f(\mathrm{dist}(x, A\Lambda)^2)$$

REMAINS TO SHOW: $\mathrm{vol}\,\hat{\vartheta} \leq C^* \|M\|$ (A)

$$\max_{x \in \hat{\vartheta}} f(\|x\|^2) - f(\mathrm{dist}(x, A\Lambda)^2) \leq C^* \|M\| \qquad \text{(B)}$$

(A): $\hat{\vartheta} = A\vartheta(\Lambda) \setminus \vartheta(A\Lambda) \subseteq \left(\underline{(1+\nu\|M\|)}\, \vartheta(\Lambda) \right) \setminus \left(\underline{(1-\nu\|M\|)}\, \vartheta(\Lambda) \right)$

$$\mathrm{vol}\,\hat{\vartheta} = \mathrm{vol}(\vartheta(\Lambda)) \left((1+\nu\|M\|)^n - (1-\nu\|M\|)^n \right) \qquad \square$$

(B) $x \in \hat{\vartheta} \subseteq A\vartheta(\Lambda) \quad \Rightarrow \quad \|x\| \leq \|A\| \mu \leq 2 \cdot \mu$

$\subseteq A \cdot \mu \cdot B^n \quad \Rightarrow \quad \mathrm{dist}(x, A\Lambda) \leq 2\mu$

$$\Rightarrow \left| f(\|x\|^2) - f(\mathrm{dist}(x, A\Lambda)^2) \right| \leq \underline{\left| \|x\| - \mathrm{dist}(x, A\Lambda) \right|} \cdot \max_{0 \leq r \leq 2\mu(\Lambda)} \left| \tfrac{\partial}{\partial r} f(r^2) \right|$$

$$\leq \quad \text{— " —} \quad \underbrace{2r|f'(r)|}_{<\infty}$$

LET $x' = (1-\nu\|M\|)^2 x \overset{\text{LEMMA}}{\in} (1-\nu\|M\|)(1+\nu\|M\|) \cdot \vartheta(A\Lambda)$

$$\left| \|x\| - \mathrm{dist}(x, A\Lambda) \right| = \left| \|x\| + \|x'\| - \|x'\| - d(x, A\Lambda) + d(x', A\Lambda) - d(x', A\Lambda) \right|$$

$$\leq \underbrace{\left| \|x\| - \|x'\| \right|}_{\leq \|x - x'\|} + \underbrace{\left| \|x'\| - \mathrm{dist}(x', A\Lambda) \right|}_{=0} + \left| \mathrm{dist}(x, A\Lambda) - \mathrm{dist}(x', A\Lambda) \right|$$

$\mathrm{dist}(x, A\Lambda) \leq \|x - x'\| + \mathrm{dist}(x', A\Lambda)$

$\Rightarrow \quad \leq \|x - x'\|$

$\leq 2\|x - x'\|$

$\leq 4 \cdot \nu \|x\| \cdot \|M\|$

$\leq 8 \cdot \nu \cdot \mu \|M\|$ $\blacksquare$