THE FLATNESS THEOREM FOR 0-SYMMETRIC BODIES (PART II)

Notation: $\mathcal{K}_0^n = \{ K \subseteq \mathbb{R}^n : K \text{ origin-symmetric cvx' body} \}$

$A \subseteq \mathbb{R}^n$, $\rho(A) := \sum_{x \in A} e^{-\pi |x|_2^2}$ $\qquad$ $[\rho(\Lambda + a) \leq \rho(\Lambda)]$

$$\alpha, \beta : \mathcal{K}_0^n \to \mathbb{R}, \qquad \alpha(K) = \sup_{\Lambda \subseteq \mathbb{R}^n \text{ lattice}} \frac{\rho(\Lambda \setminus K)}{\rho(\Lambda)}$$

$$\beta(K) = \sup_{\substack{\Lambda \text{ lattice} \\ a \in \mathbb{R}^n}} \frac{\rho((\Lambda + a) \setminus K)}{\rho(\Lambda)}$$

Key lemma from last week (BI, Lemma 1.5)

$U, V \in \mathcal{K}_0^n$, $2\alpha(U) + \beta(V) \leq 1 \Rightarrow w(U; \Lambda) \cdot \mu(V; \Lambda^*) \leq 2, \ \forall \Lambda$

In particular $U = tK$, $V = sK^*$ and $\beta(tK), \beta(sK^*) \leq \frac{1}{3}$

$$\Rightarrow \quad w(K; \Lambda) \cdot \mu(K; \Lambda) \leq 2st$$

Observation: Fix Λ and a $\qquad \frac{\rho(\Lambda + a \setminus K)}{\rho(\Lambda)} = \frac{1}{\rho(\Lambda)} \sum_{x \in (\Lambda + a) \setminus K} e^{-\pi |x|_2^2}$

$$\leq \frac{1}{\rho(\Lambda)} \sum_{x \in \Lambda + a} \|x\|_K e^{-\pi |x|_2^2} =: \int_{\mathbb{R}^n} \|x\|_K \, \sigma_{\Lambda, a}(dx), \text{ where } \frac{\rho(A \cap (\Lambda + a))}{\rho(\Lambda)}$$

γ_n

$x \notin K \Leftrightarrow \|x\|_K > 1$

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K e^{-\pi |x|^2} \, dx \qquad \beta([-1,1])$$

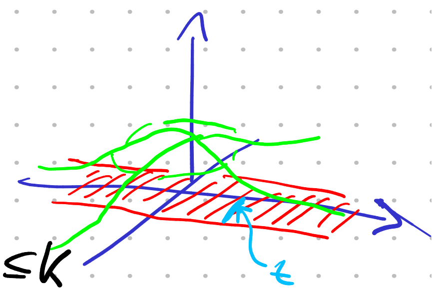
Question: How can we bound $\beta(K)$?

Lemma (BI, Lemma 2.4): Let $x^* \in (\mathbb{R}^n)^*$, $\Lambda \subseteq \mathbb{R}^n$ lattice, $a \in \mathbb{R}^n$, $t \geq 0$

$$\rho(x \in \Lambda + a : |x^*(x)| \geq t \|x^*\|_2)$$

$$< 2 e^{-\pi t^2} \rho(\Lambda)$$

$\longrightarrow 0$ 'fast' as $t \to \infty$



Suppose we have $x_1^*, \dots, x_M^* \in (\mathbb{R}^n)^*$

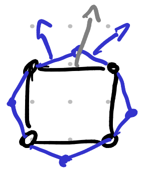
with $\quad P := \{ x : |x_j^*(x)| \leq 1, \ j = 1, \dots, M \} \subseteq K$

$$\rho((\Lambda+a)\setminus K) \le \rho((\Lambda+a)\setminus P) = \rho\Big((\Lambda+a)\setminus \bigcap_{j=1}^{M} \{x : |x_j^*(x)| \le 1\}\Big)$$

$$= \rho\Big(\bigcup_{j=1}^{M} \{x\in \Lambda+a : |x_j^*(x)| \geq 1\}\Big) \le \sum_{j=1}^{M} \rho\Big(x\in\Lambda+a : |x_j^*(x)| \le \frac{|x_j^*|_2}{\underbrace{|x_j^*|_2}}\Big) \quad \epsilon = \frac{1}{|x|}$$

$$\underset{\text{Lemma 2.4}}{\le} 2\rho(\Lambda)\sum_{j=1}^{M} e^{-\pi |x_j^*|_2^{-2}} \quad \Rightarrow \quad \beta(K) \le 2\sum_{j=1}^{M} e^{-\pi|x_j^*|_2^{-2}} \qquad (*)$$

<u>Example</u> K



$\rightsquigarrow$ Better take the normals of the cube than of K...

Talagrand's Majorizing Measure Theorem
(according to Pisier (mostly))

▷ $c_0 = \{(a_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} a_n = 0\}$ is equipped $\|a\|_\infty = \sup_k |a_k|$

▷ $\Delta : c_0 \to c_0$, $(a_k) \mapsto (a_k \cdot (1+\log k)^{-1/2})_k$

▷ Let $T : \mathbb{R}_2^n \to \mathbb{R}_K^n$ $\qquad \|x\| = \|x\|_K$

▷ We say that a pair of lin' maps (A,B) is T-admissible if

i) $\exists S_1, S_2 \le c_0$ s.th. $\Delta(S_1) \le S_2$, $B : \mathbb{R}_2^n \to S_1$, $A : S_2 \to \mathbb{R}_K^n$

ii) $T = A\Delta B$

▷ $\tilde{\ell}(T) = \inf\{\|A\|\cdot\|B\| : (A,B)\ T\text{-admissible}\}$

$$\begin{tikzcd} \mathbb{R}_2^n \arrow[rr, "T"] \arrow[d, "B"'] & & \mathbb{R}_K^n \\ S_1 \arrow[rr, "\Delta"] & & S_2 \arrow[u, "A"'] \end{tikzcd}$$

<u>THM</u> (Majorizing Measure): $\ell(T) \le \tilde{\ell}(T) \le c\cdot\ell(T)$

$\ell(T) = \int_{\mathbb{R}^n} \|Tx\|_K \, d\gamma_n(x)$

<u>Corollary</u>: Let $K \in \mathcal{K}_0^n$. There's a sequence $(x_k^*) \subseteq (\mathbb{R}_2^n)^*$ with

$\ell(K) = \int \|x\|_K \, d\gamma_n(x)$

a) $|x_k^*|_2 \le c\cdot \ell(K)\,(1+\log k)^{-1/2}$

b) $\{x : |x_k^*(x)| \le 1,\ k=1,\dots\} \subseteq K$

<u>Proof</u>: Let $T : \mathbb{R}_2^n \to \mathbb{R}_K^n$ be the identity

$\Rightarrow \exists\ B : \mathbb{R}_2^n \to c_0$, $A : \mathrm{im}(\Delta B) \to \mathbb{R}_K^n$

s.th. $x = A\Delta Bx$, $\forall x \in \mathbb{R}^n$

and $\|A\|\cdot\|B\| \le c\cdot\ell(K)$

$\|A\| = \sup_{\|a\|_\infty = 1} \|Aa\|_K$

Let $x_k^*(v) = \|A\| \cdot e_k^*(\Delta B v)$, $e_k^* : c_0 \to \mathbb{R}$, $a \mapsto a_k$

Now : a) $\|x_k^*\| \leq \|A\| \cdot \|e_k^* \circ \Delta\| \cdot \|B\|$

$\leq c \cdot \ell(K) \cdot \underline{\|e_k^* \circ \Delta\|} = c\,\ell(K) \cdot \underline{(1+\log k)^{-\frac{1}{2}}}$ ✓

b) If $|x_k^*(v)| \leq 1$, $\forall v$, then

$\underline{\|v\|_K} = \|A \Delta B v\|_K \leq \|A\| \cdot \|\Delta B v\|_\infty = \|A\| \cdot \sup_k \underbrace{|e_k^*(\Delta B v)|}_{x_k^*(v)}$

$\leq \underline{1} \Rightarrow v \in K$ ∎

Applying the corollary to (*)

$$\beta(K) \leq 2 \sum_{k=1}^{\infty} (ke)^{-\pi\left(\frac{c}{\ell(K)}\right)^2} \searrow 0 \quad \text{as } \ell(K) \searrow 0$$

$\Rightarrow$ Lemma 2 (BI) : $\forall \varepsilon > 0 \ \exists \delta \ \forall n \ \forall K \in \mathcal{K}_0^n$. $\ell(K) < \delta \Rightarrow \beta(K) < \varepsilon$

$T : \mathbb{R}_2^n \to \mathbb{R}_2^n$

$\Delta(a) = \left(a_k (1 + \log k)^{\frac{1}{2}} \right)$

$S_k = \operatorname{span}\{e_{k+1}, \dots, e_{k+n}\}$

$B : \mathbb{R}_2^n \to c_0$, $e_i \mapsto e_{k+i}$ $\quad \|B\| = 1$

$\Delta : S_k \to S_k$, $e_{k+i} \mapsto (\log(k+i) + 1)^{\frac{1}{2}} e_{k+i}$

$A : S_k \to \mathbb{R}_2^n$, $e_{k+i} \mapsto \frac{e_i}{(\log(k+i)+1)^{\frac{1}{2}}}$ $\quad [\ 1 \ \ 2 \ \ 3 \ \dots [k \ , k+1 \dots]$

$\|A\| \leq \frac{1}{(\log(k+1)+1)^{\frac{1}{2}}}$

$\|A\| \cdot \|B\| \leq \frac{1}{(1+\log k)^{\frac{1}{2}}} \xrightarrow{k \to \infty} 0$

$\Rightarrow \tilde{\ell}(T) = 0$