MM* estimate
aka ℓℓ* estimate

Based on Chapter 6 of [AGM]
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BASIC OBJECTS

Euclidean structure

$n \in \mathbb{Z}_{\geq 1}$, dimension

$\ell_2^n = (\mathbb{R}^n, \|\cdot\|_2)$, $\|x\|_2 = \sqrt{\langle x, x \rangle}$ standard Euclidean norm

$\langle x, y \rangle = x^T y$ standard scalar product

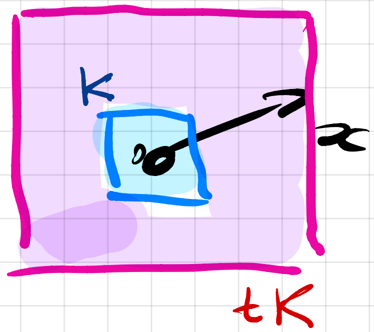
Convex bodies

$K \subseteq \mathbb{R}^n$ is called a convex body if K is compact, convex and $\underbrace{\mathrm{int}(K)}_{\text{interior}} \neq \emptyset$. $\mathcal{K}^n = \{\text{convex bodies in } \mathbb{R}^n\}$

K is called centrally symmetric if $K - p = p - K$ for some $p \in \mathbb{R}^n$ and 0-symmetric if $K = -K$.

Normed spaces & convex bodies

$0 \in \operatorname{int}(K)$, $K \in \mathcal{K}^n \Rightarrow \|x\|_K := \min\{t \geq 0 : x \in tK\}$



is called the ~~Minkowski~~ functional

$X = (\mathbb{R}^n, \|\cdot\|)$ is a normed space iff

$\|\cdot\| = \|\cdot\|_K$ for some $K \in \mathcal{K}^n$ with $K = -K$.

In this case, $K = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

is the unit ball.

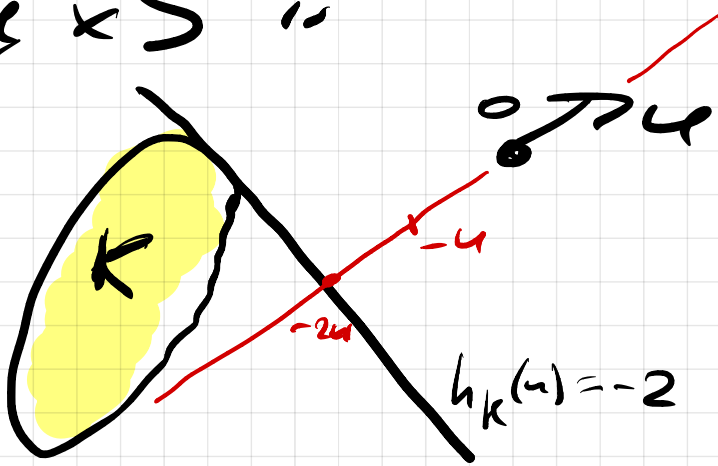
Duality for normed spaces & convex bodies

If $X = (\mathbb{R}^n, \|\cdot\|)$ is a normed space, then its dual $X^* = (\mathbb{R}^n, \|\cdot\|_*)$ is the space endowed with the so-called dual norm

$$\|y\|_* := \max_{\|x\| \le 1} \langle y, x \rangle$$

$$K \in \mathcal{K}^n \Rightarrow h_K(u) := \max_{x \in K} \langle u, x \rangle$$ is

called the support function



If $0 \in \operatorname{int}(K)$, then

$$h_K(u) = \|u\|_{K^\circ}$$, where

$K^\circ = \{ y \in \mathbb{R}^n : \langle y, x \rangle \le 1$ for all $x \in K \}$

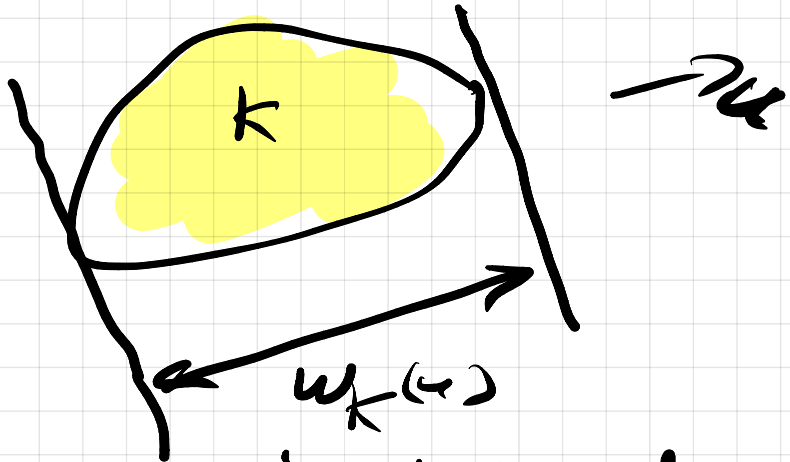
If $K = -K$, then $\|\cdot\|_{K^\circ}$ is the dual of $\|\cdot\|_K$.

MAIN RESULTS

Flatness Theorem

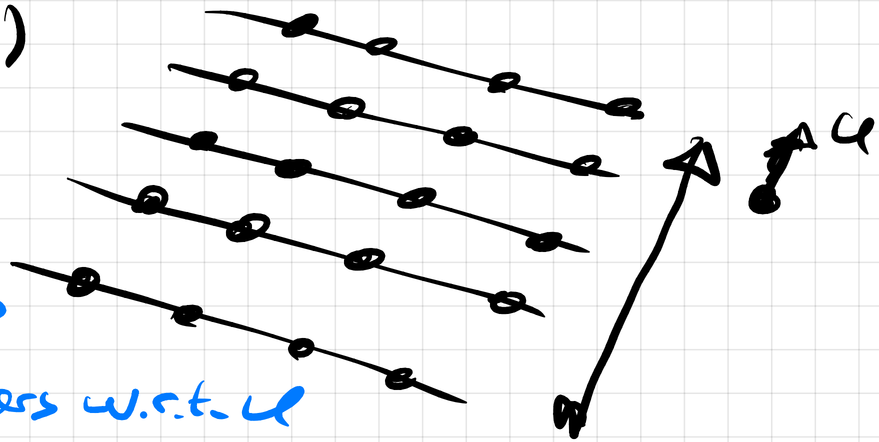
$K \in \mathcal{K}^n$ is called *hollow* if $\mathrm{int}(K) \cap \mathbb{Z}^n = \emptyset$

$$w_K(u) := h_K(u) + h_K(-u) = \max_{x \in K} \langle u, x \rangle - \min_{x \in K} \langle u, x \rangle$$



$\forall \ \|u\|_2 = 1$

$$\mathrm{lwidth}(K) := \min_{u \in \mathbb{Z}^n \setminus \{0\}} w_K(u)$$



Lattice points decomposed into layers w.r.t. u

Theorem (Khinchine 1949)

$$\mathrm{Flt}(n) := \sup_{\substack{K \in \mathcal{K}^n \\ K \text{ hollow}}} \mathrm{lwidth}(K) < \infty$$

1988 Kannan & Lovász: $\mathrm{Flt}(n) = O(n^2)$

1996 Banaszczyk: $\mathrm{Flt}(n) = O(n^{4/3} (\log n)^a)$

where $a > 0$ is fixed number

$$\mathrm{Flt}_{cs}(n) := \sup_{\substack{K \in \mathcal{K}^n \\ K \text{ hollow} \\ \text{and centrally} \\ \text{symmetric}}} \mathrm{lwidth}(K) = O(n \log n).$$

Recent breakthrough!

$\boxed{\text{2023 Rothvoß \& Reis: } \mathrm{Flt}(n) = O(n (\log n)^a) \text{ with some fixed } a > 0.}$

Current state: $\quad cn \le \Gamma\ell t(n) \le Cn(\log n)^a$

for some constants $c, C > 0$.

We don't know the exact asymptotic yet:

$\Theta(n)$, $\Theta(n \log\log n)$, $\Theta(n\sqrt{\log n})$,

$\Theta(n \log n)$, $\Theta(n(\log n)^2)$?

FUNCTIONAL ANALYTIC APPROACH

Absolute constant is a positive constant that does not depend on anything

100 is an absolute constant

2^n is not an absolute constant.

$\text{Stuff} \lesssim \text{Other\_Stuff}$ means

$\text{Stuff} \leq C \cdot \text{Other\_Stuff}$,

where C is an absolute constant.

Roadmap to the proof $\mathrm{Flt}_{cs}(n) = O(n \log n)$

If $K \in \mathcal{K}^n$ is a centrally symmetric hollow convex body and $\tilde{K}$ is an 0-symmetric convex body satisfying $\tilde{K} = A(K)$ for some affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, then

$$\mathrm{lwidth}(K) \lesssim n \, M(\tilde{K}) M^*(\tilde{K}).$$

MM^*-product vs. $\ell\ell^*$-product

If $K \in \mathcal{K}^n$ is 0-symmetric, then

$$M(K) M^*(K) \lesssim \frac{1}{n} \ell(K) \ell^*(K).$$

$\ell\ell^*$-product vs. $\mathbb{K}$-convexity constant

If $K \in \mathcal{K}^n$ is 0-symmetric and $X = (\mathbb{R}^n, \|\cdot\|_K)$, then there exists a GL_n-copy $\tilde{K}$ of K satisfying

$$\frac{1}{n}\,\ell(\tilde{K})\,\ell^*(\tilde{K}) \le \underbrace{\mathbb{K}(X)}_{\uparrow}$$

freaking technical constant.

Figiel & Tomczak's 6.4.5

$$\mathbb{K}(X) \lesssim \log d_{BM}(X, \ell_2^n) + 1$$

Pisier's Theorem 6.2.4.

$$d_{BM}(X, \ell_2^n) \leq \sqrt{n}$$

John's Theorem 2.1.3

That's the roadmap!

FUNCTIONALS INVOLVED IN THE ESTIMATES

MM^*-product

$S^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ Euclidean unit sphere

σ : invariant measure on S^{n-1} with $\sigma(S^{n-1}) = 1$.

$$M^*(K) := \int_{S^{n-1}} h_K(u)\, \sigma(du) = \frac{1}{2} \text{ mean Euclidean width of } K$$

$$M(K) := \int_{S^{n-1}} \|x\|_K\, \sigma(du) \text{ , when } 0 \in \operatorname{int}(K)$$

$$M^*(K) = M(K^0)$$

when $0 \in \operatorname{int}(K)$.

$M(K)M^*(K)$ is called the MM^*-product ll^k - product

$$\frac{1}{(2\pi)^{n/2}} e^{-\frac{\langle x, x \rangle}{2}} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}$$

is the density function of the standard Gaussian random vector in $\mathbb{R}^n$

The respective measure is denoted by γ_n, i.e.

$$\gamma_n(dx) := \frac{1}{(2\pi)^{n/2}} e^{-\frac{\langle x, x \rangle}{2}} dx = \gamma_1(dx_1) \cdots \gamma_n(dx_n)$$

$$\gamma_1(dt) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\ell^*(K) := \int_{\mathbb{R}^n} h_K(u)\, \gamma_n(du)$$

$$\ell(K) := \int_{\mathbb{R}^n} \|x\|_K\, \gamma_n(du)$$, where $0 \in \mathrm{int}(K)$

$\ell(K)\ell^*(K)$ is called the $\ell\ell^*$-product of K

$$\boxed{M(K)M^*(K) \lesssim \frac{1}{n}\,\ell(K)\ell^*(K)}$$

Easy part of the proof

K-convexity constant

$L_2(\mathbb{R}^n, \gamma_n)$ that's the space of all measurable function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $\|f\|_{L_2(\mathbb{R}^n, \gamma_n)} := \sqrt{\int_{\mathbb{R}^n} f(x)^2 \gamma_n(dx)} < \infty$

This is a Hilbert space with the scalar product

$$\langle f, g \rangle_{L_2(\mathbb{R}^n, \gamma_n)} = \int_{\mathbb{R}^n} f(x) g(x) \gamma_n(dx)$$

$(x_1, \ldots, x_n) \longmapsto c_1 x_1 + \ldots + c_n x_n$

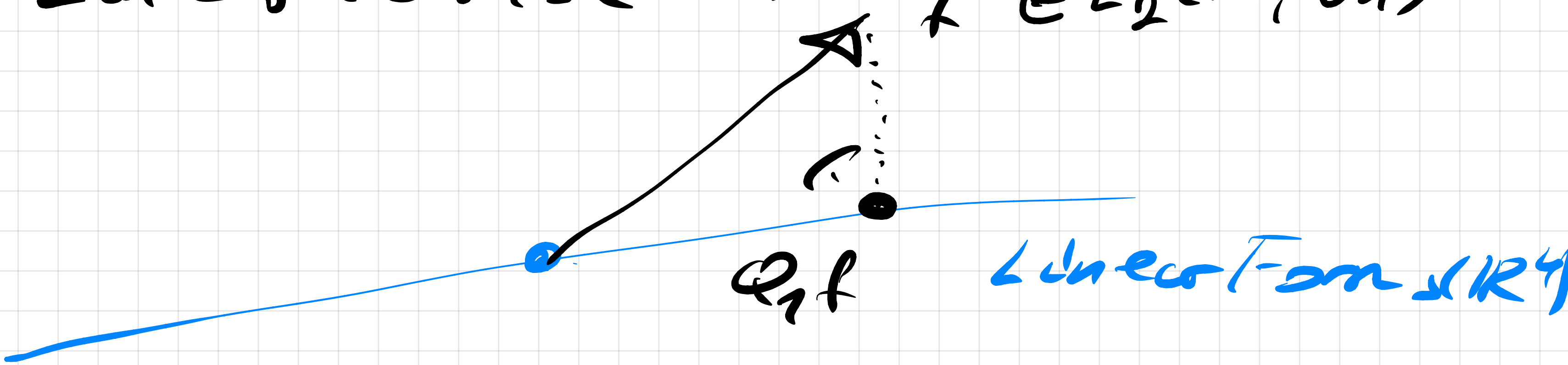
is a linear form $(c_1, \ldots, c_n \in \mathbb{R})$

$\text{LinearForms}(\mathbb{R}^n) \subset L_2(\mathbb{R}^n, \gamma_n)$

Let $Q_1 : L_2(\mathbb{R}^n, \gamma_n) \longrightarrow L_2(\mathbb{R}^n, \gamma_n)$

be the orthogonal projection

onto $\text{LinearForms}(\mathbb{R}^n)$



Given a normed space $X = (\mathbb{R}^n, \|\cdot\|)$ consider the Banach space $L_2(\mathbb{R}^n, \gamma_n; X)$ of functions

$$f : \mathbb{R}^n \to X, \quad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

satisfying

$$\|f\|_{L_2(\mathbb{R}^n, \gamma_n; X)} := \sqrt{\int_{\mathbb{R}^n} \|f(x)\|^2 \gamma_n(dx)} < \infty$$

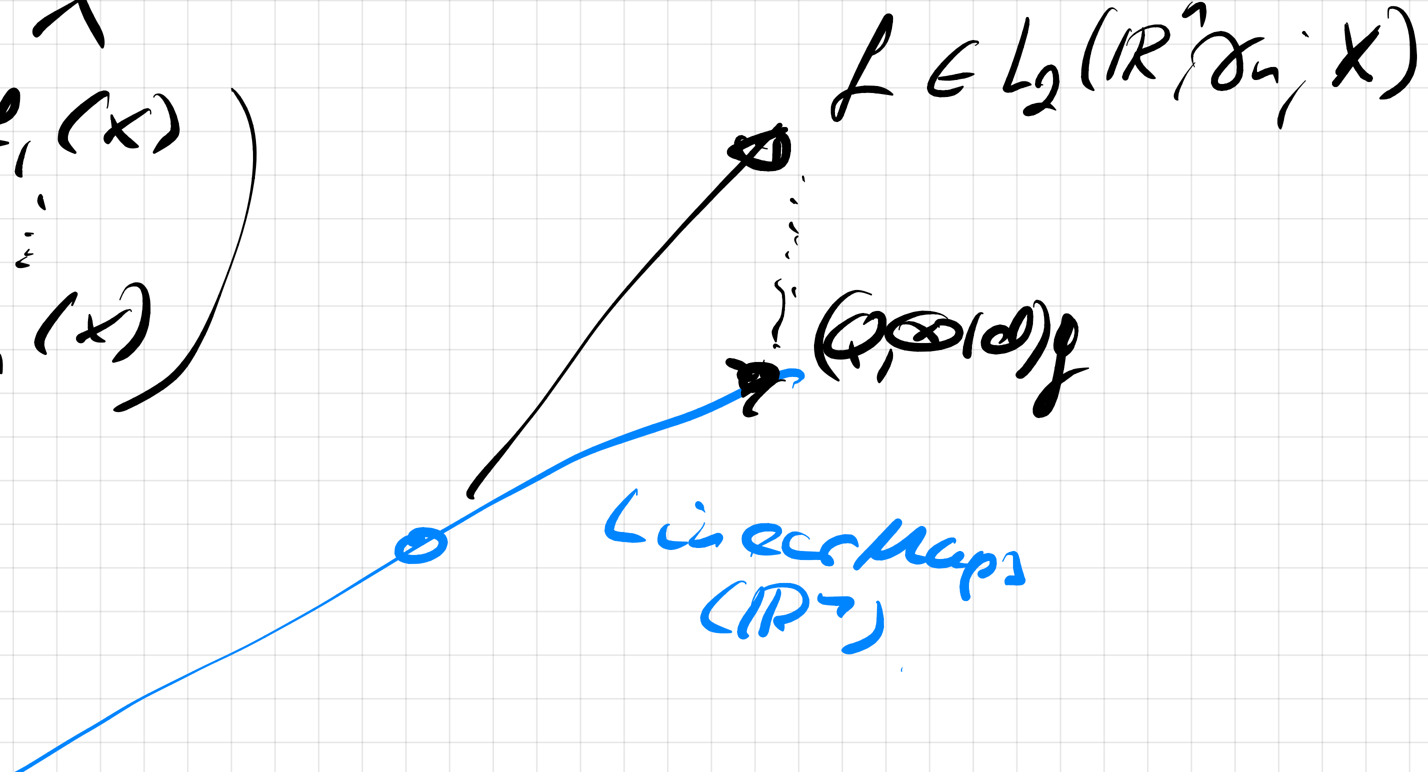
Let $Q_1 \otimes Id_X : L_2(\mathbb{R}^n, \gamma_n; X) \longrightarrow L_2(\mathbb{R}^n, \gamma_n; X)$

be the linear operator given by

$$(Q_1 \otimes Id_X)(f) = \begin{pmatrix} Q_1 f_1 \\ Q_1 f_2 \\ \vdots \\ Q_1 f_n \end{pmatrix}$$

$$f : \mathbb{R}^n \longrightarrow X$$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$



$\mathbb{K}(X) :=$ operator norm of $Q_1 \otimes \mathrm{Id}_X$.

$\Updownarrow$

$$\mathbb{K}(X) = \sup\left\{ \|(Q_1 \times \mathrm{Id}_X)(f)\|_{L_2(\mathbb{R}^n, \gamma_n; X)} : \|f\|_{L_2(\mathbb{R}^n, \gamma_n; X)} \le 1 \right\}$$

$\Updownarrow$

$$\mathbb{K}(X) = \sup\left\{ \sqrt{\int_{\mathbb{R}^n} \left\| \begin{pmatrix} (Q_1 f_1)(x) \\ \vdots \\ (Q_1 f_n)(x) \end{pmatrix} \right\|^2 \gamma_n(dx)} : \sqrt{\int_{\mathbb{R}^n} \left\| \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} \right\| \gamma_n(dx)} \le 1 \right\}$$

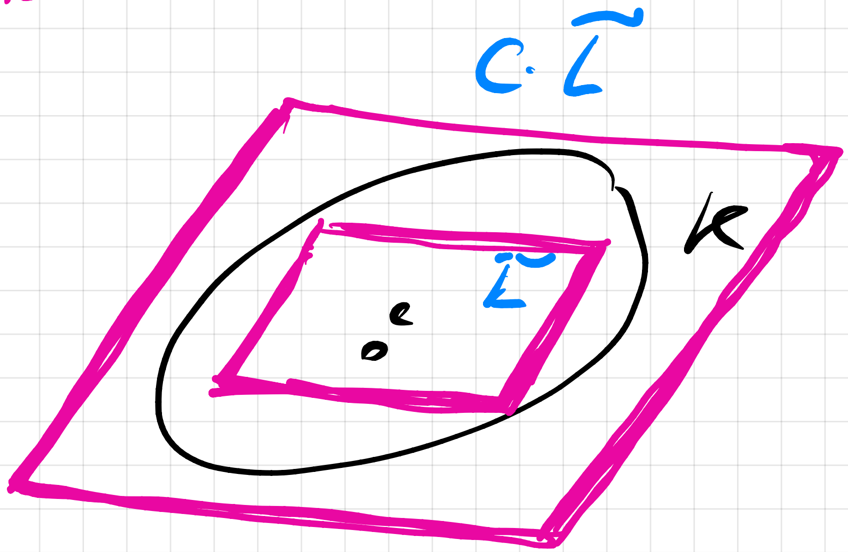
We've never spelled out Q_1. Let's do this. One use the theory of Hermite polynomials (it's an orthogonal basis of polynomials for $L_2(\mathbb{R}^n, \gamma_n)$; kind of Fourier basis).

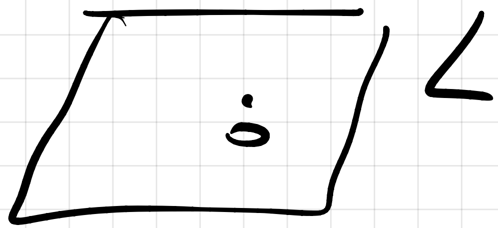
$$(Q_1 f)(x) = c_1 x_1 + \cdots + c_n x_n$$

with $c_i = \int_{\mathbb{R}^n} y_i f(y) \gamma_n(dy)$

$y = (y_1, \ldots, y_n)$.

Banach - Mazur





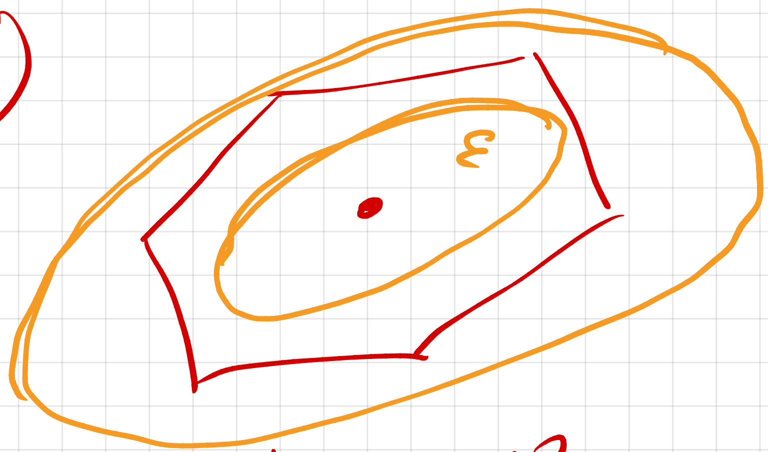
$d_{BM}(K, L)$

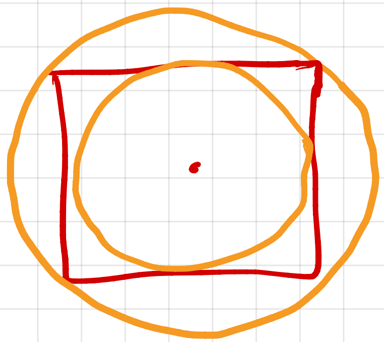
$d_{BM}(X, Y) = d_{BM}(\text{ball of } X, \text{ ball of } Y)$

normed spaces

Extremal case when $d(X, \ell_2^n) = \sqrt{n}$

$d(X, \ell_2^n)$





$K = \{x \in \mathbb{R}^n : \|x\| \le 1\}$

bijection

$d(X, \ell_2) = \min\{\lambda \ge 1 : \mathcal{E} \subseteq K \subseteq \lambda \mathcal{E}$ for some ellipsoid $\mathcal{E}\}$

MM^* vs. $\ell\ell^*$

Reminder

$\sigma(S^{n-1}) = 1$

$$M(K) = \int_{S^{n-1}} \|x\|_K \, \sigma(dx)$$

$$M^*(K) = M(K^o) = \int_{S^{n-1}} h_K(u) \, \sigma(dx)$$

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(dx)$$

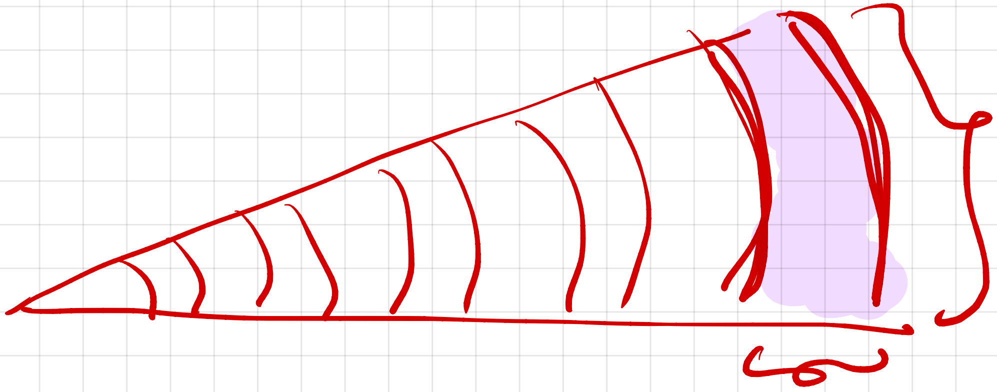
$$\gamma_n(dx) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\langle x,x \rangle}{2}} dx$$

$$\ell^*(K) = \int_{\mathbb{R}^n} h_K(x) \, \gamma_n(dx)$$

Polar coordinates

$$x = \rho \cdot u, \qquad \rho \geq 0, \quad u \in S^{n-1}$$

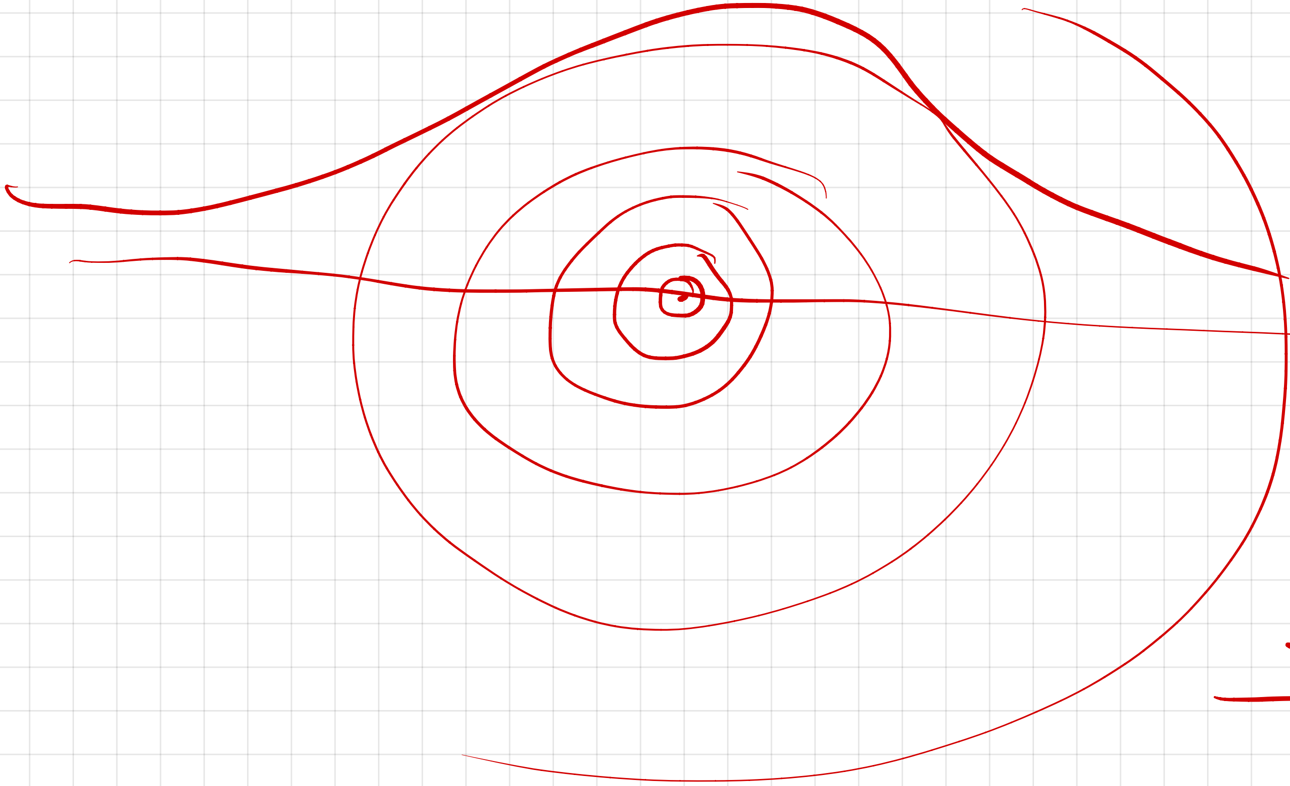
$$dx = \rho^{n-1} d\rho \cdot \underbrace{(du)}_{(n-1)\text{-dimensional volume element on the unit sphere}}$$

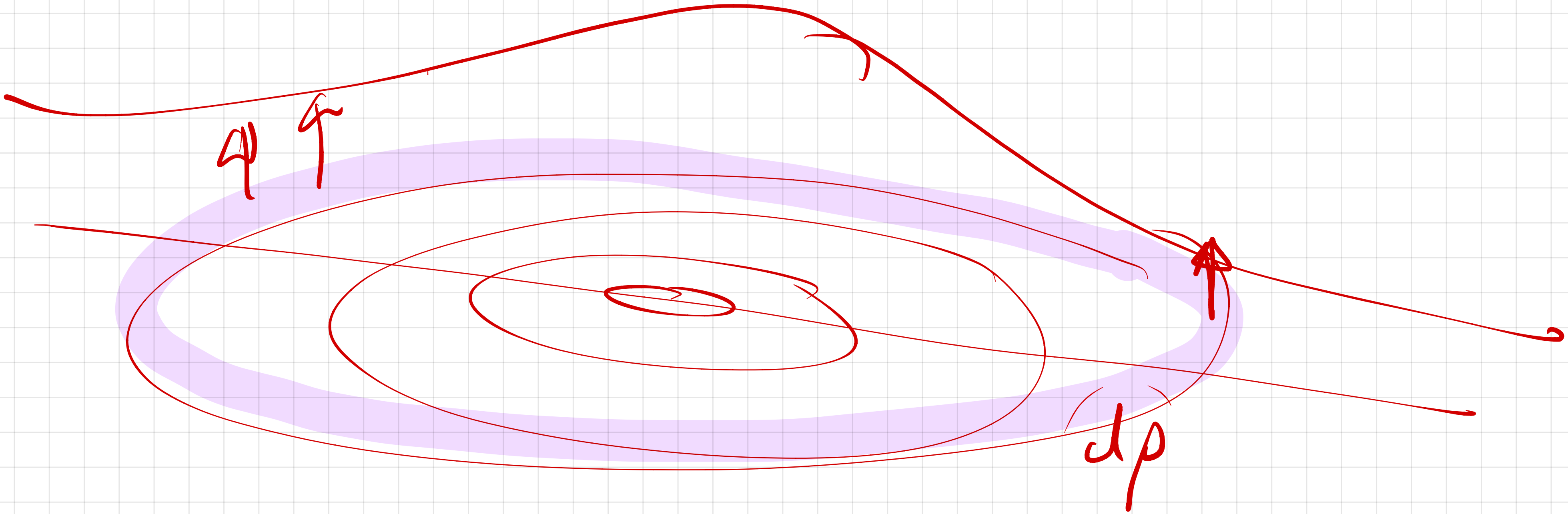


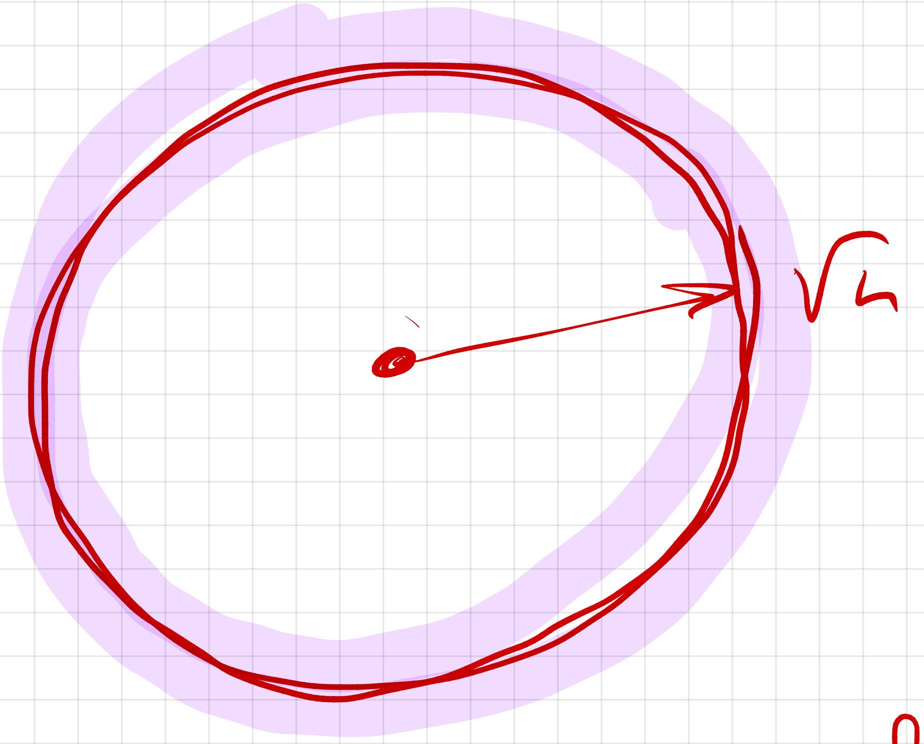
$$\gamma_n(dx) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\rho^2}{2}} \rho^{n-1} d\rho (du)$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{\rho^2}{2}} \rho^{n-1} \cdot \mathrm{Area}(S^{n-1}) d\rho\, d\sigma(u)$$

$\rho(k) = \mathbb{E}(\|X\|_k)$







$$\ell = \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(dx) \simeq \int_{\sqrt{n}\, S^{n-1}} \|x\|_K \,\mathrm{Area}(dx)$$

$$= \sqrt{n} \int_{S^{n-1}} \|x\|_K \, \sigma(dx) = \sqrt{n}\, M$$

$$\ell^* \simeq \sqrt{n}\, M^*$$

Figiel–Tomczak

$\exists$ GL_n-copy $\tilde{K}$ of K such that

$$n\, \ell(\tilde{K})\, \ell^*(\tilde{K}) \leq K(X)$$

$$X = (\mathbb{R}^n, \|\cdot\|_K).$$

$$\text{lwidth} \lesssim \min_{\substack{\tilde{K} \\ GL_n\text{-copy of } K}} (MM^*)(\tilde{K}) \lesssim n \min_{\substack{\tilde{K} \\ GL_n\text{-copy of } K}} (\ell\ell^*)(\tilde{K})$$

Banaszczyk's constant Flt(K):

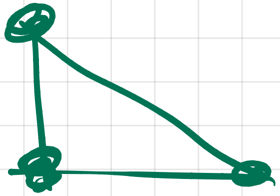
$$\mathrm{Flt}(K) = \max\{ \mathrm{lwidth}(\tilde{K}) :$$

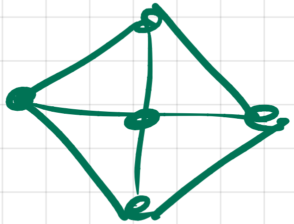
$\tilde{K}$ is a hollow n-dim convex body and $\tilde{K} = A(K)$ for some affine bijection $A: \mathbb{R}^n \to \mathbb{R}^n \}$

$K = -K$

$$\mathrm{Flt}(K) \lesssim \min_{\substack{\tilde{K} \text{ is a } GL_n(\mathbb{R}) \text{ copy} \\ \text{of } K}} M(\tilde{K})\, M^*(\tilde{K})$$

$K = \ell_2^n$-ball

$K =$ 

$K =$ 

Interesting examples

$$\mathbb{K}(X) = \| Q_1 \otimes \mathrm{Id}_X \|_{L_2(\mathbb{R}^n;\gamma_n)\otimes X \to L_2(\mathbb{R}^n;\gamma_n)\otimes X}$$

Q_1 orthogonal projection from

$L_2(\mathbb{R}^n;\gamma_n)$ onto LinearForms$(\mathbb{R}^n) \subset L_2(\mathbb{R}^n;\gamma_n)$.

$$X \simeq \tilde{X}$$

$$f \in L_2(\mathbb{R}^n;\gamma_n) \qquad \int_{\mathbb{R}^n} f(x)^2 \gamma_n(dx) < \infty$$

$$v \in X$$

$$f \otimes v := f(x)\, v : \mathbb{R}^n \longrightarrow X.$$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in L_2(\mathbb{R}^n, \gamma_n) \otimes X =: L_2(\mathbb{R}^n, \gamma_n; X)$$

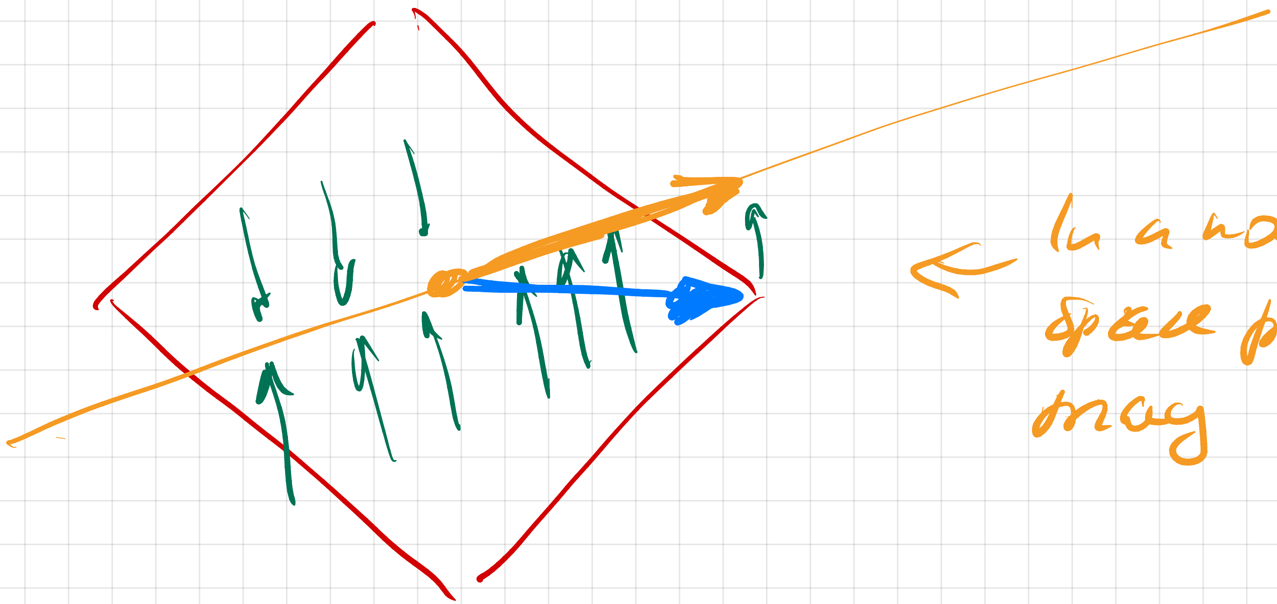
$$= L_2(X)$$

$$\|f\| = \sqrt{\int_{\mathbb{R}^n} \|f(x)\|_X^2 \, \gamma_n(dx)}$$

$$(Q_1 \otimes \mathrm{Id}) f = \begin{pmatrix} Q_1 f_1 \\ \vdots \\ Q_n f_n \end{pmatrix}$$

$L_2(\mathbb{R}^n, \gamma_n)$ is a Hilbert space.

$L_2(\mathbb{R}^n, \gamma_n) \otimes X$ is a normed space (not necessarily Hilbert).



In a non-Hilbert space projections may

Q_1 orthogonal projection so $\|Q_1\| = 1$ (the operator norm).

But $Q_1 \otimes \mathrm{Id}$ is not an orthogonal projection (it's not an operator on a Hilbert space, but an operator on a Banach space)

$$\boxed{\begin{aligned} L_2(\mathbb{R}^n, \gamma_n) \otimes X^* &= L_2(\mathbb{R}^n, \gamma_n)^* \otimes X^* \\ &= \left(L_2(\mathbb{R}^n, \gamma_n) \otimes X\right)^* \end{aligned}}$$

B Banach space $\Rightarrow$ $B^* = \{ \alpha : B \to \mathbb{R} \text{ bounded} \}$

$$\|\alpha\|_{B^*} = \sup_{\substack{x \in B \\ \|x\|_B \le 1}} \alpha(x)$$

$$\boxed{(B_1 \otimes B_2)^* = B_1^* \otimes B_2^*}$$

Is this true?

Operator theorists define different kinds of norms on $B_1^* \otimes B_2^*$ (so, the question is whether it is true for some particular type of a norm.

$$(\ell_2^n \otimes X)^* = \ell_2^n \otimes X^*$$

$U \in GL_n(\mathbb{R})$

$$\ell(U^{-1}K) = \int_{\mathbb{R}^n} \|x\|_{U^{-1}K}\, \gamma_n(dx)$$

$$= \int_{\mathbb{R}^n} \|Ux\|_K\, \gamma_n(dx) =: \ell(U)$$

$U \in \mathbb{R}^{n \times L}$

$\ell(U)$ is a norm on $\mathbb{R}^{n\times n}$

$$\ell^*(V) = \sup\{ \operatorname{trace}(VU) : U \in \mathbb{R}^{n\times n}, \ \ell(U) \le 1 \}.$$

$$\boxed{\exists\, U \in GL_n(\mathbb{R}) \text{ such that } \ell(U) = 1 \text{ and } \ell^*((U^T)^{-1}) = n.}$$

Nice self-contained lemma.