Last week: Reverse Minkowski Theorem (RMT)

$\Lambda \subseteq \mathbb{R}^n$ w/ $\det \Lambda' \geq 1$, $\forall \Lambda' \subseteq \Lambda$

$$\Rightarrow \underbrace{\sum_{x \in \Lambda} e^{-\pi \|sx\|_2^2}}_{\rho_{1/s}(\Lambda)} \leq \frac{3}{2} \qquad s = 100 \cdot \log(n)$$

$K \subseteq \mathbb{R}^n$, $\Lambda \subseteq \mathbb{R}^n$

$\mu(K;\Lambda) = \min\{\mu \geq 0 : \mu K + \Lambda = \mathbb{R}^n\}$

$= \max\{\mu \geq : \exists t : t + \mu K$ is hollow $(\Lambda\text{-})\}$

Part 1: Covering radius inequalities

Corollary of RMT: $\mu(B^n;\Lambda) \leq O(\sqrt{n} \cdot \log(n))$

Ball — stable lattice

Def: Λ is "t-stable", $t \geq 1$, if

- $\det(\Lambda')^{1/\dim \Lambda'} \geq \frac{1}{t}$, $\forall \Lambda' \subseteq \Lambda$ and $\forall \Lambda' \subseteq \Lambda^*$

Note: Λ is 1-stable $\iff$ Λ is stable

Prop 31 (RR): Λ t-stable, $Q \subseteq \mathbb{R}^n$ o.s. cvx body

$\Rightarrow \mu(Q;\Lambda) \leq O(\log(n)) \cdot \ell(Q) \cdot t$

Lemma 9: For any $t > 0$, $\Lambda \subseteq \mathbb{R}^n$, $u \in \mathbb{R}^n$ (Poisson summation)

$$|\rho_t(\Lambda + u) - t^n \det \Lambda^*| \leq t^n \det \Lambda^* \rho_{\frac{1}{t}}(\Lambda^* \setminus 0)$$

Lemma 15: Λ t-stable, $t \geq 1$

$\Rightarrow$ (a) Λ^* is t-stable

(b) $\rho_{1/st}(\Lambda) \leq 3/2$

(c) $\dfrac{\rho_{st}(\Lambda + u)}{\rho_{st}(\Lambda)} \geq 1/3 \qquad \forall u \in \mathbb{R}^n$

Pf (a) is clear

(b) Λ t-stable $\Rightarrow \det \Lambda' \geq 1$, $\forall \Lambda' \subseteq t \cdot \Lambda$

$\overset{\text{RMT}}{\Rightarrow} \rho_{\frac{1}{st}}(\Lambda) = \rho_{\frac{1}{s}}(t\Lambda) \leq 3/2$

(c) $\frac{\rho_{st}(\Lambda+u)}{\rho_{st}(\Lambda)} \underset{9}{\overset{\text{Lemma}}{\geq}} \frac{\cancel{(st)^n \det \Lambda^*} \left(1 - \rho_{\frac{1}{st}}(\Lambda^* \setminus 0)\right)}{\cancel{(st)^n \det \Lambda^*} \left(1 + \rho_{\frac{1}{st}}(\Lambda^* \setminus 0)\right)}$

$\overset{(a)+(b)}{\geq} \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}$ ∎

Lemma 30: $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall n \; \forall Q \subseteq \mathbb{R}^n$ o.s. : (Banaszczyk II)

$$\ell(Q) \leq \delta \quad \Rightarrow \quad \beta(Q) < \varepsilon$$

$\approx \sqrt{n} \cdot$ meanwidth (Q^*) $\qquad \sup_{\Lambda, u} \frac{\rho_1((u+\Lambda) \setminus Q)}{\rho_1(\Lambda)}$

Proof of 31: $\quad \mu(Q;\Lambda) \leq O(\log(n)) \cdot \ell(Q) \cdot t$

Both $\mu(Q;\Lambda)$ and $\ell(Q)$ are (-1)-homogeneous in Q.

Let $\delta > 0$ s.th. $\ell(Q) \leq \delta \Rightarrow \beta(Q) \leq \frac{1}{6}$

Wlog $\ell(Q) \leq \delta$

Suffices: $\mu(Q,\Lambda) \leq s \cdot t$

Suppose not. $\Rightarrow \exists u \in \mathbb{R}^n$ s.th. $(u+\Lambda) \cap st\, Q = \emptyset$

$\Rightarrow \rho_{st}(u+\Lambda) = \rho_{st}((u+\Lambda) \setminus st\, Q)$

$\overset{\frac{1}{3}\rho_{st}(\Lambda) \leq}{=} \rho_1\left(\left(\frac{u}{st} + \frac{\Lambda}{st}\right) \setminus Q\right) \leq \frac{1}{6} \rho_1\left(\frac{\Lambda}{st}\right) = \frac{1}{6} \rho_{st}(\Lambda)$

$\Downarrow$ ↯ ∎

$\boxed{\text{Part II}}$ Recall Flatness problem is equivalent to

$$w(K;\Lambda)\,\mu(K;\Lambda) \leq Flt(n)$$

Know $w(Q;\Lambda)\,\mu(Q;\Lambda) \leq O(n \log(n))$ if Q is o.s.

Also, for K arbitrary: $w(K;\Lambda) = \lambda_1((K-K)^*;\Lambda^*)$

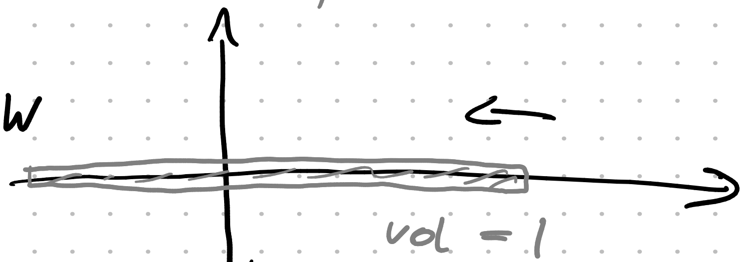
$= 2\,w(K-K;\Lambda)$

Goal: Bound $\mu(K;\Lambda)$ in terms of $\mu(K-K;\Lambda)$.

Idea: $\mu(Q;\Lambda) \geq \left(\frac{\det(\Lambda)}{\mathrm{vol}(Q)}\right)^{1/n}$ $\qquad \mathrm{vol}(\mu \cdot K) \geq \det \Lambda$

We also have:

$$\mu(Q;\Lambda) \geq \left(\frac{\det(\pi_W(\Lambda))}{\mathrm{vol}(\pi_W(Q))}\right)^{\frac{1}{\dim W}}$$



$\mu \to \infty$

$$\Rightarrow \mu(Q;\Lambda) \geq \max_W \left(\frac{\det(\pi_W(\Lambda))}{\mathrm{vol}(\pi_W(Q))}\right)^{\frac{1}{\dim W}} =: \mu_{KL}(Q;\Lambda)$$

Conj: $\mu_{KL}(K;\Lambda) \leq \mu(K;\Lambda) \leq \overset{??}{\mathcal{O}}(\log(n))\, \mu_{KL}(K;\Lambda)$

$\to \mathcal{O}(n)$

$\hookrightarrow \mathcal{O}(\log(n)^{3/2})$ if K is an ellipsoid (RMT)

Thm 2 (RR): $\mu(K;\Lambda) \leq \mathcal{O}(\log(n)^3)\, \mu_{KL}(K;\Lambda)$

Corollary: $\mathcal{F}(f(n)) \leq \mathcal{O}(n \log(n)^4)$ $\leftarrow 3$

Rogers Shephard: $\mathrm{vol}(K-K) \leq \binom{2n}{n} \mathrm{vol} K \approx 4^n \mathrm{vol} K$

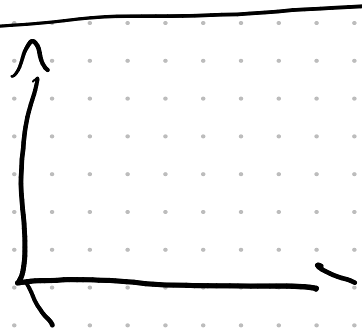
Pf: $\mu(K;\Lambda) \overset{\text{THM 2}}{\lesssim} \log(n)^3 \mu_{KL}(K;\Lambda)$

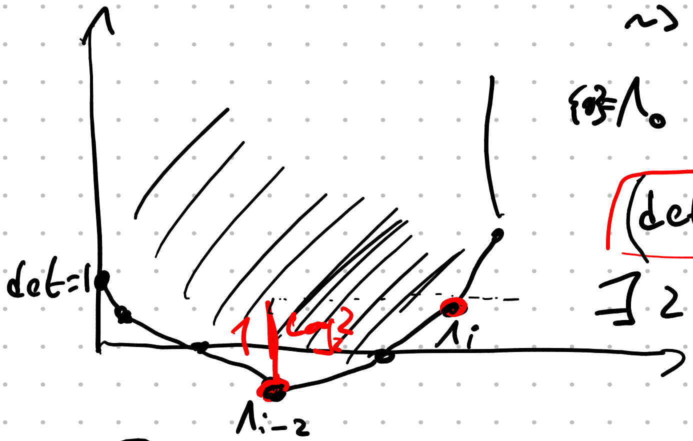
$$= \log(n)^3 \left(\frac{\det \pi_W \Lambda}{\mathrm{vol}\, \pi_W K}\right)^{\frac{1}{\dim W}}$$

Rogers, Shephard for $\pi_W K$ $\lesssim \log(n)^3 \left(\frac{\det \pi_W \Lambda}{\mathrm{vol}\, \pi_W(K-K)}\right)^{\frac{1}{\dim W}}$

$$\lesssim \log(n)^3 \mu_{KL}(K-K;\Lambda) \leq \log(n)^3 \mu(K-K;\Lambda)$$

$\to$ Combining this with Banaszczyk's inequality ∎





⇝ "well-separated filtration"

$\{0\} = \Lambda_0 \subseteq \ldots \subseteq \Lambda_m = \Lambda$

$$\boxed{\left(\det \Lambda_i\right)^{\frac{1}{\dim \Lambda_i}} \leq \frac{1}{2} \left(\det \Lambda_{i+2}\right)^{\frac{1}{\dim \Lambda_{i-2}}}}$$

$\exists$ 2-stable well-separated filtration

$$\log \det \Lambda_i \leq \log \det \Lambda_{i-2} - \log_2 2 =$$