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On the Reduction Model of Astrographic Plates

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Summary. The reduction model of astrographic plates, i.e. the dependence of the ideal coordinate measurements on the unknown star positions in any case contains further unknown parameters usually called the plate constants. Because of the error propagation through the least squares adjustment the computed plate constants deviate from the true values. The statistics of this deviation is described by the covariance matrix $\boldsymbol{C}_n$ computed as the inverse of the normal matrix. The plate constants are introduced as unknowns because they vary from plate to plate by physical reasons. The statistics of this physical variation is described by the covariance matrix $\boldsymbol{C}_p$. A method is proposed for the determination of $\boldsymbol{C}_p$ from a set of plates. The optimum reduction method then is found to be the least square adjustment of the usual observation equations combined with additional equations constraining the plate constants to follow their covariance matrix $\boldsymbol{C}_p$. The method is demonstrated by extensive numerical simulation of the actual astrometric situation.

The proposed method is applicable in a variety of analogous situations also.

Key words: data reduction – least squares method – photographic astrometry – star positions

1. Introduction

The reduction model of astrographic plates, i.e. the relationship between measured coordinates (x, y) of a star image on a plate and the standard coordinates (ξ, η), the magnitude m, and the effective wavelength λ of that star is usually written as a power series

$$x - v = \sum_i \sum_j \sum_k \sum_l p_{ijkl} \xi^i \eta^j m^k \lambda^l, \tag{1}$$

where v represents the error of the measurement x. A similar relation is valid for y. The p_{ijkl} are the plate constants. Usually only a few of them vary from plate to plate to such a degree that they have to be adjusted. All others are assumed to vanish (or to have a known value).

Unfortunately a given p_{ijkl} of Eq. 1 has no direct geometrical meaning. To overcome this the model may be written as a relationship that contains the parameters of physical meaning directly such as a scale factor or the difference between adjusted and adopted tangential point coordinates. Often this is called the photogrammetric modelling, and that of Eq. 1 the astrometric modelling, which of course only indicates what is convenient in the related domains. We will not discuss the advantages of one or the other in this paper, because the differences are of practical nature only (computational effort, geometric interpretation etc.), and are negligible in the domain of obtained star positions. As a matter of fact, the method developed in the following sections is entirely independent on the actual mathematical form of the model and is therefore applicable in a variety of analogous situations.

The main problem is to decide whether a specific plate constant should be adjusted or not. It is well known that adjusting a plate parameter creates larger errors of the computed star positions than neglecting it if the variance of this parameter caused by error propagation from the original measurements is larger than the variance of the actual physical variation. A statistical test based on this fact was presented by Eichhorn and Williams (1963). This test will be discussed and modified in Sect. 3.4 of this paper. The decision whether a plate parameter should be adjusted or not is therefore a step-function of the difference between the two variances. The discontinuity of this decision is unsatisfactory, and unfortunately the argument is close to the step for most of the plate constants commonly used. This problem is solved by the introduction of additional observation equations constraining the adjusted plate parameters to satisfy an a priori given variance approximately:

$$p_0 - w = p, \tag{2}$$

where p is the unknown value of a specific plate constant, which has to be adjusted, and for p_0 an appropriate value must be chosen (see Sect. 3.2). The deviation w of the actual value of p from the chosen mean value p_0 is interpreted here as an error of the fictitious measurement p_0. The method was used first by Moritz (1965), reviewed by Moritz (1973), and was introduced in astrometry by de Vegt and Ebner (1972). Eichhorn (1978) presented a derivation of the method from the principle of maximum likelihood in a general notation. Von der Heide (1977a) demonstrated the usefulness of such additional equations for tilt terms.

The reduction model of Eq. (1) (or another mathematical form) may contain all parameters that are physically possible if these parameters are constrained by additional observation equations. Such a reduction model is the optimum, indeed, but the original problem of the optimum model is only transferred to the choice of the optimum stochastic model for the errors of the artificial measurements introduced by the additional observation equations. The aim of this paper is to derive a method which allows to compute the desired covariance matrix that has to be used together with the additional observation equations within the adjustment.

2. The Covariance Matrices of the Plate Constants

2.1. Definition

Let $\boldsymbol{p}'$ be the vector of all adjusted parameters p_{ijkl} of one plate. This vector is composed of a term $\boldsymbol{p}_s$ that contains all known effects that are systematic to all plates, a term $\boldsymbol{p}_p$ that changes randomly from plate to plate, and a term $\boldsymbol{p}_n$ which represents the noise that is caused by the error propagation through the adjustment. The non-vanishing elements of $\boldsymbol{p}_p$ necessitate the adjustment of these plate constants.

The term $\boldsymbol{p}_s$ can be found by adjusting many plates and computing mean values or similar functions on the parameters. $\boldsymbol{p}_s$ may depend on known parameters like zenith distance, the measuring machine etc. Since $\boldsymbol{p}_s$ is known then, we will subtract if from $\boldsymbol{p}'$ and discuss the remaining stochastic part

$$\boldsymbol{p}=\boldsymbol{p}_p+\boldsymbol{p}_n$$

alone.

It is also possible to adjust the systematic part $\boldsymbol{p}_s$ in parallel to the individual part $\boldsymbol{p}_p$ if one performs a block adjustment of many plates. The singularity of the normal matrix can be avoided by use of additional Eqs. 2 (for detail see Sect. 3.2).

The covariance matrix $\boldsymbol{C}_n$ of $\boldsymbol{p}_n$ directly follows from the adjustment as the product of the inverse of the normal matrix and the reduced χ^2. Let $\boldsymbol{C}$ and $\boldsymbol{C}_p$ be the covariance matrices of $\boldsymbol{p}$ and $\boldsymbol{p}_p$, respectively. Then

$$\boldsymbol{C}=\boldsymbol{C}_p+\boldsymbol{C}_n \tag{3}$$

because $\boldsymbol{p}_p$ and $\boldsymbol{p}_n$ are entirely independent.

$\boldsymbol{C}_p$ is invariable for all plates that are exposed, developed and measured under similar conditions, $\boldsymbol{C}_n$ and $\boldsymbol{C}$, however, may slightly vary from plate to plate. Since the invariable matrix $\boldsymbol{C}_p$ contains the desired information, it is meaningful to define the mean values from an ensemble of N plates:

$$\langle \boldsymbol{C}_n\rangle=\frac{1}{N}\sum \boldsymbol{C}_n;\ \langle \boldsymbol{C}_p\rangle=\boldsymbol{C}_p;\ \langle \boldsymbol{C}\rangle=\frac{1}{N}\sum \boldsymbol{C}=\boldsymbol{C}_p+\langle \boldsymbol{C}_n\rangle. \tag{4}$$

In the case of an overlap reduction $\boldsymbol{C}_n$ is a diagonal submatrix of the whole inverse normal matrix, and $\langle \boldsymbol{C}_n\rangle$ is the mean of all diagonal submatrices belonging to the plate constants of single plates. If the vector $\boldsymbol{p}_{(r)}$ of the r-th plate is taken as the r-th row vector of a matrix $\boldsymbol{P}$ ($r=1, \ldots, N$), then

$$\boldsymbol{C}^*=\frac{1}{N}\boldsymbol{P}^T\boldsymbol{P} \tag{5}$$

is a function of the stochastic vectors $\boldsymbol{p}_{(r)}$. The expectation value of $\boldsymbol{C}^*$ is the same as that of $\langle \boldsymbol{C}\rangle$. $\boldsymbol{C}^*$ therefore may be interpreted as a measurement of $\langle \boldsymbol{C}\rangle$. With $\boldsymbol{C}^*$ and $\langle \boldsymbol{C}_n\rangle$ an approximation for $\boldsymbol{C}_p$ can be computed from Eq. 3.

In fact also the individual matrices $\boldsymbol{C}_n$ and therefore $\langle \boldsymbol{C}_n\rangle$ have a stochastic error which is introduced by the multiplication with the reduced χ^2, but the ensemble of (x, y)-measurements and (α, δ)-measurements of reference stars contains about a hundred measurements per plate if a single plate adjustment is performed. There are even more measurements in the case of a block adjustment using the overlap method. The χ^2-distribution then is a sufficiently narrow peak, such that this error of $\langle \boldsymbol{C}_n\rangle$ may be neglected against that of $\boldsymbol{C}^*$ which will be discussed in the following section.

2.2. The Variance of an Empirical Covariance Matrix

Let u and v be stochastic variables that satisfy the conditions

$$E(u)=E(v)=0;\quad E(u^2)=E(v^2)=1;\quad E(uv)=c, \tag{6}$$

where E stands for the expectation value operator.

$$w=(1-c^2)^{-1/2}(v-cu) \tag{7}$$

is a function of u and v with the properties

$$E(w)=0;\quad E(w^2)=1;\quad E(uw)=0. \tag{8}$$

Thus u and w are independent.

Now

$$F=\sum_{k=1}^{N} u_k v_k$$

is a statistic of u and v, k denoting the number of the element in the chosen ensemble. Substitution of v by w using Eq. 7 leads to

$$F=c\sum_{k=1}^{N} u^2+(1-c^2)^{1/2}\sum_{k=1}^{N} u_k w_k. \tag{9}$$

The expectation value of F follows using the Eqs. 6 and 8:

$$E(F)=cN.$$

From now on we shall assume a normal distribution for u and v. Then it is obvious from Eq. 7 that w also follows a Gaussian distribution. From Eq. 9 one has

$$F^2=c^2\left(\sum_{k=1}^{N} u_k^2\right)^2+(1-c^2)\sum_{k=1}^{N}\sum_{l=1}^{N} u_k u_l w_k w_l$$
$$+2c(1-c^2)^{1/2}\sum_{k=1}^{N}\sum_{l=1}^{N} u_k^2 u_l w_l.$$

Disregarding the factors the expectation value of the first term is the variance N^2+2N of the well known χ^2, that of the second term is N and that of the last term is zero. Thus

$$\begin{aligned} E((F-cN)^2)&=E(F^2)-c^2N^2\\ &=c^2(N^2+2N)+(1-c^2)N-c^2N^2\\ &=(1+c^2)N. \end{aligned}$$

F/N is an element of the empirically computed covariance matrix of u and v. The elements of this empirical matrix scatter around the correct (but unknown) elements of the matrix

$$\begin{pmatrix} 1 & c\\ c & 1\end{pmatrix} \tag{10}$$

with the standard deviations

$$\begin{pmatrix} (2/N)^{1/2} & ((1+c^2)/N)^{1/2}\\ ((1+c^2)/N)^{1/2} & (2/N)^{1/2}\end{pmatrix}. \tag{11}$$

If the two stochastic variables u' and v' have the covariance matrix

$$\begin{pmatrix} c_{uu} & c_{uv}\\ c_{uv} & c_{vv}\end{pmatrix} \tag{12}$$

instead of the Matrix 10, then an empirical computation of this matrix from an ensemble of N pairs (u', v') has just this expectation value (Matrix 12), and the standard deviations follow from the

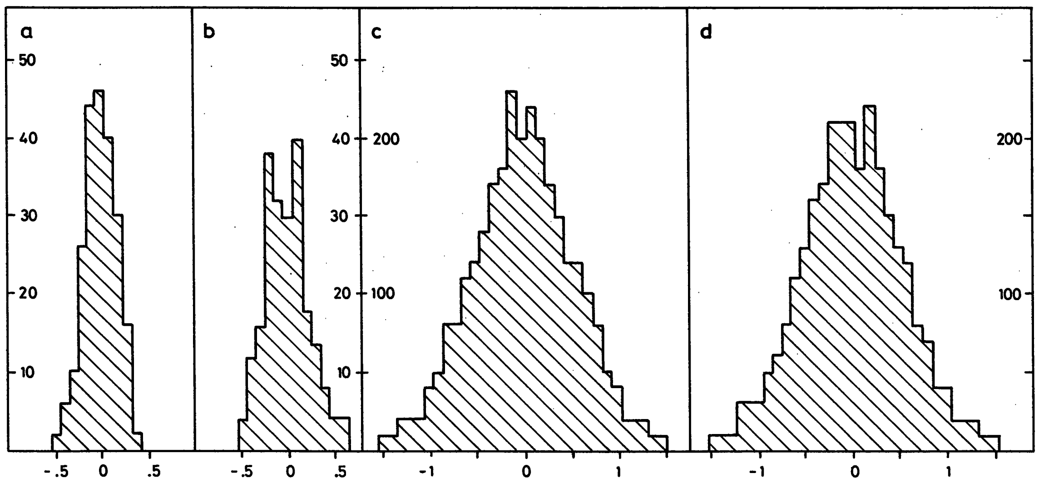

Fig. 1a–c. Distribution of the residuals obtained by an overlap reduction of 36 plates of the AGK3. **a** and **b**: α and δ coordinates of reference stars, respectively; **c** and **d**: x and y measurements of all stars

Expression 11:

$$\begin{pmatrix} c_{uu}(2/N)^{1/2} & ((c_{uu}c_{vv}+c_{uv}^2)/N)^{1/2} \\ ((c_{uu}c_{vv}+c_{uv}^2)/N)^{1/2} & c_{vv}(2/N)^{1/2} \end{pmatrix}$$

The diagonal elements of the empirical covariance matrix multiplied by the factor N/c_{uu} or N/c_{vv} respectively are χ^2-distributed.

2.3. *The Statistical Test on the Significance of a Term*

Let $\boldsymbol{D}$ be a diagonal matrix with $(\boldsymbol{C}_n)_{ii}^{-1/2}$ as the i-th diagonal element, where $(\boldsymbol{C}_n)_{ii}$ is the i-th diagonal element of $\boldsymbol{C}_n$. Then we transform the vectors $\boldsymbol{p}$, $\boldsymbol{p}_p$ and $\boldsymbol{p}_n$ by

$$\boldsymbol{q}=\boldsymbol{D}\boldsymbol{p}; \quad \boldsymbol{q}_p=\boldsymbol{D}\boldsymbol{p}_p; \quad \boldsymbol{q}_n\boldsymbol{D}\boldsymbol{p}_n.$$

Obviously follows

$$\boldsymbol{q}=\boldsymbol{q}_p+\boldsymbol{q}_n$$

and

$$E(\boldsymbol{q}_n)=0; \quad E((\boldsymbol{q}_n)_i^2)=1 \ \forall i.$$

Now we assume that the i-th component $(\boldsymbol{p}_p)_i$ of $\boldsymbol{p}_p$ is zero for all plates. We will test this hypothesis. $(\boldsymbol{p}_p)_i=0$ implies $(\boldsymbol{q}_p)_i=0$ and therefore $(\boldsymbol{q})_i=(\boldsymbol{q}_n)_i$. The $(\boldsymbol{q}_n)_i$ follow a Gaussian distribution very well, because the errors of the measurements, that are used for the adjustment, are normally distributed. An example for this distribution is shown in Fig. 1. Furthermore the region of linearity of the reduction model is much greater than the interval covered by these errors in any astrometric case. Then the results of the last section are applicable. The linearity is explained as follows: For the least squares adjustment the non-linear observation equations must be approximated by linear ones. This is usually performed by a first-order Taylor expansion around the zero-order approximations for the plate constants and standard coordinates. The inverses of the linear functions, i.e. the dependence of the standard coordinates on the plate constants and the measurements is a very good approximation to the actual non-linear inverse within the statistically allowed range of the true, but unknown locations of the star images on the plate (the measurements ± three times their standard deviations, for example).

Corresponding to $\boldsymbol{C}^*$, $\boldsymbol{C}$, $\boldsymbol{C}_n$ we have the covariance matrices $\boldsymbol{K}^*$, $\boldsymbol{K}$, and $\boldsymbol{K}_n$ for the stochastic vectors $\boldsymbol{q}$ and $\boldsymbol{q}_n$, respectively, with the relations

$$\boldsymbol{K}=\boldsymbol{D}\boldsymbol{C}\boldsymbol{D}^T; \quad \boldsymbol{K}_n=\boldsymbol{D}\boldsymbol{C}_n\boldsymbol{D}^T.$$

$\boldsymbol{K}^*$ is the empirical covariance matrix taken from the ensemble of N adjusted plates. Let $(\boldsymbol{K}^*)_{ii}$ be the i-th diagonal element of $\boldsymbol{K}^*$. If $(\boldsymbol{p}_p)_i=0$ for all plates then $N(\boldsymbol{K}^*)_{ii}$ is χ^2-distributed because of $E((\boldsymbol{q})_i)=0$ and $E((\boldsymbol{q})_i^2)=1$. If $N(\boldsymbol{K}^*)_{ii}$ exceeds a certain limit (for example that point where the probability of taking a χ^2 of this or greater values is less than 0.01) then the hypothesis, that the i-th plate parameter is insignificant is to be rejected. The number of plates for this investigation should be greater than about 25. Then the χ^2-distribution may be approximated by a normal distribution and the test can be formulated as follows: If the hypothesis of $(\boldsymbol{p}_p)_i=0$ for all plates is valid, then the probability of getting a value

$$(\boldsymbol{K}^*)_{ii}>1+3.3\,N^{-1/2}$$

is 0.01. Therefore, we reject this hypothesis if $(\boldsymbol{K}^*)_{ii}$ exceeds the quoted limit.

The significance of terms as defined here is quite different from the usual claim that the standard deviation of $(\boldsymbol{p}_p)_i$ should be greater than that of $(\boldsymbol{p}_n)_i$, because the standard deviation of $(\boldsymbol{p}_p)_i$ may be significantly greater than zero but nevertheless much smaller than the standard deviation of $(\boldsymbol{p}_n)_i$. The reason for this definition will be given in Sect. 3.1. If a systematic part $\boldsymbol{p}_s$ is determined from the same material and subtracted from the original plate constants, then a χ^2-distribution of N-1 degrees of freedom must be taken, the difference, however, is small.

2.4. *The Empirical Matrix* $\boldsymbol{C}_p^*$

The empirical covariance matrix $\boldsymbol{C}_p^*$ of $\boldsymbol{p}_p$ follows from Eq. 3:

$$\boldsymbol{C}_p^*=\boldsymbol{C}^*-\langle\boldsymbol{C}_n\rangle.$$

where $\boldsymbol{C}^*$ is taken from Eq. 5 and $\langle\boldsymbol{C}_n\rangle$ from Eq. 4. The difference between $\boldsymbol{C}^*$ and the matrix $\boldsymbol{K}^*$ defined in the last section should be noted.

If one (or more) plate constants are actually zero for all plates then C_p^* becomes negative definite or singular by a probability of 0.5 because the corresponding diagonal element of C^* scatters around that of $\langle C_n \rangle$. Such parameters must be removed from the model. Further the statistical errors of the elements of C^* and $\langle C_n \rangle$ may produce correlation factors of C_p^* that are absolutely greater than 1. This also leads to a negative definite C_p^*, although all plate constants are significant. Therefore an internal adjustment of the elements of C_p^* is necessary before it can be used for purposes like those described in Sect. 3.1. C_p^* must be used with caution because it is not a priori obvious that the components of p_p are normally distributed. This does not affect the validity of the significance test of the previous paragraph.

It should be noted that, at least in principle, the probability density function of p_p can be obtained by deconvolving the observed distribution of p in respect to the known multi-dimensional Gaussian profile of the noise distribution. It is well known that the result of such a deconvolution is extremely sensitive to errors of the original data. The simple consideration of the covariance matrices instead of the entire probability distributions is justified therefore.

2.5. Some Features of C_p and C_n

The non-diagonal elements of C_p describe the physical correlation of the plate constants, whereas those of C_n stand for the stochastical correlation. Both are entirely independent from each other. An example are the tilt terms

$$x = \ldots + p_x \xi^2 + q_x \xi\eta + \ldots$$

$$y = \ldots + p_y \xi\eta + q_y \eta^2 + \ldots$$

If there is no other physical cause for these terms than a tilt that varies stochastically from plate to plate, then p_x and p_y must be equal except for the noise of the adjustment. The same is valid for q_x and q_y. Because there is no parameter common to the models of x and y (in this case), the stochastic covariances from C_n between p_x and p_y and between q_x and q_y vanish. Against that, the physical correlation represented by the elements of C_p is complete.

The covariance matrix C_p of a photogrammetric model that really describes all aspects of the actual situation is diagonal because each of the plate constants has a physical meaning that is distinct and independent from the others. In contrast, C_n is not diagonal. This follows from the fact that the functions $f(\xi,\eta)$ and $g(\xi,\eta)$ in a photogrammetric model $x = \ldots + p \cdot f(\xi,\eta) + q \cdot g(\xi,\eta) + \ldots$ with plate parameters p and q need by no means be orthogonal. The normal matrix obtained using such a model may be ill conditioned.

If the model of Eq. 1 is used then both C_p and C_n are not diagonal, but the functions $\xi^i \eta^j m^k \lambda^l$ could be substituted by orthogonal ones in respect to the points (ξ, η, m, λ) of all stars. Then C_n would be diagonal, but not C_p because the influence of one original aspect of the geometry is spread into several terms. Since such an orthogonalization is already the solution of the adjustment (the diagonal C_n is the inverse of the normal matrix which therefore is diagonal) it is of academic interest only. In contrast, it is easy to substitute $\xi^i \eta^j m^k \lambda^l$ by functions that are orthogonal in respect to the used plate area and the known distribution functions of m and λ. Then the diagonality of C_n is destroyed by the actual random distribution of the star positions, magnitudes and effective wavelength, but it will always have a dominating diagonal, and the normal matrix is well conditioned if single plates are adjusted. On the other hand an overlap reduction causes high correlations between plate parameters of overlapping plates. Such a simple orthogonalization cannot prevent this. The normal matrix of an overlap reduction is therefore ill conditioned in any case, and a photogrammetric model may be used without great loss in matrix condition.

3. The Reduction Model

3.1. The Optimum Model

After all non-significant plate parameters (in the sense of Sect. 2.3) are removed from the model there are still some parameters left which usually are omitted, because the corresponding variances $(C_p)_{ii}$ are less than $(C_n)_{ii}$. The somewhat unsatisfactory omission of significant parameters is not necessary when the reduction model is extended to additional observation equations like Eq. 2. As covariance matrix of the corresponding fictitious measurements C_p^* must be taken. In order that C_p^* is a good approximation to the unknown C_p, the number N of plates used to derive C_p^* should be as large as possible. It is statistically correct to use the same plates first for the computation of C_p^* and then adjust them once more including the additional observation Eq. 2. This in principle is similar to the usual computation of a $\sigma_0 = (\chi^2 \text{ reduced})^{1/2}$, which is done after the first adjustment.

If the additional observation Eqs. 2 are used for the adjustment together with C_p^*, then the noise of the adjustment is substantially reduced. The covariance matrix of this noise is C_n'. For the diagonal elements obviously follows

$$(C_n')_{ii} < (C_p)_{ii} \quad \forall i.$$

The conventional requirement for a term not to be omitted therefore in any case is automatically satisfied.

Another advantage of the proposed method is that one may regard a physical correlation which is absolutely and significantly greater than zero and less than 1. A conventional model would take two strongly correlated parameters together as one parameter. For example, the tilt parameters of the last section are often adopted as $p = p_x = p_y$ and $q = q_x = q_y$. But what should be done, when there is a significant correlation that is about 0.5? This problem does not occur in the proposed method.

It is important to use the original measurements for the determination of C_p^*. The approximate tangential points of the plates used within the adjustment should be evaluated independently. A second adjustment would clearly yield small and probably insignificant tilt parameters if the tangential points were corrected using the results of a first adjustment.

The reduction model consists of the mathematical formulation (Eqs. 1 and 2) and the stochastic model given by the covariance matrices for the (x,y)-measurements, the (α,δ)-measurements and C_p^*. This reduction model is identical for a single plate adjustment and for an overlap reduction. Only the adjustment noise given by C_n' is very different. If additional Eqs. 2 are not used the reduction model of an overlap reduction will usually retain more plate parameters than are necessary for the adjustment of single plates because of the reduced adjustment noise. This was already pointed out by von der Heide (1977b).

3.2. The Fictitious Measurements p_0

The best value for the fictitious measurement $(p_0)_i$ of the i-th parameter $(p)_i$ of the reduction model obviously is the expectation value of the parameter, because the additional Eqs. 2 then sup-

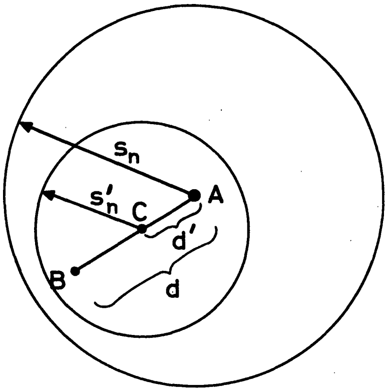

Fig. 2. Demonstration of the effect of additional observation equations for plate constants

port the observance of that expectation value. The number of plates used to determine $\langle(\boldsymbol{p})_i\rangle$ that has to be inserted as an approximation for the unknown expectation value should be as large as possible in order to reduce the error. If all plates are to be adjusted simultaneously by an overlap reduction, a term of the reduction model may be written twice, one with the parameter individually for each plate and one with a parameter that is common to all plates. Now additional observation equations $0-w_{ij}=(\boldsymbol{p})_{ij}$ are used for each plate j. The one parameter common to all plates should be adjusted without constraints. It will automatically yield an approximation for the expectation value of the parameter in question. For the implementation of common parameters within an overlap reduction see von der Heide (1978).

Eichhorn, et al. (1967) established that the plate overlap reduction technique is very sensitive in respect to model deficiencies. They simulated a systematic tilt of all plates and assumed a linear reduction model which assumes a vanishing tilt. The large systematic errors of the computed star positions caused by the simulated tilt vanish if a common tilt of all plates is introduced. Such common parameters should therefore be introduced whenever one cannot be sure that the influence on the star positions is negligible, this concerns especially higher order terms that are not written as individual terms.

3.3. Reflection upon the Additional Observation Equations

The information actually sought for by a plate reduction is a set of star positions. The plate constants are only necessary adjustment parameters. Therefore a discussion about the necessity of a term in the model or about the influence of additional observation equations should better be done in the domain of star positions than in the domain of plate constants. The transformation from plate constants to star positions is, at least in the case of an overlap reduction, so complicated, that an analytic analysis of the problem is impracticable. On the other hand, there is no evident way in which the results of numerical simulations can be converted into analytical relations, and sometimes, it is difficult to interpret them. We therefore insert this entirely heuristic paragraph.

To simplify our considerations we look at only one plate parameter $(\boldsymbol{p})_i$ and assume an orthogonalized reduction model such that the corresponding $(\boldsymbol{p}_n)_i$ is statistically independent from $(\boldsymbol{p}_n)_j$ for $j\neq i$. Now we take one plate, for which $(\boldsymbol{p}_p)_i$ is the correct, but unknown value of $(\boldsymbol{p})_i$ and look at a star on this plate. The correct position of this star is denoted by the point A in Fig. 2. The computed position differs from this point because of the errors of the (x,y)-measurements on the plate, which is by far the greatest part, but also because of error propagation from $\boldsymbol{p}_n$ into the position. We consider only the part that is introduced by the error of the i-th plate parameter. Such a separation is possible as a consequence of the assumed orthogonality of the reduction model.

The noise $(\boldsymbol{p}_n)_i$ causes an error of the star position that is characterized by the standard deviation s_n. Assuming that the parameter $(\boldsymbol{p})_i$ has equal influence in both coordinates, this adjustment noise is indicated by a circle of radius s_n around the point A in Fig. 2. On the other hand, if the model does not include the term $(\boldsymbol{p})_i$ i.e. $(\boldsymbol{p})_i=0$, then, again taking all other errors out of consideration, the adjustment produces the position at the point B. If the parameter is retained then the standard deviation s_n of the errors is larger than the distance d between the points A and B in this example.

If an additional observation equation $(\boldsymbol{p})_i=0-w$ is used combined with an appropriate variance, then the computed position lies within a two dimensional distribution with reduced standard deviation $s_n'<s_n$, centered at a point C between A and B. Now, it is no longer obvious that the model without the $(\boldsymbol{p})_i$-term is better. With decreasing variance for the additional observation equation the point C moves towards the point B, while the standard deviation s_n' of the noise decreases simultaneously.

The quality of the adjustment (or better of the model) is characterized by the mean value of the norm of the difference vector from point A to the computed position. This value is given by

$$s^2=(d')^2+(s_n')^2.$$

Denoting the variance used for the additional observation Eqs. 2 by σ_a^2 in contrast to the actually present variance $\sigma_p^2=(\boldsymbol{C}_p)_{ii}$ of the parameter, then the adjustment noise of the parameter is given by

$$(\sigma_n')^2=(\sigma_n^{-2}+\sigma_a^{-2})^{-1}, \tag{13}$$

where $\sigma_n^2=(\boldsymbol{C}_n)_{ii}$.

For heuristic purposes we assume, that the Relation 13 is linearly to be transformed into the domain of star positions, i.e.

$$(s_n')^2=s_n^2(1+(\sigma_n/\sigma_a)^2)^{-1}.$$

Further the relation between d' and σ_n' is assumed to be linear, i.e. it is described by a straight line between the points $(d,0)$ and $(0,\sigma_n)$ in the (d',σ_n')-plane. With these approximations we get an analytic expression for the quality of the model in respect to the i-th parameter at the location of the star in consideration:

$$s^2=d^2(1-(1+(\sigma_n/\sigma_a)^2)^{-1/2})+s_n^2(1+(\sigma_n/\sigma_a)^2)^{-1}. \tag{14}$$

Because of the good linearity of the transformation from plate constants to star coordinates for a single position one has $d=k_\mu(\boldsymbol{p}_p)_i$ and $s_n=k_\mu\sigma_n$, with k_μ being only a factor which varies from star to star.

The mean of Eq. 14 taken over all M stars and all plates then is the desired expression for the quality of the model to be used for the plate material. Since the correct values of $(\boldsymbol{p}_p)_i$ are distributed around the mean value 0 by a standard deviation $\sigma_p=(\boldsymbol{C}_p)_{ii}^{1/2}$, we have to integrate Eq. 14 by use of the probability-density function of $(\boldsymbol{p}_p)_i$. Instead of this, several sets of pseudo random numbers fitting this density function were generated and Eq. 14 then simply summed. The advantage of this procedure is, that it simulates the actual case, where neither the mean $\langle(\boldsymbol{p}_p)_i\rangle$ is zero

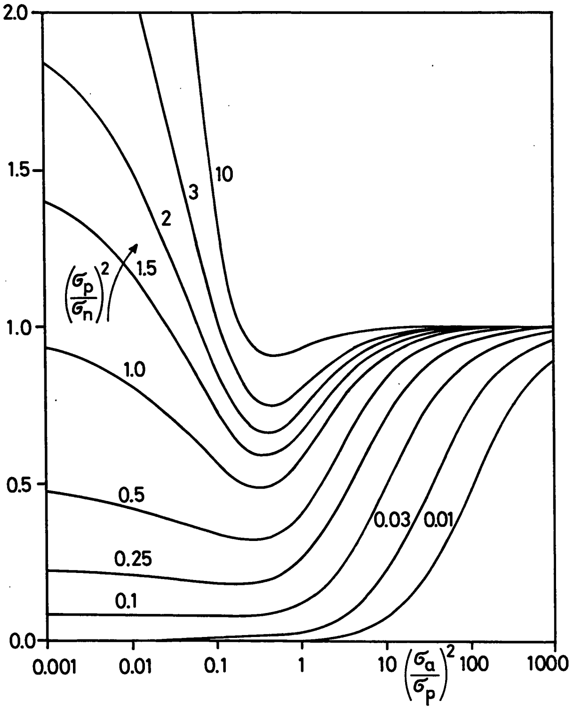

Fig. 3. Dependence of the variance of the star positions on the order of magnitude of a physical plate parameter and on the weight used for the additional observation equations. This figure is an analytical approximation

nor $(C_p^*)_{ii}^{1/2}=(C_p)_{ii}^{1/2}$. Three distributions were chosen: (a) $(p_p)_i=\sigma_p$ for all plates; (b) a Gaussian distribution; (c) a distribution, where 80 per cent of the values are exactly zero and the remaining 20 per cent are evenly distributed within the interval $[-15^{1/2}\sigma_p, +15^{1/2}\sigma_p]$. At the summation of Eq. 14 the differences of the k_μ from star to star have been neglected. The curves obtained with the three distributions do not differ significantly. Therefore only the case (b) is shown in Fig. 3. There the variance $\langle s^2\rangle$ of the computed star positions is given as a function of $(\sigma_a/\sigma_p)^2$, which is the ratio of the assumed variance of $(p_p)_i$ to the actual one (for use in conjunction with Eqs. 2), and of $(\sigma_p/\sigma_n)^2$ which is the ratio of the actual variance of the physical parameter $(p_p)_i$ to the variance of the adjustment noise. Obviously there are values of σ_a for which the $\langle s^2\rangle$ is less than both, the $\langle s^2\rangle$ of the solution with an entirely free parameter $(p)_i$ (corresponding to $\sigma_a\to\infty$), and the $\langle s^2\rangle$ of the solution without the parameter $(p)_i$ (corresponding to $\sigma_a\to 0$). The only exception is the trivial case of $\sigma_p=0$. The additional observation Eqs. 2 are therefore very valuable to reduce the systematic part of the errors of star positions, especially if σ_p has the same order of magnitude as σ_n, which indeed is true for nearly all parameters in the astrometric practice after all systematic effects have been subtracted.

We should remember that the results of Fig. 3 are based on very simple assumptions. They therefore represent only the general behavior and should not be used quantitatively. This concerns especially the position of the minima in Fig. 3. The problem whether Fig. 3 is a good approximation or not, can, at least for the case of a block adjustment, only be solved by a numerical simulation of the astrometric practice. The method and results of such an investigation are presented in Sect. 4.

3.4. Significance Test for Parameters of a Reduction Model without Additional Observation Equations

From Fig. 3 follows a simple criterion whether a parameter should be left out in the model if one does not use additional observation equations: if $\sigma_p<\sigma_n$ then $\langle s^2\rangle$ is smaller for $\sigma_a\to 0$ (omission of the parameter) than for $\sigma_a\to\infty$ (parameter retained). Since the actual, but unknown, plate parameters are independent from the errors of the adjustment, the variances of both parts are to be added (see Eq. 3). This sum can be measured (Eq. 5). The boundary between both models is given by $\sigma_p=\sigma_n$. There the diagonal elements of $\boldsymbol{C}^*$ are twice those of $\langle \boldsymbol{C}_n\rangle$. Only if a diagonal element of a computed matrix $\boldsymbol{C}^*$ (model including the parameter in question) is more than two times the corresponding element of $\langle \boldsymbol{C}_n\rangle$, which is also easily obtained, then the parameter should be retained.

This test was applied to 100 plates of the AGK 3 with the result that the two tilt terms would better have been omitted, because the diagonal elements of $\boldsymbol{C}^*$, which belong to the two parameters, are only 1.4 times the corresponding elements of $\langle \boldsymbol{C}_n\rangle$, after all systematic effects have been removed (single plate adjustment).

The proposed test is to be compared with that of Eichhorn and Williams (1963). The left hand side of their Eq. 16 means the variance of the actual difference of the star positions that are obtained by the two models. This variance contains the adjustment noise. The difference of the adjustment noise of both models as computed by inversion of the normal matrices is the right hand side of their Eq. 16. Since this noise cannot be eliminated from the left hand side, the expectation value of this side is equal to the right hand side, if the term in question does not exist physically. Therefore a factor 2 has to be inserted on the right hand side. The resulting test when applied to assemblies of more than about 25 plates is equivalent to that proposed above by the author.

4. A Test of the Method by Numerical Simulations

4.1. The Simulation

Take a plate and adjust it with an adopted reduction model. Then add artificially some known errors to the measurements, adjust again with the same model and compute the difference of the obtained star positions. That is the simulation procedure in general. Since the normal matrices of both adjustments are identical, it is advantageous to perform the adjustments in parallel. The computer program of von der Heide (1978) using the general overlap algorithm, permits the parallel computation of any practical number of simulations.

For the considered simulation a set of 36 overlapping plates from the AGK 3 were chosen instead of simulating the plates entirely as was done by von der Heide (1977b). Our simulation therefore is accommodated to practice. At each of the adjustments 9 simulations were performed in parallel and a partial inversion of the normal matrix (for the computation of $\boldsymbol{C}_n$) too.

The following reduction models were used:

$$\text{model I:}\ \begin{cases} x=p_1+p_3\xi+p_4\eta+p_5\xi+p_6\eta+p_7\xi^2 \\ y=p_2+p_3\eta-p_4\xi-p_5\eta+p_6\xi+p_7\xi\eta \end{cases}$$

model II: same as model I, but last term of y is $p_8\xi\eta$.

The parameters p_7 and p_8 were simulated by pseudo random numbers with a correlation factor of 0.5: 9 sets of 36 pairs with normal distribution and 9 corresponding sets with the probability-

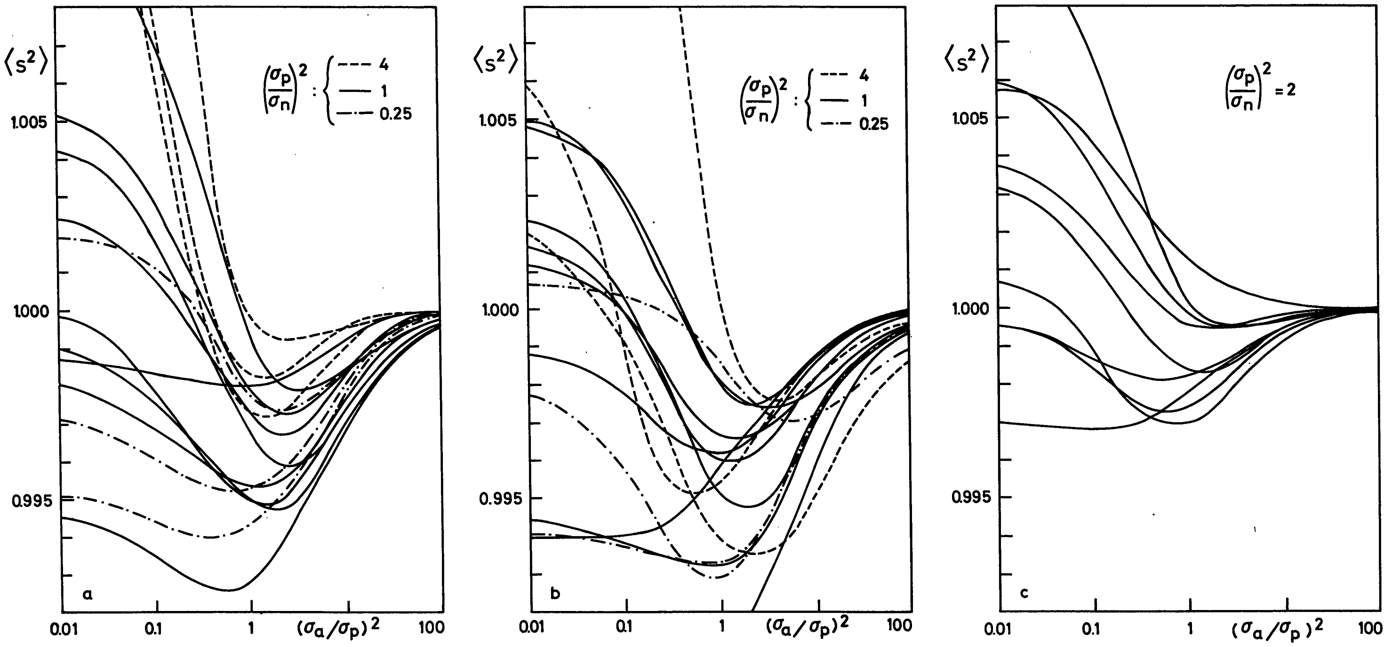

Fig. 4a–c. Same as Fig. 3, but obtained from a numerical simulation of the astrometric practice. **a** normally distributed plate parameters $(\boldsymbol{p}_p)_i$; **b** non-normal distribution of the $(\boldsymbol{p}_p)_i$ as described in the text; **c** computed by use of the model I instead of model II and normally distributed $(\boldsymbol{p}_p)_i$

density $\rho(p)$=constant for $|p| \leqq 15^{1/2}\sigma_p$ and $\rho(p)=0$ elsewhere for 20 per cent (by pseudo-random selection of the 36 plates) and $p=0$ for the remaining 80 per cent. The standard deviation was chosen as 0.25, 1, and 4 times the standard deviation of the corresponding component of p_n (adjustment with model II). With these values of p_7 and p_8 the simulated terms for x and y in any case are computed from $p_7\xi^2$ and $p_8\xi\eta$, respectively. The correlated pseudo random numbers are computed from uncorrelated ones by use of Eq. 7. The noise of the (x, y)- and (α, δ)-measurements was added by normally distributed pseudo random numbers with the standard deviation that is used for the computation of the weights in the adjustment. By reason of simplicity only p_7 and p_8 were generated as pseudo random numbers whilst p_1 through p_6 remained unchanged. The physical meaning of this simulation therefore is not exactly a tilt.

The additional observation Eqs. 2 only change the block diagonal of the matrix of the reduced normal equations. Therefore these equations were produced without use of such equations and stored. Then Eqs. 2 were introduced and the normal equations solved changing the standard deviation σ_a stepwise.

4.2. The Results of the Investigation

The analysis of the simulations was performed by computing the differences $(\Delta\alpha \cos\delta, \Delta\delta)$ for the 2203 stars of the block. From these differences the variance $\langle s^2\rangle = \langle(\Delta\alpha\cos\delta)^2 + (\Delta\delta)^2\rangle$ follows, which is drawn in Fig. 4 as a function of $(\sigma_a/\sigma_p)^2$ and of $(\sigma_p/\sigma_n)^2$. Figure 4 therefore directly corresponds to Fig. 3. In contrast to Fig. 3 this time $\langle s^2\rangle$ contains all the noise of the star positions which in practice is present. Clearly the error of the measurement of a star image on the plate causes by far the greatest error of the position of the corresponding star in comparison to the error propagation from plate constants into that position. It is, in the case of the AGK3, about 0\".2 (as the mean taken from two plates). Assuming densities of 20 reference stars and 200 field stars per plate and standard deviations $\sigma_{xy} = 0\overset{''}{.}28$ for a single measurement of a star image and $\sigma_{\alpha\delta} = 0\overset{''}{.}1$ for the position of the reference stars, the model accuracy follows from the results of von der Heide (1977b) as about 0\".05 for the above model I and 0\".04 for that model without the tilt terms p_7 and p_8, which is equivalent to model I plus Eqs. 2 for p_7 and p_8 with very small σ_a. The resulting variance of the errors of the star positions is the sum of both terms: 0.0425 or 0.0416 (arc s)^2, respectively. The difference is only 2 per cent in the domain of star positions, although it is about 40 per cent in the domain of plate constants (which causes local systematic errors of the star positions). It is therefore not surprising that the influence of the Eqs. 2 (used here only for one or two of the 7 or 8 plate constants) is about 0.5 per cent as can be seen from the scale of $\langle s^2\rangle$ in Fig. 4. The influence is considerably larger when single plates are adjusted instead of this overlap solution.

The conformity of the curves of Fig. 3 and Fig. 4 is excellent. The position of the minima varies as a consequence of statistical fluctuations. This variation is larger than one would expect from the χ^2-distribution of $\langle(\boldsymbol{p}_p)_i^2\rangle$ for 36 plates. The reason is that $\langle s^2\rangle$ is not simply a function of $\langle(\boldsymbol{p}_p)_i^2\rangle$ but of all $(\boldsymbol{p}_p)_i$ in a very complicated way. The optimum value for σ_a obviously is σ_p. Because a too low value of σ_a may cause large errors when $\sigma_p > \sigma_n$, it may be better to take a larger value for σ_a, if σ_p is computed from a small number of plates. Similarly it may be advantageous to take $\sigma_a < \sigma_p$, when $\sigma_p < \sigma_n$.

Figure 4b also shows that additional observation equations are valuable in the case of a non-Gaussian distribution of the physical plate parameters. Nevertheless there is one problem: The actual distribution of plate parameters from the AGK3 shows no significant difference from a normal distribution except a very small number of plates, where the parameters lie up to several hundred times the standard deviation aside the mean.

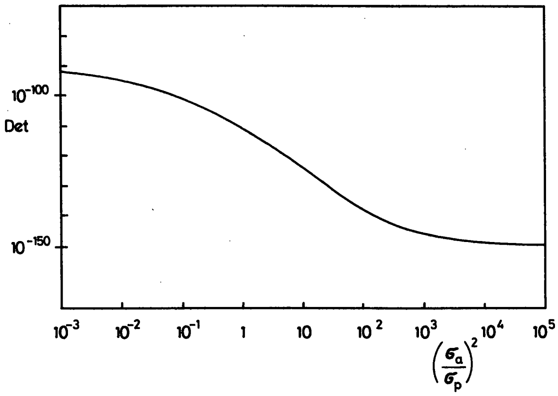

Fig. 5. Dependence of the determinant of the scaled normal matrix on the weight used for the additional observation equations for the parameters p_7 and p_8 of model II

Because of this very large effect it is easy to detect these plates. The overlap program of von der Heide (1978) allows for this fact by applying a different and very large σ_a specially for such plates. Of course the number of such escaping plates must be very small compared to the total number of plates, otherwise the presented results are not applicable.

The results obtained with model I are shown in Fig. 4c. The possible gain in systematic accuracy seems to be smaller than that using model II. Although this conclusion statistically is not significant, it is probable because the parameters p_7 and p_8 were simulated with a correlation factor of 0.5, which was taken into account in the case of model II. The results for $(\sigma_p/\sigma_n)^2=1$ of Fig. 4a and those for $(\sigma_p/\sigma_n)^2=2$ of Fig. 4c base on the same set of simulated plate constants. The parameter is different because σ_n^2 for p_7 of model I is about half that for p_7 and p_8 of model II.

The additional observation equations for plate parameters increase the diagonal elements of the matrix of the reduced system of normal equations. Before solving this system, the matrix is scaled such that all diagonal elements are equal to one and the symmetry is retained. Then the determinant does not depend on an arbitrary factor of the observation equations which is advantageous if the determinants of different adjustments are to be compared. The dependence of this determinant on the variance used for the additional observation equations for p_7 and p_8 of model II is drawn in Fig. 5 for our testblock of 36 plates. This diagram demonstrates the sensivity of the determinant on single plate parameters.

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