

# PHYS 410 Homework 2

Gavin Pringle, 56401938

October 28, 2024

## Introduction

In this homework assignment, the fourth-order Runge-Kutta method for computing numerical solutions of ODEs is explored. This is done in stages, culminating in creating a MATLAB function that numerically integrates an ODE using an algorithm that automatically varies the step size of the integrator in order to achieve a relative error tolerance.

First, a function `rk4step` is written that computes a single fourth-order Runge-Kutta step for a system of coupled first-order ODEs, returning the approximate values of the dependent variables after a defined time step. The function `rk4step` is then used in the function `rk4` which computes the solution of an initial value problem over a range of values for the independent variable, done by taking multiple fourth-order Runge-Kutta steps in a loop. Lastly, the function `rk4ad` is written which finds the numerical solution of an initial value problem by comparing the results of fourth-order Runge-Kutta steps of different sizes and then varying the step size as until the error in the approximation is below a specified relative tolerance.

## Review of Theory

### Casting systems of ODEs in first-order form

In order to solve complicated ODEs numerically, it is useful to first cast them in a canonical form that is easier for a computer program to understand. Any ODE defining the function  $y(t)$  that is of the form

$$f(t, y, y', y'', y^{(3)}, \dots, y^{(N)}) = 0$$

can be rewritten as a system of  $N$  coupled first-order ODEs for the functions  $y_i(t)$ ,  $i = 1, 2, 3, \dots, N$ :

$$y'_i(t) \equiv \frac{dy_i}{dt}(t) = f_i(t, y_1, y_2, y_3, \dots, y_N) \quad (1)$$

where  $f_i$  are known functions of  $t$  and  $y_i$ . This is equivalent to

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \quad \text{where} \quad \mathbf{y} \equiv (y_1, y_2, y_3, \dots, y_N) \quad (2)$$

For example, the function  $y^{(4)}(t) = f(t)$  can be written as

$$y'_3 = f, \quad y' = y_1, \quad y'_1 = y_2, \quad y'_2 = y_3$$

### The fourth-order Runge-Kutta step

The fourth-order Runge-Kutta step for numerically solving a system of  $N$  coupled first-order ODEs is defined as:

$$y_i(t_0 + h) = y_i(t_0) + \frac{h}{6}(f_{0,i} + 2f_{1,i} + 2f_{2,i} + f_{3,i}) \quad (3)$$

with the terms  $f_{0,i}$   $f_{1,i}$   $f_{2,i}$   $f_{3,i}$  given by

$$f_{0,i} = f_i(t_0, y_{0,i}) \quad (4)$$

$$f_{1,i} = f_i\left(t_0 + \frac{h}{2}, y_{0,i} + \frac{h}{2}f_{0,i}\right) \quad (5)$$

$$f_{2,i} = f_i\left(t_0 + \frac{h}{2}, y_{0,i} + \frac{h}{2}f_{1,i}\right) \quad (6)$$

$$f_{3,i} = f_i(t_0 + h, y_{0,i} + hf_{2,i}) \quad (7)$$

where each  $f_i$  on the right-hand sides of the above equations are defined in (1) and  $h$  is the step size. This can be understood as a weighted sum of four numerical approximations to the solution of the ODE. A single fourth-order Runge-Kutta step is accurate to  $O(\Delta t^5)$ .

## Numerical Approach

### Consecutive fourth-order Runge-Kutta steps

In order to compute the numerical solution to an ODE (or in canonical form, a system of first-order ODEs), multiple consecutive fourth-order Runge-Kutta steps must be taken. The MATLAB implementation of a single fourth-order Runge-Kutta step is shown in Appendix A as `rk4step.m`, while Appendix C shows the MATLAB implementation of a complete fourth-order Runge-Kutta ODE integrator as `rk4.m`. In `rk4.m`, equation (3) is repeatedly computed using the previous output of equation (3) as an input. The step size `dt` is given by the difference at the current time-step between independent variable values at which the solution of the ODE is to be computed at, passed as `tspan`.

It is important to note that while a single fourth-order Runge-Kutta step is accurate to  $O(\Delta t^5)$ , the full numerical solution to an ODE given by `rk4.m` is accurate to  $O(\Delta t^4)$ , since error accumulates linearly throughout the integration.

### Adaptive step sizing

The Runge-Kutta integrator can be made more accurate by varying the step size `dt` at each step depending on an estimation of the error accumulated in that step. To show how the per-step error can be estimated using time-steps of multiple lengths, consider (for each dependent variable  $y$  in the system of ODEs) a "coarse" step of length  $\Delta t$  (producing the output  $y_C$ ) and two "fine" steps of length  $\Delta t/2$  (producing the output  $y_F$ ).

$$\begin{aligned} y_C(t_0 + \Delta t) &\approx y_{\text{exact}}(t_0 + \Delta t) + k(t_0)\Delta t^5 \\ y_F(t_0 + \Delta t) &\approx y_{\text{exact}}(t_0 + \Delta t) + k(t_0)\left(\frac{\Delta t}{2}\right)^5 + k\left(t_0 + \frac{\Delta t}{2}\right)\left(\frac{\Delta t}{2}\right)^5 \\ &\approx y_{\text{exact}}(t_0 + \Delta t) + 2k(t_0)\left(\frac{\Delta t}{2}\right)^5 \end{aligned}$$

In the above equations,  $k(t)$  is some function of time. Subtracting  $y_F$  and  $y_C$  at the advanced time, we get

$$y_C(t_0 + \Delta t) - y_F(t_0 + \Delta t) \approx \frac{15}{16}k(t_0)\Delta t^5 \approx \frac{15}{16}e_C \quad (8)$$

which yields an estimate for the local solution error  $e_C$ .

This error estimation is implemented in the MATLAB function `rk4ad.m` (Appendix F), which functions as an adaptive step size fourth-order Runge-Kutta ODE integrator. In `rk4ad.m`, similar to `rk4.m`, the function is written to compute the values of the ODE defined by the argument `fcn` at the times defined in `tspan`, with the initial values of the dependent variables defined by `y0`. However, the additional argument

`reltol` is also passed which directs the function to iteratively decrease the step sizes in between values of `tspan` until the step-size to step-size error described above is below `reltol`. A lower bound for the step-size is defined by the constant `floor = 1.0e-4`.

At each time-step in `rk4ad.m`, the coarse numerical solution is computed using the distance to the next time-step and the fine numerical solution is computed using two time-steps equalling half the distance to the next time step. These solutions are then compared as shown in (8). If the estimated solution error is above `reltol`, the process is repeated where each fine step is divided into two even finer steps and the solution across the four time steps is compared to the solution across the two time steps. If the new estimated error is below `reltol`, the process stops and the function moves on to computing the solution at the next time-step, and if not the process of successively halving the step-size continues until either `reltol` is achieved or the step-size limit `floor` is reached.

## Implementation

### ODE test functions

In order to test the functions `rk4step.m`, `rk4.m`, and `rk4ad.m`, sample ODEs must be used. The first of the two ODEs used in this assignment to test the ODE integrators is the ODE for the simple harmonic oscillator with unit angular frequency. This ODE is given by the following equation:

$$\frac{d^2x}{dt^2}(t) = -x(t) \quad (9)$$

with the initial conditions

$$x(0) = 0, \quad \frac{dx}{dt}(0) = 1$$

In the canonical first-order form this ODE can be written as

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1$$

where

$$x_1 = x, \quad x_2 = \frac{dx}{dt}$$

The function `fcn_sho` was written to implement this ODE in MATLAB in order to pass it to `rk4.m` and `rk4ad.m`:

```
function dxdt = fcn_sho(t, x)
    dxdt = zeros(2,1);
    dxdt(1) = x(2);
    dxdt(2) = -x(1);
end
```

The second of the two ODEs used is the ODE for the unforced Van der Pol oscillator, shown below

$$\frac{d^2x}{dt^2}(t) = -x(t) - a(x(t)^2 - 1)\frac{dx}{dt}(t) \quad (10)$$

In this assignment,  $a = 5$  and the following initial conditions are used:

$$x(0) = 1, \quad \frac{dx}{dt}(0) = -6$$

In the canonical first-order form this ODE can be written as

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1(t) - a(x_1(t)^2 - 1)x_2(t)$$

where

$$x_1 = x, \quad x_2 = \frac{dx}{dt}$$

The function `vdp_sho` was written to implement this ODE in MATLAB in order to pass it to `rk4.m` and `rk4ad.m`:

```
function dxdt = fcn_vdp(t, x)
    global a;
    dxdt = ones(2,1);
    dxdt(1) = x(2);
    dxdt(2) = -x(1) - a*(x(1)^2 - 1)*x(2);
end
```

## Results

### rk4step.m Output

To test the functionality and accuracy of the single fourth-order Runge-Kutta step function `rk4step.m`, the script `trk4step.m` was written, shown in Appendix B. The simple harmonic oscillator ODE is implemented as described previously, and the function `rk4step` is called repeatedly in a loop with an increasing step size defined by the array `dt = linspace(0.01, 0.3, 1000)` and the parameters `x0 = [0; 1]` and `t0 = 0`. The error between computed value of  $y$  for each step size and the exact value given by  $y(t + \Delta t) = \sin(t + \Delta t)$  was then plotted.

By guessing and checking, the constant  $C = 0.0083$  was found that causes the curve defined by  $C\Delta t^5$  to align with the curve given by the error between the computed solutions and the exact solutions. Since this curve is proportional to  $\Delta t^5$  as shown in Figure 1, we can conclude that a single step of the fourth-order Runge-Kutta integrator is accurate to  $O(\Delta t^5)$ .

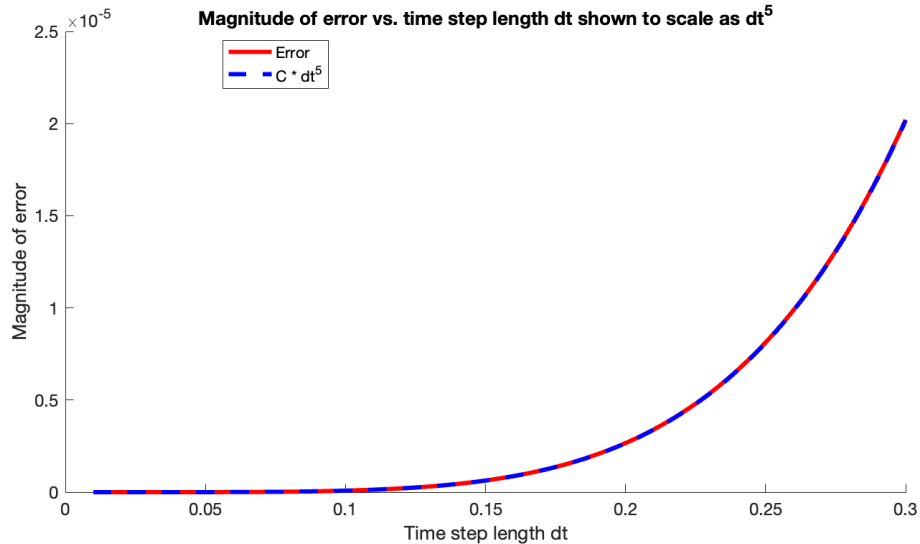


Figure 1: Exact error of Runge-Kutta steps of various sizes for the simple harmonic oscillator ODE at  $t_0 = 0$ . Alignment with the curve  $0.0083 \times dt^5$  shows the error is proportional to  $\Delta t^5$ .

## rk4.m Output

The function `rk4` was written to numerically integrate an ODE by taking consecutive Runge-Kutta steps, as described previously. To test this function, two scripts were written, the first being `trk4.sho.m`, which calls `rk4` to integrate the simple harmonic oscillator ODE. This was done with the initial conditions defined by  $x_0 = [0; 1]$  on the interval  $0 \leq t \leq 3\pi$  at discretization levels  $l = 6, 7, 8$ . Since the exact solution to this IVP is given by  $y(t) = \sin(t)$ , the exact errors were calculated for each discretization level.

For fourth order convergence, the magnitude of the error given by the numerical solutions is expected to decrease by a factor of  $(2\Delta l)^4$  for each increase  $\Delta l$  above a baseline  $l$ . If we scale the errors at higher discretization levels by dividing by this value, we should see near convergence for the error curves if our approximation is accurate to  $O(\Delta t^4)$ . We can see in Figure 2 that is the case as we expect. Refer to Appendix D for the MATLAB code for `trk4.sho.m`.

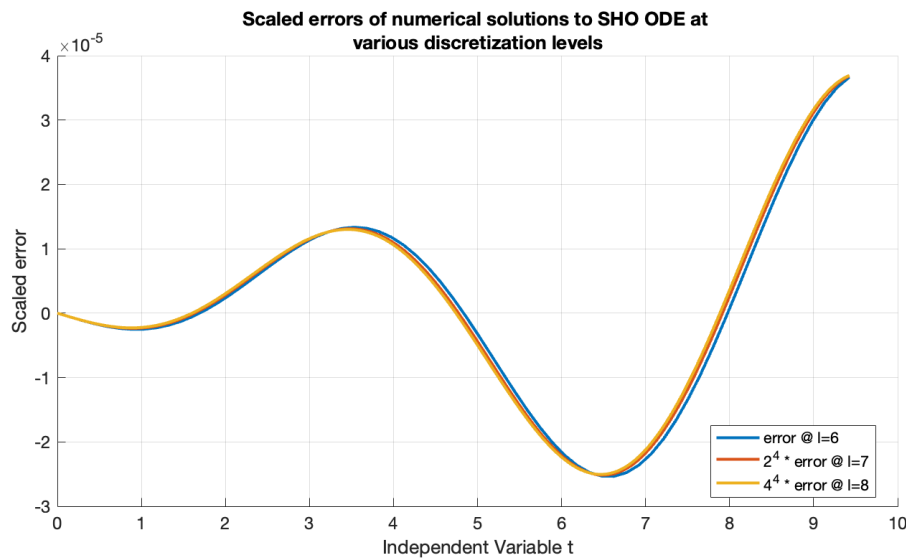


Figure 2: Fourth-order convergence shown by near alignment of the exact errors of numerical solutions at different discretization levels, each scaled to show expected fourth-order convergence behaviour.

The second script written to test the function `rk4` is `trk4.vdp.m`, which calls `rk4` to integrate the unforced Van der Pol oscillator ODE. This was done with the initial conditions defined by  $x_0 = [1; -6]$  on the interval  $0 \leq t \leq 100$  at discretization level 12. The numerical solution for the function  $x(t)$  is plotted in Figure 3 and the phase space evolution  $\frac{dx}{dt}(x)$  is plotted in Figure 4. In view of the [Wikipedia page](#) for the Van der Pol oscillator, we can confirm that this is the expected output which provides evidence that `rk4` has been implemented correctly. Refer to Appendix E for the MATLAB code for `trk4.vdp.m`.

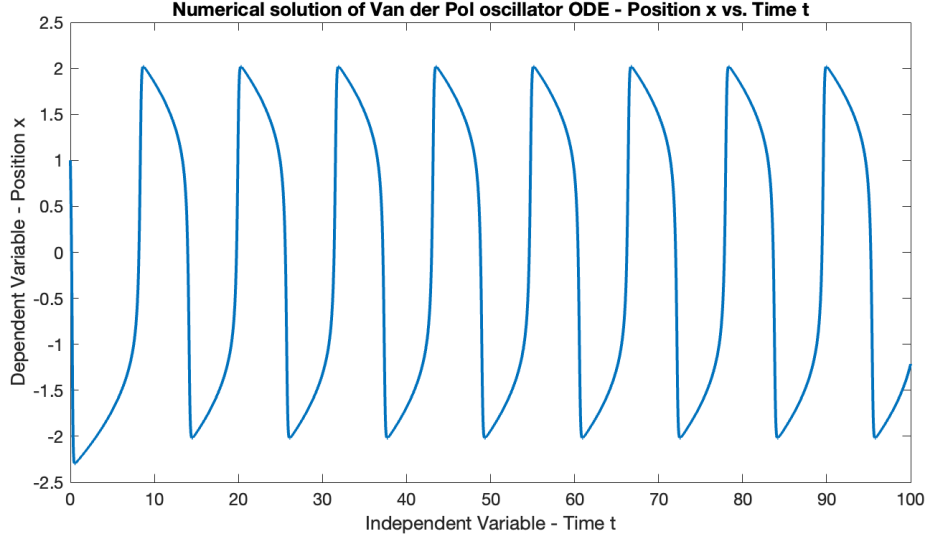


Figure 3: Numerical solution of the Van der Pol oscillator ODE computed by `rk4.m` on the domain  $0 \leq t \leq 100$  at discretization level 12.

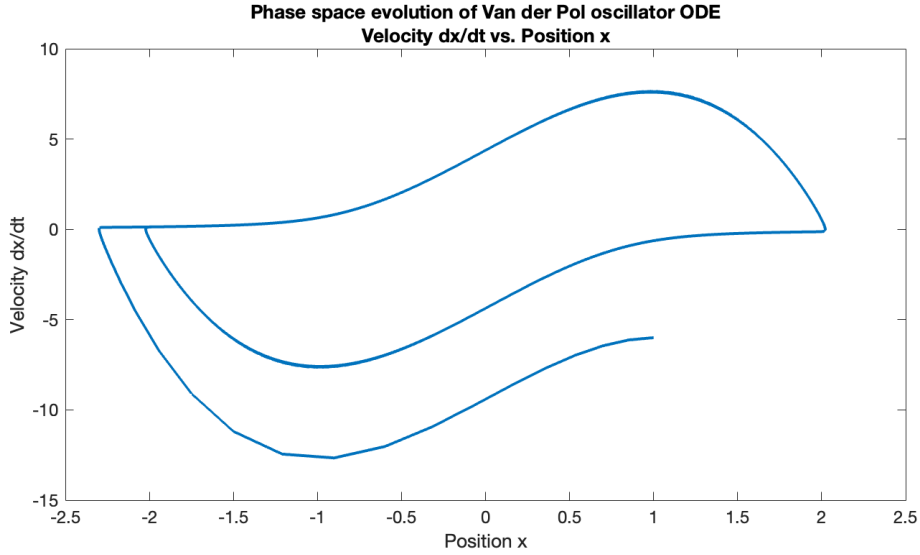


Figure 4: Phase space evolution of the Van der Pol oscillator ODE computed by `rk4.m` on the domain  $0 \leq t \leq 100$  at discretization level 12.

### `rk4ad.m` Output

The fourth-order Runge-Kutta ODE integrator described previously is implemented in the function `rk4ad`. To test this function, two scripts were again written, the first being `trk4.sho.m`, which calls `rk4` to integrate the simple harmonic oscillator ODE. This script uses the function parameter `x0 = [0; 1]` and `tspan = linspace(0.0, 3.0 * pi, 65)` to integrate the function four times with the relative tolerances  $1.0\text{e-}5$ ,  $1.0\text{e-}7$ ,  $1.0\text{e-}9$ ,  $1.0\text{e-}11$ . Figure 5 shows the exact error of each approximation throughout the domain of  $t$  upon which the ODE was integrated. It is clear that the numerical approximations rapidly become more accurate as `reltol` decreases, as expected. Refer to Appendix E for the MATLAB code for `trk4ad.sho.m`.

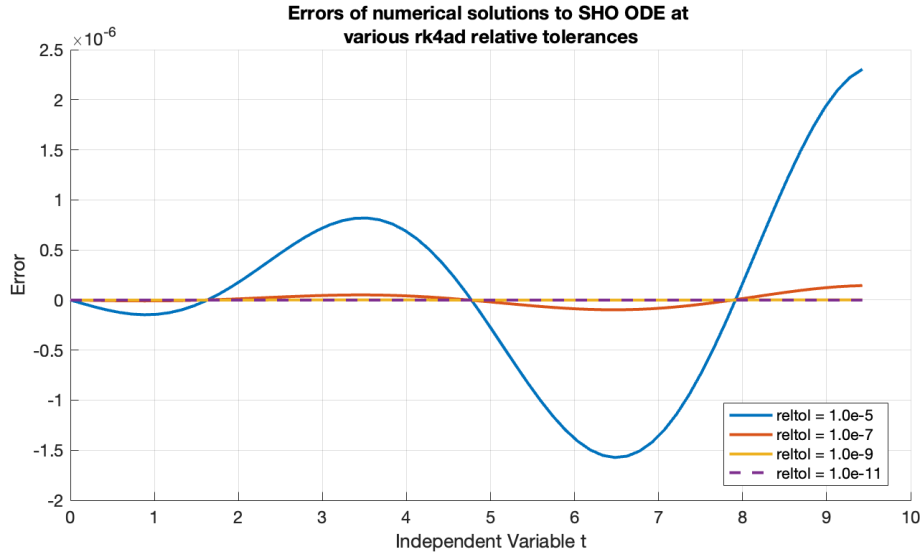


Figure 5: Errors of numerical solutions to the simple harmonic oscillator ODE at various `rk4ad.m` relative tolerances `reltol`, on the domain defined by `tspan = linspace(0.0, 3.0 * pi, 65)`.

As for `rk4.m`, the second script written to test the function `rk4ad` is `trk4ad.vdp.m` which calls `rk4ad` to integrate the unforced Van der Pol oscillator ODE. This was done with the initial conditions defined by `x0 = [1; -6]` on the interval defined by `tspan = linspace(0.0, 100, 4097)` with `reltol = 1.0e-10`. The numerical solution for the function  $x(t)$  is plotted in Figure 6 and the phase space evolution  $\frac{dx}{dt}(x)$  is plotted in Figure 7. These plots look quite similar to Figures 3 and 4, indicating that the function `rk4ad` again produces an accurate solution that is only more precise than function `rk4ad`.

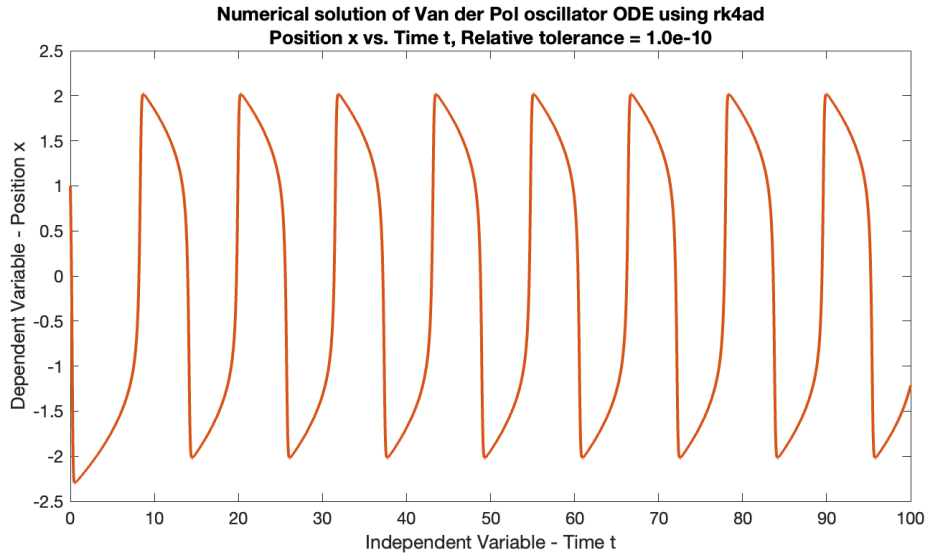


Figure 6: Numerical solution of the Van der Pol oscillator ODE computed by `rk4ad.m` on the domain defined by `tspan = linspace(0.0, 100, 4097)` with `reltol = 1.0e-10`.

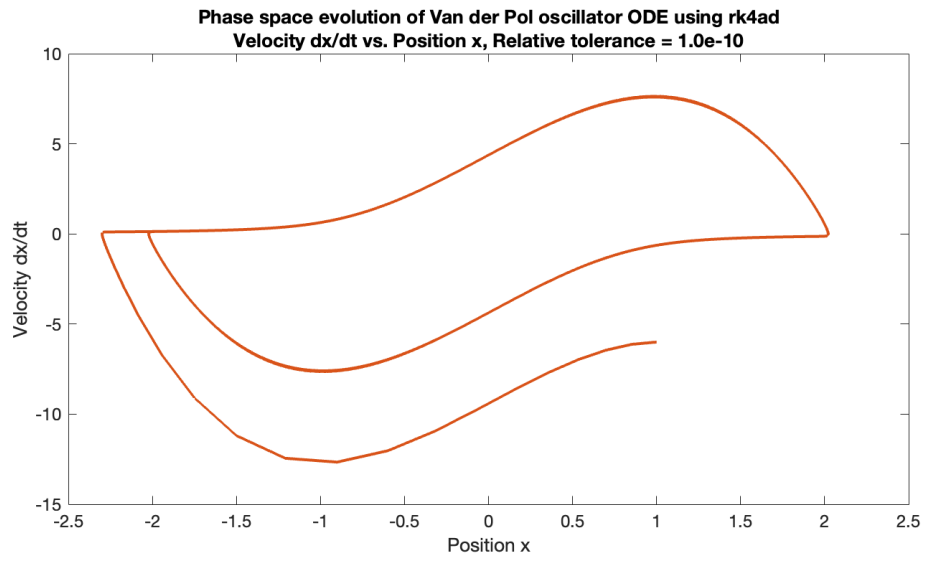


Figure 7: Phase space evolution of the Van der Pol oscillator ODE computed by `rk4ad.m` on the domain defined by `tspan = linspace(0.0, 100, 4097)` with `reltol = 1.0e-10`.

## Conclusions



## Appendix A - rk4step.m Code

```
1 %% Problem 1 — Single Fourth Order Runge–Kutta Step
2
3 % Function that computes a single fourth order Runge–Kutta Step.
4 %
5 % Inputs
6 %     fcn:      Function handle for right hand sides of ODEs (returns
7 %              length–n column vector).
8 %     t0:      Initial value of independent variable.
9 %     dt:      Time step.
10 %     y0:      Initial values (length–n column vector).
11 %
12 % Output
13 %     yout:     Final values (length–n column vector)
14 function yout = rk4step(fcn, t0, dt, y0)
15     % Compute terms in RK step
16     f0 = fcn(t0, y0);
17     f1 = fcn(t0 + dt/2, y0 + (dt/2)*f0);
18     f2 = fcn(t0 + dt/2, y0 + (dt/2)*f1);
19     f3 = fcn(t0 + dt, y0 + dt*f2);
20     % Add terms to compute full RK step
21     yout = y0 + (dt/6)*(f0 + 2*f1 + 2*f2 + f3);
22 end
```

## Appendix B - trk4step.m Code

```

1 %% Problem 1 – Test Script
2
3 close all; clear; clc;
4
5 format long;
6
7 % Function that computes right hand sides of ODEs for simple harmonic
8 % oscillator with unit angular frequency. For use in rk4step, rk4, and
9 % rk4ad.
10 % Governing DE:  $x'' = -x$ 
11 % Canonical first order dependent variables:  $x_1 = x, x_2 = x'$ 
12 % System of Equations:  $x_1' = x_2, x_2' = -x_1$ 
13 %
14 % Inputs
15 %     t:      Independent variable at current time-step
16 %     x:      Dependent variables at current time-step (length-n column
17 %              vector).
18 %
19 % Outputs
20 %     dxdt:   Computes the derivatives of x1 and x2 at the current
21 %              time-step (length-n column vector).
22 function dxdt = fcn_sho(t, x)
23     dxdt = zeros(2,1);
24     dxdt(1) = x(2);
25     dxdt(2) = -x(1);
26 end
27
28 % Function parameters for exact solution of  $x(t) = \sin(t)$ 
29 x0 = [0; 1]; % Initial conditions
30 t0 = 0; % Initial time
31 % Vector of linearly increasing time-step lengths
32 dt = linspace(0.01, 0.3, 1000);
33
34 % Run Runge-Kutta step at various time steps
35 xout = zeros(2, length(dt));
36 for i = 1:length(dt)
37     xout(:,i) = rk4step(@fcn_sho, t0, dt(i), x0);
38 end
39
40 % Calculate the error at each time step length using the known exact
41 % solution
42 errors = abs(xout(1,:) - sin(dt));
43
44 % Plot error as a function of dt and compare to  $C \cdot dt^5$ 
45 hold on;
46 plot(dt, errors, "Color", 'r', "LineWidth", 3);
47 C = 8.3e-3;
48 plot(dt, C*dt.^5, "--", "Color", 'b', "LineWidth", 3);
49 title("Magnitude of error vs. time step length dt shown to scale as dt^5");
50 xlabel("Time step length dt");
51 ylabel("Magnitude of error");
52 legend(["Error", "C * dt^5"], 'location', 'best');
53 ax = gca;
54 ax.FontSize = 12;

```

## Appendix C - rk4.m Code

```
1 %% Problem 2 – Runge–Kutta System of ODEs Integrator
2
3 % Function that numerically computes the solution to a system of ODEs
4 % over a given period of time using a fourth–order Runge–Kutta method.
5 %
6 % Inputs
7 %     fcn:      Function handle for right hand sides of ODEs (returns
8 %              length–n column vector)
9 %     tspan:    Vector of output times (length nout).
10 %     y0:       Initial values (length–n column vector).
11 %
12 % Outputs
13 %     tout:     Vector of output times.
14 %     yout:     Output values (nout x n array. The ith column of yout
15 %              contains the nout values of the ith dependent variable).
16 function [tout yout] = rk4(fcn, tspan, y0)
17     % Number of equations in ODE system
18     n = max(size(y0));
19     % Number of time–steps
20     nout = max(size(tspan));
21
22     % Initialize array for output values
23     yout = zeros(nout, n);
24     yout(1,:) = y0.';
25
26     % Integrate ODE
27     for i = 2:nout
28         % Step size for the current step
29         dt = tspan(i) – tspan(i–1);
30         % Compute the values of the dependent variables at the next step
31         yout(i,:) = rk4step(fcn, tspan(i–1), dt, yout(i–1,:)).';
32     end
33
34     % Generate array of output values
35     tout = tspan;
36 end
```

## Appendix D - trk4\_sho.m Code

```

1 %% Problem 2 – Test Script – Simple Harmonic Oscillator
2
3 close all; clear; clc;
4
5 format long;
6
7 % Function that computes right hand sides of ODEs for simple harmonic
8 % oscillator with unit angular frequency. For use in rk4step, rk4, and
9 % rk4ad.
10 % Governing DE:  $x'' = -x$ 
11 % Canonical first order dependent variables:  $x_1 = x, x_2 = x'$ 
12 % System of Equations:  $x_1' = x_2, x_2' = -x_1$ 
13 %
14 % Inputs
15 %     t:      Independent variable at current time-step
16 %     x:      Dependent variables at current time-step (length-n column
17 %              vector).
18 %
19 % Outputs
20 %     dxdt:   Computes the derivatives of x1 and x2 at the current
21 %              time-step (length-n column vector).
22 function dxdt = fcn_sho(t, x)
23     dxdt = zeros(2,1);
24     dxdt(1) = x(2);
25     dxdt(2) = -x(1);
26 end
27
28 % Function parameters for exact solution of  $x(t) = \sin(t)$ 
29 x0 = [0; 1]; % Initial conditions
30 t0 = 0; tf = 3*pi; % Start and end times
31
32 % Vector of output times for each discretization level
33 tspan6 = linspace(t0, tf, 2^6 + 1);
34 tspan7 = linspace(t0, tf, 2^7 + 1);
35 tspan8 = linspace(t0, tf, 2^8 + 1);
36
37 % Compute ODE numerical solution at each discretization level
38 [tout6 xout6] = rk4(@fcn_sho, tspan6, x0);
39 [tout7 xout7] = rk4(@fcn_sho, tspan7, x0);
40 [tout8 xout8] = rk4(@fcn_sho, tspan8, x0);
41
42 % Plot the solutions at each discretization level
43 fig1 = figure(1);
44 hold on
45 plot(tout6, xout6(:,1), "LineWidth", 2);
46 plot(tout7, xout7(:,1), "LineWidth", 2);
47 plot(tout8, xout8(:,1), "LineWidth", 2);
48 title("Numerical solutions to SHO ODE at various discretization levels");
49 xlabel("Independent Variable t");
50 ylabel("Dependent Variable x");
51 legend(["l = 6", "l = 7", "l = 8"], 'location', 'best');
52 ax = gca;
53 ax.FontSize = 12;
54
55 % Compute the errors at each time step for each discretization level

```

```

56 errors6 = xout6(:,1) - sin(tout6).';
57 errors7 = xout7(:,1) - sin(tout7).';
58 errors8 = xout8(:,1) - sin(tout8).';
59
60 % Plot the scaled errors for each discretization level
61 fig2 = figure(2);
62 hold on
63 plot(tout6, errors6, "LineWidth", 2);
64 plot(tout7, 2^4*errors7, "LineWidth", 2);
65 plot(tout8, 4^4*errors8, "LineWidth", 2);
66 grid on
67 title({"Scaled errors of numerical solutions to SHO ODE at ", ...
68       "various discretization levels"});
69 xlabel("Independent Variable t");
70 ylabel("Scaled error");
71 legend(["error @ l=6", "2^4 * error @ l=7", "4^4 * error @ l=8"], ...
72        'location', 'best');
73 ax = gca;
74 ax.FontSize = 12;

```

## Appendix E - trk4\_vdp.m Code

```

1 %% Problem 2 – Test Script – Van der Pol oscillator
2
3 close all; clear; clc;
4
5 format long;
6
7 % Function that computes right hand sides of ODEs for Van der Pol
8 % Oscillator. Following Tsatsos: https://arxiv.org/pdf/0803.1658
9 %
10 % Governing DE:  $x'' = -x - a(x^2 - 1)x'$ 
11 % Canonical first order dependent variables:  $x_1 = x$ ,  $x_2 = x'$ 
12 % System of Equations:
13 %       $x_1' = x_2$ 
14 %       $x_2' = -x_1 - a(x_1^2 - 1)x_2$ 
15 %
16 % Inputs
17 %      t:      Independent variable at current time-step
18 %      x:      Dependent variables at current time-step (length-n column
19 %              vector).
20 %
21 % Outputs
22 %      dxdt:   Computes the derivatives of x1 and x2 at the current
23 %              time-step (length-n column vector).
24 function dxdt = fcn_vdp(t, x)
25     global a;
26     dxdt = ones(2,1);
27     dxdt(1) = x(2);
28     dxdt(2) = -x(1) - a*(x(1)^2 - 1)*x(2);
29 end
30
31 % Function parameters
32 x0 = [1; -6];      % Initial conditions
33 t0 = 0; tf = 100;  % Start and end times
34 global a; a = 5;    % Adjustable parameter
35
36 % Discretization level
37 level = 12;
38 tspan = linspace(t0, tf, 2^level + 1);
39
40 % Compute ODE numerical solution
41 [tout xout] = rk4(@fcn_vdp, tspan, x0);
42
43 % Plot position vs time
44 fig1 = figure(1);
45 plot(tout, xout(:,1), "LineWidth", 2)
46 title("Numerical solution of Van der Pol oscillator ODE – Position x vs.
47       Time t");
48 xlabel("Independent Variable – Time t");
49 ylabel("Dependent Variable – Position x");
50 ax = gca;
51 ax.FontSize = 12;
52
53 % Plot phase space evolution
54 fig2 = figure(2);
55 plot(xout(:,1), xout(:,2), "LineWidth", 2)
56 title({"Phase space evolution of Van der Pol oscillator ODE", ...

```

```
56         "Velocity dx/dt vs. Position x"});
57 xlabel("Position x");
58 ylabel("Velocity dx/dt");
59 ax = gca;
60 ax.FontSize = 12;
```

## Appendix F - rk4ad.m Code

```

1 %% Problem 3 – Adaptive Fourth Order Runge–Kutta System of ODEs Integrator
2
3 % Function that numerically computes the solution to a system of ODEs
4 % over a given period of time using a fourth–order Runge–Kutta method
5 % with adaptive steps sizes to ensure a relative tolerance is reached.
6 %
7 % Inputs
8 %     fcn:      Function handle for right hand sides of ODEs (returns
9 %              length–n column vector)
10 %     tspan:    Vector of output times (length nout vector).
11 %     reltol:   Relative tolerance parameter.
12 %     y0:      Initial values (length–n column vector).
13 %
14 % Outputs
15 %     tout:     Output times (length–nout column vector, elements
16 %              identical to tspan).
17 %     yout:     Output values (nout x n array. The ith column of yout
18 %              contains the nout values of the ith dependent variable).
19 function [tout yout] = rk4ad(fcn, tspan, reltol, y0)
20     % Number of equations in ODE system
21     n = max(size(y0));
22     % Number of time–steps
23     nout = max(size(tspan));
24     % Lower bound on step size
25     floor = 1.0e–4;
26
27     % Initialize array for output values
28     yout = zeros(nout, n);
29     yout(1,:) = y0.';
30
31     % Integrate ODE
32     for i = 2:nout
33         % Compute coarse rk4step arguments
34         tprev = tspan(i–1);
35         yprev = yout(i–1,:).';
36         dt = tspan(i) – tspan(i–1);
37
38         % Compute fine and coarse approximations for y(tprev + dt)
39         yc = rk4step(fcn, tprev, dt, yprev);
40         if dt/2 < floor
41             % If fine step is lower than floor, cannot narrow down any
42             % further
43             yout(i,:) = yc.';
44             continue;
45         end
46         yhalf = rk4step(fcn, tprev, dt/2, yprev);
47         yf = rk4step(fcn, tprev + dt/2, dt/2, yhalf);
48
49         % Check if error meets relative tolerance parameter
50         if abs((yc – yf)/yf) < reltol
51             yout(i,:) = yf.';
52             continue;
53         else
54             % Iteratively compute yf at repeatedly halved dt sizes
55             % until reltol is met or floor is reached
56             j = 2;

```



```

56         while dt/(2^j) > floor % Decrease step size by half each
           iteration
57             yc = yf;
58             yf = yprev;
59             for k = 0:2^j - 1 % Number of steps to get to tprev + dt
60                 yf = rk4step(fcn, tprev + k*dt/(2^j), dt/(2^j), yf);
61             end
62
63             if abs((yc - yf)/yf) < reltol
64                 break;
65             end
66
67             j = j + 1;
68         end
69         yout(i,:) = yf.';
70     end
71 end
72
73 % Generate array of output values
74 tout = tspan;
75 end

```

## Appendix G - trk4ad\_sho.m Code

```

1 %% Problem 3 – Test Script – Simple Harmonic Oscillator
2
3 close all; clear; clc;
4
5 format long;
6
7 % Function that computes right hand sides of ODEs for simple harmonic
8 % oscillator with unit angular frequency. For use in rk4step, rk4, and
9 % rk4ad.
10 % Governing DE:  $x'' = -x$ 
11 % Canonical first order dependent variables:  $x_1 = x, x_2 = x'$ 
12 % System of Equations:  $x_1' = x_2, x_2' = -x_1$ 
13 %
14 % Inputs
15 %     t:      Independent variable at current time-step
16 %     x:      Dependent variables at current time-step (length-n column
17 %              vector).
18 %
19 % Outputs
20 %     dxdt:   Computes the derivatives of x1 and x2 at the current
21 %              time-step (length-n column vector).
22 function dxdt = fcn_sho(t, x)
23     dxdt = zeros(2,1);
24     dxdt(1) = x(2);
25     dxdt(2) = -x(1);
26 end
27
28 % Function parameters for exact solution of  $x(t) = \sin(t)$ 
29 x0 = [0; 1]; % Initial conditions
30 tspan = linspace(0.0, 3.0 * pi, 65); % Vector of output times
31
32 % Compute ODE numerical solution at each relative tolerance
33 [tout5 xout5] = rk4ad(@fcn_sho, tspan, 1.0e-5, x0);
34 [tout7 xout7] = rk4ad(@fcn_sho, tspan, 1.0e-7, x0);
35 [tout9 xout9] = rk4ad(@fcn_sho, tspan, 1.0e-9, x0);
36 [tout11 xout11] = rk4ad(@fcn_sho, tspan, 1.0e-11, x0);
37
38 % Plot the solutions at each relative tolerance
39 fig1 = figure(1);
40 hold on
41 plot(tout5, xout5(:,1), "LineWidth", 2);
42 plot(tout7, xout7(:,1), "LineWidth", 2);
43 plot(tout9, xout9(:,1), "LineWidth", 2);
44 plot(tout11, xout11(:,1), "LineWidth", 2);
45 title("Numerical solutions to SHO ODE from rk4ad at various relative
46       tolerances");
47 xlabel("Independent Variable t");
48 ylabel("Dependent Variable x");
49 legend(["reltol = 1.0e-5", "reltol = 1.0e-7", "reltol = 1.0e-9", ...
50        "reltol = 1.0e-11"], 'location', 'best');
51 ax = gca;
52 ax.FontSize = 12;
53
54 % Compute the errors at each time step for each discretization level
55 errors5 = xout5(:,1) - sin(tout5).';

```

```

55 errors7 = xout7(:,1) - sin(tout7).';
56 errors9 = xout9(:,1) - sin(tout9).';
57 errors11 = xout11(:,1) - sin(tout11).';
58
59 % Plot the errors for each relative tolerance
60 fig2 = figure(2);
61 hold on
62 plot(tout5, errors5, "LineWidth", 2);
63 plot(tout7, errors7, "LineWidth", 2);
64 plot(tout9, errors9, "LineWidth", 2);
65 plot(tout11, errors11, "--", "LineWidth", 2);
66 grid on
67 title({"Errors of numerical solutions to SHO ODE at ", ...
68       "various rk4ad relative tolerances"});
69 xlabel("Independent Variable t");
70 ylabel("Error");
71 legend(["reitol = 1.0e-5", "reitol = 1.0e-7", "reitol = 1.0e-9", ...
72        "reitol = 1.0e-11"], 'location', 'best');
73 ax = gca;
74 ax.FontSize = 12;

```

## Appendix H - trk4ad\_vdp.m Code

```

1 %% Problem 3 – Test Script – Van der Pol oscillator
2
3 close all; clear; clc;
4
5 format long;
6
7 % Function that computes right hand sides of ODEs for Van der Pol
8 % Oscillator. Following Tsatsos: https://arxiv.org/pdf/0803.1658
9 %
10 % Governing DE:  $x'' = -x - a(x^2 - 1)x'$ 
11 % Canonical first order dependent variables:  $x_1 = x$ ,  $x_2 = x'$ 
12 % System of Equations:
13 %       $x_1' = x_2$ 
14 %       $x_2' = -x_1 - a(x_1^2 - 1)x_2$ 
15 %
16 % Inputs
17 %      t:      Independent variable at current time-step
18 %      x:      Dependent variables at current time-step (length-n column
19 %              vector).
20 %
21 % Outputs
22 %      dxdt:   Computes the derivatives of x1 and x2 at the current
23 %              time-step (length-n column vector).
24 function dxdt = fcn_vdp(t, x)
25     global a;
26     dxdt = ones(2,1);
27     dxdt(1) = x(2);
28     dxdt(2) = -x(1) - a*(x(1)^2 - 1)*x(2);
29 end
30
31 % Function parameters
32 x0 = [1; -6]; % Initial conditions
33 tspan = linspace(0.0, 100, 4097); % Vector of output times
34 global a; a = 5; % Adjustable parameter
35 reltol = 1.0e-10; % Relative tolerance
36
37 % Compute ODE numerical solution
38 [tout xout] = rk4ad(@fcn_vdp, tspan, reltol, x0);
39
40 % Plot position vs time
41 fig1 = figure(1);
42 plot(tout, xout(:,1), "LineWidth", 2, "Color", "#D95319")
43 title({"Numerical solution of Van der Pol oscillator ODE using rk4ad", ...
44       "Position x vs. Time t, Relative tolerance = 1.0e-10"});
45 xlabel("Independent Variable – Time t");
46 ylabel("Dependent Variable – Position x");
47 ax = gca;
48 ax.FontSize = 12;
49
50 % Plot phase space evolution
51 fig2 = figure(2);
52 plot(xout(:,1), xout(:,2), "LineWidth", 2, "Color", "#D95319")
53 title({"Phase space evolution of Van der Pol oscillator ODE using rk4ad", ...
54       "Velocity dx/dt vs. Position x, Relative tolerance = 1.0e-10"});
55 xlabel("Position x");

```

```
56 ylabel("Velocity dx/dt");
57 ax = gca;
58 ax.FontSize = 12;
```