Introduction to Fourier Series

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MATLAB

In this topic, we will be using MATLAB as well as performing calculations by hand.

If you have not already got your MATLAB license from IT Help, you should **do so immediately**.

Aims for this week

- Learn about the underlying idea of **Fourier Series**.
- Recognise a **periodic** function and determine its angular frequency.
- Represent a piecewise function using **Heaviside step functions**.
- Calculate the Fourier coefficients using MATLAB.

Jean-Baptiste Joseph Fourier (1768-1830)

“[Mathematics] brings together phenomena the most diverse, and discovers the hidden analogies which unite them”

The Analytical Theory of Heat

He was also Napoleon’s scientific adviser during the 1798 Egyptian campaign.

Introduction

Fourier series

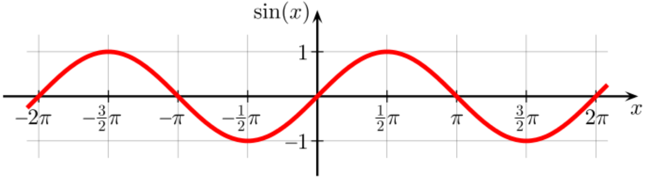
Fourier series is a technique by which **periodic** functions may be represented or approximated by combinations of **simple sine and cosine waves**. We can then determine how important each frequency is to the overall function.

This technique finds applications in many areas of engineering where you might wish to analyse a signal:

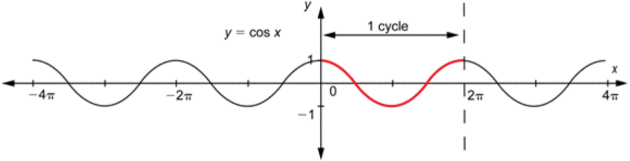
- image processing
- audio compression
- seismic wave analysis
- x-ray crystallography
- material spectroscopy

Periodic functions: Sine and Cosine

Sine, cosine and tangent are examples of **periodic** functions, meaning that they repeat a pattern forever.



The **period** T of both sine and cosine is 2π as this is the minimum time required before the pattern begins to repeat.



Trigonometry and radians

We will **never** use degrees to measure the angular input to sine and cosine.

We will **always** use radians - make sure your calculator is set to it!

So for sine and cosine...

- Instead of a full period every 360°, a full period is 2π
- Instead of a half-period every 180°, a half-period is π.

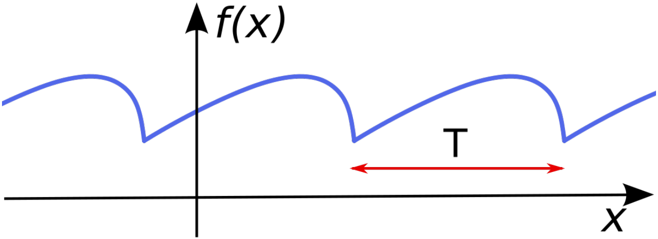
Periodic functions

Periodic function

A function $f(t)$ is periodic with period T if for all values of t, and for any integer m:

$$f(t + mT) = f(t)$$

The minimum time required for one full cycle is the **period** T.



Periodic functions

The number of full cycles per unit of time (usually seconds) is called the **frequency** and given by $f = T^{-1}$.

However it is often useful to use the **angular frequency** ω, measured in radians per second.

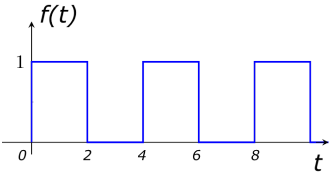
Angular Frequency

$$\omega = \frac{2\pi}{T} \qquad \text{or} \qquad T = \frac{2\pi}{\omega}$$

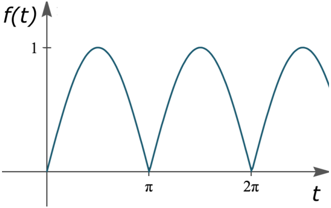
As the period of oscillation T increases, the frequency (both angular and regular) decreases and vice versa.

Exercise: Periodic functions

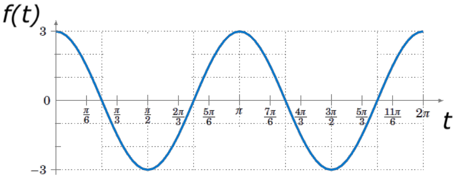
Determine the period and angular frequency of the following periodic functions:



(a) Pulse wave



(b) $|\sin(t)|$



(c) $3\cos(2t)$

Periodic functions: pure sine and cosine waves

Note that if we have a sine or cosine wave of the form:

$$f(t) = a\sin(mt) \quad \text{or} \quad f(t) = a\cos(mt)$$

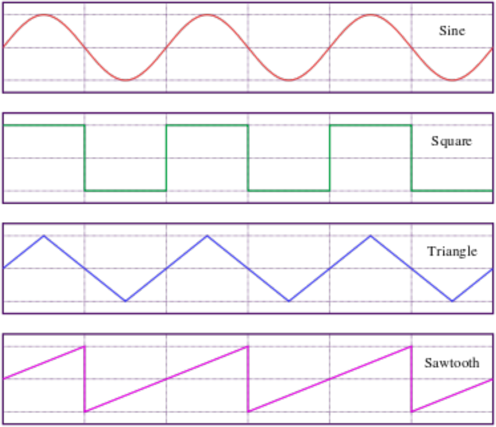
where a and $m > 0$ are constants,

Then the angular frequency is just:

$$\omega = m$$

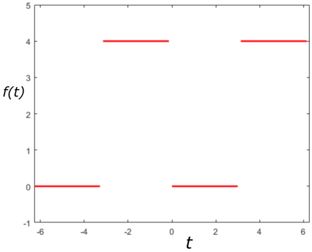
What about other kinds of signals?

Often when analysing audio, electrical, or other signals, we encounter (possibly discontinuous) periodic waveforms:



Motivation for Fourier Series

To analyse the frequencies present it can be useful to approximate these signals by fitting a continuous curve:



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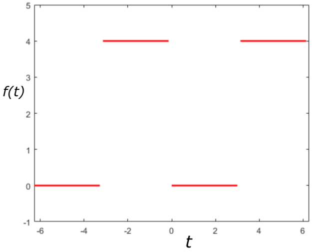
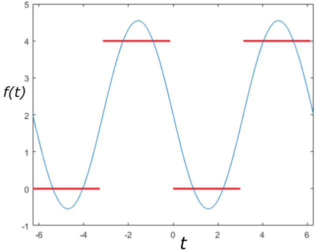




This is the core idea of Fourier Series:

We can represent (almost) any periodic function by some **combination of sine and cosine waves of different frequencies**.

Fourier Series

We can construct any **periodic** function $f(t)$ with period T (angular frequency $\omega = 2\pi/T$) by adding the right amount sine and cosine waves together:

$$\begin{aligned} f(t) = \frac{1}{2}a_0 \quad &+ \quad a_1 \cos(\omega t) + a_2 \cos(2\omega t) + a_3 \cos(3\omega t) + \dots \\ &+ \quad b_1 \sin(\omega t) + b_2 \sin(2\omega t) + b_3 \sin(3\omega t) + \dots \end{aligned}$$

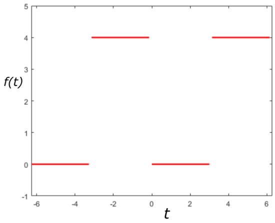
This is now (in general) an infinite sum. We can use sigma notation to write it as:

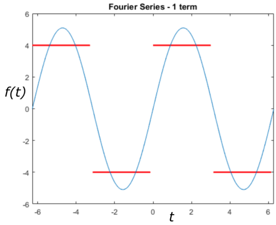
Fourier Series for a general periodic function

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

Fourier Series

As more terms in the Fourier Series are calculated (with higher frequencies), a more accurate approximation is found:





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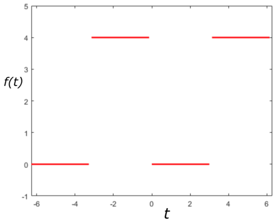

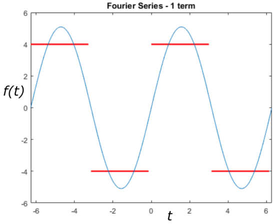

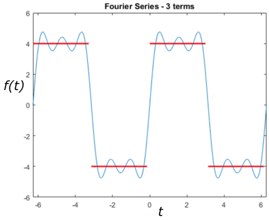

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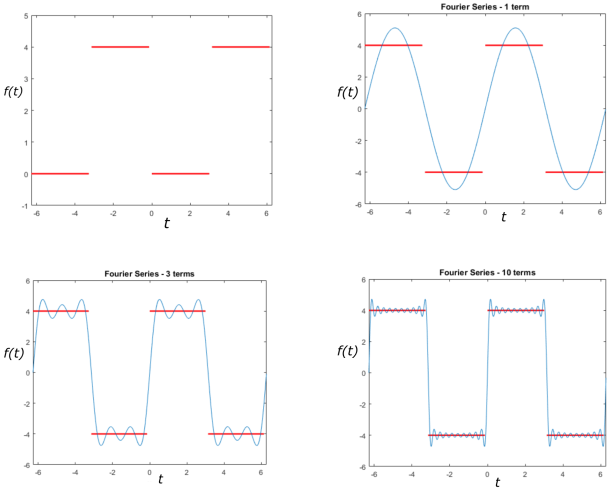


Fourier coefficients

To determine the Fourier series of our signal $f(t)$, in addition to the angular frequency ω (sometimes written as ω_0 and called the fundamental frequency.) we will need to know how much of each frequency is required. So we need to find the values of the **Fourier coefficients** (numbers) a_0, a_1, a_2, ... and b_1, b_2, b_3, ...

From applications in electronics, $\frac{a0}{2}$ is called the DC level. In simple cases, it can be found by calculating the average value of the graph of $f(t)$.

But how do we determine these constants in general? They are given by integrals that we will calculate using MATLAB.

Fourier coefficients

Fourier Series for a general periodic function

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

Where ω is the angular frequency of the function $f(t)$, and:

$$a_0 = \frac{2}{T}\int_0^T f(t)\ \mathrm{d}t$$

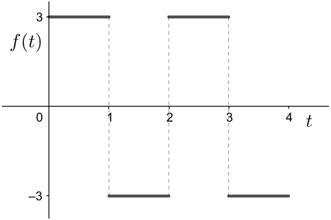
$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\ \mathrm{d}t$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\ \mathrm{d}t$$

Piecewise and Step Functions

So now, given a periodic signal function, we know how to obtain the angular frequency, and the Fourier coefficients by integration. Then we can assemble the Fourier series from these elements.

But. . . how do we describe this signal function in MATLAB?



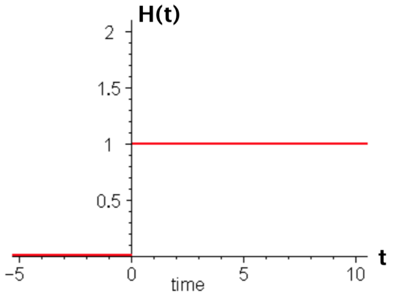
This is a **piecewise** function, and we can construct them using a combination of **step functions** that “switch” the constituent parts of the behaviour “on” and “off” at the necessary times.

Heaviside Step Function

The **Heaviside step function** $H(t)$ (also known as the unit step function $U(t)$) is defined by:

Heaviside Step Function $H(t)$

$$H(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t > 0. \end{cases}$$

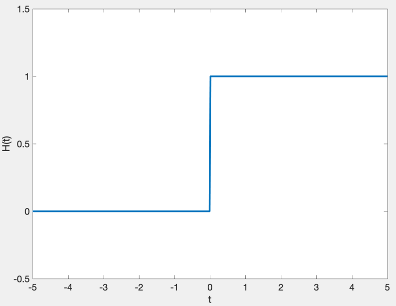


It "switches" on from a value of 0 to 1 at the moment when $t = 0$.

Heaviside Step Function

To input this in MATLAB:

```
syms t;
h=heaviside(t);
fplot(t,h);
```



Heaviside Functions

Combine multiple step functions to switch signals on and off.

Example:

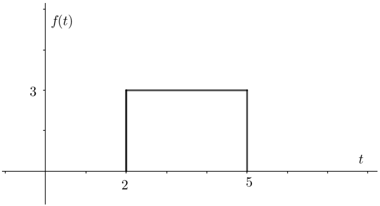
$$
\begin{aligned}
f(t) &= 3H(t-2) - 3H(t-5) \\
&= 3\Big(H(t-2) - H(t-5)\Big)
\end{aligned}
$$

Combine multiple step functions to switch signals on and off.

Example:

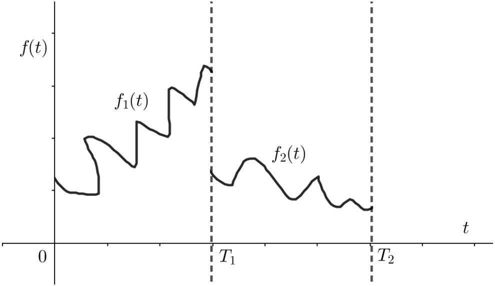
$$
\begin{aligned}
f(t) &= 3H(t-2) - 3H(t-5) \\
&= 3\Big(H(t-2) - H(t-5)\Big)
\end{aligned}
$$

- Signal of constant value 3.
- Begins at time $t = 2$
- Ends at time $t = 5$



Heaviside Functions

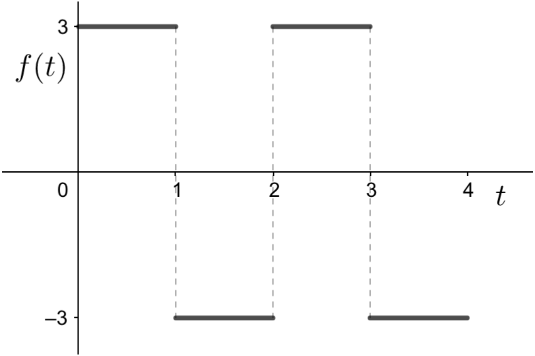
For a general function f that behaves like f_1 for the interval $[0, T_1]$, then changes to act like f_2 during the next interval $[T_1, T_2]$ before switching off:



$$f(t) = f_1(t)\Big(H(t) - H(t - T_1)\Big) + f_2(t)\Big(H(t - T_1) - H(t - T_2)\Big)$$

Example: Square Wave

Consider the following square wave signal.



What is the period and angular frequency?

Example: Square Wave

The period is the shortest interval required before the pattern repeats. We can see that this is equal to:

$$T = 2$$

Hence the angular frequency is given by:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

(**Note:** we can see that the average value of this function is zero, so the DC level and a_0 must be zero.)

Example: Square Wave

Now to determine the Fourier series in MATLAB.

First, we need to describe this function using Heaviside functions so we can declare it in MATLAB.

In the first period $0 < t < 2$, this function has two behaviours:

- A constant signal of $f(t) = 3$ switches on at $t = 0$ and off at $t = 1$
- A constant signal of $f(t) = -3$ switches on at $t = 1$ and off at $t = 2$

thus, it is described by:

$$f(t) = 3\Big(H(t) - H(t-1)\Big) + (-3)\Big(H(t-1) - H(t-2)\Big)$$

Example: Square Wave

So now we have:

- The signal function itself:

$$f(t) = 3\Big(H(t) - H(t-1)\Big) + (-3)\Big(H(t-1) - H(t-2)\Big)$$

- The angular frequency: $\omega = \pi$
- The first period: $0 < t < 2$

Everything we need to determine the Fourier series!

Open the worksheet:

`Lecture10ArbFourierExample.mlx`

Example: Square Wave

Define the function:

```
f = 3 * (heaviside(t) - heaviside(t - 1)) +
    (-3) * (heaviside(t - 1) - heaviside(t - 2))
```

Input T and ω:

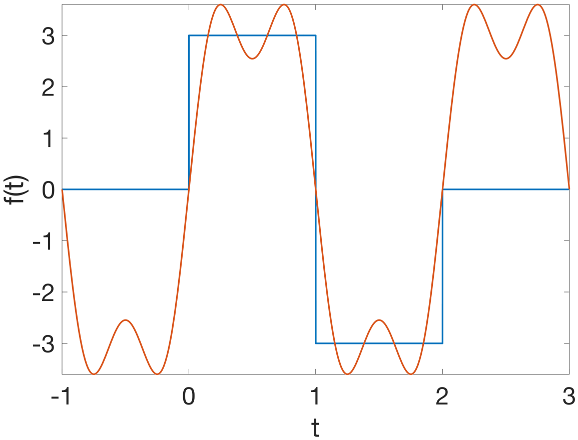
```
T = 2        w = pi
```

We need to use the new formula for the integals, e.g.

```
a1 = int(f * cos(w*t), t, 0, T)*2/T
b3 = int(f * sin(3 * w * t), t, 0, T)*2/T

FourApprox = a0/2 + a1 * cos(w * t) +
  a2 * cos(2 * w * t) + a3 * cos(3 * w * t) + ...
```

Example: Square Wave



Example: Square Wave

If we were to evaluate the integrals for general k, we would find:

$$a_k = 0 \quad \text{for all integers } k$$

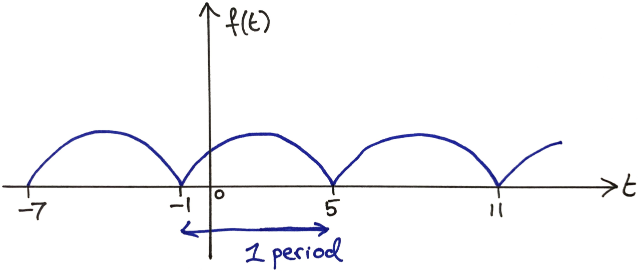
and

$$b_k = \frac{6}{k\pi}\left\{1 - (-1)^k\right\}$$

Thus the (infinite) Fourier series is given by:

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \\ &= \frac{6}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - (-1)^n\right)\sin(n\pi t) \end{aligned}$$

Note: In many cases, the function does not start and end its behaviour nicely at $t = 0$ and $t = T$. For example:



In this case, it would clearly be easier to integrate over the period $[-1, 5]$, rather than $[0, 6]$. Fortunately, **we are allowed to choose any interval** of length T to integrate over!

Arbitrary limits

This means that in the square wave example we have just done, we could have integrated over the range $[-1, 1]$ or $[-2, 0]$, or $[1, 3]$ etc.

Any interval of length equal to $T = 2$ would suffice.

It doesn't *have* to be $[0, 2]$.

However: you must ensure that when defining the function $f(t)$ in MATLAB (using Heaviside functions), that you have described the function *in this region* that you wish to integrate over!