Bio Stats II : Lecture 2, Probability

Bolker 2008, Chapter 4

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This Week...

1/28: Working with data
1/29: Lab 2
1/30: Probability review

Objectives

- Review probability laws
- Review definitions of expected value and variance of random variables
- Present common probability distributions

Why does variability matter?

Variability affects any ecological system.

Noise affects ecological data in two ways:
- measurement error
- process noise

Measurement error is variability in our measurements.
- leads to large confidence intervals and low power

Process noise (process error), variability in the system.
- demographic stochasticity
- environmental stochasticity

We are interested in understanding patterns in our data.
- use probability to describe relationships between processes and data.

Often assume that our data is generated by some stochastic process whose expected value is a function of covariates we are interested in.

Basic probability theory

The *sample space* is the set of all possible outcomes that could occur.

e.g. for a regular six-sided die

$$s\{1, 2, 3, 4, 5, 6\}$$

Probability of an event A is the frequency with which that event occurs.

e.g.

$$P(1) = 1/6$$

Laws of Probability

1. Law of total probability
The probabilities of all possible outcomes of an observation or experiment add to 1.0

$$P(\text{heads}) + P(\text{tails}) = 1.0$$

2. Probability of A *or* B, or $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

3. Mutually exclusive vs. independent events

- two mutually exlusive events cannot be independent
- mutually exclusive $\implies P(A \cap B) = 0$
- independence $\implies P(A \cap B) = P(A) \cdot P(B) \neq 0$

Laws of Probability

4. General multiplication rule

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1, A_2) \ldots$$

5. Conditional probability

$P(A|B)$, is the probability that A happens if we know or assume B happens.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional probability leads to Bayes' rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B \cap A) = P(B|A) \cdot P(A)$$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

This is mostly termed with A being the model (hypothesis) and B being the data.
i.e. what is the probability of a hypothesis given the data.

$$P(H|D) = \frac{P(D|H) \cdot P(H)}{P(D)}$$

with $P(D) = \sum P(D|H) \cdot P(H)$

Random Variables

A random variable is a numerical valued function defined over a sample space.

The probability distribution describes how the frequency of occurrence varies across the sample space.

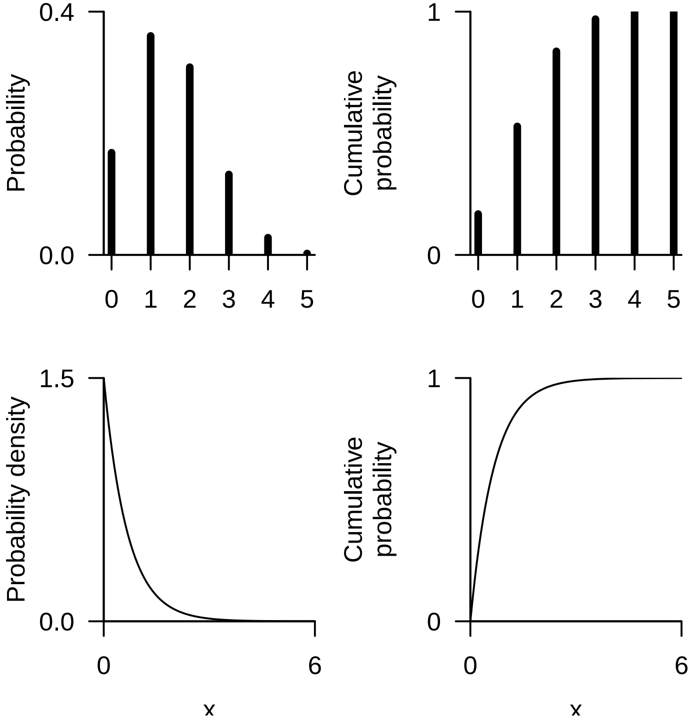
For discrete variables, characterized by $f(x)$,
- the **probability distribution function** (discrete variables)

$$f(x) = Prob(X = x)$$

(for continuous variables, $f(x)$ is the **probability density function**)

Both types of variables are also described by the **cumulative distribution function**, $F(x)$

$$F(x) = P(X \leq x)$$



0.4
0.0
Probability
0 1 2 3 4 5
1
0
Cumulative probability
0 1 2 3 4 5
1.5
0.0
Probability density
0 6
x
1
0
Cumulative probability
0 6
x

Expected Value of Random Variable X

Discrete random variables

$$\mu = E(X) = \sum_{i=0}^{\infty} x_i P(X = x_i)$$

Continuous random variables

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Variance of a Random Variable X, $E[(X-\mu)^2]$

Discrete random variables

$$Var(X) = \sum_{i=0}^{\infty} (x_i - E(x_i))^2 \, P(X = x_i)$$

Continuous random variables

$$Var(X) = \int_{-\infty}^{\infty} (x_i - E(x_i))^2 f(x) dx$$

In general

$$Var(X) = E(X^2) - (E(X))^2 = E((X-\mu)^2)$$

Variances are additive.

$$Var(X \pm Y) = Var(X) + Var(Y) \pm 2Cov(X, Y)$$

The *standard deviation* of a distribution is $\sqrt{Var}$
The *coefficient of variation* (CV) is $\sqrt{Var}/\mu$

Summary of probability distributions

Binomial

Describes the number of successes from a fixed number of trials.
Two possible outcomes on each trial, success or failure.
Probability of success is the same in each trial.

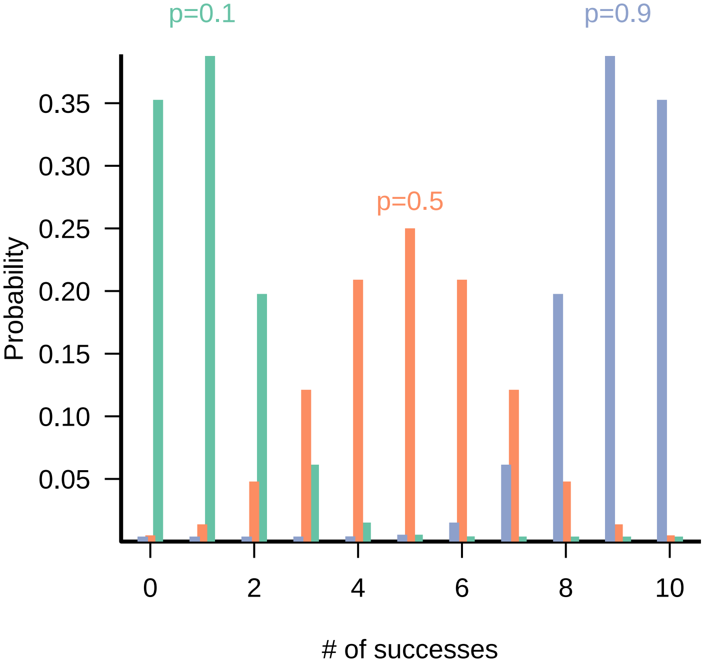
Range: discrete, $0 \le x \le N$
Distribution:

$$\binom{N}{x} p^x (1-p)^{N-x}$$

R: `dbinom` `pbinom` `qbinom` `rbinom`
Parameters:
- p [real, 0-1], probability of success [`prob`]
- N [positive integer], number of trials [`size`]

Mean: Np
Variance: $Np(1-p)$
CV: $sqrt(1-p)/(Np)$
Conjugate prior: Beta

Binomial



p=0.1
p=0.9
p=0.5
Probability
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
2
4
6
8
10
# of successes

Multinomial

Extension of binomial trials to three or more possible outcomes.
$X = (X_1, X_2, \ldots, X_k)$

Range: discrete, $0 \leq x_i \leq N$
Distribution:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \binom{N}{x_1, x_2, \ldots, x_k} \prod_{i=1}^{k} p_i^{x_i}$$

R: `dbinom pbinom qbinom rbinom`
Parameters:

- p_i [real, 0-1], $\sum_{i=1}^{k} p_i = 1$
- N [positive integer], number of samples

$E(X_i) = Np_i$
$Var(X_i) = Np_i(1 - p_i)$
$Cov(X_i, X_j) = -Np_i p_j$, $i \neq j$

Poisson

Describes events which occur randomly and independently in time.

Limit of a binomial distribution in which:
$N \to \infty, p \to 0$ while $Np = \mu$ is fixed.

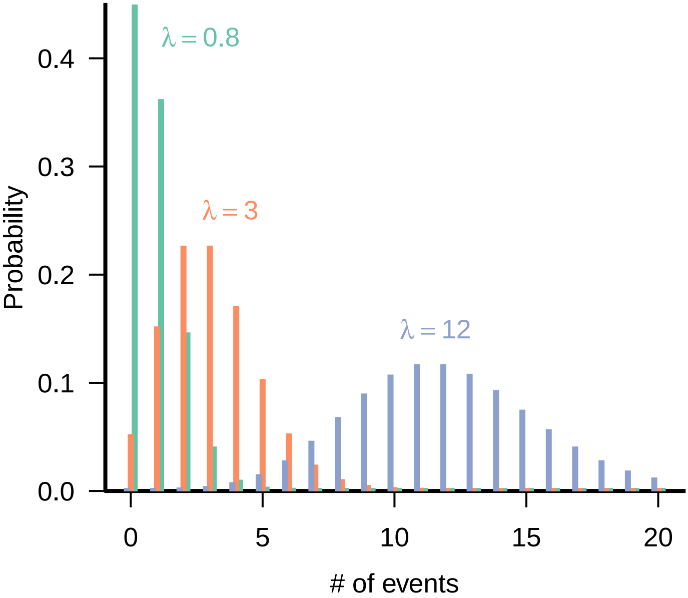
Distribution of "rare events" (i.e., $p \to 0$).

Range: discrete ($0 \leq x$)
Distribution:

$$\frac{e^{-\lambda}\lambda^n}{n!} \textbf{ or } \frac{e^{-rt}(rt)^n}{n!}$$

`R`: `dpois`, `ppois`, `qpois`, `rpois`
Parameters: λ (real, positive), expected number per sample [`lambda`] **or** r (real, positive), expected number per unit effort, area, time, etc. (*arrival rate*)
Mean: λ (**or** rt)
Variance: λ (**or** rt)
CV : $1/\sqrt{\lambda}$ (**or** $1/\sqrt{rt}$)
Conjugate prior: Gamma

Poisson



Negative Binomial

For binomial trials, the number of failures before n successes.

In ecology, most often used because it is discrete like the Poisson but the variance can be greater than the mean (*overdispersed*).

Range: discrete, $x \geq 0$
Distribution:

$$P(X = x) = \frac{(n + x - 1)!}{(n - 1!)x!} p^n (1 - p)^x$$

$$\textbf{or } \frac{\Gamma(k + x)}{\Gamma(k)x!} (k/(k + \mu))^k (\mu/(k + \mu))^x$$

Parameters:
p ($0 < p < 1$) probability per trial [`prob`]
or μ (real, positive) expected number of counts [`mu`]
n (positive integer) number of successes awaited [`size`]
or k (real, positive), overdispersion parameter [`size`]
(= shape parameter of underlying heterogeneity)

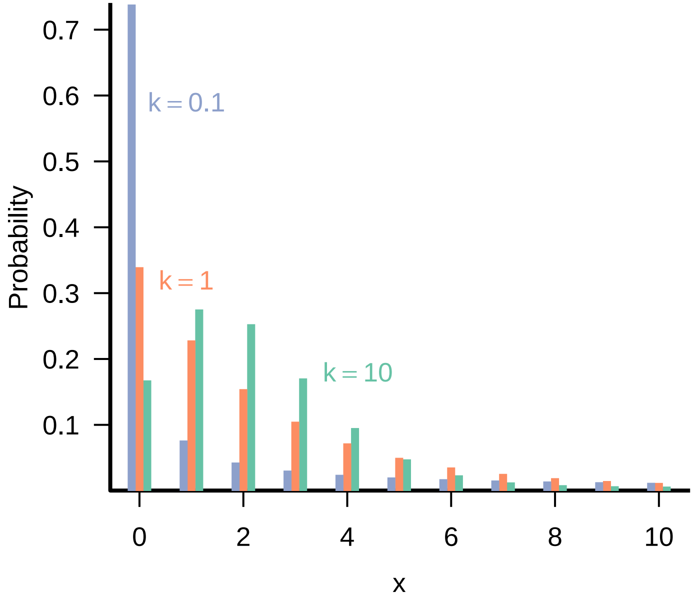
Negative Binomial

`R`: `dnbinom`, `pnbinom`, `qnbinom`, `rnbinom`
Mean: $\mu = n(1-p)/p$
Variance: $\mu + \mu^2/k = n(1-p)/p^2$
CV: $\sqrt{\frac{(1+\mu/k)}{\mu}} = 1/\sqrt{n(1-p)}$
Conjugate prior: No simple conjugate prior (Bradlow et al. 2002)

To use the 'ecology' parameterization in `R` you *must* name `mu` explicitly.

The negative binomial is also the result of a Poisson sampling process where λ is Gamma-distributed.

Negative Binomial ($\mu = 2$ all cases)



Continuous Probability Distributions

Uniform distribution

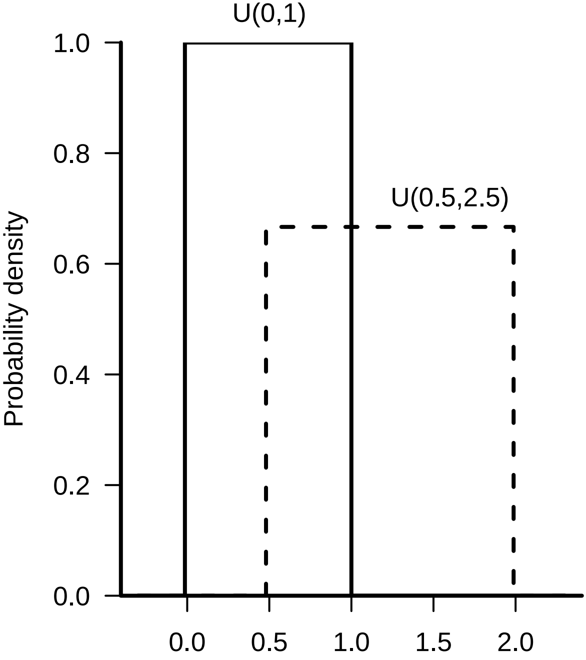
Constant probability across a range with limits a and b

Standard uniform, $U(0, 1)$, frequently used as building block.

Range: $a \leq x \leq b$
Distribution: $1/(b-a)$
`R`: `dunif`, `punif`, `qunif`, `runif`
Parameters: minimum (a) and maximum (b) limits (real)
[`min`, `max`]

Mean: $(a+b)/2$
Variance: $(b-a)^2/12$
CV: $(b-a)/((a+b)\sqrt{3})$

Uniform distribution



Normal Distribution

Arises from adding things together.

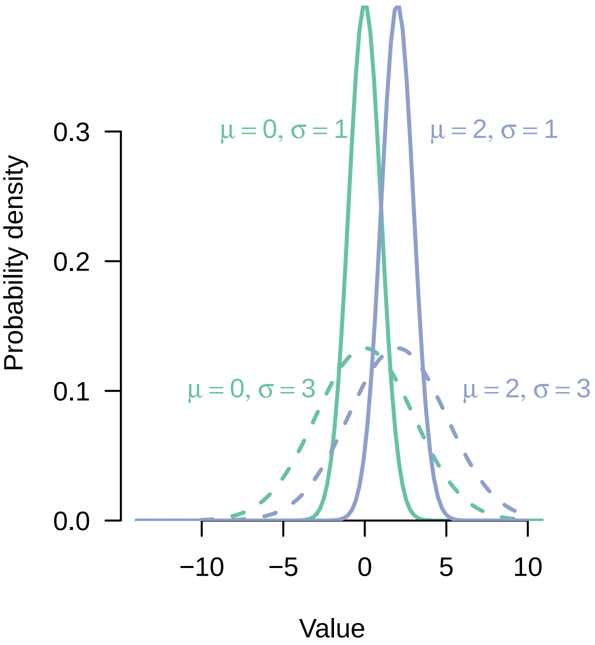
Sum of a large number of independent samples from the same distribution is approximately normal.

Limit of many distributions (binomial, Poisson, negative binomial, Gamma).

Range: all real values
Distribution: $\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
R: `dnorm`, `pnorm`, `qnorm`, `rnorm`
Parameters:
- μ (real), mean [`mean`]
- σ (real, positive), standard deviation [`sd`]

Mean: μ
Variance: σ^2
CV: σ/μ
Conjugate prior: Normal (μ); Gamma ($1/\sigma^2$)

Normal distribution



Gamma

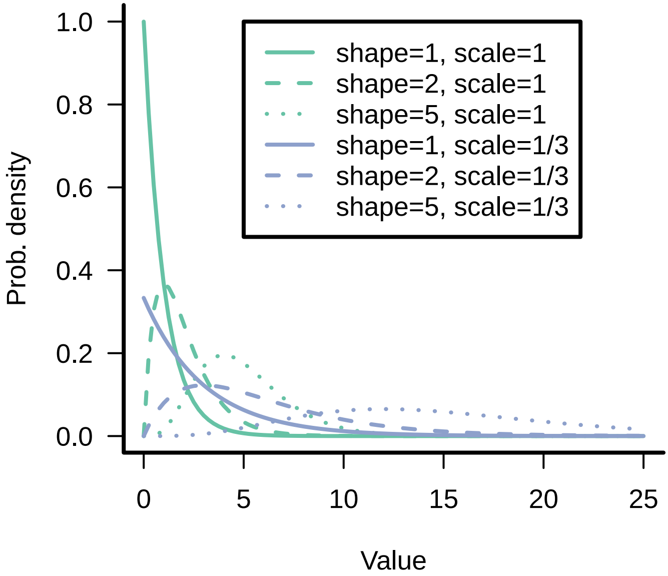
Distribution of waiting times until a certain number of events occurs.

Continuous counterpart to the negative binomial.

Gamma is very useful. Continuous positive variable with large variance and (possible) skew.

Range: positive real values
R: `dgamma`, `pgamma`, `qgamma`, `rgamma`
Distribution: $\frac{1}{s^a \Gamma(a)} x^{a-1} e^{-x/s}$
Parameters:
s (real, positive), scale: length per event [`scale`]
or r (real, positive), rate $= 1/s$; rate at which events occur [`rate`]
a (real, positive), shape: number of events [`shape`]
Mean: as or a/r
Variance: as^2 or a/r^2
CV: $1/\sqrt{a}$

Gamma



shape=1, scale=1
shape=2, scale=1
shape=5, scale=1
shape=1, scale=1/3
shape=2, scale=1/3
shape=5, scale=1/3
1.0
0.8
0.6
0.4
0.2
0.0
Prob. density
0
5
10
15
20
25
Value

Beta

Continuous distribution related to the binomial.

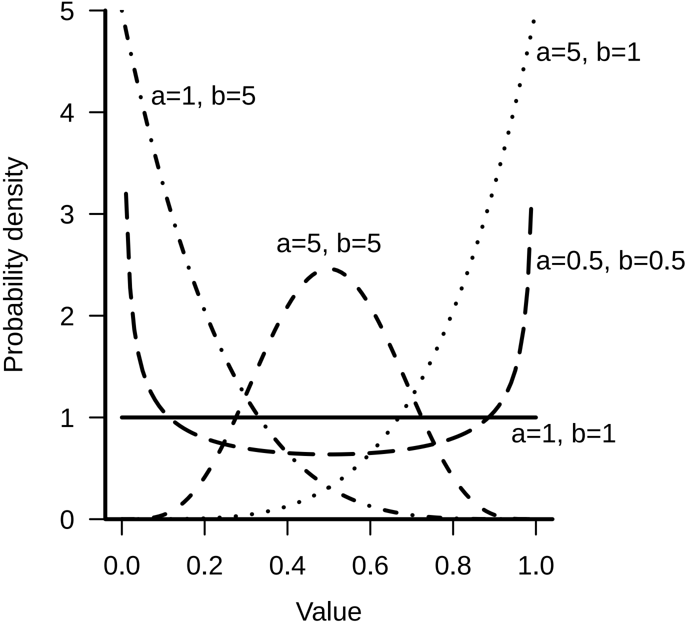
Distribution of *probability* of success in a binomial trial with $a-1$ successes and $b-1$ failures.

Very useful in modeling probabilities or proportions.

Range: real, 0 to 1
R: `dbeta`, `pbeta`, `qbeta`, `rbeta`
Density: s$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$
Parameters:
- a (real, positive), shape 1: number of successes $+1$ [`shape1`]
- b (real, positive), shape 2: number of failures $+1$ [`shape2`]

Mean: $a/(a+b)$
Mode: $(a-1)/(a+b-2)$
Variance: $ab/((a+b)^2)(a+b+1)$
CV: $\sqrt{(b/a)/(a+b+1)}$

Beta



5
4
3
2
1
0
Probability density
0.0
0.2
0.4
0.6
0.8
1.0
Value
a=1, b=5
a=5, b=1
a=5, b=5
a=0.5, b=0.5
a=1, b=1

Lognormal

Not a continuous analogue or limit of some discrete distribution.

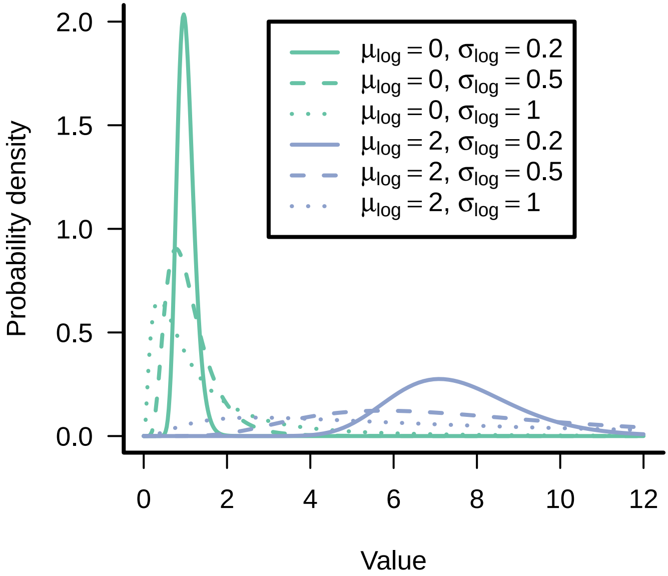
Justification: as for normal, but for *product* of many iid variables.

Used in many situations where Gamma also fits, continuous, positive distribution with long tail or variance $>$ mean.

Range: positive real values
R: `dlnorm`, `plnorm`, `qlnorm`, `rlnorm`
Density: $\frac{1}{\sqrt{2\pi}\sigma x} e^{-(\log x - \mu)^2/(2\sigma^2)}$
Parameters:
- μ (real): mean of the logarithm [`meanlog`]
- σ (real): standard deviation of the logarithm [`sdlog`]

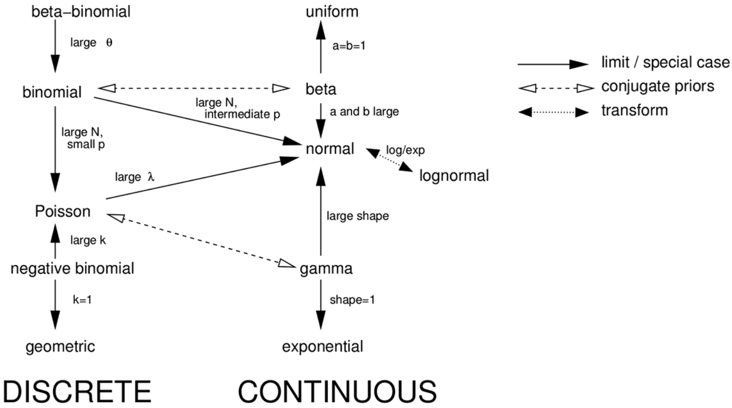
Mean: $\exp(\mu + \sigma^2/2)$
Variance: $\exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$
CV: $\sqrt{\exp(\sigma^2) - 1}$ ($\approx \sigma$ when $\sigma < 1/2$)

Lognormal



$\mu_{log} = 0, \sigma_{log} = 0.2$
$\mu_{log} = 0, \sigma_{log} = 0.5$
$\mu_{log} = 0, \sigma_{log} = 1$
$\mu_{log} = 2, \sigma_{log} = 0.2$
$\mu_{log} = 2, \sigma_{log} = 0.5$
$\mu_{log} = 2, \sigma_{log} = 1$
Probability density
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8
10
12
Value

Relationships among distributions



Other common distributions

Discrete

- Geometric (negative binomial with $k = 1$)
- Beta-binomial (binomial but with p being beta distributed)
- Hypergeometric (useful for sampling without replacement, finite population)
- Multivariate hypergeometric (similar to the multinomial)

Continuous

- Exponential (distribution of waiting times for a single event)
- Pareto (quantity whose log is exponentially distributed, power laws!)
- Chi square (distribution of a sum of squared standard normals)
- Student's t (ratio of a standard normal and the square root of a scaled chi square)
- F (ratio of two scaled chi-squares)
- Dirichlet (generalization of beta, for a vector that must sum to 1)
- Wishart (generalization of gamma, for a symmetric non-negative definite matrix)

Delta Method

Calculating expected values and variances of (nonlinear) functions of continuous (differentiable) random variables using Taylor series expansion.

$$e^x \qquad\qquad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Exponential Function

Exponential Function
(Taylor's Version)

Delta Method

Calculating expected values and variances of (nonlinear) functions of continuous (differentiable) random variables using Taylor series expansion.

Let x_i be a random variable with mean $\mu_i (i = 1, \ldots, n)$. Given some function $g(x_1, x_2, \ldots, x_n)$, say, $g(\underset{\sim}{x})$, then

1. $E\left(g(\underset{\sim}{x})\right) \doteq$

$$g(\underset{\sim}{\mu}) + \frac{1}{2}\sum_{i=1}^{n} Var(X_i)\left(\frac{\partial_g^2}{\partial x_i^2}\right)_{|\mu} + \sum_{i<j}\sum Cov(x_i, x_j)\left(\frac{\partial_g^2}{\partial x_i \partial x_j}\right)_{|\mu}$$

2. $Var\left(g(\underset{\sim}{x})\right) \doteq$
$$\sum_{i=1}^{n} Var(x_i)\left(\frac{\partial_g}{\partial x_i}\right)^2_{|\mu} + 2\sum_{i<j}\sum Cov(x_i, x_j)\left(\frac{\partial g}{\partial x_i}\right)_{|\mu}\left(\frac{\partial g}{\partial x_j}\right)_{|\mu}$$

3. $Cov\left[g(\underset{\sim}{x}), h(\underset{\sim}{x})\right] = \sum_i \sum_j Cov(x_i, x_j)\left(\frac{\partial g}{\partial x_i}\right)_{|\mu}\left(\frac{\partial g}{\partial x_j}\right)_{|\mu}$

$|\mu$ denotes evaluation of derivative at the values of μ.

Next Time...

2/04: Linear regression review
2/05: Lab 3
2/06: Likelihood