Bio Stats II : Lecture 2, Probability

Bolker 2008, Chapter 4

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This Week…

1/21: Introduction, Statistical Rethinking
1/22: Lab 1
1/23: Probability review

Objectives

- Review probability laws
- Review definitions of expected value and variance of random variables
- Present common probability distributions

Why does variability matter?

Variability affects any ecological system.

Noise affects ecological data in two ways:
- measurement error
- process noise

Measurement error is variability in our measurements.
- leads to large confidence intervals and low power

Process noise (process error), variability in the system.
- demographic stochasticity
- environmental stochasticity

We are interested in understanding patterns in our data.
- use probability to describe relationships between processes and data.

Often assume that our data is generated by some stochastic process whose expected value is a function of covariates we are interested in.

Basic probability theory

The *sample space* is the set of all possible outcomes that could occur.

e.g. for a regular six-sided die

$$s\{1,2,3,4,5,6\}$$

Probability of an event A is the frequency with which that event occurs.

e.g.

$$P(1) = 1/6$$

Laws of Probability

1. Law of total probability
   The probabilities of all possible outcomes of an observation or experiment add to 1.0

$$P(\text{heads}) + P(\text{tails}) = 1.0$$

2. Probability of A *or* B, or $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

3. Mutually exclusive vs. independent events

- two mutually exlusive events cannot be independent
- mutually exclusive $\implies P(A \cap B) = 0$
- independence $\implies P(A \cap B) = P(A) \cdot P(B) \neq 0$

Laws of Probability

4. General multiplication rule

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1, A_2) \dots$$

5. Conditional probability

$P(A|B)$, is the probability that A happens if we know or assume B happens.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional probability leads to Bayes' rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B \cap A) = P(B|A) \cdot P(A)$$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

This is mostly termed with A being the model (hypothesis) and B being the data. i.e. what is the probability of a hypothesis given the data.

$$P(H|D) = \frac{P(D|H) \cdot P(H)}{P(D)}$$

with $P(D) = \sum P(D|H) \cdot P(H)$

Random Variables

A random variable is a numerical valued function defined over a sample space.

The probability distribution describes how the frequency of occurrence varies across the sample space.

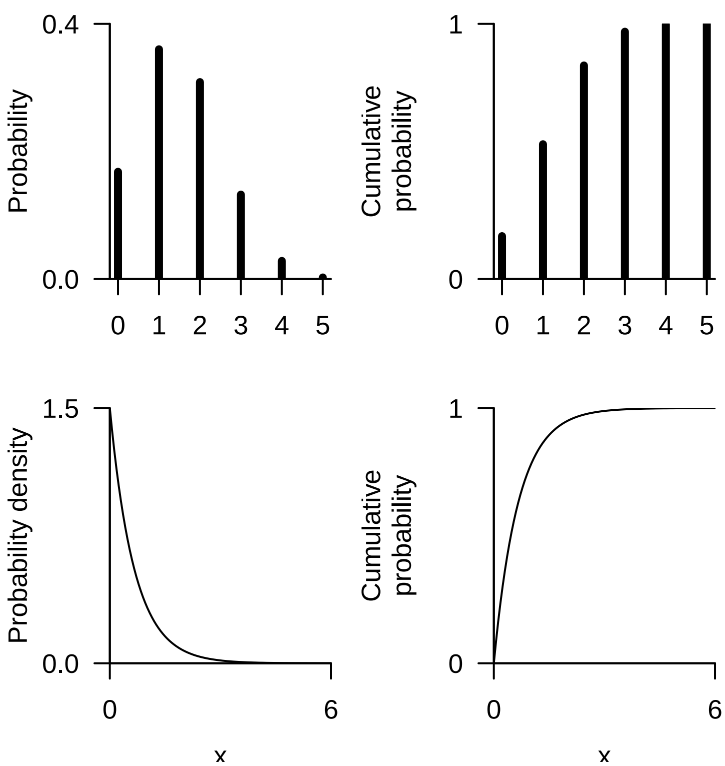
For discrete variables, characterized by $f(x)$,
- the **probability distribution function** (discrete variables)

$$f(x) = Prob(X = x)$$

(for continuous variables, $f(x)$ is the **probability density function**)

Both types of variables are also described by the **cumulative distribution function**, $F(x)$

$$F(x) = P(X \leq x)$$



Expected Value of Random Variable X

Discrete random variables

$$\mu = E(X) = \sum_{i=0}^{\infty} x_i P(X = x_i)$$

Continuous random variables

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Variance of a Random Variable X, $E[(X-\mu)^2]$

Discrete random variables

$$Var(X) = \sum_{i=0}^{\infty} \left(x_i - E(x_i)\right)^2 P(X = x_i)$$

Continuous random variables

$$Var(X) = \int_{-\infty}^{\infty} (x_i - E(x_i))^2 f(x) dx$$

In general

$$Var(X) = E(X^2) - (E(X))^2 = E((X-\mu)^2)$$

Variances are additive.

$$Var(X \pm Y) = Var(X) + Var(Y) \pm 2Cov(X,Y)$$

The *standard deviation* of a distribution is $\sqrt{Var}$
The *coefficient of variation* (CV) is $\sqrt{Var}/\mu$

Summary of probability distributions

Binomial

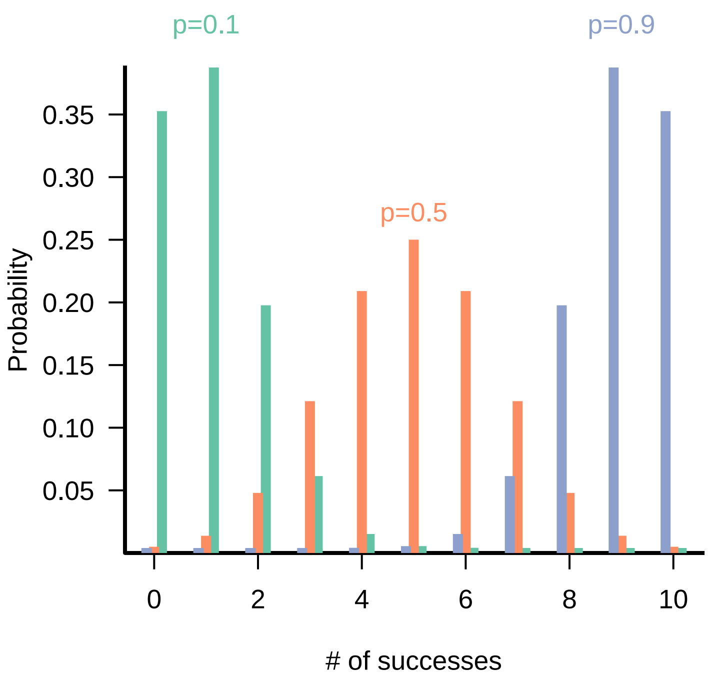
Describes the number of successes from a fixed number of trials.
Two possible outcomes on each trial, success or failure.
Probability of success is the same in each trial.

Range: discrete, $0 \leq x \leq N$
Distribution:

$$\binom{N}{x} p^x (1-p)^{N-x}$$

`R`: `dbinom pbinom qbinom rbinom`
Parameters:
- p [real, 0-1], probability of success [`prob`]
- N [positive integer], number of trials [`size`]
Mean: Np
Variance: $Np(1-p)$
CV: $sqrt(1-p)/(Np)$
Conjugate prior: Beta

Binomial



Multinomial

Extension of binomial trials to three or more possible outcomes.
$X = (X_1, X_2, \ldots, X_k)$

Range: discrete, $0 \le x_i \le N$
Distribution:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \binom{N}{x_1, x_2, \ldots, x_k} \prod_{i=1}^{k} p_i^{x_i}$$

R: `dbinom pbinom qbinom rbinom`
Parameters:
- p_i [real, 0-1], $\sum_{i=1}^{k} p_i = 1$
- N [positive integer], number of samples

$E(X_i) = Np_i$
$Var(X_i) = Np_i(1 - p_i)$
$Cov(X_i, X_j) = -Np_i p_j$, $i \neq j$

Poisson

Describes events which occur randomly and independently in time.

Limit of a binomial distribution in which:
$N \to \infty, p \to 0$ while $Np = \mu$ is fixed.

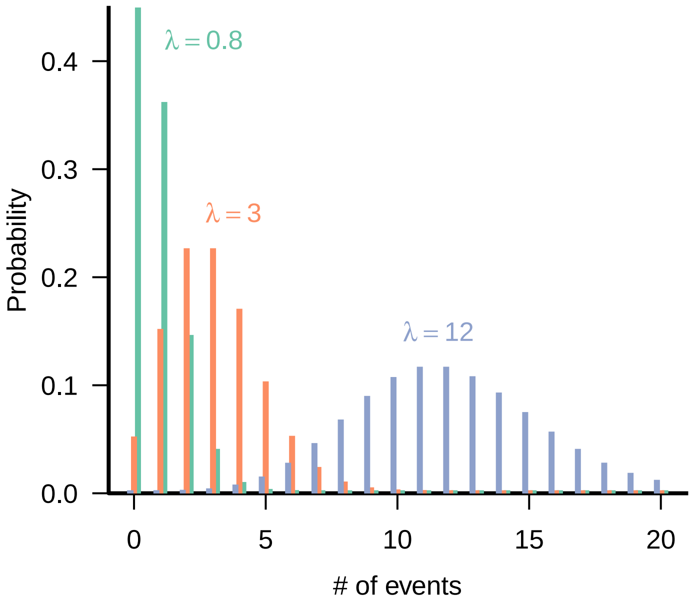
Distribution of "rare events" (i.e., $p \to 0$).

Range: discrete ($0 \leq x$)
Distribution:

$$\frac{e^{-\lambda}\lambda^n}{n!} \textbf{or} \frac{e^{-rt}(rt)^n}{n!}$$

R: `dpois`, `ppois`, `qpois`, `rpois`
Parameters: λ (real, positive), expected number per sample [`lambda`] **or** r (real, positive), expected number per unit effort, area, time, etc. (*arrival rate*)
Mean: λ (**or** rt)
Variance: λ (**or** rt)
CV : $1/\sqrt{\lambda}$ (**or** $1/\sqrt{rt}$)
Conjugate prior: Gamma

Poisson



Negative Binomial

For binomial trials, the number of failures before n successes.

In ecology, most often used because it is discrete like the Poisson but the variance can be greater than the mean (*overdispersed*).

Range: discrete, $x \geq 0$
Distribution:

$$P(X = x) = \frac{(n+x-1)!}{(n-1!)x!} p^n (1-p)^x$$

$$\textbf{or}\ \frac{\Gamma(k+x)}{\Gamma(k)x!} (k/(k+\mu))^k (\mu/(k+\mu))^x$$

Parameters:
p $(0 < p < 1)$ probability per trial [`prob`]
or μ (real, positive) expected number of counts [`mu`]
n (positive integer) number of successes awaited [`size`]
or k (real, positive), overdispersion parameter [`size`]
(= shape parameter of underlying heterogeneity)

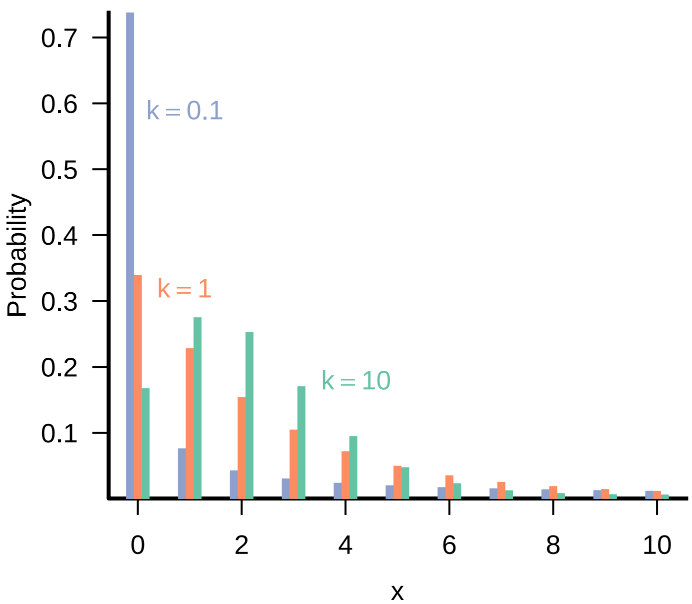
Negative Binomial

`R`: `dnbinom`, `pnbinom`, `qnbinom`, `rnbinom`
Mean: $\mu = n(1-p)/p$
Variance: $\mu + \mu^2/k = n(1-p)/p^2$
CV: $\sqrt{\frac{(1+\mu/k)}{\mu}} = 1/\sqrt{n(1-p)}$
Conjugate prior: No simple conjugate prior (Bradlow et al. 2002)

To use the 'ecology' parameterization in `R` you *must* name `mu` explicitly.

The negative binomial is also the result of a Poisson sampling process where λ is Gamma-distributed.

Negative Binomial ($\mu = 2$ all cases)



Continuous Probability Distributions

Uniform distribution

Constant probability across a range with limits a and b

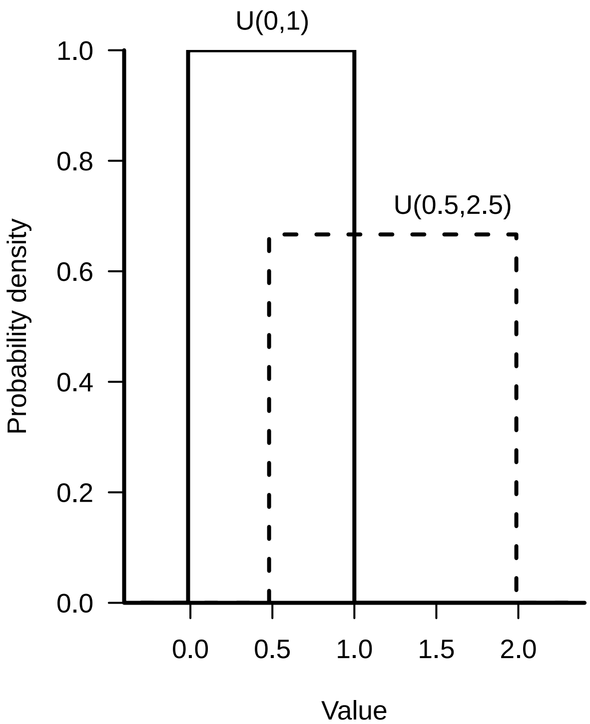
Standard uniform, $U(0,1)$, frequently used as building block.

Range: $a \leq x \leq b$
Distribution: $1/(b-a)$

R: `dunif`, `punif`, `qunif`, `runif`
Parameters: minimum (a) and maximum (b) limits (real)
[`min`, `max`]

Mean: $(a+b)/2$
Variance: $(b-a)^2/12$
CV: $(b-a)/((a+b)\sqrt{3})$

Uniform distribution



Normal Distribution

Arises from adding things together.

Sum of a large number of independent samples from the same distribution is approximately normal.

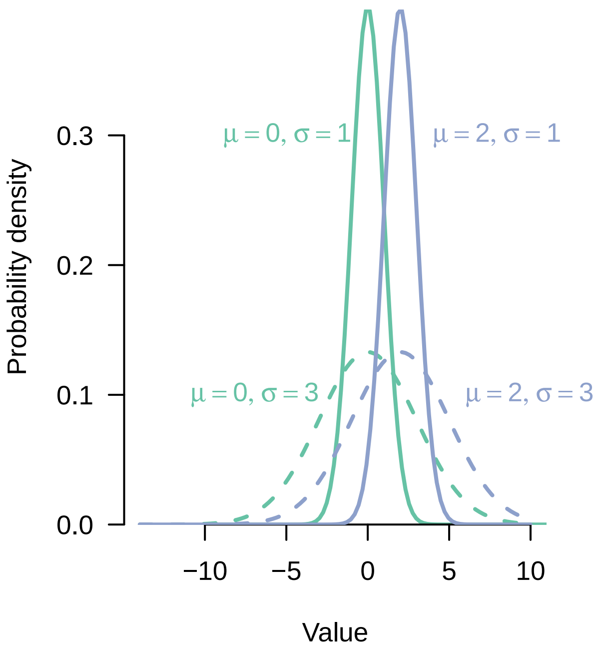
Limit of many distributions (binomial, Poisson, negative binomial, Gamma).

Range: all real values
Distribution: $\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
R: `dnorm`, `pnorm`, `qnorm`, `rnorm`

Parameters:
- μ (real), mean [`mean`]
- σ (real, positive), standard deviation [`sd`]

Mean: μ
Variance: σ^2
CV: σ/μ
Conjugate prior: Normal (μ); Gamma ($1/\sigma^2$)

Normal distribution



Gamma

Distribution of waiting times until a certain number of events occurs.

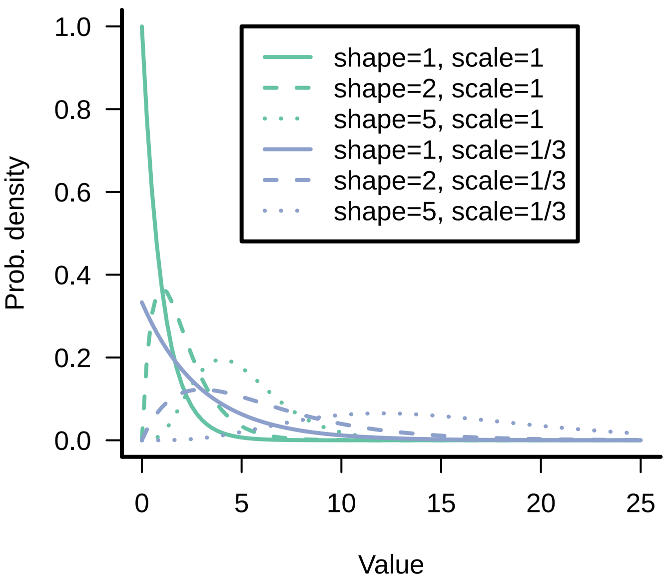
Continuous counterpart to the negative binomial.

Gamma is very useful. Continuous positive variable with large variance and (possible) skew.

Range: positive real values
`R`: `dgamma`, `pgamma`, `qgamma`, `rgamma`
Distribution: $\frac{1}{s^a \Gamma(a)} x^{a-1} e^{-x/s}$
Parameters:
s (real, positive), scale: length per event [`scale`]
or r (real, positive), rate $= 1/s$; rate at which events occur [`rate`]

a (real, positive), shape: number of events [`shape`]
Mean: as or a/r
Variance: as^2 or a/r^2
CV: $1/\sqrt{a}$

Gamma



Beta

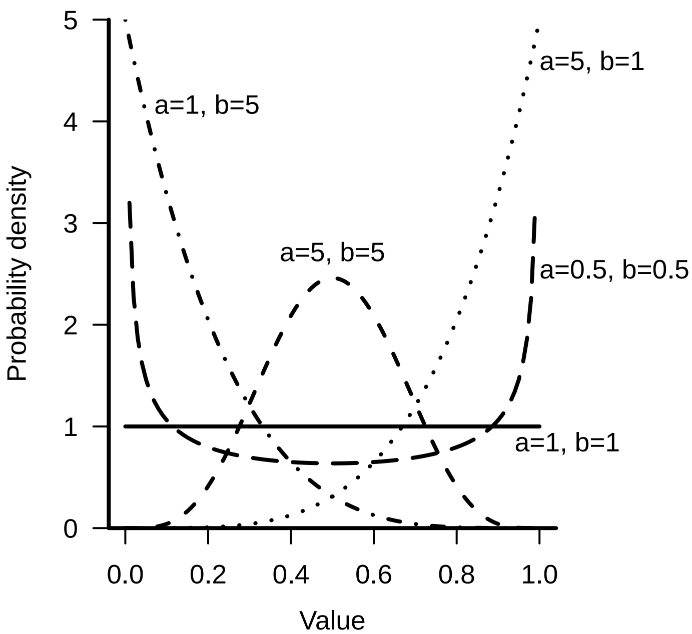
Continuous distribution related to the binomial.

Distribution of *probability* of success in a binomial trial with $a-1$ successes and $b-1$ failures.

Very useful in modeling probabilities or proportions.

Range: real, 0 to 1
`R`: `dbeta`, `pbeta`, `qbeta`, `rbeta`
Density: s$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$
Parameters:
- a (real, positive), shape 1: number of successes $+1$ [`shape1`]
- b (real, positive), shape 2: number of failures $+1$ [`shape2`]
Mean: $a/(a+b)$
Mode: $(a-1)/(a+b-2)$
Variance: $ab/((a+b)^2)(a+b+1)$
CV: $\sqrt{(b/a)/(a+b+1)}$

Beta



Lognormal

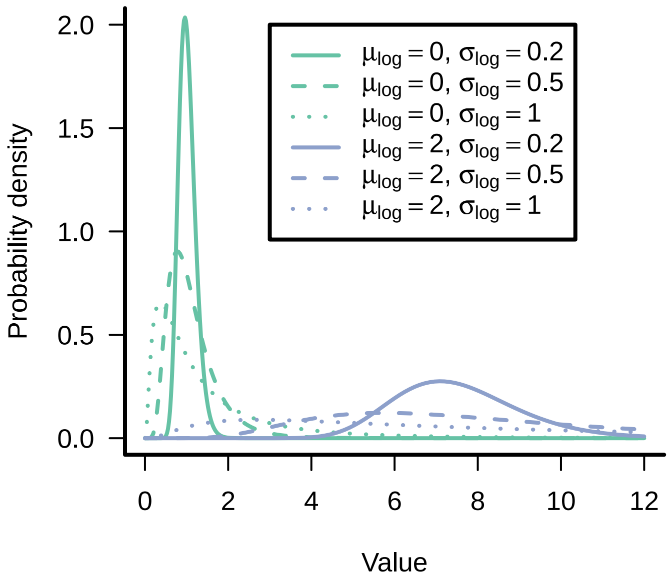
Not a continuous analogue or limit of some discrete distribution.

Justification: as for normal, but for *product* of many iid variables.

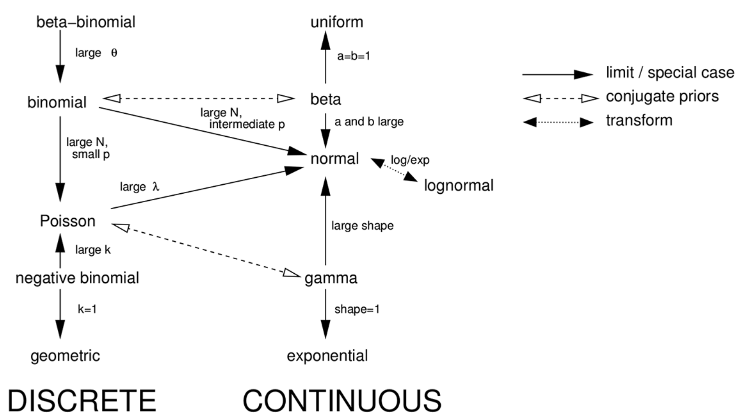
Used in many situations where Gamma also fits, continuous, positive distribution with long tail or variance > mean.

Range: positive real values
R: `dlnorm`, `plnorm`, `qlnorm`, `rlnorm`
Density: $\frac{1}{\sqrt{2\pi}\sigma x} e^{-(\log x - \mu)^2/(2\sigma^2)}$
Parameters:
- μ (real): mean of the logarithm [`meanlog`]
- σ (real): standard deviation of the logarithm [`sdlog`]
Mean: $\exp(\mu + \sigma^2/2)$
Variance: $\exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$
CV: $\sqrt{\exp(\sigma^2) - 1}$ ($\approx \sigma$ when $\sigma < 1/2$)

Lognormal



Relationships among distributions



Other common distributions

Discrete

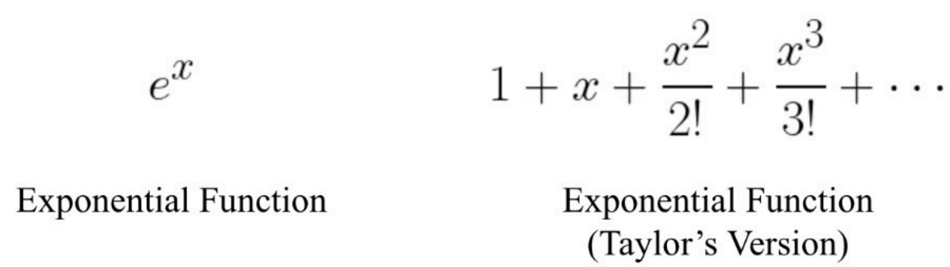
- Geometric (negative binomial with $k = 1$)
- Beta-binomial (binomial but with p being beta distributed)
- Hypergeometric (useful for sampling without replacement, finite population)
- Multivariate hypergeometric (similar to the multinomial)

Continuous

- Exponential (distribution of waiting times for a single event)
- Pareto (quantity whose log is exponentially distributed, power laws!)
- Chi square (distribution of a sum of squared standard normals)
- Student's t (ratio of a standard normal and the square root of a scaled chi square)
- F (ratio of two scaled chi-squares)
- Dirichlet (generalization of beta, for a vector that must sum to 1)
- Wishart (generalization of gamma, for a symmetric non-negative definite matrix)

Delta Method

Calculating expected values and variances of (nonlinear) functions of continuous (differentiable) random variables using Taylor series expansion.



Delta Method

Calculating expected values and variances of (nonlinear) functions of continuous (differentiable) random variables using Taylor series expansion.

Let x_i be a random variable with mean $\mu_i (i = 1, \dots, n)$. Given some function $g(x_1, x_2, \dots, x_n)$, say, $g(\underset{\sim}{x})$, then

1. $E\left(g(\underset{\sim}{x})\right) \doteq g(\underset{\sim}{\mu}) + \frac{1}{2}\sum_{i=1}^{n} Var(X_i)\left(\frac{\partial^2_g}{\partial x_i^2}\right)_{|\mu} + \sum_{i<j}\sum Cov(x_i, x_j)\left(\frac{\partial^2_g}{\partial x_i \partial x_j}\right)_{|\mu}$

2. $Var\left(g(\underset{\sim}{x})\right) \doteq \sum_{i=1}^{n} Var(x_i) \left(\frac{\partial_g}{\partial x_i}\right)^2_{|\mu} + 2\sum_{i<j}\sum Cov(x_i, x_j) \left(\frac{\partial g}{\partial x_i}\right)_{|\mu} \left(\frac{\partial g}{\partial x_j}\right)_{|\mu}$

3. $Cov\left[g(\underset{\sim}{x}), h(\underset{\sim}{x})\right] = \sum_i \sum_j Cov(x_i, x_j) \left(\frac{\partial g}{\partial x_i}\right)_{|\mu} \left(\frac{\partial g}{\partial x_j}\right)_{|\mu}$

$|\mu$ denotes evaluation of derivative at the values of μ.

Next Time...

1/28: Data exploration, checking
1/29: Lab 2
1/30: Linear regression review