Advanced Population Modeling : Process-based Model building & Matrix Population Models

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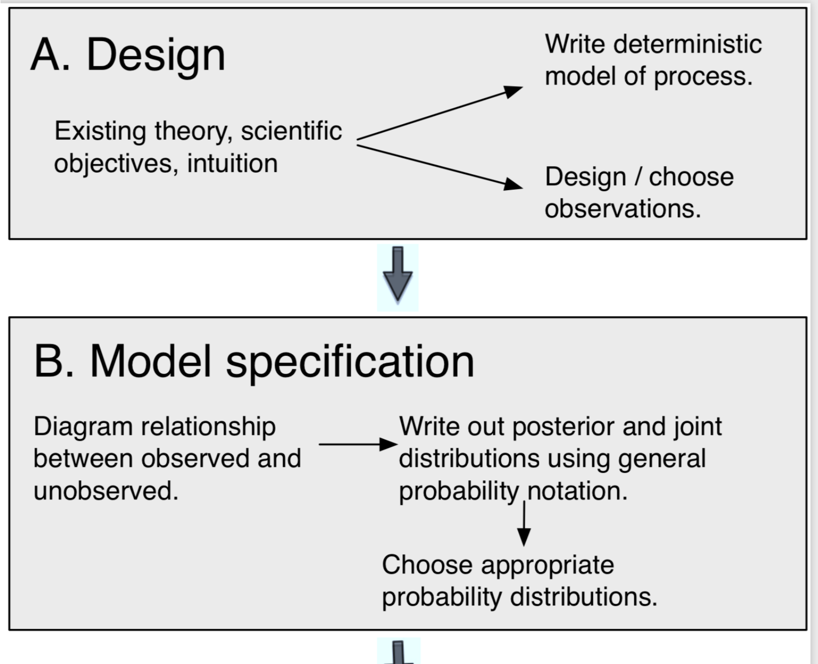
2025-09-11

Process-based model building



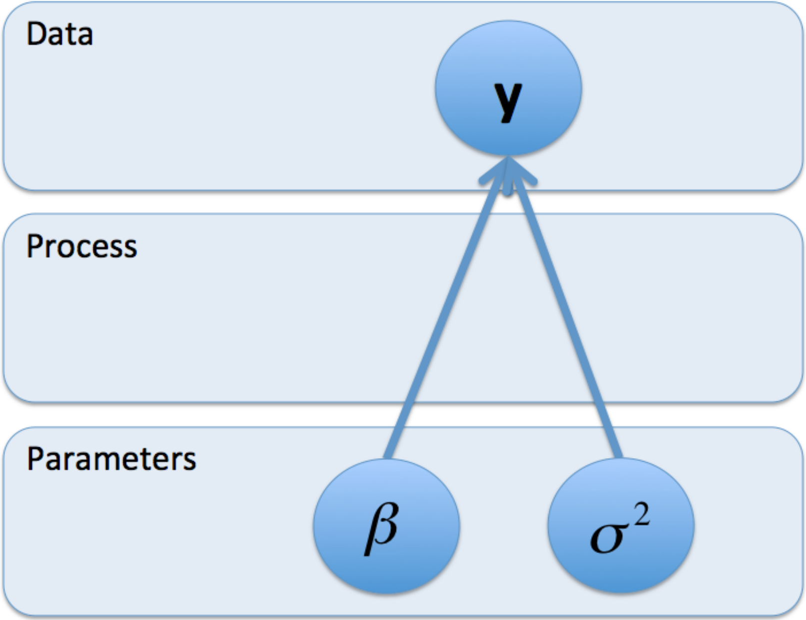
Process-based model building

Contrast between empirical model of data and

'Data Generating Process'

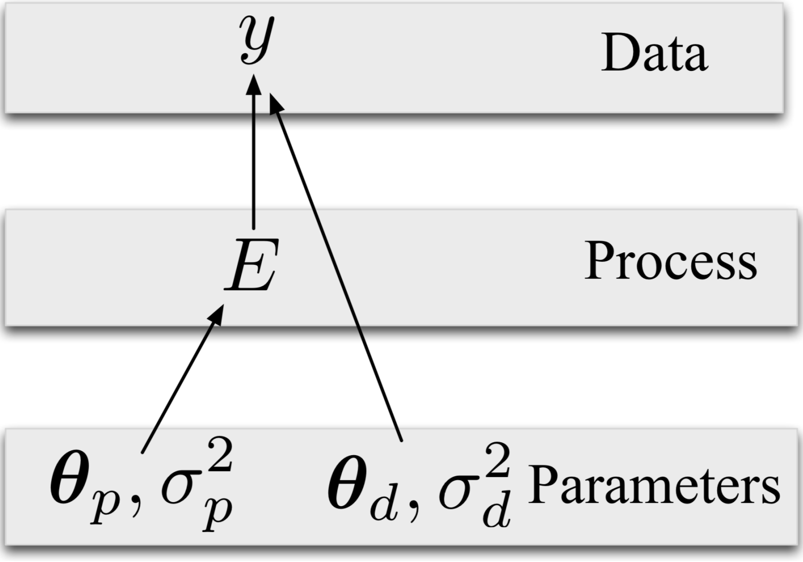
Understanding of how observations link to population processes, & how model parameters relate to these relationships

Simple model for parameters influencing data



(from Hobbs & Hooten 2015)

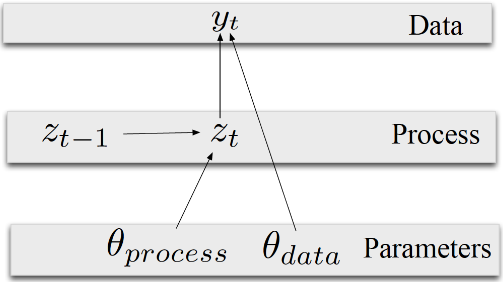
Process-based model



(from Hobbs & Hooten 2015)

A broadly applicable approach to modeling dynamic processes in ecology

$$[\mathbf{z}, \boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data} | \mathbf{y}] \propto \prod_{t=2}^{T} [y_t | \boldsymbol{\theta}_{data}, z_t] \, [z_t | \boldsymbol{\theta}_{process}, z_{t-1}] \, [\boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}, z_1]$$



(from Hobbs & Hooten 2015)

Matrix Population Modeling

Matrix Population Models are convenient way to represent population processes & project population dynamics.

Can be expanded to many cases.

Many fisheries stock assessment models can be written in matrix form.

Caveat is largely linear processes
- can include nonlinear functions in matrices

Matrices

Matrix **Order** is the dimension of the matrix.
e.g. a matrix order (m x n) contains mn elements, has m rows and n columns:

$$A_{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Matrix form:

Square matrix:

$$\begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix}$$

Row matrix:

$$\begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \end{bmatrix}$$

Column matrix:

$$\begin{bmatrix} y_{1,1} \\ y_{2,1} \\ \cdots \\ y_{m,1} \end{bmatrix}$$

Multiplication

scalar multiplication:

$$A = \alpha \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} \alpha a_{1,1} & \alpha a_{1,2} \\ \alpha a_{2,1} & \alpha a_{2,2} \end{bmatrix}$$

So for:

$$A = \begin{bmatrix} 4 & -8 & 6 \\ 2 & -10 & 4 \end{bmatrix} \quad \frac{1}{2}A = \begin{bmatrix} 2 & -4 & 3 \\ 1 & -5 & 2 \end{bmatrix}$$

Matrix Multiplication

Consider A of order (m, n) and B of order (n, r).

Product $AB = C$ is order (m, r) with elements $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$

To be able to do matrix multiplication, number of columns in first matrix must equal the number of rows in the second matrix.

Let

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -2 \\ 1 & -5 \\ 3 & 2 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 4 \times 2 + 2 \times 1 + 6 \times 3 & 4 \times (-2) + 2 \times (-5) + 6 \times 2 \\ 2 \times 2 + (-1) \times 1 + 4 \times 3 & 2 \times (-2) + (-1) \times (-5) + 4 \times 2 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 4\times 2 + 2\times 1 + 6\times 3 & 4\times(-2) + 2\times(-5) + 6\times 2 \\ 2\times 2 + (-1)\times 1 + 4\times 3 & 2\times(-2) + (-1)\times(-5) + 4\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+2+18 & -8-10+12 \\ 4-1+12 & -4+5+8 \end{bmatrix} = \begin{bmatrix} 28 & -6 \\ 15 & 9 \end{bmatrix}$$

Matrix multiplication is *not* commutative.

BA is of order $(3,3)$. $AB \neq BA$

Matrix multiplication

Other things to know:
Matrix multiplication is associative: $(AB)C = A(BC)$
Matrix multiplication is distributive: $A(B + C) = AB + AC$
Scalar multiplication commutative, associative, & distributive.

Transpose of a product: $(AB)^{'} = B^{'}A^{'}$ Use exponentiation operator to denote repeated multiplication: $A^3 = A \cdot A \cdot A$

Population projection using matrices

Leslie Matrix of population age structure:

Vector N_t is the numbers at age. i.e.

$$N_t = \begin{bmatrix} n_{0,t} \\ n_{1,t} \\ \dots \\ n_{J,t} \end{bmatrix}$$

Life table matrix consists of the survival from one age to the next on the off-diagonals, and the fecundity of mature individuals on the first row.

$$X = \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_{J-1} & f_J \\ S_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & S_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & S_2 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & S_{J-1} & S_J \end{bmatrix}$$

Population projection

$$N_{t+1} = AN_t = \begin{bmatrix} n_{1,t} \cdot f_1 + n_{2,t} \cdot f_2 + \cdots + n_{J,t} \cdot f_J \\ n_{0,t} \cdot S_0 \\ n_{1,t} \cdot S_1 \\ \cdots \\ n_{J-1,t} \cdot S_{J-1} + n_{J,t} \cdot S_J \end{bmatrix}$$

Age-structured Fisheries Model

$$N_{1,t+1} = R_{t+1}$$
$$N_{a+1,t+1} = N_{a,t}e^{-(F_{a,t}+M)}$$

$$N_{A,t+1} = N_{A-1,t}e^{-(F_{A-1,t}+M)} + N_{A,t}e^{-(F_{A,t}+M)}$$

Matrix form Age-structured Fisheries Model

$$\mathbf{N}_{t+1} = \begin{bmatrix} R_{t+1} \\ 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ e^{-(F_{0,t}+M)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{-(F_{1,t}+M)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{-(F_{2,t}+M)} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{-(F_{A-1,t}+M)} & e^{-(F_{A,t}+M)} \end{bmatrix} \begin{bmatrix} N_{0,t} \\ N_{1,t} \\ N_{2,t} \\ N_{3,t} \\ \cdots \\ N_{A,t} \end{bmatrix}$$

Building block approach

- multiple matrices applied sequentially
- each representing a different population process
- (birth, age increment, survival, movement, etc.)

(Newman et al. 2015 Chap 2)

Population Projection Examples

1. Coho salmon (Newman Chapter 2)
2. SEAK Steller sea lions
3. BCB Bowhead whales (HW #1)

SE AK Steller sea lions

Counts of pups & non-pups over time

Estimates of fecundity & survival from mark-resight

Create a stage-structured projection model

Two life stages, pups (Age 0) and non-pups (Ages 1+)

Observation Models

Thus far defined population model projections

Want to confront population processes with our data

- e.g. to estimate population parameters

Additional model for how data relate to population state variables

Observation model

e.g. Steller sea lion counts

- Observations are counts of pups and non-pups from aerial photographic surveys
- Believe counts are proportional to the abundance
- Some fraction not observed, likely different for pups and non-pups

$$\begin{bmatrix} E[y_{pups,t}] \\ E[y_{np,t}] \end{bmatrix} = \begin{bmatrix} q_p & 0 \\ 0 & q_{np} \end{bmatrix} \begin{bmatrix} N_{pups,t} \\ N_{np,t} \end{bmatrix}$$

Next Week

NO CLASS (GF at ICES conference).

reading:
Quinn 2003, Nat Res Mod
Thorson & Minto 2015, ICES JMS