Cache-Oblivious Search Trees via Trees of Small Height

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ALCOM-FT

Result : New Search Tree

$$\{\ 1,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 13\ \}$$

$$\Downarrow$$

6	4	8	1	–	3	5	–	–	7	–	–	11	10	13

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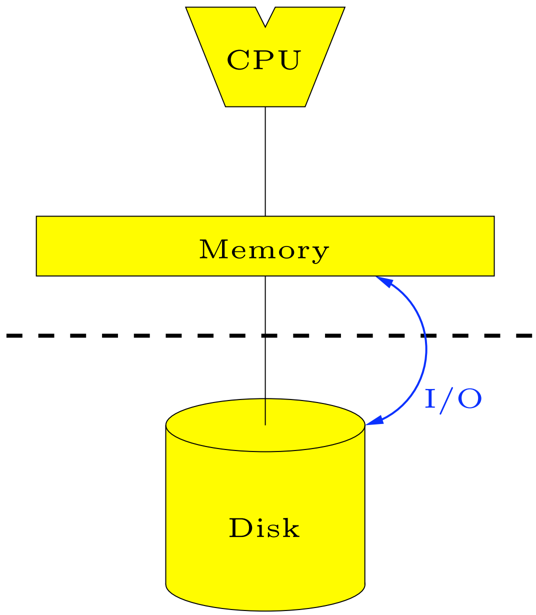
Outline

- Models of Computation
  - I/O Model
  - Cache-Oblivious Model
- Cache-Oblivious Search Trees
  - Static
  - Dynamic
- Experiments
  - Memory Layouts of Trees
- Summary

I/O Model



N = problem size
M = memory size
B = I/O block size

Aggarwal and Vitter 1988

- Bottleneck $\equiv$ I/Os between the two highest memory levels
- B-trees support searches and updates in $O(\log_B N)$ I/Os
- $\Theta\left(\frac{M}{B}\right)$-way merge-sort achieves optimal $\Theta\left(\frac{N}{B}\log_{M/B}\frac{N}{B}\right)$ I/Os

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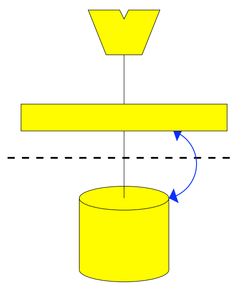
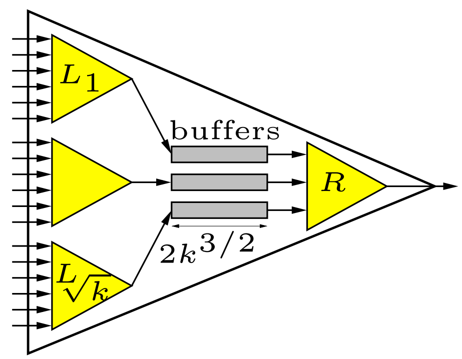
Cache-Oblivious Model

- I/O model
- Algorithms do not know the parameters B and M
- Optimal off-line cache replacement strategy



Examples

- Scanning, Linear time selection
- Matrix-transposition, FFT, Funnel-sorting



Lemma

Optimal cache-oblivious algorithm implies optimal algorithm on each level of a fully associative multi-level cache using LRU

Frigo et al. 1999

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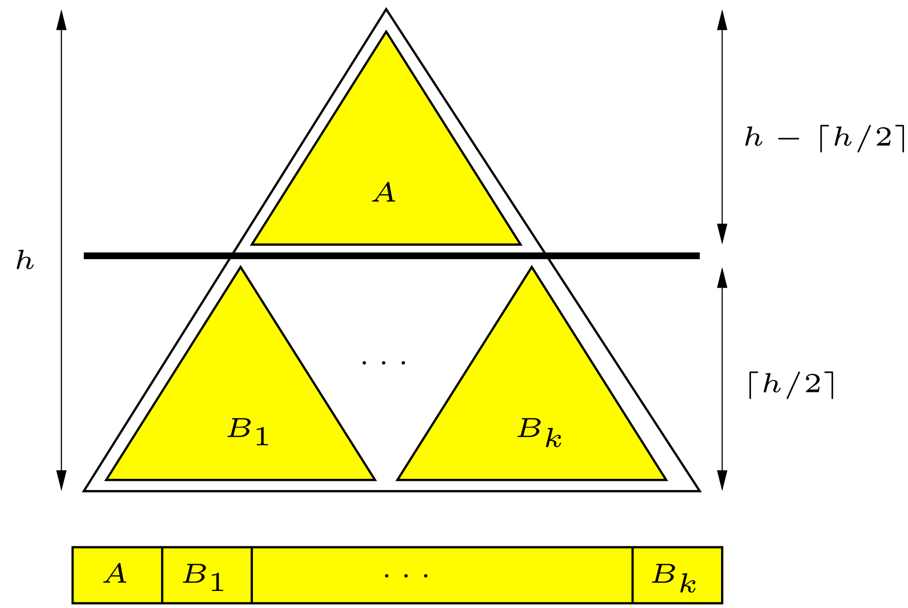
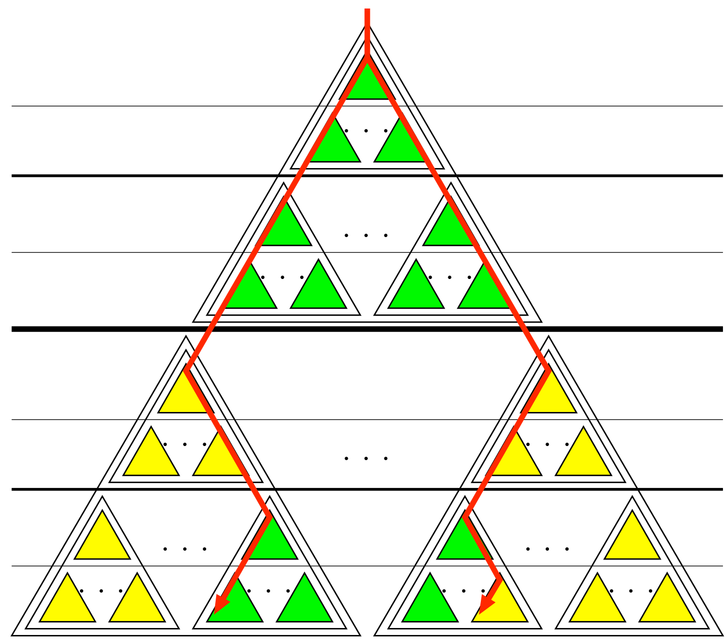
Static Cache-Oblivious Trees

Recursive memory layout $\equiv$ van Emde Boas layout





Degree $O(1)$

Searches use $O(\log_B N)$ I/Os

Range reportings use

$O\left(\log_B N + \frac{k}{B}\right)$ I/Os

Prokop 1999

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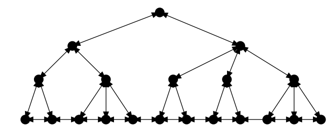
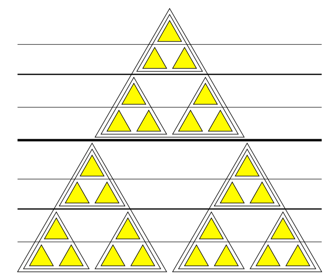
Dynamic Cache-Oblivious Trees

Search	$O(\log_B N)$
Range Reporting	$O\left(\log_B N + \frac{k}{B}\right)$
Updates	$O\left(\log_B N + \frac{\log^2 N}{B}\right)$

- Pointer Based Strongly Weight Balanced B-trees
- Dynamic van Emde Boas Layout
- Packed Memory Management
- Two Levels of Indirection

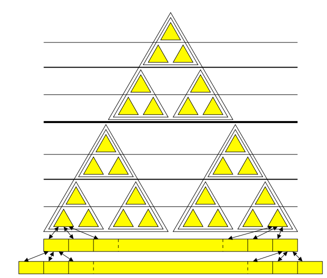


Arge and Vitter 1996



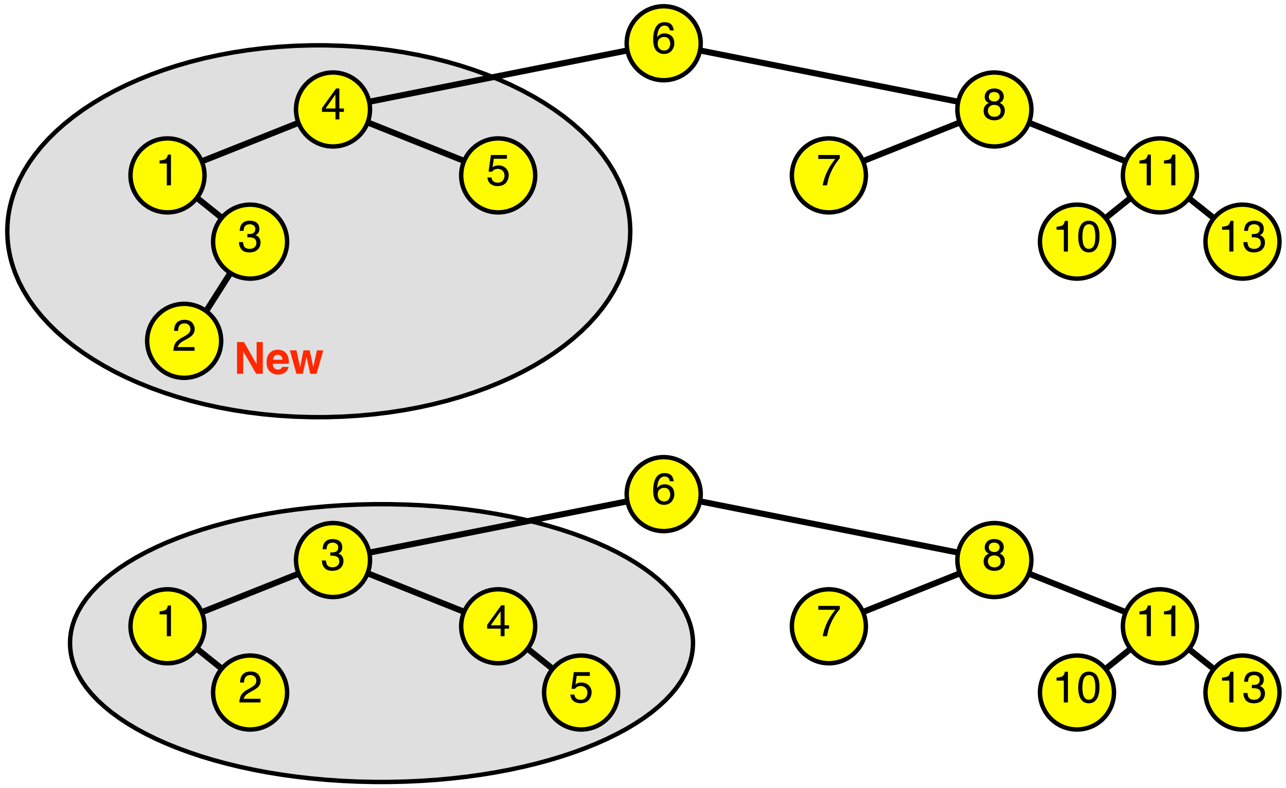
Prokop 1999

Itai et al. 1981



Bender, Demain, Farach-Colton 2000

Binary Trees of Small Height

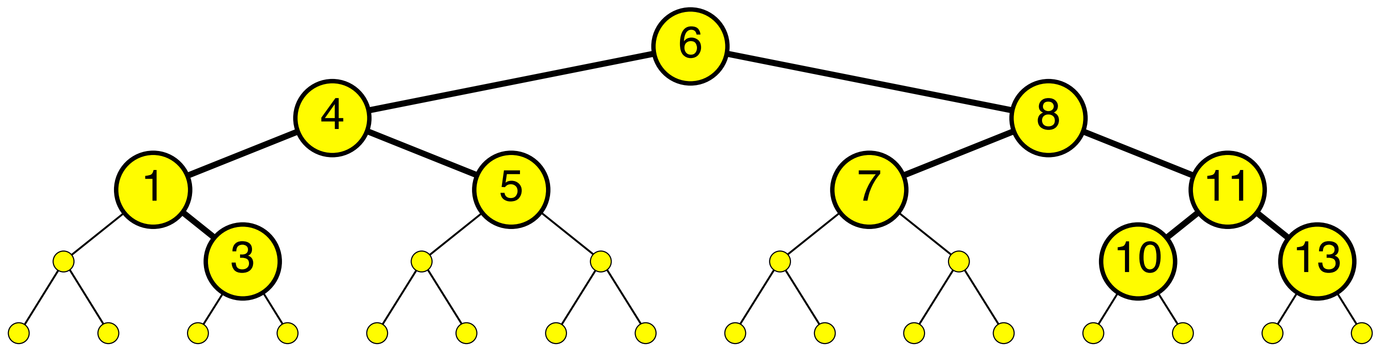


- If an insertion causes non-small height then rebuild subtree at nearest ancestor with sufficient few descendents
- Insertions require amortized time $O(\log^2 N)$

Andersson and Lai 1990

Dynamic Cache-Oblivious Trees

- Embed a dynamic tree of small height into a complete tree
- Static van Emde Boas layout



Search	$O(\log_B N)$
Range Reporting	$O\left(\log_B N + \frac{k}{B}\right)$
Updates	$O\left(\log_B N + \frac{\log^2 N}{B}\right)$

New

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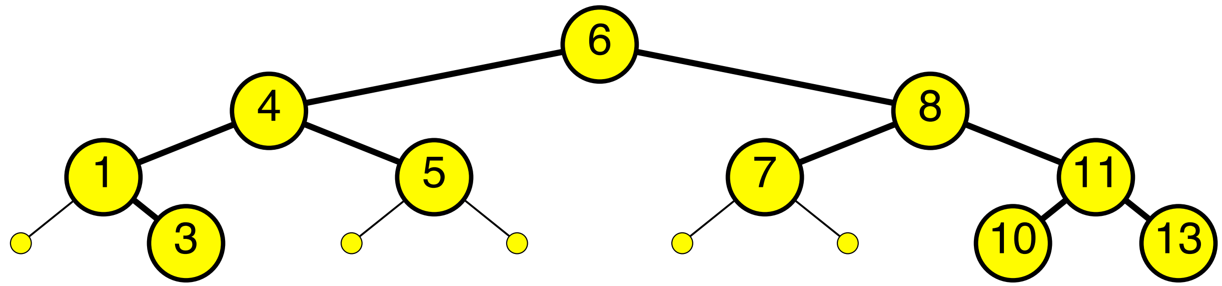
Example



$$\Downarrow$$

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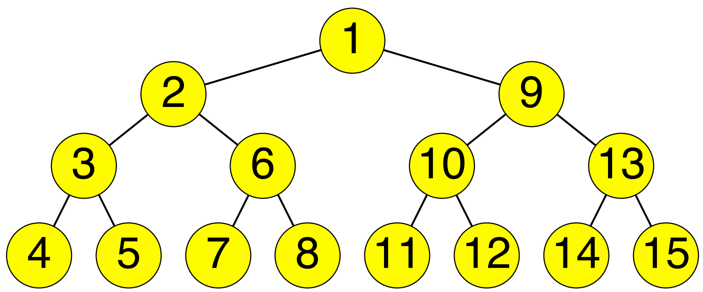
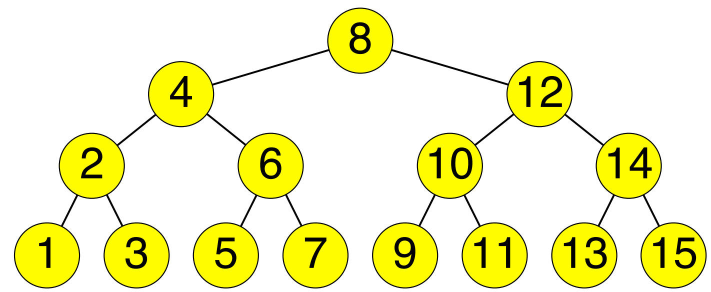
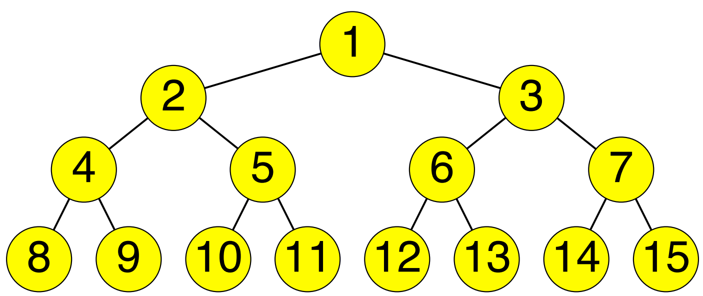
Memory Layouts of Trees



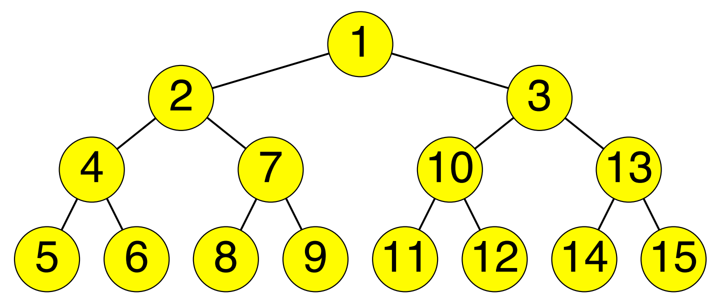
DFS



inorder

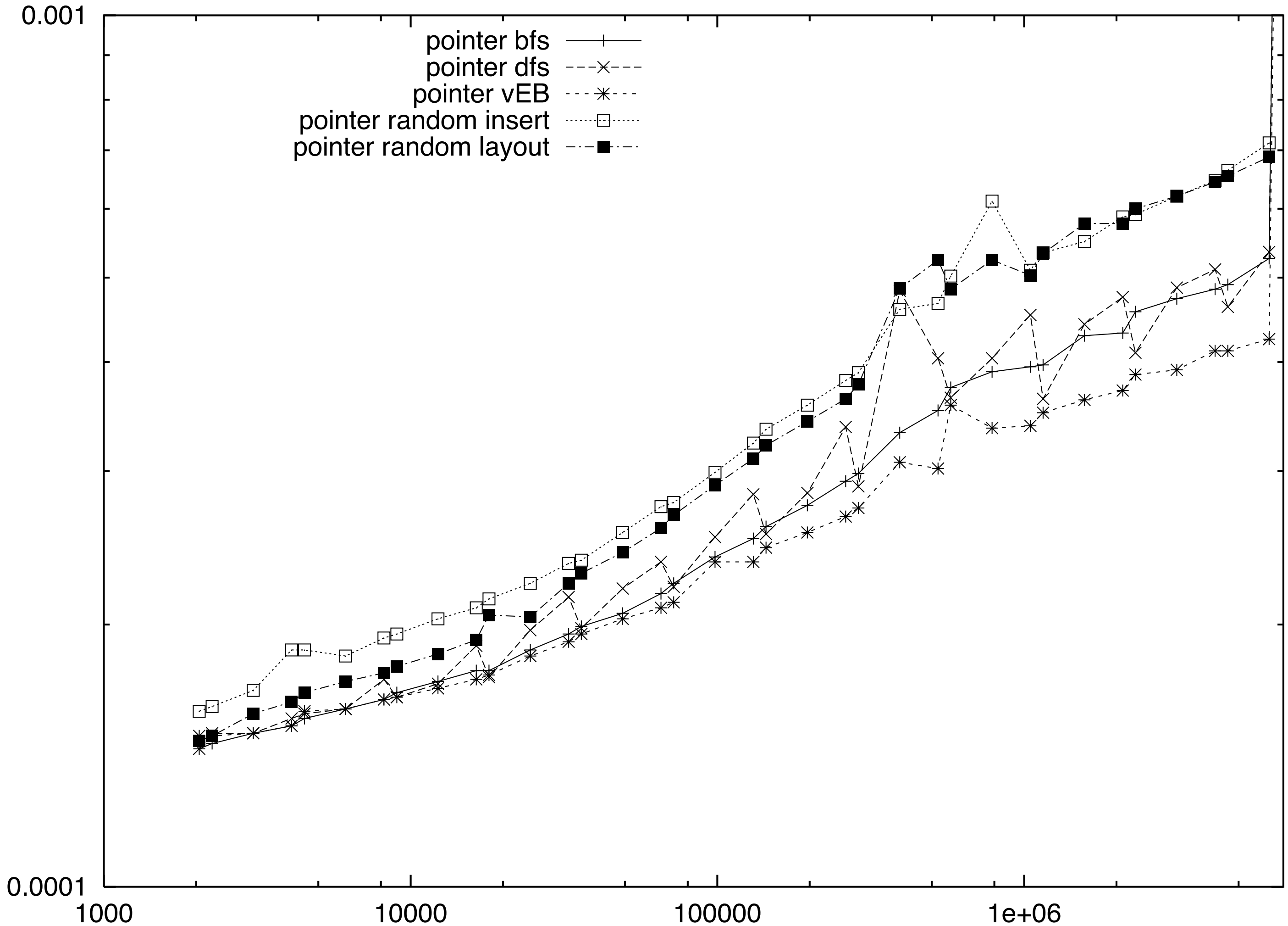


BFS



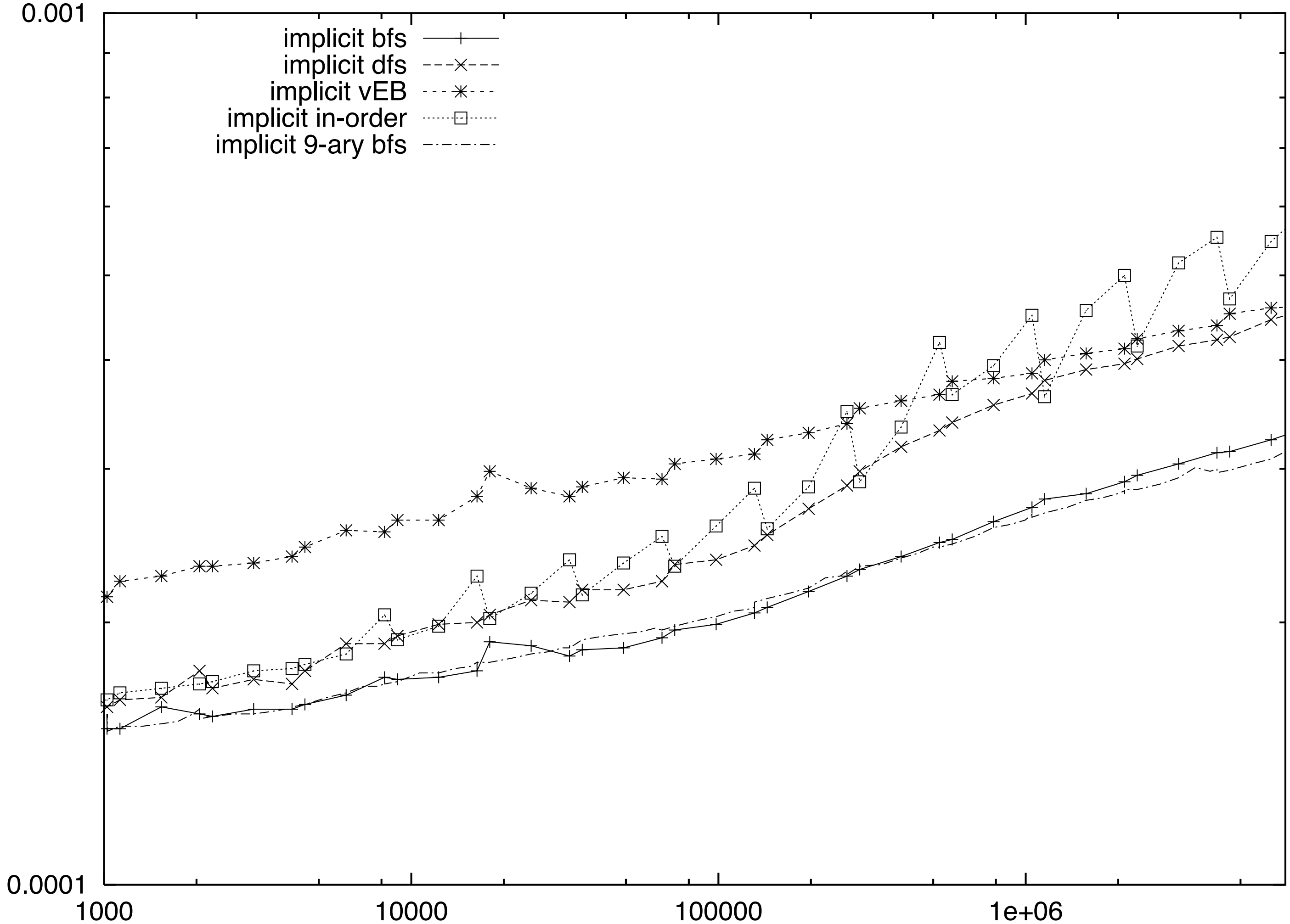
van Emde Boas

Searches in Pointer Based Layouts



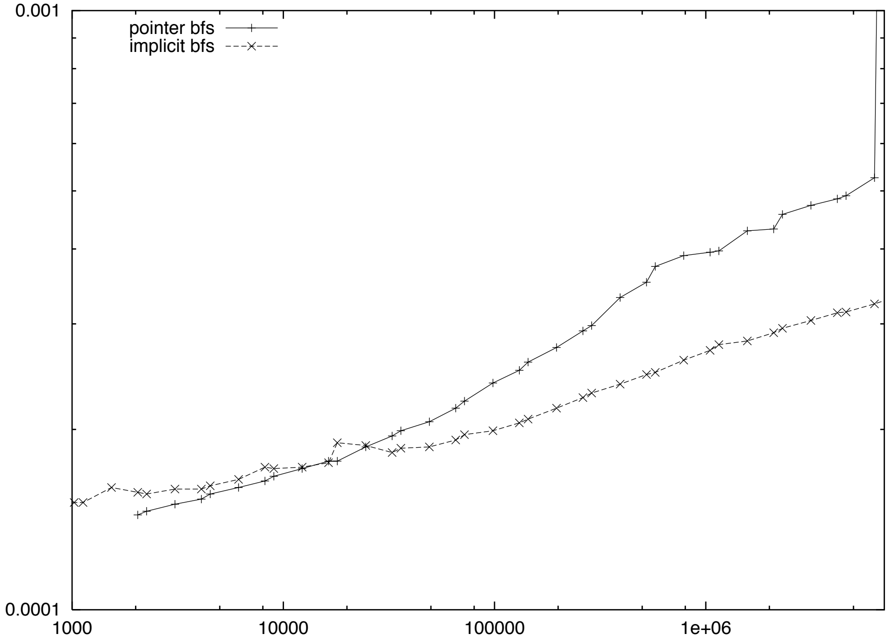
- van Emde Boas layout wins, followed by the BFS layout

Searches with Implicit Layouts

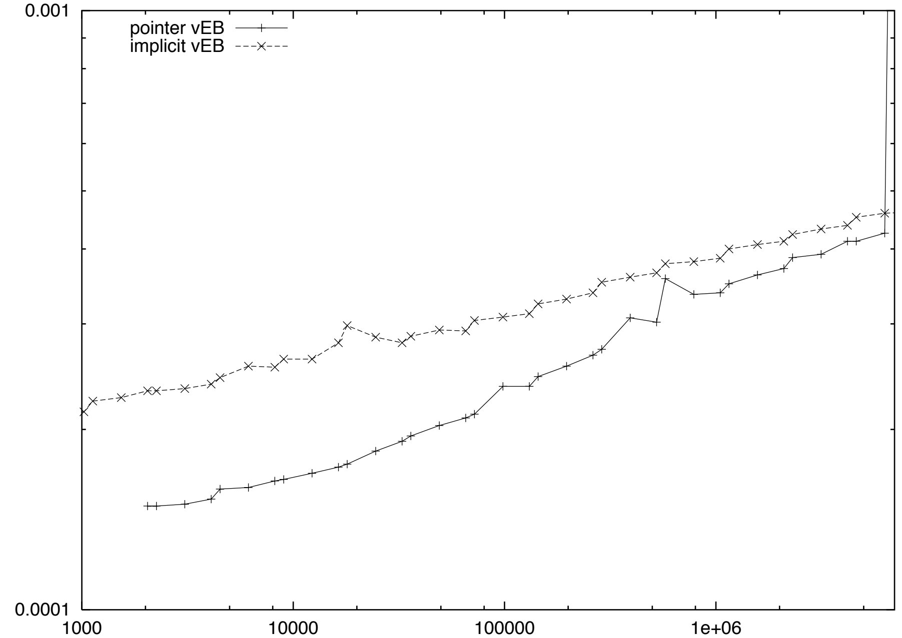


- BFS layout wins due to simplicity and caching of topmost levels
- van Emde Boas layout requires quite complex index computations

Implicit vs Pointer Based Layouts



BFS layout



van Emde Boas layout

- Implicit layouts become competitive as n grows

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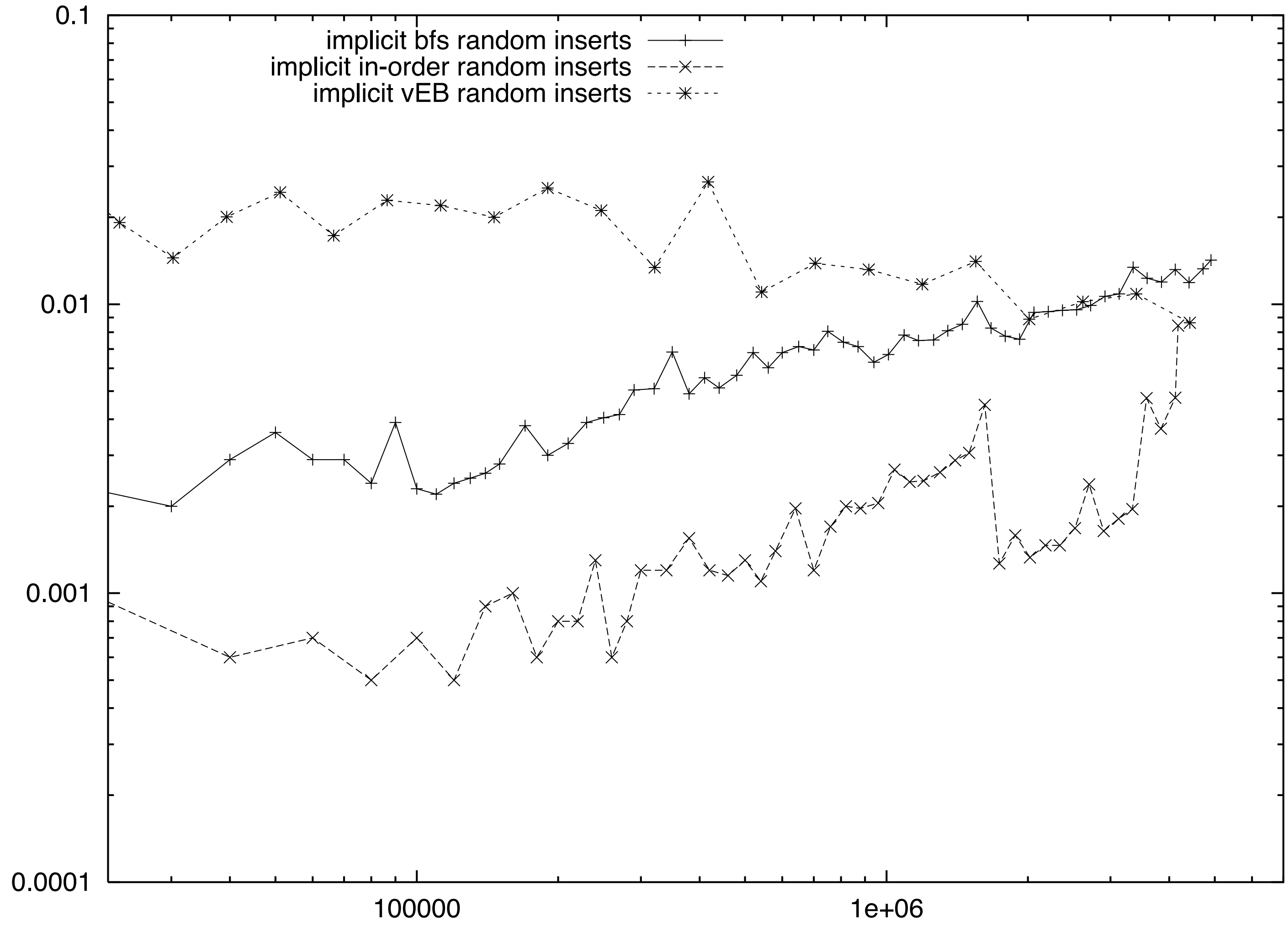
Insertions in Implicit Layouts



- Insertions are rather slow (factor 10-100 over searches)

Summary

- New simple cache-oblivious search trees

Search	$\mathrm{O}(\log_B N)$
Range Reporting	$\mathrm{O}\left(\log_B N + \frac{k}{B}\right)$
Updates	$\mathrm{O}\left(\log_B N + \frac{\log^2 N}{B}\right)$

- Update time $\mathrm{O}(\log_B N)$ by one level of indirection (implies sub-optimal range reporting)
- Importance of memory layouts
- van Emde Boas layout gives good cache performance
- Computation time is important when considering caches

6	4	8	1	–	3	5	–	–	7	–	–	11	10	13