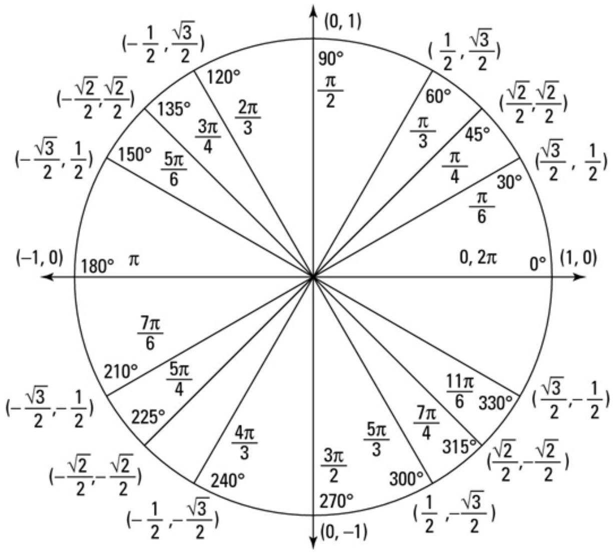
MATH 1060: Unit 1 Review Sheet

Quick Review



Things to Remember

- $\sqrt[a]{x^b} = x^{b/a}$
- SOHCAHTOA
- Conjugate of $\sqrt{a} \pm b$ is $\sqrt{a} \mp b$

1.3: Exponential Functions

$f(x) = b^x$; $b \neq 1$ and b is a positive number

domain: $(-\infty, \infty)$
range: $(0, \infty)$
natural exponential function: $f(x) = e^x$
Laws of Exponents:

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

1.3: Inverse Functions

inverse function: Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$
Properties of Inverse Functions:

- The domains and ranges of f and f^{-1} are switched.
- The graph of f^{-1} is the graph of f reflected about the line $y = x$.
- $f^{-1}(x)$ is the inverse of $f(x)$, but $(f(x))^{-1}$ is the reciprocal of $f(x)$.

Check for inverse: If the function passes the horizontal line test, the function is one-to-one and has an inverse
How to find the inverse:

1. Switch x and y.
2. Solve for y.
3. Substitute $f^{-1}(x)$ for y.

1.3: Logarithmic Functions

$$f(x) = \log_b(x)$$

Exp/Log: Exponential and logarithmic functions are inverses

- $b^{\log_b x} = x$ for $x > 0$
- $\log_b(b^x) = x$ for all x

Natural Logarithmic Function: $f(x) = \ln(x)$
Laws of Logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^r) = r\log_b(x)$

Change of Basis Rules:

- $b^x = e^{x\ln(b)}$
- $\log_b(x) = \frac{\ln(x)}{\ln(b)}$ for $x > 0$

1.4: Trig Functions and Inverses

Trigonometric Identities:

- $\csc(\theta) = \frac{1}{\sin(\theta)}$
- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Inverse trig functions take a *number* as an input and output an *angle*
Example: For $\cos^{-1}\left(\frac{1}{2}\right)$, Ask yourself, "What angle would give $cos(\theta) = \frac{1}{2}$?"
Important Note: Be careful that the angle you give is within the domain for the inverse trig function.

Function	Range
$y = \sin^{-1}(x)$	$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \cos^{-1}(x)$	$y \in [0, \pi]$
$y = \tan^{-1}(x)$	$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \csc^{-1}(x)$	$y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$
$y = \sec^{-1}(x)$	$y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{3})$
$y = \cot^{-1}(x)$	$y \in (0, \pi)$

2.1: The Idea of Limits

$$\text{Average Velocity} = v_{avg} = \frac{s(t_1)-s(t_0)}{t_1-t_0}$$
$$\text{Secant Line} = m_{sec} = \frac{s(t_1)-s(t_0)}{t_1-t_0}$$

Note:

- average velocity $\rightarrow$ secant line
- instantaneous velocity $\rightarrow$ tangent line
- secant lines approach the tangent line
- slopes of secant lines approach the slops of the tangent line at the point $(a, s(a))$.

$$\text{Inst Velocity} = v_{inst} = \lim_{t\to a} v_{avg} = \lim_{t\to a} \frac{s(t)-s(a)}{t-a}$$
$$\text{slope of tangent line} = m_{tan} = \lim_{t\to a} m_{sec}$$
$$= \lim_{t\to a} \frac{s(t)-s(a)}{t-a}$$

2.2: Definitions of Limits

$$\lim_{x\to a} f(x) = L$$

If the limit exists, it depends on the value of f *near* a, *not* the value of $f(a)$.

Right-sided limit: $\lim_{x\to a^+} f(x) = L$

Left-sided limit: $\lim_{x\to a^-} f(x) = L$

Theorem 2.1: Assume f is defined for all x near a except possibly at a. Then

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x).$$

2.3: Computing Limits

Limit Laws: Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where c is a real number and $n > 0$ is an integer.

- $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$
- $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$
- $\lim_{x\to a}(f(x))^{1/n} = \left(\lim_{x\to a} f(x)\right)^{1/n}$, provided $f(x) > 0$, for x near a, if n is even.

Limits of Polynomial and Rational Functions: Assume p and q are polynomials and a is a constant.

- $\lim_{x\to a} p(x) = p(a)$
- $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$.

2.3: Computing Limits Continued

One Sided Limits: You can still use direct substituion!

Direct Sub Doesn't Work?
If you get $\frac{0}{0}$ I.F. by direct substitution, write $\frac{0}{0}$ I.F. then try the following:

- algebraically manipulate
- factor and cancel out terms
- multiply by the conjugate

The Squeeze Theorem: Assume the functions f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Important Inequalities:

- $-1 \leq \sin(\theta) \leq 1$
- $-1 \leq \cos(\theta) \leq 1$

How to Use Squeeze Theorem:

1. Use one of the two above inequalities
2. multiply/divide/subtract/add to all terms in the inequality to get the middle to look like what you want to take the limit of
3. Take the limit of the left hand side of the inequality
4. Take the limit of the right hand side of the inequality
5. If these limits match, then the limit of the middle is also the same

Useful Trig Limits:

- $\lim_{x\to 0} \sin(x) = 0$
- $\lim_{x\to 0} \cos(x) = 1$

2.4: Infinite Limits

$$\lim_{x\to a} f(x) = \infty \qquad \text{OR} \qquad \lim_{x\to a} f(x) = -\infty$$

Finding Infinite Limits:

1. Try direct substitution first.
2. if you get $\frac{0}{0}$, see section 2.3.
3. if you get $\frac{\text{nonzero number}}{0}$,
   - try plugging in numbers reeeeeeally close to the right and left of a.
   - if $\lim_{x\to a^-} f(x) = \infty = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x) = \infty$. (Same for $-\infty$).

vertical asymptote: if $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, or $\lim_{x\to a^-} f(x) = \pm\infty$, then the line $x = a$ is a vertical asymptote of f.

Finding Vertical Asymptotes:

1. Find the values where the denominator$= 0$ but the numerator$\neq 0$. You will usually have to factor.
2. Prove that you have a vertical asymptote using limits. Take the limit of the function as x approaches each value from the left and right. At least one limit should be infinite.

2.5 Limits at Infinity

$$\lim_{x\to\infty} f(x) = L$$

horizontal asymptote: The line $y = b$ is a *horizontal asymptote* of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{OR} \qquad \lim_{x\to-\infty} f(x) = b$$

I.F.: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

Limits at Infinity of Powers and Polynomials:

- $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = \infty$; n is *even.*
- $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$; n is *odd.*
- $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$
- $\lim_{x\to\pm\infty} p(x) = \infty$ or $-\infty$ depends on the degree of the polynomial and sign of leading coefficient.

2.5: Limits at Infinity (Continued)

Technique for Rational Functions:

1. Choose the highest power of x in the denominator.
2. Divide every term in the numerator and denominator by the highest power of x in the denominator.
3. Take the limit of each term. Recall that the limit as $x \to \pm\infty x^{-n} = 0$

VERY IMPORTANT NOTE:

- $\sqrt{x^2} = |x| = x$ if $x > 0$ (when $x \to \infty$)
- $\sqrt{x^2} = |x| = -x$ if $x < 0$ (when $x \to -\infty$)

slant asymptote: When the degree of numerator is *ONE MORE* than degree of denominator,

1. Use long division to divide the numerator by the denominator.
2. The equation of the line that is the slant asymptote is the quotient from your long division.

2.6: Continuity

continuous at a point: A function f is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

Conditions for Continuity of f at a:

- $f(a)$ is defined. (a is in the domain of f)
- $\lim_{x\to a} f(x)$ exits
- $\lim_{x\to a} f(x) = f(a)$

Types of Discontinuities:

- removable
- jump
- infinite
- oscillating

continuous from the right at a number a: A function f is continuous from the right at a number a if

$$\lim_{x\to a^+} f(x) = f(a)$$

continuous from the left at a number b: A function f is continuous from the left at a number b if

$$\lim_{x\to b^-} f(x) = f(b)$$

continuous on an interval: A function is continuous on an interval if it is continuous at every number in the interval

continuous function: A continuous function is continuous at every point of its domain.

Continuity Theorems:

- The inverse of a continuous function is continuous.
- The composition of continuous functions is continuous.

2.6: Continuity (Continued)

Functions that are Continuous of their Domains

- polynomials $(-\infty, \infty)$
- rational functions (everywhere except denominator=0)
- root functions (inside of root ≥ 0)
- trig functions
- inverse trig functions
- exponential functions $(-\infty, \infty)$

The Intermediate Value Theorem (IVT)

If

- f continuous on $[a, b]$
- $f(a) < L < f(b)$

Then,

- $a < c < b$
- $f(c) = L$

2.7: $\delta - \epsilon$ Proof

limit: If for every number $\epsilon > 0$, there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

$$\lim_{x\to a} f(x) = L$$

Steps for Proving $\lim_{x\to a} f(x) = L$

1. Write down what $f(x), L$, and a are.
2. Find δ in your scratch work using $|f(x) - L| < \epsilon$. This is not part of your proof and δ should be in terms of ϵ. We try to algebraically get $|f(x) - L|$ to look like a multiple of $|x - a|$.
3. Write your proof using the following sentences with your values for δ, a, and L plugged in:
   - Given $\epsilon > 0$, let $\delta =$
   - If $0 < |x - a| < \delta =$, then *SCRATCH WORK to show $|f(x) = L| < \epsilon$*.
   - By the definition of a limit, $\lim_{x\to a} f(x) = L$.

3.1: Introducing the Derivative

derivative: the slope of the tangent line

How to find equation of tangent line:

1. Find slope of the tangent line using:

$$m_{tan} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$
OR
$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

2. Use point-slope formula to find equation

$$y - y_1 = m_{tan}(x - x_1)$$

Note that $y_1 = f(x_1)$ if not given

How to find equation of normal line:

1. Find slope of the tangent line using:

$$m_{tan} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$
OR
$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

2. Find the slope of the normal line using:

$$m_{norm} = -\frac{1}{m_{tan}}$$

3. Use point-slope formula to find equation

$$y - y_1 = m_{norm}(x - x_1)$$

Note that $y_1 = f(x_1)$ if not given

position: $s(t)$
velocity: derivative of position

$$v(a) = s'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

3.2: The Derivative as a Function

derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

How to find equation of tangent line:

1. Find derivative of function
2. Plug in given x_1 value to derivative to get slope of tangent line

$$m_{tan} = f'(x_1)$$

3. Use point-slope formula to find equation

$$y - y_1 = m_{tan}(x - x_1)$$

Note: $y_1 = f(x_1)$ if not given

differentiable at a If $f'(a)$ exists
differentiable on an open interval if f is differentiable at every number in the interval.
Theorem:

- Differentiability $\implies$ Continuity
- Continuity $\not\Longrightarrow$ Differentiability

Differentiability Fails:

- discontinuity
- corner
- vertical tangent, cusps

$f(x)$	$f'(x)$
increasing	positive
decreasing	negative
horizontal tangent	zero (root)
not diff at a	$f'(a)$ is undefined

Graph Note: The derivative of a graph of degree n is a graph of degree $n-1$. So, the derivative is one degree lower than the original function.

3.2: The Derivative as a Function (Continued)

Functions and their Derivative Graphs

- Look for *horizontal tangent lines* first and match these x-coordinates to *zeros* on the derivative graph.
- Look for *points of discontinuity* and match these to *holes or gaps* in the derivative graph.
- Look for other values of x where the function is *not differentiable*. The derivative graph will *not be defined* there.
- Look for the intervals of *increase* on the original graph. This tells you when the derivative graph is *above the x-axis*.
- Look for the intervals of *decrease* on the original graph. This tells you when the derivative graph is *below the x-axis*.