1.3: Inverse, Exponential, and Logarithmic Functions

Exponential Functions

exponential function:

$$f(x) = b^x \text{ where } b \neq 1 \text{ and } b \text{ is a positive real number}$$

Properties of Exponential Functions:

(1) Domain: $\{x : -\infty < x < \infty\}$ or $(-\infty, \infty)$ since b^x is defined for all real numbers

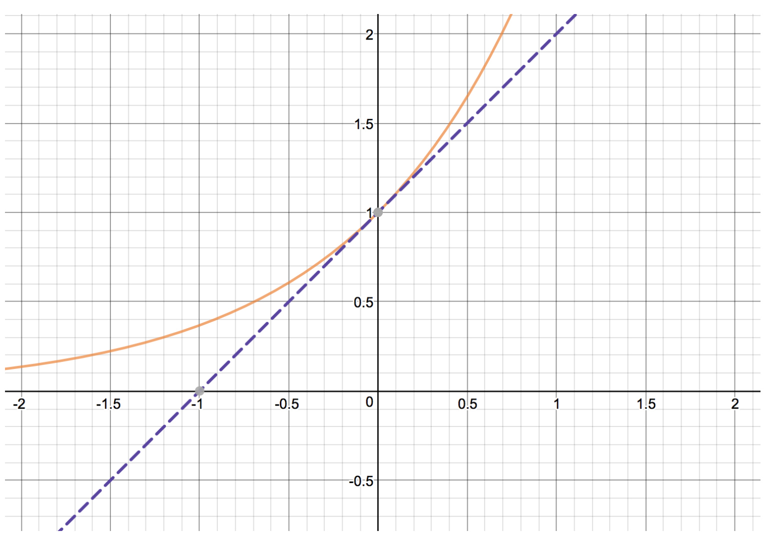
(2) Range: $\{y : 0 < y < \infty\}$ or $(0, \infty)$ since $b^x > 0$ for all values of x

(3) If $b > 1$, then f is an increasing function.
i.e. $b = 2$ means that $2^x > 2^y$ whenever $x > y$.

(4) If $0 < b < 1$, then f is a decreasing function.
i.e. $b = \frac{1}{2}$ means that $f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$

The Natural Exponential Function: $f(x) = e^x$ where $e = 2.71\cdots$

Why is the natural exponential function so special? Because at $x = 0$, the slope of the tangent line of $f(x) = e^x$ is 1. Also, $e^0 = 1$. So, the value and that slope of the tangent line are both 1 at $x = 0$.



Laws of Exponents

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

Examples: Use the laws of exponents to simplify the expressions

1. $(x^2 y^3)^5$
2. $\left(\sqrt{3}\right)^{1/2} \left(\sqrt{12}\right)^{1/2}$
3. $\left(\frac{x^{-2}}{x^8}\right)^{-2}$

Inverse Functions

inverse function: Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(x) = x$.

Properties of Inverse Functions:

- The domains and ranges of f and f^{-1} are switched.
- The graph of f^{-1} is the graph of f reflected about the lines $y = x$.
- $f^{-1}(x)$ is the inverse of $f(x)$, but $(f(x))^{-1}$ is re reciprocal of $f(x)$.

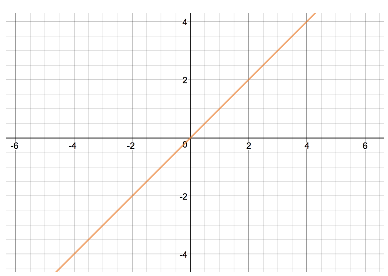
But how do we know if an inverse exists or not? The inverse of a function only exists when the function is one-to-one on the domain.

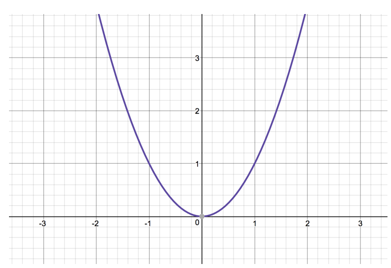
one-to-one: A function f is one-to-one on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D; i.e. if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

How do we check if a function is one-to-one? You can check if a function is one-to-one with the horizontal line test.

horizontal line test: Every horizontal line intersects the graph of a one-to-one function at most once; i.e. if you hold a horizontal line against the function, no part of the line crosses twice.

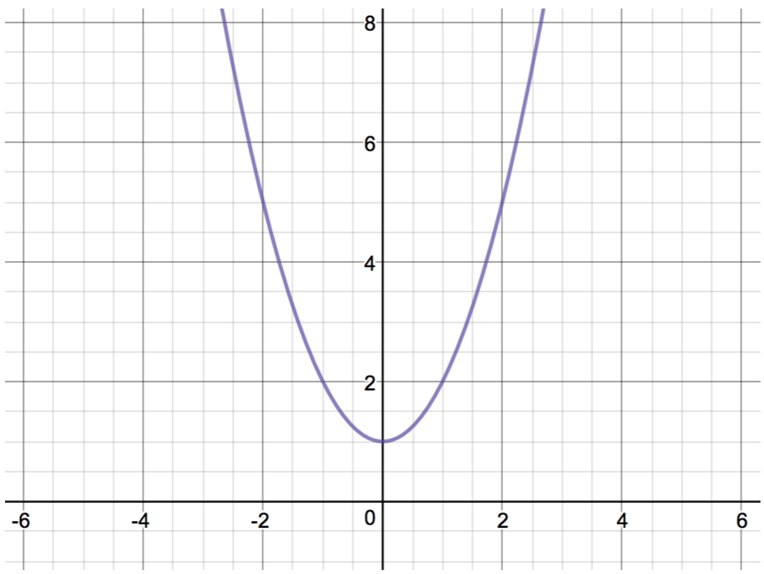
Example: The first function shown below, $f(x) = x$, is one-to-one. The second function shown below, $f(x) = x^2$, is not one-to-one.





If a function is not one-to-one, is there a way to split the interval so certain parts of the function is one-to-one? Yes, we can restrict the domain so the function is one-to-one on certain intervals.

Example: Determine the largest possible intervals on which the function is one-to-one.
$f(x) = x^2 + 1$.



How to find the inverse:

(1) Switch x and y.
(2) Solve for y.
(3) Substitute $f^{-1}(x)$ for y.

Example: Find the formula for f^{-1} and state the domain and range of f and f^{-1}.
$f(x) = x^2 + 1$ where $x \geq 0$.

Existence of Inverse Functions: Let f be a one-to-one function on a domain D with range R. Then, f has a *unique* inverse, f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

where x is in D and y is in R.

Example: Show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ for

$$f(x) = x^2 + 1 \text{ where } x \geq 0.$$

We know from the previous example that $f^{-1}(x) = \sqrt{x-1}$.

Logarithmic Functions

Now, we are going to combine what we learned about exponential functions and inverse functions.
Note that when $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is one-to-one. So, we know that $f(x) = b^x$ has an inverse. The inverse of $f(x) = b^x$ is the logarithmic function with base b.

logarithmic function with base b: This is the inverse function of $f(x) = b^x$ and looks like

$$f^{-1}(x) = \log_b x$$

We can write this as $y = \log_b x$ and realize that the logarithmic function asks the question, "What is the exponent y of the base b that gives x?"

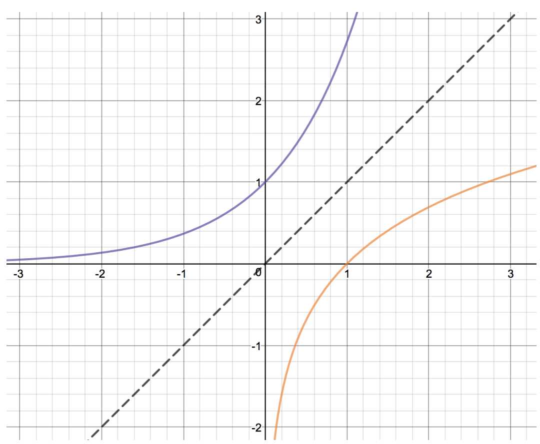
Inverse Relations for Exponential and Logarithmic Functions:
For any base $b > 0$, $b \neq 1$, we have the following:

- $b^{\log_b x} = x$ for $x > 0$
- $\log_b(b^x) = x$ for all x

The Natural Logarithmic Function: $f(x) = \ln(x)$ is the logarithmic function with base e.

Note the following relationship:

- $y = \ln(x)$ and $y = e^x$ are inverses of one another
- $e^{\ln(x)} = x$
- $\ln(e^x) = x$



Laws of Logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^r) = r\log_b(x)$

Example: Solve each equation for x.

1. $\ln(10) - \ln(7 - x) = \ln(x)$

2. $5^{3x-1} = 12$

Sometimes it is useful to change bases, especially with exponential functions. For example, suppose you want to express b^x in the form e^y, so you want to go from base b to base e. Fortunately, we have simple rules for this.

Change of Basis Rules:

- $b^x = e^{x\ln(b)}$
- $\log_b(x) = \frac{\ln(x)}{\ln(b)}$ for $x > 0$

Example: Convert the following expressions to the indicated base.

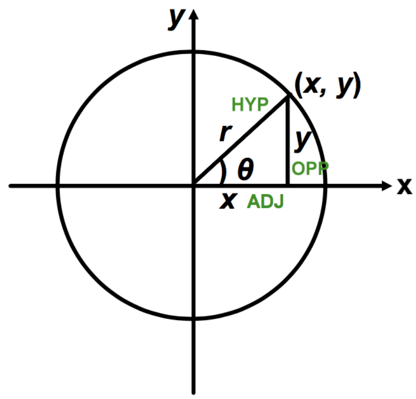
1. 5^{3x-1} using base e.

2. $\log_3(x+1)$

1.4: Trig Functions and Their Inverses

Trig Functions

For a general circle, we have:



SOHCAHTOA

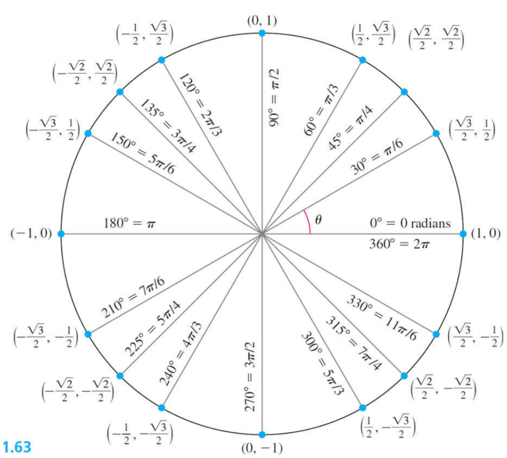
Therefore, we have that

$\sin(\theta) = \frac{y}{r}$ $\quad$ $\csc(\theta) = \frac{r}{y}$

$\cos(\theta) = \frac{x}{r}$ $\quad$ $\sec(\theta) = \frac{r}{x}$

$\tan(\theta) = \frac{y}{x}$ $\quad$ $\cot(\theta) = \frac{x}{y}$

We wall know that the unit circle has radius $r = 1$.



1.63

Example: Find $\cos(\theta)$ and $\tan(\theta)$ given $\sin(\theta) = \frac{3}{5}$ and θ is in $[\pi/2, \pi]$.

Trigonometric Identities:

Reciprocal Identities:

- $\csc(\theta) = \frac{1}{\sin(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities:

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$

Double Angle Formulas:

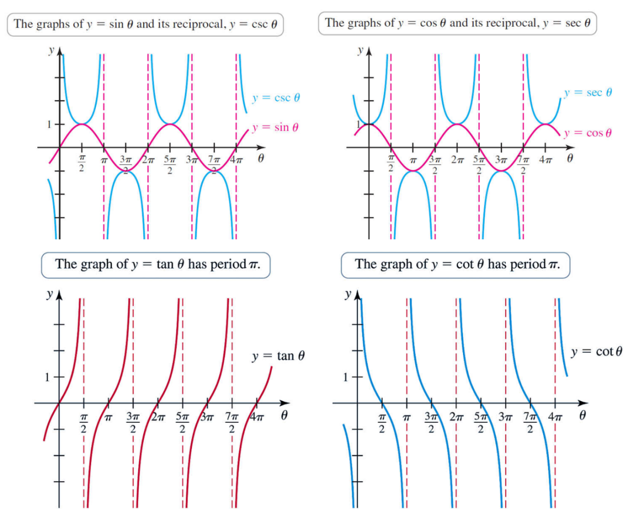
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Half-Angle Formulas:

- $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$
- $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

Example: Solve $\cos(3x) = \sin(3x)$ on $0 \le x < 2\pi$

Graphs of Trig Functions



Inverse Trig Functions

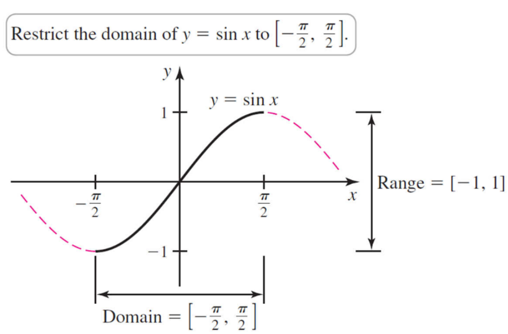
Since all the six basic trif functions are periodic, none are one-to-one. However, we can restrict their domains to create one-to-one versions, like we have done before.

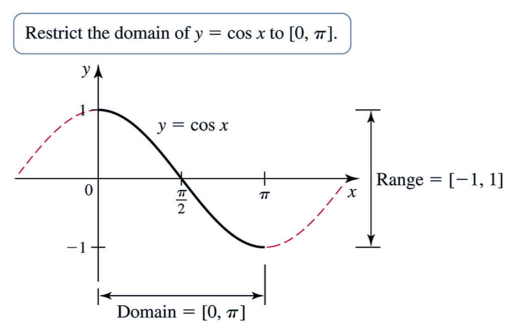
Sometimes, we prefix the trig function name with "arc" to indicate the inverse trig function:

Example: $y = \sin^{-1}(x) = \arcsin(x) \leftrightarrow \sin(y) = x$

Note: It is important to recognize the difference between an inverse and a reciprocal of a trig function.

Inverse: $\cos^{-1}(x)$ **Reciprocal:** $(\cos(x))^{-1} = \frac{1}{\cos(x)}$





Recall that trig functions take an *angle* as an input and output a *number*. However, inverse trig functions take a *number* as an input and output an *angle*.

Note: You must memorize the restricted domains because the angle you give as an evaluation of an inverse trig function must be in the domain for that restricted trig function.

Example: Find the exact value of each expression.

a) $\cos^{-1}\left(\frac{1}{2}\right)$.
b) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.
c) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.
d) $\cos(\cos^{-1}(-1))$
e) $\cos^{-1}(\cos(\frac{7\pi}{6}))$

Example: Draw a right triangle to simplify the expression. Assume $x > 0$.
$\sin(\cos^{-1}(\frac{x}{2}))$

Inverse Trig Functions Summary

Function	**Range**
$y = \sin^{-1}(x)$	$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \cos^{-1}(x)$	$y \in [0, \pi]$
$y = \tan^{-1}(x)$	$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \csc^{-1}(x)$	$y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$
$y = \sec^{-1}(x)$	$y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{3})$
$y = \cot^{-1}(x)$	$y \in (0, \pi)$

2.1: The Idea of Limits

Average Velocity

Let's say you travel thirty miles in your car in half an hour. Therefore, your average velocity is 30 miles/ 0.5 hours = 60 mi/hr. However, even though your average velocity may be 60 mi/hr, your instantaneous velocity (as seen on your speedometer) varies a lot.

Average Velocity: the change in position divided by the elapsed time.

$$v_{avg} = \frac{s(t_1)-s(t_0)}{t_1-t_0}$$

Example: A rock is launched vertically upward from the ground with a speed of 96 ft/s. Neglecting air resistance, the position function of the rock after t seconds is

$$s(t) = -16t^2 + 96t$$

The position s is measured in feet with $s = 0$ corresponding to the ground. Find the average velocity between $t = 1$s and $t = 3$s.

Figure 2.1 shows the position of the rock on the time interval $0 \le t \le 3$. The graph is *not* the path of the rock. The rock travels up and down on a vertical line.

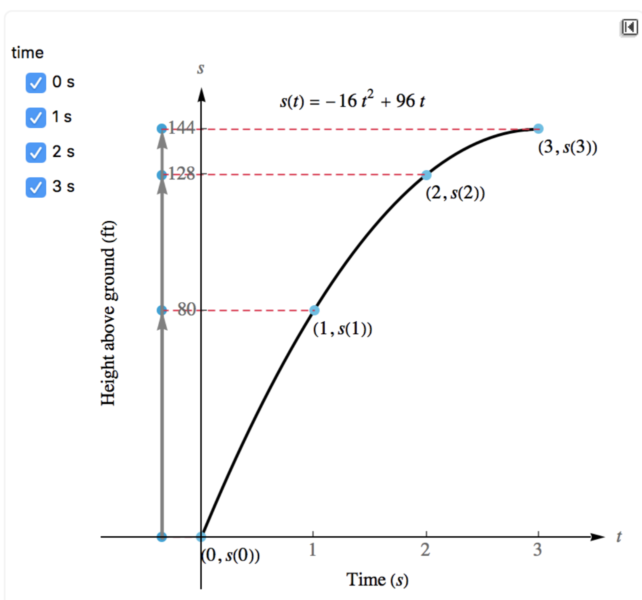


Figure 2.1

Therefore, the average velocity of the rock over the interval $[1, 3]$ is

Note that the average velocity is the slope of the line joining $(1, s(1))$ and $(3, s(3))$ on the graph of the position function.

Here is an important observation: As shown in **Figure 2.2**, the average velocity is the slope of the line joining the points $(1,\ s(1))$ and $(3,\ s(3))$ on the graph of the position function.

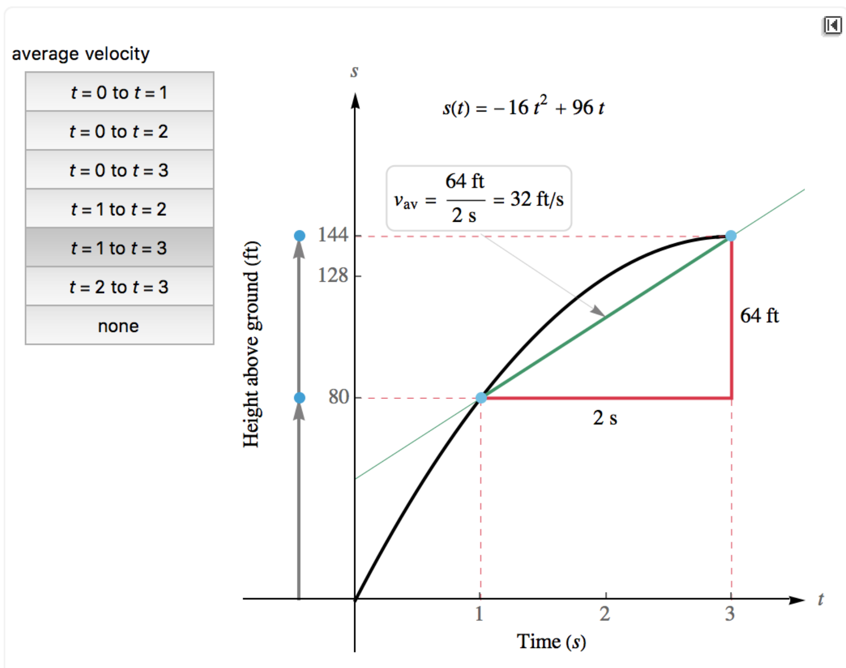


Figure 2.2

Secant Line: any line joining two points on a curve. The slope of a secant line over $[t_0, t_1]$ is

$$m_{sec} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Instantaneous Velocity

Instead of using the position of the object at two distinct points in time to calculus the average velocity, we want to compute the instantaneous velocity at a single point in time. We can find this by computing the average velocities over intervals that decrease in length. So, as t_1 approaches t_0, the average velocities typically approach a unique number- the instantaneous velocity. This single number is also called a *limit*.

Instantaneous Velocity: the limit of the average velocity as time approaches a single point a.

$$v_{inst} = \lim_{t \to a} v_{avg} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Example: Estimate the instantaneous velocity of the rock in the previous example at the single point $t = 1$.

We can compute the average velocities in smaller and smaller intervals and see what vale they tend towards to get our instantaneous velocity at $t = 1$.

Table 2.1

Time interval	Average velocity
[1, 2]	48 ft/s
[1, 1.5]	56 ft/s
[1, 1.1]	62.4 ft/s
[1, 1.01]	63.84 ft/s
[1, 1.001]	63.984 ft/s
[1, 1.0001]	63.9984 ft/s

So, we can say a reasonable estimate for the instantaneous velocity at $t = 1$ is 64 ft/s. This can be written as

Slope of the Tangent Line

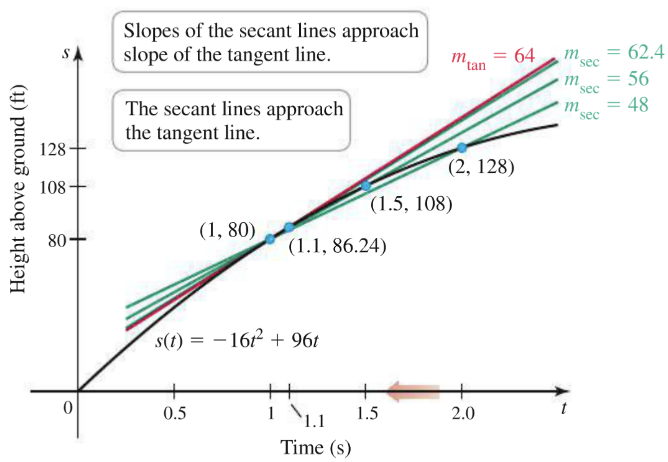
We just saw that the average velocities approach a limit as t approaches a single point a. The same thing happens with the secant lines we observed. The secant lines approach the same limit as t approaches a single point a. As t approaches a, two things happen:

1. The secant lines approach a unique line called the *tangent line*.
2. The slopes of the secant lines, m_{sec}, approach the slope of the tangent line, m_{tan}, at the point $(a, s(a))$. Therefore, we can express the slope of the tangent line as a limit as well:

$$m_{tan} = \lim_{t\to a} m_{sec} = \lim_{t\to a} \frac{s(t)-s(a)}{t-a}$$

Notice that this is a same limit that defines the instantaneous velocity. So, the instantaneous velocity at $t = a$ is the slope of the line tangent to the position function at $t = a$.

From our example before, we can see this:



2.2: Definitions of Limits

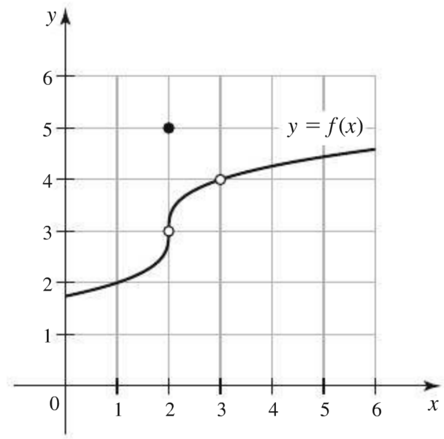
Limit of a Function: Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

Note: Informally, we say that $\lim_{x \to a} f(x) = L$ if $f(x)$ gets closer and closer to L as x gets closer and closer to a. If the limit exists, it depends on the value of f *near* a, *not* the value of $f(a)$.

Example: Use the graph of f to determine the following values, if possible:



a) $f(1)$ and $\lim_{x \to 1} f(x)$

b) $f(2)$ and $\lim_{x \to 2} f(x)$

c) $f(3)$ and $\lim_{x \to 3} f(x)$.

Example: Create a table of values of $f(x) = \frac{\sqrt{x}-1}{x-1}$ corresponding to values of x near 1. Then make a conjecture about the value of $\lim_{x\to 1} f(x)$.

Table 2.2

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x) = \frac{\sqrt{x}-1}{x-1}$	0.5131670	0.5012563	0.5001251	0.5000125	0.4999875	0.4998751	0.4987562	0.4880885

So, we conjecture that $f(x)$ approaches 0.5 as x approaches 1. So, we can say that

One-Sided Limits

The limit we have looked at $\lim_{x\to a} f(x) = L$ is a two-sided limit since we have to look at when x approaches a for values less than a and for values greater than a. We can look at either the left or the right hand side of the limit.

Right-sided limit: Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, then we write

$$\lim_{x\to a^+} f(x) = L$$

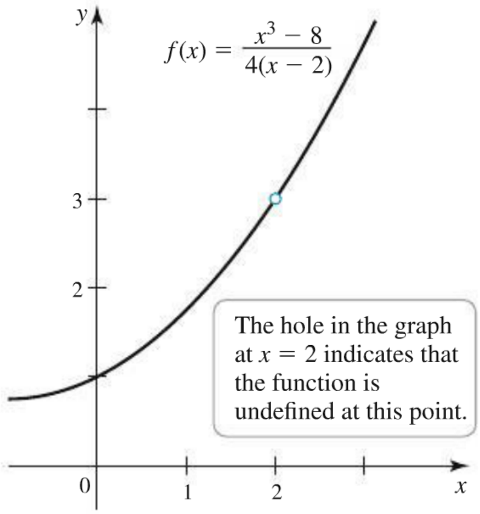
and say the limit of $f(x)$ as x approaches a from the right equals L

Left-sided limit: Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, then we write

$$\lim_{x\to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L

Example: Let $f(x) = \frac{x^3-8}{4(x-2)}$. Use tables and graphs to make a conjecture about the values of $\lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2} f(x)$, if they exist.

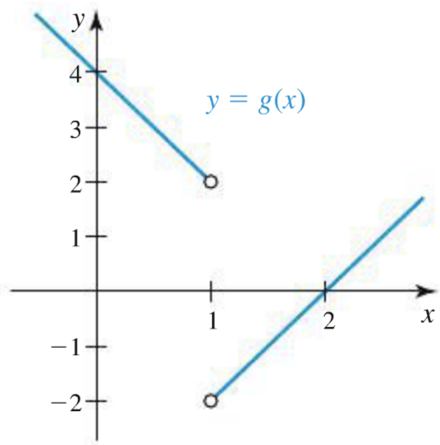


- $\lim\limits_{x\to 2^+} f(x)$
- $\lim\limits_{x\to 2^-} f(x)$
- $\lim\limits_{x\to 2} f(x)$

Theorem 2.1: Relationship Between One-Sided and Two-Sided Limits
Assume f is defined for all x near a except possibly at a. Then $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.

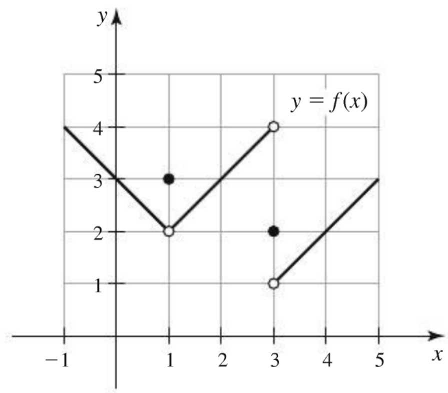
Note: If either $\lim\limits_{x\to a^+} f(x) \neq L$ or $\lim\limits_{x\to a^-} f(x) \neq L$ (or both), then $\lim\limits_{x\to a} f(x) \neq L$.
Also, if either $\lim\limits_{x\to a^+} f(x)$ DNE or $\lim\limits_{x\to a^-} f(x)$ DNE, then $\lim\limits_{x\to a} f(x)$ DNE.

Example: Use the graph of $g(x) = \frac{2x^2-6x+4}{|x-1|}$ to find the values of $\lim_{x\to 1^-} g(x)$, $\lim_{x\to 1^+} g(x)$, and $\lim_{x\to 1} g(x)$.



- $\lim_{x\to 1^-} g(x)$
- $\lim_{x\to 1^+} g(x)$
- $\lim_{x\to 1} g(x)$

Example: Use the graph of $f(x)$ to find the following values or state that they do not exists. If a limit does not exist, explain why.



- $f(1)$
- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$
- $f(3)$
- $\lim_{x\to 3^-} f(x)$
- $\lim_{x\to 3^+} f(x)$
- $\lim_{x\to 3} f(x)$
- $f(2)$
- $\lim_{x\to 2^-} f(x)$
- $\lim_{x\to 2^+} f(x)$
- $\lim_{x\to 2} f(x)$

2.3: Techniques for Computing Limits

Limits of Linear Functions

Linear functions are of the form $f(x) = mx + b$. You can evaluate the limit of a linear function by direct substitution of a into x.

Limits of Linear Functions: Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b$$

Example: Evaluate the following limits:

- $\lim_{x \to 3} \left(\frac{1}{2}x - 7\right)$

- $\lim_{x \to 2} 6$

Limit Laws: Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number and $n > 0$ is an integer.

- **Sum:** $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- **Difference:** $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- **Constant Multiple:** $\lim_{x \to a}(cf(x)) = c \lim_{x \to a} f(x)$
- **Product:** $\lim_{x \to a}(f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
- **Quotient:** $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
- **Power:** $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$
- **Root:** $\lim_{x \to a}(f(x))^{1/n} = \left(\lim_{x \to a} f(x)\right)^{1/n}$, provided $f(x) > 0$, for x near a, if n is even.

Example: Suppose $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$, and $\lim_{x\to 2} h(x) = 8$. Use the limit laws to compute each limit

- $\lim_{x\to 2} \frac{f(x)-g(x)}{h(x)}$

- $\lim_{x\to 2}(6f(x)g(x) + h(x)$

- $\lim_{x\to 2}(g(x))^3$

Limits of Polynomial and Rational Functions

Polynomials and rational functions can also be evaluated by direct substitution, as long as the denominator of the rational function does not evaluate to 0.

Limits of Polynomial and Rational Functions: Assume p and q are polynomials and a is a constant.

- $\lim\limits_{x\to a} p(x) = p(a)$
- $\lim\limits_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$.

Example: Evaluate the limit $\lim\limits_{x\to 1}(5x^2 + 3x + 7)$.

Example: Evaluate the limit $\lim\limits_{x\to 2} \frac{3x^2-4x}{5x^3-36}$.
Notice that the denominator evaluated at 2 is $5(2)^3 - 36 = 4 \neq 0$. So, we can use our theorem.

Example: Evaluate $\lim\limits_{x\to 2} \frac{\sqrt{2x^3+9}+3x-1}{4x+1}$.
Notice that the denominator evaluated at 2 is $4(2) + 1 = 9 \neq 0$.

One Sided Limits

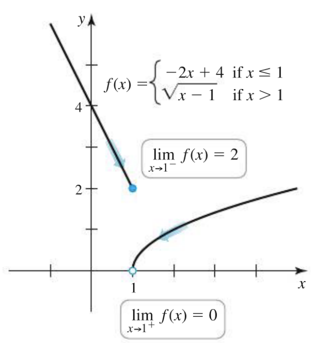
The Limit Laws above also hold for one-sided limits ($\lim_{x\to a^+}$ and $\lim_{x\to a^-}$), except for the last law, the Root Law- we just have to modify it a bit.

Limit Laws for One-Sided Limits: The sum, difference, constant multiple, product, quotient, and power laws for limits hold with $\lim_{x\to a}$ replaced with either $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$. The Root Law is modified as follows: Assume $n > 0$ is an integer:

- $\lim_{x\to a^+} (f(x))^{1/n} = \left(\lim_{x\to a^+} f(x)\right)^{1/n}$, provided $f(x) \geq 0$, for x near a with $x > a$, if n is even.
- $\lim_{x\to a^-} (f(x))^{1/n} = \left(\lim_{x\to a^-} f(x)\right)^{1/n}$, provided $f(x) \geq 0$, for x near a with $x < a$, if n is even.

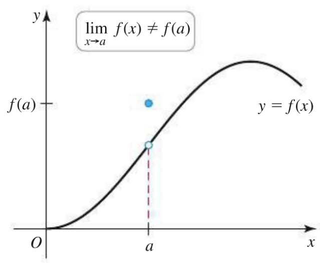
Example: Let $f(x) = \begin{cases} -2x+4 & x \leq 1 \\ \sqrt{x-1} & x > 1 \end{cases}$.

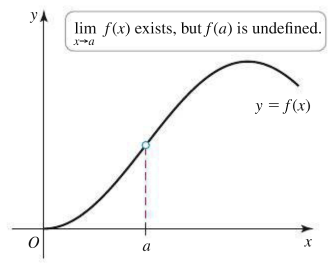
Find the values of $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to} f(x)$, or state that they do not exist.



For $x \leq 1$, we have that $f(x) = -2x + 4$. Therefore, For $x > 1$, note that $x - 1 > 0$. Therefore,

What if direct substitution doesn't work? If we get a 0 in the denominator using direct substitution on a rational function, we have to use another technique to evaluate the limit since we can't divide by 0. This happens when we have a hole in our graph- so when $f(a) \neq \lim_{x\to a} f(x)$ or $f(a)$ is undefined. Such as:





Example: Evaluate $\lim_{x\to 2} \frac{x^2-6x+8}{x^2-4}$.

Notice that if we try to use direct substitution, we get $(2)^2 - 4 = 4 - 4 = 0$ in our denominator. So, to avoid this, we factor the numerator and denominator.

Example: Evaluate $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$.

Notice that if we try to use direct substitution, we get $(1) - 1 = 0$ in our denominator. So, to avoid this, we multiply the numerator and denominator by the conjugate of the numerator in hopes to cancel out some factors.

Remember: If you try to use direct substitution, but get 0 in the denominator of your rational function, you should try to factor or multiply by the conjugate.

An Important Limit

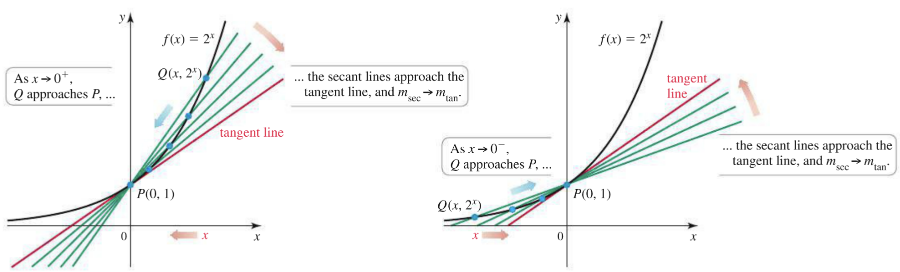
So, now we know how to compute limits using direct substitution, the limit laws, and algebraic manipulation. However, there are important limits for which these techniques won't work. An example of this is when we trying to find the slope of a line tangent to the graph of an *exponential function*.

Example: Estimate the slope of the line tangent to the graph of $f(x) = 2^x$ at the point $P(0, 1)$.

To do this, we will use what we learned in Section 2.1- that the limit of the slopes of the secant lines is the slope of the tangent line. Let's first pick another point on the function to create a secant line. We choose $Q(x, 2^x)$. So, the slope of the secant line joining points P and Q is

$$m_{sec} = \frac{2^x - 1}{x - 0} = \frac{2^x - 1}{x}$$

If we want to find the tangent line at the point $P(0, 1)$ (note that $x = 0$), we want to find the limit of m_{sec} as $x \to 0$.



We know that $\lim_{x\to 0} \frac{2^x-1}{x}$ exists only if it has the same value as $\lim_{x\to 0^+} \frac{2^x-1}{x}$ and $\lim_{x\to 0^-} \frac{2^x-1}{x}$. This isn't an easy limit to evaluate, so we use a similar method as we did in Section 2.2. The following is for positive values of x near 0, so $x \to 0^+$.

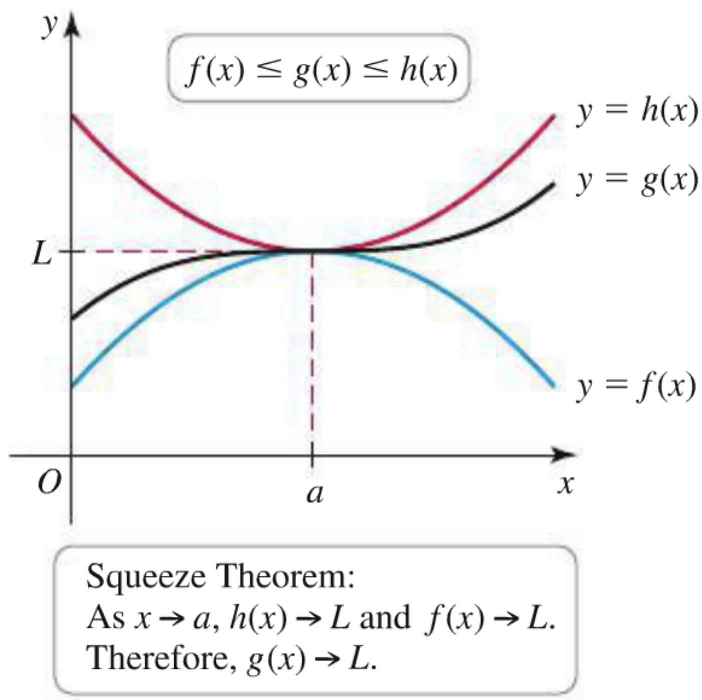
x	1.0	0.1	0.01	0.001	0.0001	0.00001
$m_{sec} = \frac{2^x - 1}{x}$	1.000000	0.7177	0.6956	0.6934	0.6932	0.6931

A similar computation show that the negative values of x near 0, $x \to 0^-$ shows that m_{sec} also approaches 0.693. Since both sides are the same, we can saw that the slope of the tangent line to $f(x) = 2^x$ at $x = 0$ is approximately 0.693.

$$m_{tan} = \lim_{x\to 0} m_{sec} = \lim_{x\to 0} \frac{2^x-1}{x} = \lim_{x\to 0^+} \frac{2^x-1}{x} = \lim_{x\to 0^-} \frac{2^x-1}{x} \approx 0.693$$

The Squeeze Theorem

This is another way to calculate limits and comes in useful when trying to take the limit of trig functions multiplied by a variable. Suppose the function f and h have the same limit L at a and assume the function g is trapped between f and h. The Squeeze theorem says that g must also have the limit L at a.



The Squeeze Theorem: Assume the functions f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

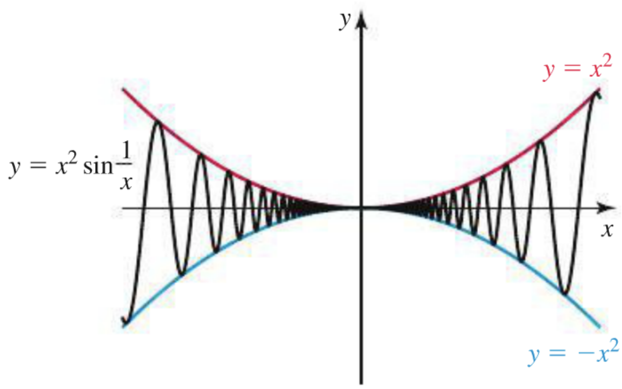
Remember: For the squeeze theorem, it is important to note the following for any real number θ:

- $-1 \leq \sin(\theta) \leq 1$
- $-1 \leq \cos(\theta) \leq 1$

Example: Use the Squeeze Theorem to verify that $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

We know that $-1 \le \sin\theta \le 1$. Let $\theta = 1/x$ for $x \ne 0$, if follows that:

We know that $\lim_{x\to 0}(-x^2) = -(0)^2 = 0$ and $\lim_{x\to 0} x^2 = (0)^2 = 0$. Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2$, we know that $\lim_{x\to 0} x^2 \sin(1/x) = 0$.



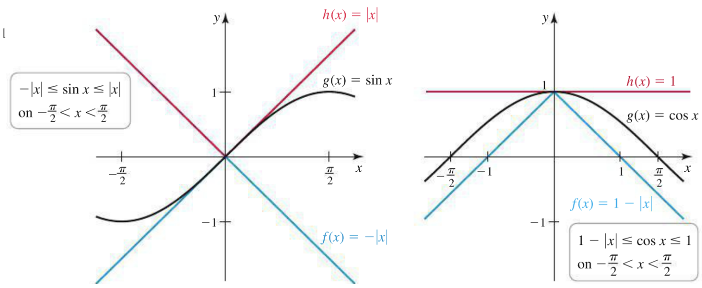
Trigonometric Limits

The Squeeze Theorem is used to evaluate two important limits that play a crucial role in establishing fundamental properties of the trignometric functions. These limits are:

- $\lim_{x \to 0} \sin(x) = 0$
- $\lim_{x \to 0} \cos(x) = 1$

We can verify these limits using the Squeeze Theorem and the fact that they are bounded by

- $-|x| \le \sin x \le |x|$
- $1 - |x| \le \cos x \le 1$



Example: Evaluate $\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}$. Hint: Use $\sin^2 x + \cos^2 x = 1$.

Example: Evaluate $\lim_{x\to 0} \frac{1-\cos(2x)}{\sin x}$. Hint: Use $\cos(2x) = \cos^2 x - \sin^2 x$.

2.4: Infinite Limits

Infinite Limit: An infinite limit occurs when function values increase or decrease without bound near a point.

$$\lim_{x \to a} f(x) = \infty \qquad \text{OR} \qquad \lim_{x \to a} f(x) = -\infty$$

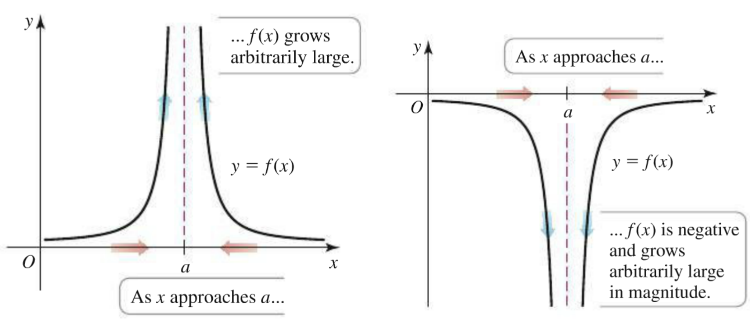
More mathematically, we have:
Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficient close (but not equal to) a, we write

$$\lim_{x \to a} f(x) = \infty$$

and say the limit of $f(x)$ as x approaches a is infinity.
If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal to a), we write

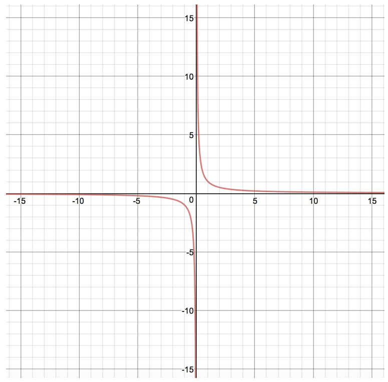
$$\lim_{x \to a} f(x) = -\infty$$

and say the limit of $f(x)$ as x approaches a is negative infinity.



Note: Technically, infinite limits do not exist. However, with infinite limits, they are special in that we like to indicate *how* the limit does not exist.

Example: Let $y = \frac{1}{x}$. What happens as $x \to 0^+$?



x	$1/x$
$1/2$	
$1/4$	
$1/10$	
$1/100$	
$1/1000$	

So, the values of $1/x$ increase without bound as x approaches 0 from the right.

So, we have that

- $\lim\limits_{x \to 0^-} \frac{1}{x} =$
- $\lim\limits_{x \to 0^+} \frac{1}{x} =$
- $\lim\limits_{x \to 0} \frac{1}{x}$

Finding Infinite Limits

Note the difference between:

- *indeterminate form:* 0/0
- *undefined:* nonzero number/0

So, when you calculate the limit of a quotient and there is a zero in the denominator when you first try direct substitution, you should find out either:

- *Is there a zero in the numerator?*
  This would give the indeterminate form, 0/0. So, we can do a number of things to evaluate the limit that we learned in Section 2.3:
  - simplify the quotient
  - factor the numerator and denominator and cancel out any commong factors
  - multiply by the conjugate
  - use the Squeeze Theorem
  - more advanced methods that we will learn later!
- *Is the numerator a nonzero number?*
  Technically, this limit DNE, but we have to check if it is an infinite limit. So, we have to determine if the limit is ∞ or $-\infty$ or different infinities coming from both sides (like in example above).

Example: Find the limits:

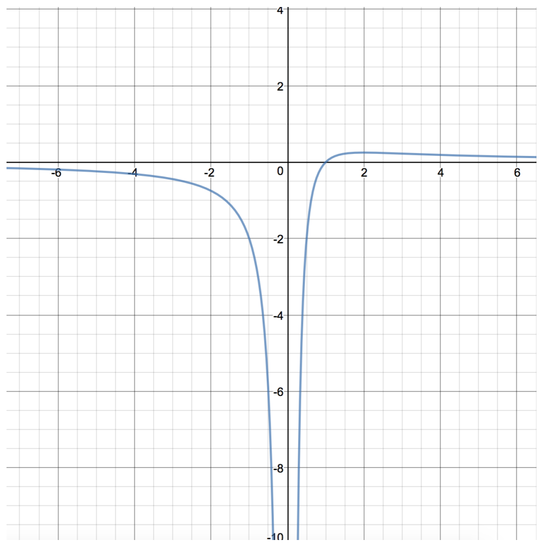
- $\lim\limits_{x \to 0^-} \frac{x^2-3x+2}{x^3-2x^2}$

  We first try direct substitution:
  $$\lim_{x \to 0^-} \frac{x^2-3x+2}{x^3-2x^2} = \frac{0^2-0+2}{0^3-0^2} = \frac{2}{0}$$

  So, we have a zero in our denominator and a nonzero number in our numerator. So, we know that we are going to have a limit that DNE. However, let's check if we have a limit tending to ∞ or $-\infty$. We can try this by plugging in a *really* small number for x that is slightly less than 0 since we are coming from the left.

- $\lim\limits_{x \to 0^+} \frac{x^2-3x+2}{x^3-2x^2}$.
  Similarly, we will now plug in a *really* small number for x that is slightly greater than 0 since we are coming from the right.

- $\lim_{x\to 0} \frac{x^2-3x+2}{x^3-2x^2} = -\infty$
  since $\lim_{x\to 0^-} \frac{x^2-3x+2}{x^3-2x^2} = -\infty$ and $\lim_{x\to 0^+} \frac{x^2-3x+2}{x^3-2x^2} = -\infty$



Vertical Asymptotes

Vertical Asymptote: If the limit of a function $f(x)$ as x approaches a from the left, right, or both is an infinite limit, then the line $x = a$ is a vertical asymptote of the curve $y = f(x)$.
More mathematically, if $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, or $\lim_{x\to a^-} f(x) = \pm\infty$, then the line $x = a$ is a vertical asymptote of f.

Example: Our first example of the day was $f(x) = \frac{1}{x}$. $x = 0$ is a vertical asymptote of f because $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

Note: So, we only have to show an infinite limit from one side or the other to say a function has a vertical asymptote at $x = a$.

Finding Vertical Asymptotes:

1. Find the values where the denominator$= 0$ but the numerator$\neq 0$.
2. Prove that you have a vertical asymptote using limits. Take the limit of the function as x approaches each value from the left and right. At least one limit should be infinite.

Example: Find the vertical asymptotes of $f(x) = \frac{x+1}{2x^2+x-3}$.

$$f(x) = \frac{x+1}{2x^2+x-3} = \frac{x+1}{(x-1)(2x+3)}$$

Example: Evaluate the limits

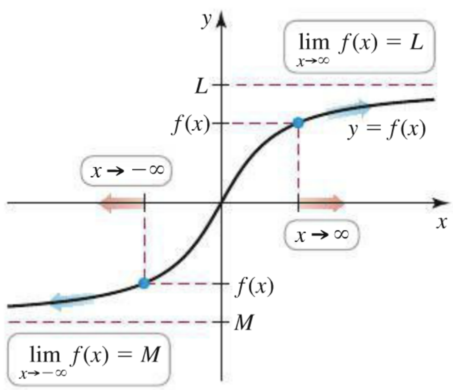
- $\lim\limits_{\theta \to \frac{\pi}{2}^+} \tan\theta$

- $\lim\limits_{\theta \to \frac{\pi}{2}^-} \tan\theta$

2.5: Limits at Infinity

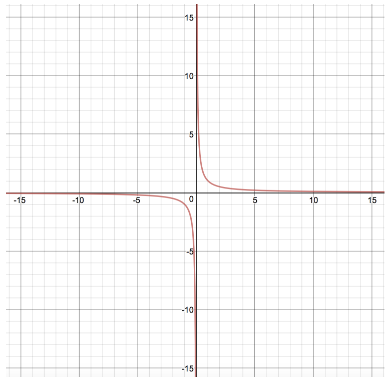
Limits at Infinity: occur when the independent variable becomes large in magnitude. These determine the end behavior of a function. Mathematically, if $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write $\lim_{x\to\infty} f(x) = L$. We say the limit of $f(x)$ as x approaches infinity is L.

Horizontal Asymptote: When $\lim_{x\to\infty} f(x) = L$, the line $y = L$ is a horizontal asymptote of f. The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$ is similar. So, the line $y = M$ is also a horizontal asymptote.



Note: When evaluating limits, we may see an indeterminate form. If you see one of these indeterminate forms, you must apply algebraic techniques to express the limit without the indeterminate form in order to evaluate the limit.
Indeterminate Forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

Example: Let $f(x) = \frac{1}{x}$.



What happens as $x \to \infty$?
As $x \to \infty$, $\frac{1}{x} \to 0$. So, we have that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.
What happens as $x \to -\infty$?
As $x \to -\infty$, $\frac{1}{x} \to 0$. So, we have that $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$.
So, $y = 0$ is a horizontal asymptote.

Note: Whenever you take a limit at infinity and get a number L, you have a horizontal asymptote of $y = L$.

Vertical vs Horizontal Asymptote:

- The line $x = a$ is a *vertical asymptote* of the curve $y = f(x)$ if either
$$\lim_{x\to a^-} f(x) = \pm\infty \qquad \text{OR} \qquad \lim_{x\to a^+} f(x) = \pm\infty$$
- The line $y = b$ is a *horizontal asymptote* of the curve $y = f(x)$ if either
$$\lim_{x\to\infty} f(x) = b \qquad \text{OR} \qquad \lim_{x\to-\infty} f(x) = b$$

End Behavior

Limits at Infinity of Powers and Polynomials:

Let n be a positive integer and let p be a polynomial.

- $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = \infty$ when n is *even*.
- $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = -\infty$ when n is *odd*.
- $\lim\limits_{x\to\pm\infty} \frac{1}{x^n} = \lim\limits_{x\to\pm\infty} x^{-n} = 0$
- $\lim\limits_{x\to\pm\infty} p(x) = \infty$ or $-\infty$ depending on the degree of the polynomial p and the sign of the leading coefficient.

Example: Determine the limits as $x \to \pm\infty$ of the following functions.

- $p(x) = 3x^4 - 6x^2 + x - 10$

- $q(x) = -2x^3 + 3x^2 - 12$

Technique for Rational Functions:

1. Choose the highest power of x in the denominator.
2. Divide every term in the numerator and denominator by the highest power of x in the denominator.
3. Take the limit of each term. Recall that the limit as $x \to \pm\infty x^{-n} = 0$

Example: Evaluate the limit.
$\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7}$.

Example: Evaluate the limit.
$\lim_{x\to-\infty} \frac{2x^3+7}{x^3-x^2+x+7}$.

Example: Evaluate the limits.
$\lim_{x\to\infty} \frac{10x^3-3x^2+8}{\sqrt{25x^6+x^4+2}}$.

Notes: We can also have infinite limits at infinity.

infinite limit at infinity: If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty$$

The limits $\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$, and $\lim\limits_{x\to-\infty} f(x) = -\infty$ are defined similarly.

Example: Evaluate the limit.
$\lim\limits_{x\to-\infty} (3x^{-7} + x^3)$

Example: Evaluate the limit.
$\lim\limits_{x\to\infty} (3x^{-7} + x^3)$

Slant Asymptotes

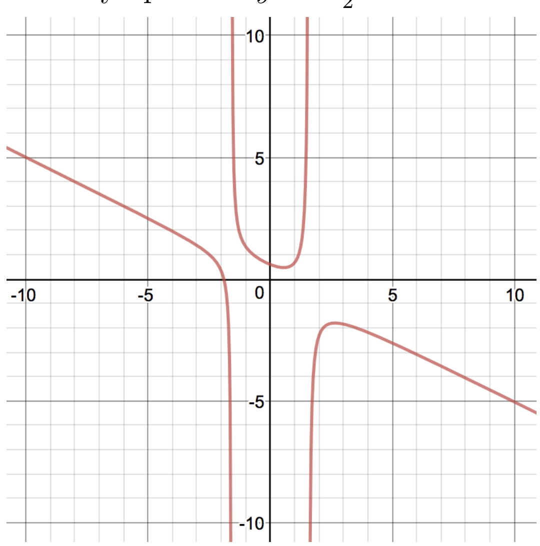
Slant Asymptote: If the graph of a function f approaches a line (with finite and nonzero slope) as $x \to \pm\infty$, then that line is a slant asymptote.

How to find a Slant Asymptote:

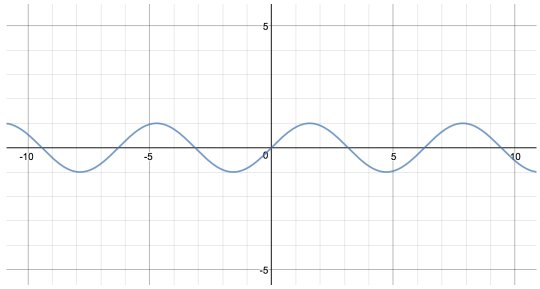
1. For rational functions, check if the degree of the numerator is more than the degree of the denominator.
2. Use long division to divide the numerator by the denominator.
3. The equation of the line that is the slant asymptote is the quotient from your long division.

Example: Find the slant asymptote of $\lim\limits_{x\to\infty} \frac{x^3-2x+3}{5-2x^2}$.

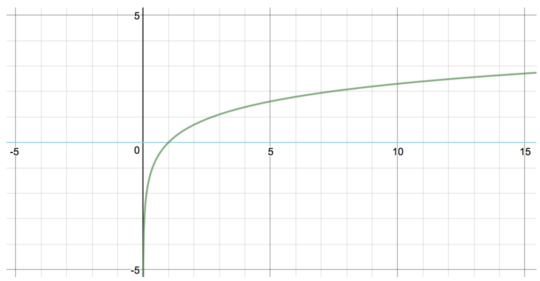
Therefore, the slant asymptote is $y = -\frac{1}{2}x$



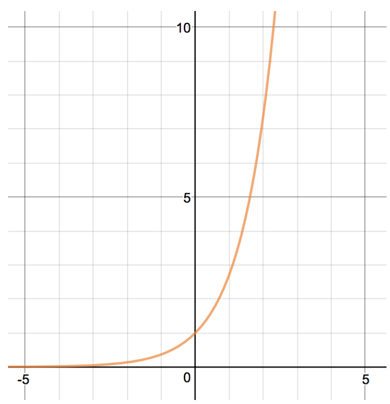
End Behavior of Other Functions



$\lim_{x\to\infty}$ DNE because $\sin(x)$ oscillates between -1 and 1. The same is true for $\cos(x)$.



$$\lim_{x\to 0^+} \ln(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \ln(x) = \infty$$



$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0$$

End Behavior and Asymptotes of Rational Functions:

Suppose $f(x) = \frac{p(x)}{q(z)}$ is a rational function where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2 x^2 + a x_1 + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2 x^2 + b x_1 + b_0$$

with $a_m \neq 0$ and $b_n \neq 0$.

- **Degree of numerator less than degree of denominator.**
  If $m < n$, then $\lim\limits_{x\to\pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of f.
- **Degree of numerator equals degree of denominator.**
  If $m = n$, then $\lim\limits_{x\to\pm\infty} f(x) = \frac{a_m}{b_n}$, and $y = \frac{a_m}{b_n}$ is a horizontal asymptote of f.
- **Degree of numerator greater than degree of denominator.**
  If $m > n$, then $\lim\limits_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.
- **Slant asymptote.**
  If $m = n + 1$, then $\lim\limits_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptotes, but f has a slant asymptote.
- **Vertical asymptote.**
  Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

2.6: Continuity

Continuity at a Point

continuous at a point: A function f is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$.

point of discontinuity: If f is not continuous at a, then a is a point of discontinuity.

Conditions for Continuity of f at a:

- $f(a)$ is defined. (a is in the domain of f)
- $\lim_{x \to a} f(x)$ exits
- $\lim_{x \to a} f(x) = f(a)$

Note: If any item in the list fails to hold, the function fails to be continuous at a.

Example: Determine whether the functions are continuous at the given value a.

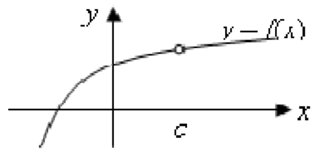
- $f(x) = \frac{x^2+x+1}{x^2+2x}$; $a = 2$.

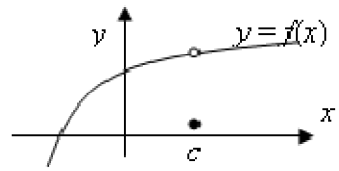
- $f(x) = \frac{x^2+x+1}{x^2+2x}$; $a = -2$.

Types of Discontinuity

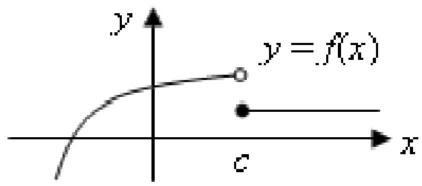
If a function f is not continuous at a number c, we say that f has a discontinuity at c.

- **Removable:** The function has a hole at c.

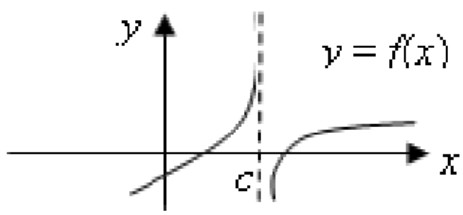




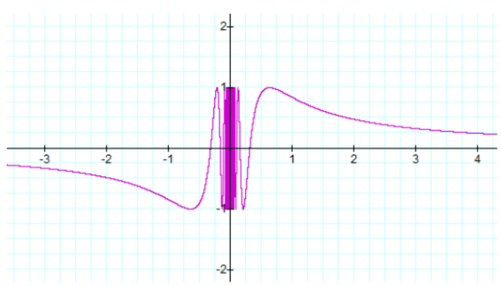
- **Jump:** The function jumps from one value to another at c.



- **Infinite:** There is a vertical asymptote at c.



- **Oscillating:** The function oscillates too much to have a limit at c.



Continuity on an Interval

Intuitively, a function is continuous if it is connected; that is, it has no "hole" or break at the point in which we are interested. (You can draw it without lifting your pencil).

continuous from the right at a number a: A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

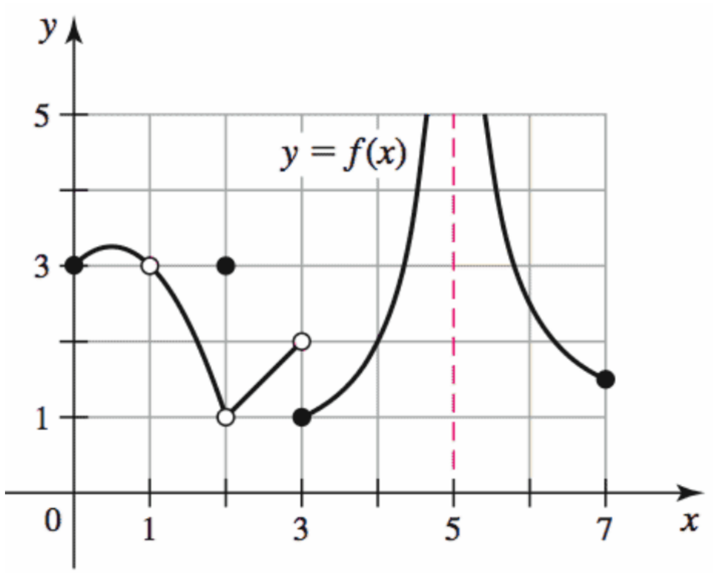
continuous from the left at a number b: A function f is continuous from the left at a number b if

$$\lim_{x \to b^-} f(x) = f(b)$$

continuous on an interval: A function is continuous on an interval if it is continuous at every number in the interval.

Note: Continuous at an endpoint of an interval is understood to mean continuous from the left or from the right.

Example: Identify values of x on the interval $(0, 7)$ at which f is discontinuous and state the type of discontinuity at each. Then, determine the intervals of continuity for f.



Continuous Functions

continuous function: A continuous function is continuous at every point of its domain.

Continuity Rules: If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

- $f + g$
- $f - g$
- cf
- fg
- f/g provided $g(a) \neq 0$
- $(f(x))^n$

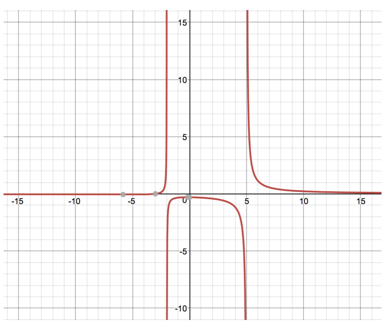
Continuity Theorems:

- The inverse of a continuous function is continuous.
- The composition of continuous functions is continuous.

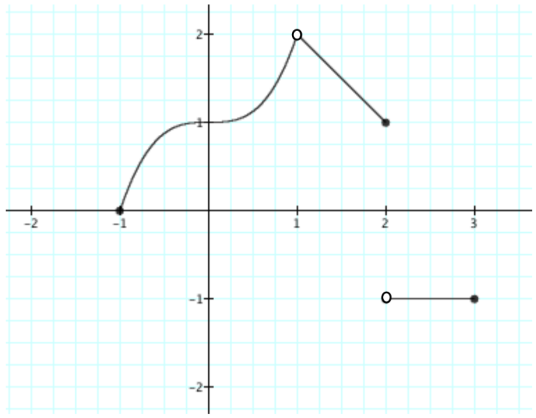
Functions that are Continuous of their Domains

- polynomials
- rational functions
- root functions
- trig functions
- inverse trig functions
- exponential functions

Example: Determine the intervals on which the function g is continuous.
$g(x) = \frac{x+3}{x^2-3x-10}$



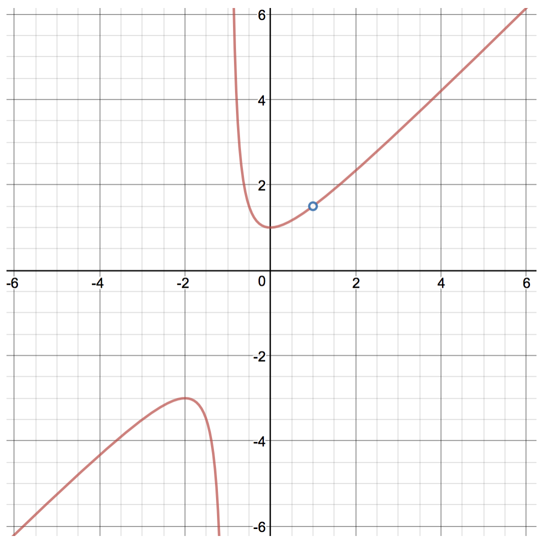
Example: Conider the graph $y = f(x)$ on its domain $[-1, 3]$.



- Is $f(-1)$ defined?
- Does $\lim_{x \to -1^+} f(x)$ exist?
- Does $\lim_{x \to -1^+} f(x) = f(-1)$?
- Is $f(x)$ continuous at $x = -1$?
- What value should be assigned to $f(1)$ to make $f(x)$ continuous at $x = 1$?

Example: Show that the function f has a removable discontinuity at $s = 1$. Determine the value of the constant a for which the function g is continuous at $s = 1$.

$f(s) = \frac{s^3-1}{s^2-1}$ $\qquad$ $g(s) = \begin{cases} \frac{s^3-1}{s^2-1} & s \neq 1 \\ a & s = 1 \end{cases}$



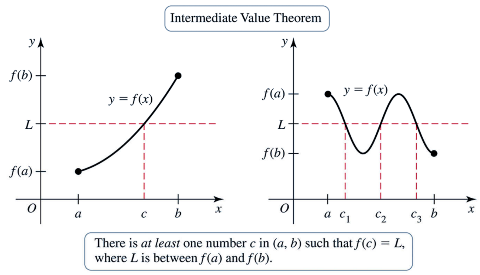
Limits of Composite Functions:

- If g is continuous at a and f is continuous at $g(a)$, then
$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$
- If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then
$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Example: Evaluate the limit.

$\lim_{x\to 4} \sqrt{\frac{x^3-2x^2-8x}{x-4}}$

The Intermediate Value Theorem (IVT)
Suppose a function f is continuous on the closed interval $[a, b]$ and let L be any number strictly between $f(a)$ and $f(b)$. Then, there exists at least one number c in (a, b) such that $f(c) = L$.



Example: Show that the equation $f(x) = x^4 + x - 3 = 0$ has a solution on the interval $[1, 2]$.

1. Let's see what the function values are at the endpoints of the interval.

2. Verify the condition of the IVT is satisfied.

3. Use the IVT to produce the result.

2.7: The Precise Definition of a Limit

The $\epsilon - \delta$ Definition of a Limit

limit: Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly a itself. Then, we say that the limit of $f(x)$ as x approaches a is L, denoted

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Finding δ graphically given $f, L, a,$ and $\epsilon > 0$

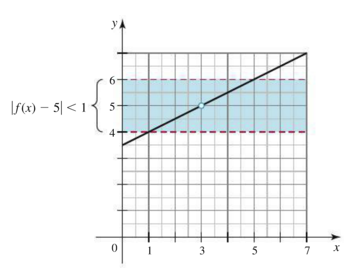
When given a graph, use the $x - values$ on either side of a for the open interval about a.
Assume that your δ should be rounded to the same number of decimal places. When using a graph instead of algebraic methods, even if the true value of δ is irrational, we will state the answer as "$\delta =$".

Example: We have a linear function $f(x)$ with $\lim_{x \to 3} f(x) = 5$. For each value of $\epsilon > 0$, determine a value of $\delta > 0$ satisfying the statement

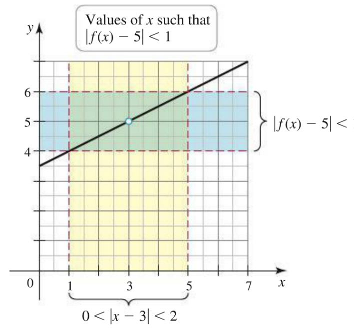
$$|f(x) - 5| < \epsilon \text{ whenever } 0 < |x - 3| < \delta$$

- $\epsilon = 1$
  First, let's state everything we know:
  - $L =$
  - $a =$

  So, graphically, since $\epsilon = 1$, we want $f(x)$ to be less than 1 unit from 5. So, $f(x)$ should be between 4 and 6.

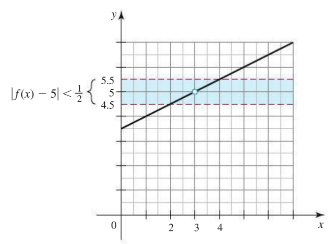


To determine our corresponding value of δ, we draw the vertical lines passing through the points where the horizontal lines and the graph of f intersect.

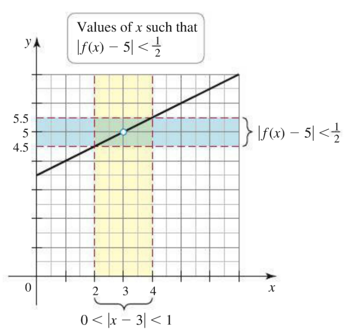


Therefore, we can see that the vertical lines intersect the f function at $x = 1$ and $x = 5$. So, for $\epsilon = 1$, we let $\delta = 2$ or any smaller positive value.

- $\epsilon = \frac{1}{2}$
  Now, $f(x)$ must be between 4.5 and 5.5 .



We again draw vertical lines where these horizontal lines and the function intersect and get that $\delta = 1$ because x is less than 1 unit away from 3 on the x-axis.



Note: If we keep making $\epsilon > 0$ smaller and smaller, our δ window will also become smaller and we get closer to the value of the limit.

Proving the Limit

We're big kids, so we can find δ without a graph and then use that δ to show that the limit does in fact equal L.

Steps for Proving $\lim_{x \to a} f(x) = L$

1. Write down what $f(x), L$, and a are.
2. Find δ in your scratch work using $|f(x) - L| < \epsilon$. This is not part of your proof and δ should be in terms of ϵ. We try to algebraically get $|f(x) - L|$ to look like a multiple of $|x - a|$.
3. Write your proof using the following sentences with your values for δ, a, and L plugged in:
   - Given $\epsilon > 0$, let $\delta =$
   - If $0 < |x - a| < \delta =$, then *SCRATCH WORK* to show $|f(x) = L| < \epsilon$.
   - By the definition of a limit, $\lim_{x \to a} f(x) = L$.

Example: Let $f(x) = 5x + 1$. Use the $\delta - \epsilon$ definition of a limit to prove $\lim_{x \to 2} f(x) = 11$.

- $f(x) =$
- $L =$
- $a =$

Scratch Work:

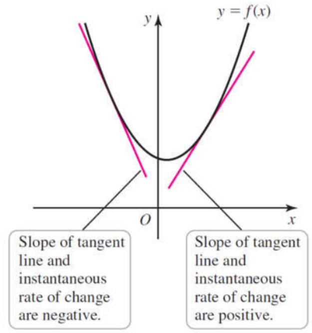
Example: Prove $\lim_{x \to -5} \left(4 - \frac{3x}{5}\right) = 7$.

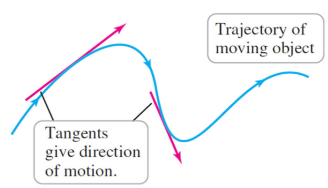
- $f(x) =$
- $L =$
- $a =$

3.1: Introducing the Derivative

The Slope of a Tangent Line to a Curve

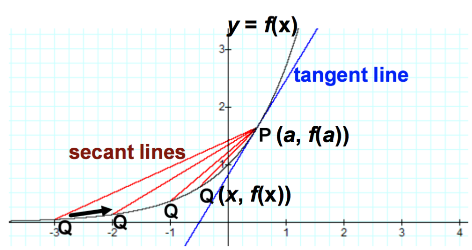
As we have seen before, we know that the slope of the tangent line is the instantaneous rate of change at that point. Now, we are going to think of the slopes of the tangent lines as they change along a curve as the values of a new function that we call the **derivative**. Physically, if you think of a curve as the trajectory of a moving object, then the tangent line at a point indicates the direction of motion at that point.





Recall: What is the relationship between tangent lines and secant lines?

The tangent line is the limit of the secant lines.



Recall that the equation for the slope of the secant line between points P and Q is

$$m_{sec} = \frac{f(x)-f(a)}{x-a}$$

So, the **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope

$$m_{tan} = \lim_{x \to a} m_{sec} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists. This slope is also called the **derivative** of f at a, denoted $f'(a)$.

Notes:

- **derivative = slope of the tangent line**
- "slope of the curve" means "slope of the tangent line"
- Since the derivative is the slope of the tangent line, it is interpreted as the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.
- Useful formula: $y - y_1 = m(x - x_1)$ is the point-slope formula. We will use this to find the equation of a line where we know the slope (m) and a point $((x_1, y_1))$.

Example: Find the equation of the tangent line to $f(x) = x^3 - x$ at the point $(2, 6)$.

- Find the slope of the tangent line.

- Find the equation of the tangent line using point-slope formula.

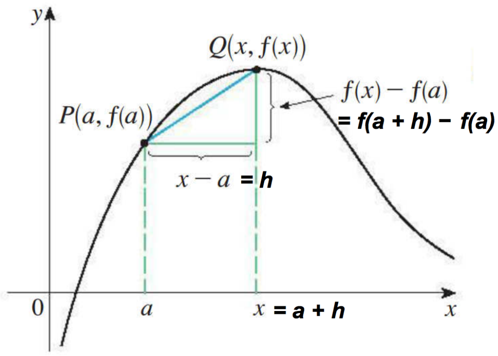
Example: Find the equation of the tangent line to $f(x) = -2x^2 + 8$ at the point $(2, 8)$.

- Find the slope of the tangent line

- Find the equation of the tangent line using the point-slope formula.

The Limit Definition of a Derivative

If we let h be the distance from x to a, we get another form of the limit definition of a derivative (aka: another formula for the slope of the tangent line)



The **derivative:** of a function f at a number a, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Note: Sometimes it's easier to use one form of the definition than the other.

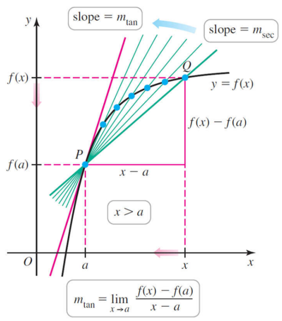
Example: Find the slope of the tangent line to $f(x) = x^3 - x$ at the point $(2, 6)$.

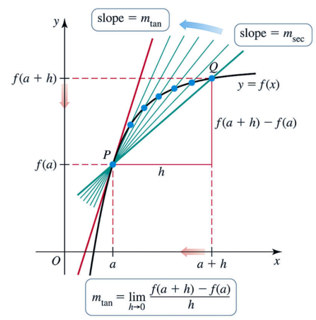
Example: Find the slope of the tangent line to $f(x) = -2x^2 + 8$ at the point $(2, 8)$.

To recap, the **derivative of f at a**, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \qquad \text{or} \qquad f'(a) = \lim_{h\to a} \frac{f(a+h)-f(a)}{h}$$

If $f'(a)$ exists, we say that f is **differentiable** at a.





Example: Find the slope of the tangent line to $f(t) = \frac{2t+1}{t+3}$ at the point $(2, 1)$.

Example: Using the slope of the tangent line you just found in the last example, find the equation of the tangent line at the point $(2, 1)$.

If we were asked to find the slope of the line **normal** to the curve at point a, we would find the slope of the tangent line as normal, then use the following formula to find the slope of the line normal to the point a,

$$m_{norm} = -\frac{1}{m_{tan}}$$

So, we just take the negative reciprocal of the slope of the tangent line.

Example: Find the equation of the normal line of $f(t) = \frac{2t+1}{t+3}$ at the point $(2, 1)$.

Note: You can't just replace the normal slope into the equation of the tangent line! You **HAVE** to go through point-slope formula again.

Velocity

Suppose an object moves according to an equation of motion, which we will call our position function, $s = f(t)$, where s is the position of the object at time t.
On the interval $t = a$ to $t = a + h$, the change in position, or **displacement**, is

$$\text{displacement} = f(a + h) - f(a)$$

We know from physics that average velocity is displacement over change in time, so we have

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

So, we can define **velocity** (instantaneous velocity), $v(a)$, at time $t = a$, to be the limit of these average velocities as the elapsed time, h, approaches 0.

$$\text{velocity} = v(a) = s'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Note: So, the velocity the derivative of position.

Example: If a ball is thrown into the air with a velocity of 40 ft/s from an initial height of 0 feet, its height (in feet) after t seconds is given by $s(t) = 40t - 16t^2$.

- Find the velocity when $t = 2$.

- When will the ball hit the ground?
  The ball will hit the ground when the position function equals 0. So, we need to solve $s(t) = 0$

3.2: The Derivative as a Function

In the last section, we learned how to calculate the slope of the tangent line at a fixed input of a function, so the derivative at a certain point $(a, f(a))$. We will now consider a more general point of view in which we replace the fixed input with a variable input.

Given any number x for which the slope of the tangent line exists, we can assign the value of the derivative $f'(x)$. So, the derivative $f'(x)$ can now be regarded as a new function. This way, we can get our derivative for our entire curve. If we need to find the derivative (slope of the tangent line) at a certain x-value, we can just plug that x-value into $f'(x)$.

The **derivative** of f is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

Derivative Notation

The derivative of the function $y = f(x)$ may be denoted by:

$$f'(x) = y' = y'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_2 f(x)$$

- Differentiation operators: $\frac{d}{dx}$, D, and D_x
- Leibniz Notation: $\frac{dy}{dx}$ and $\frac{df}{dx}$

Notation for evaluating the derivative at a number a:

$$f'(a) = y'(a) = \frac{dy}{dx}|_{x=a} = \frac{df}{dx}|_{x=a}$$

Example: Consider the function $f(x) = \frac{1}{2x+1}$

- Find the derivative of the function using the definition of the derivative.

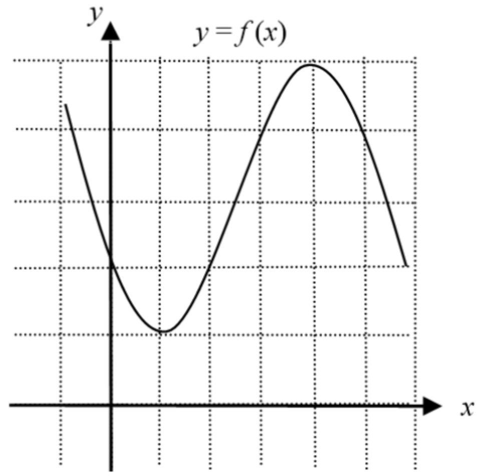
- State the domain of the function and the domain of its derivative

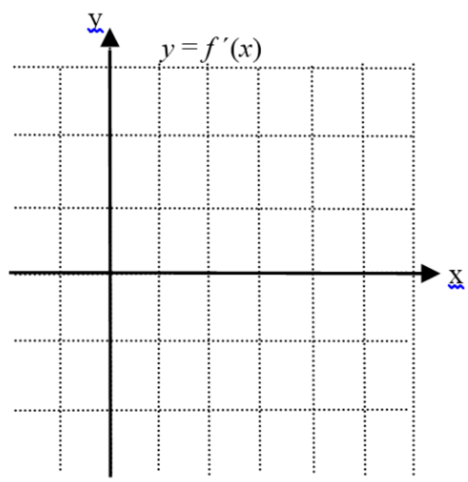
Example: Consider the function $f(x) = \sqrt{3x+1}$.

- Find f' using the definition of a derivative.

- Find the equation of the tangent line to f at $x = 1$.
  To do this, we already have the equation of the derivative, $f'(x)$, over our curve f. So, to find the slope of the tangent line at $x = 1$, we just plug $x = 1$ into $f'(x)$.

Example: Given the graph of f below, sketch the graph of f'.





Differentiability

If $f'(a)$ exists, then we say f is **differentiable** at a.

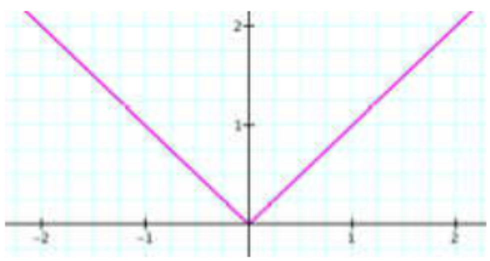
A function f is **differentiable on an open interval** (a, b) (or (a, ∞) or $(-\infty, a) or (-\infty, \infty)$) if f is differentiable at every number in the interval.

Theorem:

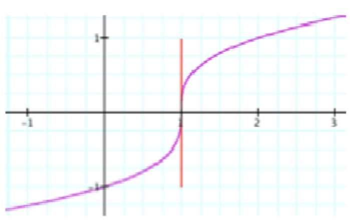
- Differentiability $\implies$ Continuity
- Continuity $\not\Longrightarrow$ Differentiability

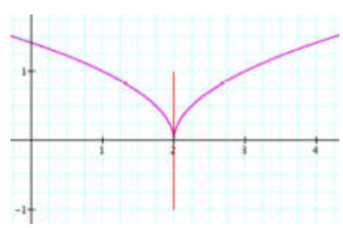
How can a function fail to be differentiable?
In general, if the graph of a function has a *corner* or *kink* in it, then the graph of the function has no tangent at this point and the function is not differentiable here.

- **discontinuity:** any discontinuity causes the derivative to not exist
- **corner:** tn trying to compute $f'(a)$, we find that the left and right limits are different

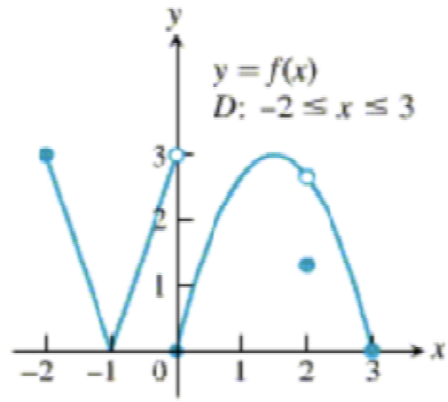


- **vertical tangent:** the slope of the tangent approaches infinity (or negative infinity) from both sides, including cusps





Example: Consider the graph of f below:



- Find the values of x in $(-2, 3)$ at which f is not continuous.

- Find the values of x in $(-2, 3)$ at which f is not differentiable.

Derive the Graph of f' from the Graph of f

Characteristic of $f(x)$	**Characteristic of** $f'(x)$
increasing	positive
decreasing	negative
horizontal tangent line	zero (root)
not differentiable at a	$f'(a)$ is undefined (break in graph)

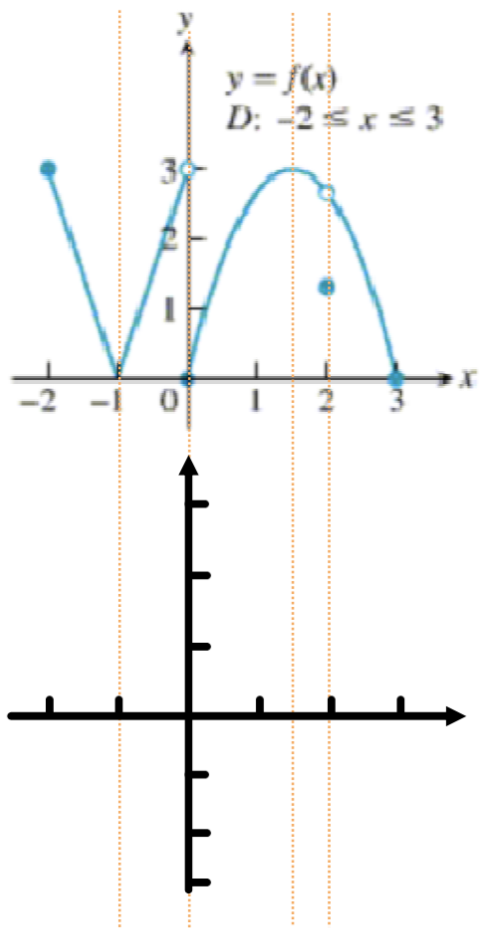
Note:

- The derivative of a linear graph is a constant graph (a horizontal line).
- The derivative of a quadratic graph is a linear graph.
- The derivative of a cubic graph is a quadratic graph.
- The derivative of a graph of degree n is a graph of degree $n-1$. So, the derivative is one degree lower than the original function.

Functions and their Derivative Graphs

- Look for *horizontal tangent lines* first and match these x-coordinates to *zeros* on the derivative graph.
- Look for *points of discontinuity* and match these to *holes or gaps* in the derivative graph.
- Look for other values of x where the function is *not differentiable*. The derivative graph will *not be defined* there.
- Look for the intervals of *increase and decrease* on the original graph. This tells you when the derivative graph is *above or below the x-axis*, respectively.

Example: Given the graph of f, sketch the derivative function f'.



3.3: Rules of Differentiation

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Proof. Let $f(x) = c$. Then, by the limit definition of a derivative, we have that

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 = 0 \end{aligned}$$

Therefore, the derivative of a constant function is 0. $\square$

Note: This makes sense because we know the derivative is the slope of the tangent line of a function. A constant function is a horizontal line. The slope of a horizontal line at any point is 0.

Example: Find the derivative of $f(x) = 2$.

Power Functions

$$\frac{d}{dx}(x) = 1$$

Proof. Let $f(x) = x$. Then, by the limit definition of a derivative, we have that

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h) - (x)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 = 1\end{aligned}$$

Therefore, the derivative of a constant function is 1. $\square$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example: Find the derivative of each function using proper notation.

- $f(x) = x^{11}$
- $y = \frac{1}{x}$
- $f(x) = \sqrt{x}$

Constant Multiple Rule

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) = cf'(x)$$

Example: Differentiate $f(x) = 5x^9$.

Sum and Difference Rules

If f and g are both differentiable, then

- $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$
- $\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$

Example: Differentiate the functions.

- $p(x) = 3.22x^3 - 3x^2 + 9.98x - 30$

- $B(y) = cy^{-6} + \pi$ where c is a real number.

Example: Suppose the line tangent to the graph of f at $x = 2$ is $y = 4x + 1$ and suppose the line tangent to the graph of g at $x = 2$ has slope 3 and passes trhough $(0, -2)$. Find an equation of the line tangent to the following curves at $x = 2$.

Before we begin, let's state what we know

- Tangent line to f at $x = 2$: $y = 4x + 1$
- Tangent line to g at $x = 2$: $y - (-2) = 3(x - 0) \implies y = 3x - 2$
- Derivative is the slope of the tangent line.

1. $h = f(x) + g(x)$

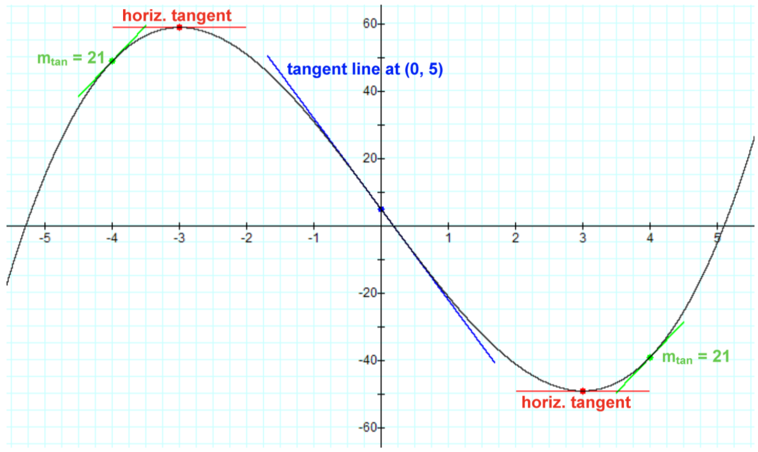
2. $h = f(x) - 2g(x)$

Example: Consider the function $f(x) = x^3 - 27x + 5$.

1. Find the equation of the tangent line to f at $x = 0$.

2. Find the points on the curve where the tangent is horizontal.

3. Find the values of x for which the tangent lines to f are parallel to the lines $21x - y = 2$.


horiz. tangent
m_{tan} = 21
tangent line at (0, 5)
horiz. tangent
m_{tan} = 21

Exponential Functions

$$\frac{d}{dx}[e^x] = e^x$$

This is because e is the number such that $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$.

Example: Differentiate $h(t) = \sqrt[4]{t} - 4e^t$.
First, we rewrite $h(t)$ as $h(t) = t^{1/4} - 4e^t$.

Example: $k(r) = e^r + r^e$.
Don't freak out when you see e or π! They're just NUMBERS, so you should treat them as such.

Higher-Order Derivatives

If f is a differentiable function, then its derivative f' may also be differentiable.
The derivative of the function f' is called the **second derivative** and is denoted $(f')' = f''$.
Other notation: $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2 f(x)$

Note: WE may also be also to continue taking higher derivatives.

$$f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

$$f^{(4)}(x) = y^{(4)} = \frac{d^4y}{dx^4}$$

Example: Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the following functions:

1. $f(x) = x^5 - 2x^3 + 10$

2. $f(x) = 2e^x$