# Math 1b: Calculus, Series, and Differential Equations

*Harvard University, Spring 2011*

## List of Worksheets

# Density and Approximation

1. According to Wikipedia, the population density in Cambridge in the year 2000 was 15,000 people per square mile. The area of Cambridge is about 7 square miles. What was the population of Cambridge in the year 2000?

2. A seaside village, Playa del Carmen, is in the shape of a rectangle with sides of 4 miles and 6 miles. The sea lies along a 6-mile long side. People prefer to live near the water, so the density of people is given by $\rho(x) = 1000(20 - x^2)$ people per square mile, where $x$ is the distance from the seaside.[1] **Without using any calculus**, how could you **approximate** the population of the village?

If you come up with a method, here are some questions about it:

- How accurate is your method? If you needed a quick approximation that didn't require too much computation, could you come up with one? (Try!) What if you needed a really good approximation and were willing to spend a ton of time on computation?
- How flexible is your method? For example, if it was discovered in the next census that the density function $\rho(x)$ had changed to $500(40 - \sqrt{x})$, would your method still work?

[1] The symbol $\rho$ is the Greek letter "rho" and is pronounced "row" (as in "Row, row, row your boat").

# Density and the Definite Integral

1. (Problem Set 1, #3) Pizza Pythagoras is known for its "pizza wedges," which are shaped like right triangles with sides of 3, 4, and 5 inches. Parmesan cheese is sprinkled on each wedge so that the density of cheese is given by $\rho(x)$ ounces per square inch, where $x$ measures distance (in inches) from the 3-inch side of the slice. Approximate the amount of Parmesan on each wedge.

2. People in the Boston area like to live near the city center, so the population density around Boston is $\rho(r) = \frac{36{,}000}{r^2+2r+1}$ people per square mile, where $r$ is the distance in miles to the center of Boston. We'd like to find the number of people who live within 5 miles of the center of Boston.

   (a) Show in a sketch how you would **slice** the region in question.

   (b) How would you **approximate** the number of people who live in the $k$-th slice?

   (c) Write a general Riemann **sum** that approximates the number of people who live within 5 miles of the center of Boston.

   (d) By **taking an appropriate limit**, find a definite integral that gives the number of people who live within 5 miles of the center of Boston.

3. A plot of land is shaped like an equilateral triangle with sides of length 2 miles. One side of the plot lies along a river, and vegetation density varies with $x$, distance to the river. Suppose vegetation density is given by $\rho(x)$.

   (a) Write a general Riemann sum that approximates the amount of vegetation in the plot.

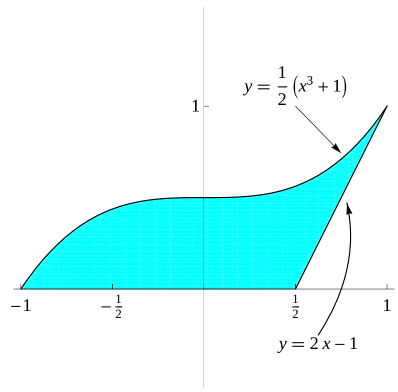
   (b) Find a definite integral that gives the exact amount of vegetation in the plot.

# Area and Volume

1. Find the area under the curve $y = \ln x$ from $x = 1$ to $x = e$.

2. Find the area enclosed by $y = -2x$, $y = x^3$, and $y = 1$.

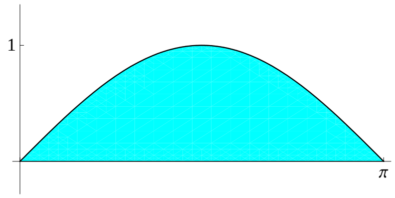
3. (a) What shape do you get if you rotate the following region about the $x$-axis?




(b) Write down an integral or sum of integrals that expresses the volume of the solid from (a).

# Volumes of Revolution

1. Here is one loop of the sine curve.

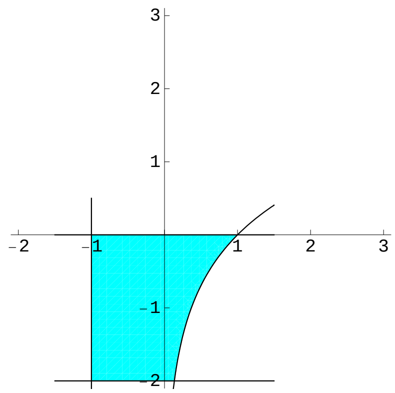



   (a) If you rotate this region about the $x$-axis, what shape do you get? What is its volume? You may leave your answer as an integral.

   (b) If you rotate the region about the horizontal line $y = -1$, what shape do you get? What is its volume? You may leave your answer as an integral.

   (c) If you rotate the region about the vertical line $x = 4$, what shape do you get? What is its volume? You may leave your answer as an integral.

2. Let $\mathcal{R}$ be the region enclosed by the $x$-axis and the graphs of $x = -1$, $y = -2$, and $y = \ln x$.




(a) Write an integral giving the volume generated when $\mathcal{R}$ is rotated about the line $y = 2$. (You need not evaluate your integral.)

(b) Write an integral giving the volume generated when $\mathcal{R}$ is rotated about the line $x = 2$. (You need not evaluate your integral.)

3. How might you describe a bagel as a solid of revolution? (That is, what sort of region would you rotate, and what line would you rotate it about?)

4. Let $\mathcal{R}$ be the region enclosed by the circle of radius 3 centered at the origin. Using vertical slices, find the volume generated when $\mathcal{R}$ is rotated about the line $x = 4$. (Please evaluate your integral.)

   *Note:* It is also possible to find the volume using horizontal slices, and you might want to try that for extra practice.

# Integration by Substitution

**Basic antiderivatives.** You should know the following.

If $n \neq -1$, $\int u^n \, du =$ ______________________. $\quad \int \cos u \, du =$ ______________________.

$\int \frac{1}{u} \, du =$ ______________________. $\quad \int e^u \, du =$ ______________________.

$\int \sin u \, du =$ ______________________. $\quad \int \frac{1}{1+u^2} \, du =$ ______________________.

1. **Warm-up.**

   (a) What is $\frac{d}{dt} e^{-t}$?

   Evaluate the following integrals.

   (b) $\int \frac{1}{5} \sin u \, du$.

   (c) $\int \frac{1}{e^t} \, dt$.

   (d) $\int_1^4 \sqrt{u} \, du$.

2. Evaluate the following integrals using substitution.

   (a) $\int \frac{\sin(\ln x)}{5x} \, dx$.

   (c) $\int \frac{e^x}{\sqrt{2e^x + 5}} \, dx$.

   (b) $\int \sin x \cos^5 x \, dx$.

   (d) $\int_1^4 3x^2 \sqrt{1 + x^3} \, dx$.

3. In this problem, you will evaluate the integral $\int_0^3 \sqrt{9-x^2}\,dx$ using a clever substitution.

   (a) Actually, you can easily evaluate the integral by interpreting it as a signed area. Do this.

   We'll use the substitution $x = 3\sin u$ to do the integral. (You'll soon see why this is a good choice.)

   (b) If $x = 3\sin u$, what is $\sqrt{9-x^2}$ in terms of $u$? Simplify as much as possible. (Your final answer should *not* include any square root signs. If it does, you may find the trigonometric identity $\sin^2 u + \cos^2 u = 1$ helpful. Also, remember that $\sqrt{a^2} = |a|$.)

   The substitution $x = 3\sin u$ is a good choice exactly because it enables us to get rid of the square root.

   (c) If $x = 3\sin u$, what is $dx$ in terms of $u$ and $du$?

   (d) If $x = 3\sin u$, what are the new endpoints of integration in terms of $u$?

   (e) Rewrite the integral $\int_0^3 \sqrt{9-x^2}\,dx$ in terms of $u$.

   (f) Evaluate the integral in (e). (You will need the fact that $\int \cos^2 u\,du = \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right) + C$, which you will show in Problem Set 6, #3.) Does it agree with your answer to (a)?

   Notice that this substitution is quite different from the ones in #2:

   - In #2, we chose $u$ so that $du$ also appeared in the integrand. Here, we chose $u$ just so we could simplify the integrand, even though $du$ certainly didn't appear.
   - Instead of writing $du$ in terms of $x$ and $dx$, we actually did the opposite — we wrote $dx$ in terms of $u$ and $du$.

   The substitution $x = 3\sin u$ is an example of a **trigonometric substitution**: we substituted a trigonometric function for $x$ even though the original integral had no trigonometric functions in it.

4. *In the previous problem, we chose $u$ for the sole purpose of getting rid of the $\sqrt{9-x^2}$, even though we didn't see $du$ in the integrand. This turns out to be a handy strategy in other situations: if some part of an integral is annoying you, call it $u$ and see if that makes things any simpler! This is not guaranteed to help, but it's certainly worth a try if you're stuck.*

   Evaluate $\displaystyle\int \frac{1}{2+\sqrt{x}}\,dx$.

5. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{x}{x+3}\,dx$.

   (b) $\displaystyle\int \frac{3}{9+16x^2}\,dx$.

   (c) $\displaystyle\int \frac{2x+1}{1+x^2}\,dx$.

# Integration by Parts

Evaluate the following integrals.

1. $\int x \cos x \; dx.$

2. $\int x e^x \; dx.$

3. $\int x \ln x \; dx.$

4. $\int_1^e \ln x \; dx.$

5. $\int x^2 \cos 2x \; dx.$

6. $\int e^x \cos x \; dx.$

7. $\int_0^1 \arctan x \; dx.$

8. $\int \cos \sqrt{x} \; dx.$

9. $\int \sin 5x \sin 3x \; dx.$

10. You are given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u} \; du = 3, \int_1^2 g(u) \; du = 4, \int_1^4 g(u) \; du = 5, g(1) = 2, g(2) = -2.$$

(a) Evaluate $\int_1^2 (\ln x) g'(x) \; dx.$

(b) Evaluate $\int_1^2 x g(x^2) \; dx.$

# Integration Techniques

1. Which of the following is easiest to evaluate? Evaluate it.

   (a) $\displaystyle\int \frac{5x-4}{x^2-x-2}\,dx.$

   (b) $\displaystyle\int \frac{5x-4}{(x-2)(x+1)}\,dx.$

   (c) $\displaystyle\int \frac{5x}{x^2-x-2}\,dx - \int \frac{4}{x^2-x-2}\,dx.$

   (d) $\displaystyle\int \frac{3}{x+1}\,dx + \int \frac{2}{x-2}\,dx$

   How do the four choices relate to each other?

2. (a) Evaluate $\displaystyle\int \frac{1}{y^2-1}\,dy.$

   (b) Evaluate $\displaystyle\int \frac{y^2}{y^2-1}\,dy.$

3. Write down the form of the partial fraction expansion for the following integrals. (Don't actually solve for the coefficients.)

   (a) $\displaystyle\int \frac{x+5}{x^2-5x-6}\,dx.$

   (b) $\displaystyle\int \frac{5x^2+7x}{x^2+4x+3}\,dx.$

4. Evaluate $\int \frac{1}{x^2+2x+2}\,dx$.

5. (Problem Set 6, #2) Evaluate **three of the the following four** integrals.

(a) $\int \ln(3x+2)\,dx$.

(b) $\int e^{-x^2}\,dx$.

(c) $\int e^{\sqrt{x}}\,dx$.

(d) $\int \ln\sqrt{x}\,dx$.

6. In each part, decide which method of integration you would use. If you would use substitution, what would $u$ be? If you would use integration by parts, what would $u$ and $dv$ be? If you would use partial fractions, what would the partial fraction expansion look like? (Don't solve for the coefficients.)

(a) $\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$.

(b) $\int (\ln x)^2\,dx$.

(c) $\int e^x \sin x\,dx$.

(d) $\int \frac{x}{x^2-1}\,dx$.

(e) $\int xe^{x^2}\,dx$.

(f) $\int \frac{x^2}{x^2+4x+3}\,dx$.

(g) $\int \frac{e^t}{1+e^t}\,dt$.

(h) $\int \arcsin x\,dx$.

# 3-dimensional Density Problems

1. A cone with height 8 inches and radius 6 inches is filled with flavored slush. When the cup is held upright with the pointed end resting on a table, the density of flavoring syrup in the cup varies with height above the table. Suppose $\rho(x)$ gives the number of ounces of syrup per cubic inch, where $x$ is the distance from the table top. Write an integral giving the total amount of syrup in the cup.

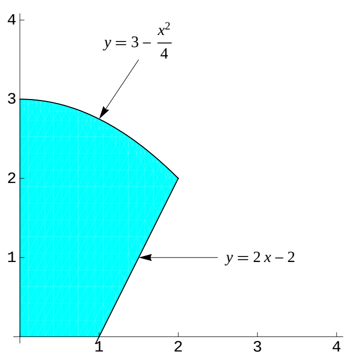
2. Suppose the density of a planet is given by the function $\rho(r) = \dfrac{40000}{1 + 0.0001r^3}$ kilograms per cubic kilometer, where $r$ is the distance in kilometers from the center of the planet. Find the total mass of the planet if its radius is 8000 km. (You need not evaluate your integral.)

3. A cylindrical candle of height 50 mm and radius 12 mm is formed by repeatedly dipping a wick of radius 1 mm into hot wax and then allowing the new layer of wax to dry. The density of each new layer is slightly different, so the density of the candle varies with the distance to the wick. Suppose $\rho(x)$ gives the density in grams per cubic mm of the wax, where $x$ measures the distance to the wick (in mm).

   (a) Write an integral giving the mass of the candle. (The mass of the wick is negligible.)

   (b) Show that, if $\rho(x) = \frac{1}{x+1}$, then the mass of the candle is $1100\pi$ grams.

4. We can model a muffin as a solid of revolution, obtained by rotating the following region about the $y$-axis. Due to a poor recipe, the chocolate chips in our muffin tend to sink to the bottom. The amount of chocolate in the muffin is given by $\rho(y) = 5 - y$ grams per cubic inch, where $y$ represents the distance to the bottom of the muffin. Find the total amount of chocolate in the muffin.



# Work

**Definition.** If a constant force $F$ acting in the direction of motion of an object causes a displacement $d$, then the work done by the force on the object is defined to be the product of the force and displacement; that is, work = force · displacement.

1. For no particular reason, you decide to lift a very heavy rock. If the rock has a mass of 40 kg and you lift it 2 m, how much work do you do in lifting it?

2. A 50 m chain with mass density 2 kg/m is attached to a drum hung from the ceiling. The ceiling is high enough so that the bottom of the chain doesn't touch the floor. Write an integral giving the amount of work required to wind the chain around the drum. (Assume that the height of the drum is negligible.) You may express your answer in terms of the gravitational constant $g$, and you need not evaluate your integral.

3. A 100 kg bag of sand is hoisted 50 m at a rate of 5 m/s. Because of a hole in the bag, 2 kg of sand is lost each second. Find the work done.

4. An ant-hill in the Amazon rainforest has the shape of a perfect hemisphere, with its base on the forest floor. The ant-hill has a 1 m radius and a uniform density of 50 kg/m$^3$. How much work have the ants done to assemble this ant-hill, lifting materials from the level of the forest floor? You may leave your answer as an integral in terms of $g$.

# Improper Integrals

1. (a) Does $\int_1^\infty \frac{1}{x^2}\,dx$ converge or diverge? If it converges, evaluate it.

   (b) Does $\int_1^\infty \frac{1}{x}\,dx$ converge or diverge? If it converges, evaluate it.

2. Using #1, can you conclude anything about whether the following integrals converge or diverge? (Try to figure this out without evaluating the integrals!)

   (a) $\int_1^\infty \frac{1}{x^3}\,dx$?

   (b) $\int_1^\infty \frac{1}{x^{1/2}}\,dx$?

   (c) $\int_1^\infty \frac{1}{x^{3/2}}\,dx$?

3. In each of the following, decide whether the improper integral converges or diverges, and use the Comparison Theorem to justify your answer.

   (a) $\displaystyle\int_1^\infty \frac{x}{\sqrt{3+x^5}}\,dx.$

   (b) $\displaystyle\int_1^\infty \frac{1}{x^2+\sqrt{x}}\,dx.$

4. Does the improper integral $\displaystyle\int_0^\infty \sin x\,dx$ converge or diverge? If it converges, what is its value? How about $\displaystyle\int_{-\infty}^0 \sin x\,dx$?

5. Decide whether the following improper integrals converge or diverge. Explain your reasoning thoroughly.

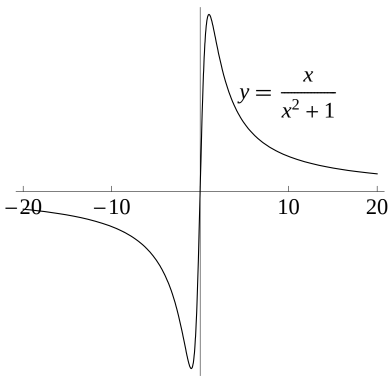
   (a) $\displaystyle\int_5^\infty \frac{1}{x}\,dx.$

   (b) $\displaystyle\int_4^\infty \frac{x^2}{x^3-1}\,dx.$

   (c) $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^3}\,dx.$

# Improper Integrals, Continued

1. Does the improper integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ converge or diverge? If it converges, to what number does it converge?




2. True or false: For any function $f(x)$, $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to\infty}\left(\int_{-a}^{a} f(x)\,dx\right)$.

3. Does the improper integral $\int_{-1}^{\infty} \frac{1}{x^2}\,dx$ converge or diverge? If it converges, to what number does it converge?

An integral is improper if

1. the interval of integration is unbounded and/or
2. the integrand is unbounded somewhere on the interval of integration

The basic strategy for dealing with an improper integral is:

1. Identify all improprieties.
2. If necessary, split up the integral into a sum of integrals so that every impropriety is an endpoint of integration and each new integral has at most one impropriety.
3. Compute each of these new improper integrals as a (one-sided) limit of definite integrals.
4. If any one of the new integrals diverges, the original improper integral diverges. If all of the new integrals converge, sum to find out what the original converges to.

   ☞ **Caution**: This means that divergent summands can never "cancel" one another!

4. Carry out steps 1 and 2 of the above strategy on the following integrals.

   (a) $\displaystyle\int_{-2}^{2} \frac{x}{x^2-1}\, dx.$

   (b) $\displaystyle\int_{-\infty}^{\infty} \sin x\, dx.$

5. Suppose $f(x)$ is a function which is defined, continuous, and positive on $[1,\infty)$, and you are told that $\lim_{x\to\infty} f(x) = 1$. Does $\displaystyle\int_{1}^{\infty} f(x)\, dx$ converge or diverge, or is there not enough information to tell?

Determine whether the following integrals converge or diverge. Explain your reasoning. (You may use the above strategy to actually compute the integral or use the Comparison Theorem to compare to an integral that is easier to compute.)

6. $\displaystyle\int_0^\infty e^{-x^2}\,dx.$

7. $\displaystyle\int_1^\infty \frac{1}{x^4+2}\,dx.$

8. $\displaystyle\int_0^\infty \frac{1}{e^x+x}\,dx.$

9. $\displaystyle\int_{-\infty}^\infty \frac{1}{x^3}\,dx.$

# Taylor Approximation, Continued

1. If you want to find the degree $n$ Taylor polynomial approximation $a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ for $f(x)$ centered at $c$, write a formula for the coefficient $a_k$.

2. Find the degree 15 Taylor polynomial approximation for $f(x) = \cos x$ centered at 0.

3. Rewrite the following sums in $\cdots$ notation. (This means: Write down at least the first 3 terms $+\cdots+$ the last term.)

   (a) $\displaystyle\sum_{k=1}^{100} \frac{1}{2k}$.

   (b) $\displaystyle\sum_{k=3}^{15} \frac{(-1)^k}{k}$.

   (c) $\displaystyle\sum_{k=0}^{25} \frac{x^{2k+1}}{k+1}$.

   (d) $\displaystyle\sum_{k=1}^{26} \frac{x^{2k-1}}{k}$.

4. Rewrite the following sums in sigma notation.

(a) $\frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \cdots + \frac{1}{100^3}$.

(b) $\sqrt{2} - \sqrt{4} + \sqrt{6} - \sqrt{8} + \sqrt{10} - \cdots - \sqrt{120}$.

(c) $1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + \cdots + x^{30}$.

(d) $x - x^3 + x^5 - x^7 + x^9 - \cdots + x^{101}$.

(e) $x - \frac{x^3}{2} + \frac{x^5}{3} - \frac{x^7}{4} + \cdots - \frac{x^{31}}{16}$.

5. (a) Rewrite your answer to #2 in sigma notation.

(b) Find the degree 25 Taylor polynomial approximation for $f(x) = \cos x$ centered at 0. Write it in both sigma and $\cdots$ notation.

(c) Use the degree 25 Taylor polynomial approximation to get an approximation for $\cos 0.1$.

# Taylor Series

1. What is the Taylor series expansion of $\cos x$ centered at 0? Write it in both sigma and $\cdots$ notation.

2. What is the Taylor series expansion of $\sin x$ centered at 0? Write it in both sigma and $\cdots$ notation.

We hope that, by using a "polynomial of infinite degree," we end up with something that is not just an approximation for our function but is actually equal to the function. We don't really know if this is true yet, but let's see how it would be useful to have such an alternate representation of a function.

For the remaining problems, take on faith that $\sin x$ is actually equal to its Taylor series expansion about 0.

3. Write an "infinite polynomial" representation of:

   (a) $\sin x^3$.

   (b) $\int \sin x^3 \, dx$. (Note: The antiderivatives of $\sin x^3$ are not elementary functions!)

4. Find the degree 100 Taylor polynomial approximation of $\sin x^3$.

5. Let $f(x) = \sin x^3$. What is $f^{(100)}(0)$? $f^{(51)}(0)$?

# Definition of Convergence

1. Does the series $1 + 1 + 1 + \cdots$ converge or diverge? If it converges, what is its sum?

2. Does the series $1 + (-1) + 1 + (-1) + \cdots$ converge or diverge? If it converges, what is its sum?

3. (a) Does the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converge or diverge? If it converges, what is its sum?

   (b) Draw a diagram showing the first 6 partial sums of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

4. In this problem, you'll investigate the series $\sum_{k=1}^{\infty} \frac{2}{(2k)^2 - 1}$.

   (a) Write out the first 4 terms of the series.

   (b) Find the first 4 partial sums of the series. Express them as simplified fractions.

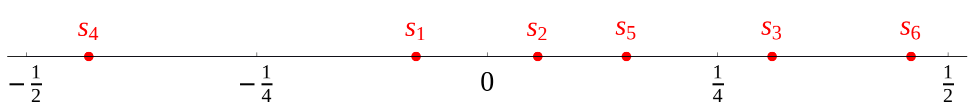
   (c) Looking at your answer to (b), guess a formula for the $n$-th partial sum.

   (d) Based on your guess from part (c), does the series converge or diverge? If it converges, what is its sum?

5. (a) Draw a partial sum diagram showing the first 6 partial sums of the series $\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$.

   (b) Based on your diagram, do you think the series $\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$ converges or diverges?

6. Here is a diagram showing the first 6 partial sums of an unknown series $\sum_{k=1}^{\infty} a_k$.




   (a) From the diagram, determine whether each of $a_1, a_2, \ldots, a_6$ is positive, negative, or zero.

   (b) Which of $a_1, a_2, \ldots, a_6$ is biggest in magnitude?

# Nth Term Test, Comparison Test

1. Suppose you know that the series $\sum_{k=1}^{\infty} a_k$ converges. Let $s_n = a_1 + a_2 + \cdots + a_n$, the $n$-th partial sum. For each of the following, determine whether the statement must be true, must be false, or could be either true or false.

   (a) $\lim_{n\to\infty} a_n = 0$.

   (b) $\lim_{n\to\infty} s_n = 0$.

   (c) $\sum_{k=5}^{\infty} a_k$ converges.

2. Suppose $a_1, a_2, \ldots$ are numbers and $\lim_{k\to\infty} a_k = 0$. Does it necessarily follow that the infinite series $\sum_{k=1}^{\infty} a_k$ converges?

3. The series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is known as the harmonic series.

   (a) What does the Nth Term Test tell you about the harmonic series?

   (b) Does the harmonic series converge or diverge?

4. (a) Does the series $1+\frac{1}{2}+\frac{1}{2}+\underbrace{\frac{1}{4}+\cdots+\frac{1}{4}}_{\text{4 terms}}+\underbrace{\frac{1}{8}+\cdots+\frac{1}{8}}_{\text{8 terms}}+\underbrace{\frac{1}{16}+\cdots+\frac{1}{16}}_{\text{16 terms}}+\cdots$ converge or diverge?

(b) Does the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\underbrace{\frac{1}{8}+\cdots+\frac{1}{8}}_{\text{4 terms}}+\underbrace{\frac{1}{16}+\cdots+\frac{1}{16}}_{\text{8 terms}}+\underbrace{\frac{1}{32}+\cdots+\frac{1}{32}}_{\text{16 terms}}+\cdots$ converge or diverge?

5. Does the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2^k+k}$ converge or diverge?

6. Does the series $\displaystyle\sum_{k=10^{10}}^{\infty}\frac{1}{100000000k}$ converge or diverge?

7. Decide whether each of the following series converges, and use the Comparison Test to justify your answer mathematically.

(a) $\displaystyle\sum_{k=100}^{\infty}\frac{5+3\sin k}{k}$.

(b) $\displaystyle\sum_{k=37}^{\infty}\frac{5+3\sin k}{2^k}$.

# Geometric Series

1. If you suffer from allergies, your doctor may suggest that you take Claritin once a day. Each Claritin tablet contains 10 mg of loratadine (the active ingredient). Because of the way the human body metabolizes loratadine, no matter how much is in your body at a given time, only $\frac{1}{8}$ of that amount remains 24 hours later.[(1)]

   (a) If you take one Claritin tablet each morning for 2 successive days, how much loratadine is in your body right after you take the 2nd tablet?

   (b) If you take one Claritin tablet every morning for a week, how much loratadine is in your body right after you take the 3rd tablet? 7th tablet? (Don't try to simplify your computations; just write out an arithmetic expression.)

   (c) If you take Claritin for years and years, will the amount of loratadine in your body level off? Or will your bloodstream be pure loratadine?

[(1)] This estimate comes from the fact that the average half-life of loratadine is known to be 8 hours.

**Definition.** A geometric series is a series that can be written in the form $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$.

2. Let $s_n$ be the $n$-th partial sum of the geometric series $a + ar + ar^2 + ar^3 + \cdots$. That is, $s_n$ is the sum of the first $n$ terms. In this problem, we'll consider only geometric series with $a \neq 0$.

   (a) Write a closed form expression for $s_n$ if $r = 1$. (A "closed form expression" is one without $\cdots$ or $\sum$; if you had values for $a$ and $n$, you could type it directly into a calculator to evaluate it.)

   (b) Write a closed form expression for $s_n$, assuming $r \neq 1$.

   (c) For what values of $a$ and $r$ does the geometric series $a + ar + ar^2 + \cdots$ converge? For what values does it diverge? When it converges, what is its sum?

   *This result is incredibly important; you should feel free to cite it from now on. (When using it, be sure to explain what $a$ and $r$ are for your geometric series.)*

3. (a) For what values of $x$ does the geometric series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$ converge? For what values of $x$ does it diverge? When it converges, what does it converge to?

   (b) How would you graph $\sum_{k=0}^{\infty} x^k$?

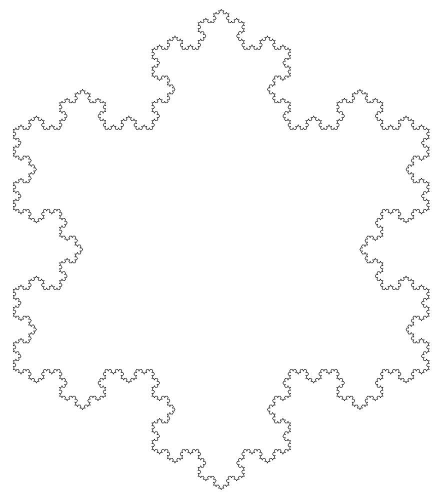
4. Remember that a geometric series is a series of the form $a + ar + ar^2 + \cdots = \sum_{k=0}^{\infty} ar^k$. Which of the following series are geometric? For each geometric series, identify $a$ and $r$, and say whether the series converges or diverges; if it converges, find its sum.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 2^{k+1}}{3^k}$.

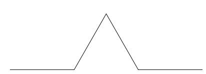
   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

   (c) $\displaystyle\sum_{n=4}^{\infty} \frac{2}{3^{2n}}$.

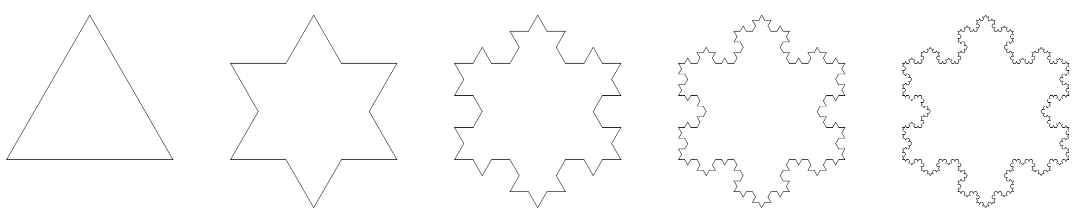
5. The following figure is called the Koch snowflake.



Here's how it is made. Let $P_0$ be an equilateral triangle whose sides have length 1. Replace each line segment in $P_0$ with this shape (called the "motif"). Each line segment in the motif is 1/3 of the length of the original line segment.



This gives you a new polygon $P_1$; each line segment in $P_1$ has length 1/3. To get $P_2$, replace each line segment in $P_1$ with the motif. Now, each line segment in $P_2$ has length 1/9. If you do this again, you get $P_3$, which is composed of line segments of length 1/27. If you continue this process forever, you end up with the Koch snowflake.



$P_0$ $\quad$ $P_1$ $\quad$ $P_2$ $\quad$ $P_3$ $\quad$ $P_4$

(a) How many sides does polygon $P_k$ have?

(b) What is the length of each side in polygon $P_k$?

(c) Write a series which expresses the total area of the Koch snowflake.

(d) What is the area of the Koch snowflake?

(e) What is the perimeter of polygon $P_k$? What does this tell you about the perimeter of the Koch snowflake? (This question has nothing to do with series.)

# $p$-series, Comparing Series to Integrals

1. Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge or diverge?

2. We have already seen that the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. (We showed this by comparing to another series.) Can you show this by comparing to an improper integral?

3. For what values of $p$ is the $\underline{p\text{-series}}$ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ convergent? For what values of $p$ is it divergent?

   *From now on, you may simply cite the result of this problem. When you do so, be sure to identify the value of $p$ you are considering.*

4. *Now that you know about geometric series and p-series, you can use the Direct Comparison Test in many more situations than you could before.*

   Use the Direct Comparison Test to explain why the following series converge or diverge.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{|\cos k|}{k^3}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{3 + \sin k}{0.9^k}$.

5. Here is a new definition: a series $\sum a_k$ is called absolutely convergent if the series $\sum |a_k|$ is convergent. Decide whether or not each of the following series is absolutely convergent. (Notice that, when you decide whether $\sum a_k$ is absolutely convergent, you're not looking at $\sum a_k$ at all; instead, you're looking at the series $\sum |a_k|$, whose terms are all non-negative.)

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3 + \sin k}{0.9^k}$.

# Asymptotics

1. Does the series $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2^k - 100}$ converge or diverge?

**Definition.** We say that $b_k$ grows faster than $a_k$ as $k \to \infty$ if $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ or, equivalently, $\displaystyle\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$. We write this as $a_k \ll b_k$.

Here are some common expressions listed according to their growth rates:

$$\ln k \ll k^m \ll k^n \ll a^k \ll b^k \ll k! \ll k^k \text{ where } 0 < m < n \text{ and } 1 < a < b \tag{1}$$

Here are some more useful facts:

- A sum is asymptotic to its "largest" term (where largest means the one that grows fastest).
- If $b_k \gg a_k$, then $b_k \gg ca_k$ for any constant $c$. For instance, since $k^3 \gg k^2$, $k^3 \gg 50000k^2$.

2. Asymptotically simplify the following expressions as much as you can. (That is, find a simpler expression that is asymptotic to the given one, and try to make your simpler expression as simple as possible.)

   (a) $\dfrac{k^{1000} + k^{999} + 1.1^k}{\ln k + k^4}$.

   (b) $\dfrac{k! + \pi k^{17} + 10000^k}{10^{10} \ln k + k^k + k!}$.

   (c) $\dfrac{2^k (k^3 + k^2)}{3^k}$.

   (d) $\dfrac{1}{k!(k^9 - 7k^2)^{5/3}}$.

3. Use asymptotics to help you decide whether the following series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{5\sqrt{k}}{k^2+2k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^4+2^k-7\ln k}{(3^k-10k^2)k!}$.

4. Asymptotically simplify each expression.

(a) $\displaystyle\frac{5+3\sin^2 k}{k^3+k!}$.

(b) $\displaystyle\frac{(k+\sin k)\ln k}{5^k\cdot k^3}$.

5. Decide whether each of the following series converges, and justify your conclusion mathematically.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3(e-2\cos k)}{\sqrt{k^7-5k}}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{(100+5k^2+k^{11})^{1/3}}$

# Ratio Test

**Definition.** A series $\sum a_k$ is called <u>absolutely convergent</u> if the series of absolute values $\sum |a_k|$ is convergent.

**Theorem.** If a series $\sum a_k$ is absolutely convergent, then it is convergent.

The opposite (converse) of the theorem is not true: there are convergent series which are not absolutely convergent. These series are called <u>conditionally convergent</u>.

1. (a) Suppose you have a series $\sum_{k=1}^{\infty} a_k$ whose terms are nonzero (but the series could have both positive and negative terms), and you know that $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 5$. What does this tell you about the magnitude (absolute value) of the terms as $k$ gets really big? Does the series converge or diverge, or is there not enough information to tell?

   (b) Suppose you have a series $\sum_{k=1}^{\infty} b_k$ whose terms are all positive, and you know that $\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \frac{2}{3}$. Does the series converge or diverge, or is there not enough information to tell?

   (c) Suppose you have a series $\sum_{k=1}^{\infty} c_k$ whose terms are nonzero (but the series could have both positive and negative terms), and you know that $\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = \frac{2}{3}$. Does the series $\sum_{k=1}^{\infty} |c_k|$ converge or diverge, or is there not enough information to tell? What does this tell you about the series $\sum_{k=1}^{\infty} c_k$?

2. What does the Ratio Test tell you about the following series?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{3^k}$.

(b) $\displaystyle\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1000^k}{k!}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

(d) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

3. (a) What is the Taylor series expansion of $e^x$ about 0?

   (b) What does the Ratio Test say about the convergence of the series you found in (a)?

4. What does the Ratio Test tell you about the series $\displaystyle\sum_{k=1}^{\infty} \frac{(3k)!}{5^{2k}}$?

# Power Series

1. In Problem Set 20, #5, you looked at the Taylor series of $\ln(1+x)$ centered at 0, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

   (a) What did the Ratio Test tell you about this Taylor series?

   (b) For what values of $x$ was the Ratio Test inconclusive? How did you determine whether this Taylor series converged for those values of $x$?

   (c) What are all values of $x$ for which this Taylor series converges?

2. A long time ago (Problem Set 14, #2), you found that the Taylor series of $\frac{1}{1-x}$ centered at 0 is $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$.

   (a) The series $\sum_{k=0}^{\infty} x^k$ happens to be geometric. For what values of $x$ does it converge? For what values of $x$ does it diverge? When it converges, what does it converge to?

   (b) Graph $\sum_{k=0}^{\infty} x^k$.

A power series centered at the number $c$ is a series of the form $\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ where $x$ is a variable and the $a_k$ are constants.

3. Which of the following are power series?

(a) $\sum_{k=1}^{\infty} \sqrt{k}(x-3)^{1+2k}$.

(b) $\sum_{k=1}^{\infty} \sqrt{k}(x-3)^{1-2k}$.

(c) $\sum_{k=0}^{\infty} \sqrt{x}(x-5)^{k}$.

(d) $\sum_{k=5}^{\infty} \frac{(x-k)^k}{k}$.

**Theorem on convergence of a power series.** For a given power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ centered at $c$, there are 3 possibilities:

1. The series converges only when $x = c$.
2. The series converges absolutely for all $x$.
3. There is a positive number $R$ such that the series converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$. $R$ is called the radius of convergence. (Note that this doesn't say anything about what happens when $|x-c| = R$.)

If (1) is true, we say that the radius of convergence of the series is 0; if (2) is true, we say that the radius of convergence is $\infty$. So, every power series has a radius of convergence.

**Terminology.** The interval of convergence of a power series is the set of $x$ for which the power series converges. The largest open interval of convergence is the open interval $(c-R, c+R)$.

4. (a) What is the radius of convergence of the power series $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k}$ (the power series from #1)? What is the interval of convergence? What is the largest open interval of convergence?

(b) What is the radius of convergence of the power series $\sum_{k=0}^{\infty} x^k$ (the power series from #2)? What is the interval of convergence? What is the largest open interval of convergence?

5. (a) Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k^2 \cdot 7^k}$.

(b) Find the largest open interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k^2 \cdot 7^k}$.

(c) Find the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k^2 \cdot 7^k}$.

(d) Draw the interval of convergence on a number line.

**Definition.** If a function $f(x)$ is *equal* to a power series $\sum_{k=0}^{\infty} a_k (x-c)^k$ (possibly only on a limited domain around $c$), we say that $\sum_{k=0}^{\infty} a_k (x-c)^k$ is a power series representation of $f(x)$ (at $c$).

6. (a) Find a power series representation of the function $f(x) = \frac{1}{1+4x^2}$. For what $x$ can we be sure that this representation is valid?

(b) Find a power series representation of the function $g(x) = \dfrac{x}{1+4x^2}$. For what $x$ can we be sure that this representation is valid?

(c) In fact, the series you found in the two previous parts should have both been geometric. Use your knowledge of geometric series to verify that your answers are correct.

7. Find a power series representation of $\dfrac{x^3}{1+27x^3}$. For what $x$ can we be sure that this representation is valid?

8. In this problem, you'll look at the power series $\displaystyle\sum_{n=0}^{\infty} n!x^n$.

(a) Write out the first few terms of the power series. Does the series converge when $x = 0$?

(b) Find all values of $x$ for which the series $\displaystyle\sum_{n=0}^{\infty} n!x^n$ converges. (Make sure your answer is consistent with your answer to #8.)

(c) What is the radius of convergence of this power series?

# Power Series Representations of Functions

**Theorem.** If the power series $\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ has radius of convergence $R$ where $R > 0$ or $R = \infty$, then the function $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ is differentiable on $(c-R, c+R)$. On that interval,

1. $f'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$.
2. $\int f(x)\, dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1} = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots$.

The power series in (1) and (2) both have radius of convergence $R$. (Note: Although the radius of convergence remains unchanged, the interval of convergence may change.)

1. Find a power series representation of $\frac{1}{(1-x)^2}$ centered at 0. (Hint: $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$.) What is the radius of convergence of the power series you have found?

2. (a) Find a power series representation for $\ln(1+x)$ centered at 0. (Hint: What is the derivative of $\ln(1+x)$?)

   (b) What is the radius of convergence for the power series you have found? What is the interval of convergence?

(c) You showed in Problem Set 15, #5 that the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges. What is its sum?

**Theorem.** If $f(x)$ has a power series representation centered at $c$ (that is, $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ for $|x-c| < R$), then that power series must be the Taylor series of $f$ at $c$.

3. (a) Taking for granted that $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ for all $x$, find the Taylor series of $f(x) = x \sin(x^3)$ at 0.

(b) What is the radius of convergence of the power series you found in (a)?

(c) What is the degree 15 Taylor polynomial approximation of $f(x)$ centered at 0? (Please write out all terms.)

(d) What is $f'''(0)$? $f^{(4)}(0)$?

4. In Problem Set 14, #1, you found that the Taylor series of $e^x$ centered at 0 is $\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1+x+\frac{x^2}{2!}+\cdots$.

(a) What is the interval of convergence of the power series?

(b) Define $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1+x+\frac{x^2}{2!}+\cdots$ on the interval you found in (a). (That is, the domain of $f$ is the interval of convergence of the power series.) Write a power series representation of $f'(x)$. Please give your answer in both sigma and $\cdots$ notation. For what $x$ is this representation valid?

(c) It is a fact[1] that there is exactly one function $h(x)$ such that $h'(x) = h(x)$ and $h(0) = 1$. Using this fact, show that $e^x$ is equal to its Taylor series.

*From now on, you may use the fact that $e^x$ is equal to its Taylor series for all $x$.*

[1] We will understand this fact better when we study differential equations.

5. Consider the function $f(x) = \dfrac{1}{1-4x^2}$.

(a) Find a power series representation of $\displaystyle\int \frac{1}{1-4x^2}\,dx$. For what $x$ is your representation valid? (Give the largest open interval.)

(b) If possible, use your answer to (a) to write down a series that converges to $\displaystyle\int_0^{1.5} \frac{1}{1-4x^2}\,dx$. If it is not possible, explain why.

# More Series Problems

1. (a) Give a power series representation of $e^x$. For what $x$ is it valid? (How do we know that the power series is actually equal to $e^x$?)

   (b) Use (a) to find a series whose sum is $e$.

   (c) Explain why the series in (b) converges.

   (d) Suppose you use the first 5 terms of the series you found in (b) to approximate $e$. Write a series that gives the exact error in this approximation.

   (e) By comparing the error to a geometric series, put an upper bound on the error (that is, find a number that you can demonstrate is bigger than the error). Try to make your upper bound as close as possible. (After all, it's a lot more useful to say the error is at most 0.1 than it is to say that the error is at most 1000).

   *Hint: First factor out* $\frac{1}{5!}$.

(f) Suppose you instead use the first 7 terms of the series to approximate $e$. Find an upper bound on the error.

(g) Suppose you use the first $n$ terms of the series to approximate $e$. Find an upper bound on the error.

(h) Suppose you want to approximate $e$ with an error of at most $10^{-5}$. Explain how you could use the previous part to find such an approximation.

2. (a) Find the degree 2 Taylor polynomial approximation of $(1+x)^{1/3}$ centered at 0.

(b) Does it seem like a good idea to use the approximation you've just found to approximate $1.01^{1/3}$? $1001^{1/3}$? Why or why not?

(c) Let $q$ be any constant. Find the degree 2 Taylor polynomial approximation of $(1+x)^q$ centered at 0.

(d) What happens when $q = -1$? Is this what you expect?

(e) What happens when $q = 1$? $q = 2$?

3. Let $f(x) = \sum_{k=0}^{\infty} \frac{(x+4)^{2k+1}}{(k+3)9^k}$.

(a) Find the radius of convergence of the power series defining $f$.

(b) Which of the following quantities are defined? Which are easy to compute exactly? Which can you easily find a series representation of?

i. $f(6)$.

ii. $f(-6)$.

iii. $f^{(50)}(-2)$.

iv. $f^{(100)}(0)$.

v. $f^{(25)}(-4)$.

vi. $f^{(10)}(4)$.

vii. $f'(-3)$.

viii. $\displaystyle\int_{-2}^{0} f(x)\, dx$.

ix. $\displaystyle\int_{-4}^{-3} f(x)\, dx$.

# Differential Equations

1. Mr. Moneybags decides to open a bank account with an opening deposit of \$1000. Suppose that the account earns a nominal annual interest rate of 6%, compounded annually.[1] Assuming Mr. Moneybags completely ignores the account after he opens it (so he doesn't make any deposits or withdrawals), how much money does the account have $t$ years after Mr. Moneybags opens it? Can you graph his money vs. time?

2. Suppose that Mr. Moneybags had instead deposited his money in a bank that offered monthly compounding (but everything else was the same). That is, the bank has 12 compounding periods a year. How much money would the account have after $t$ years?

3. Continuous compounding is defined to be the limit as $n \to \infty$ of having $n$ compounding periods a year. If Mr. Moneybags had used a bank with continuous compounding, how much money would he have after $t$ years?

[1] For economists, there are two uses of the phrase "nominal interest rate"; one is trying to account for the effects of inflation. We're using it in the other sense, which will be easier to understand a bit later.

4. In the situation of the previous problem, we say that Mr. Moneybags gets a "6% nominal annual interest rate, compounded continuously." If we are modeling his money $M(t)$ as a continuous function, what is the relationship between $\frac{dM}{dt}$ and $M$?

5. \$1000 is deposited in a bank account. The account has a nominal annual interest rate of 6%, compounded continuously. Money is being deposited continuously at a rate of \$600 per year.[(2)] How could you model this situation?

In #6 - #9, write a differential equation that reflects the situation. Include initial conditions if the information is given.

6. The population of a certain country increases at a rate proportional to the population size. Let $P = P(t)$ be the population at time $t$.

7. A yellow rubber duck is dropped out of the window of an apartment building from a height of 80 m. Let $s = s(t)$ be the height of the duck above the ground at time $t$. (Gravity is the acceleration $-9.8 \text{ m/s}^2$.)

   *Hint:* Write an equation involving a second derivative.

[(2)] In reality, you cannot deposit money continuously, but it's convenient to use a continuous model.

8. Ferdinand is trying to fill a bucket from a faucet. Unfortunately, he doesn't realize that there is a small hole in the bottom of the bucket. Water flows in to the bucket from the faucet at a constant rate of 0.75 quarts per minute, and it flows out of the hole at a rate proportional to the amount of water $W(t)$ already in the bucket (due to the increased water pressure).

9. (a) Harry Potter is practicing his magical Refilling Charm on a mug of pumpkin juice. He uses the Refilling Charm to add juice to the mug at a rate inversely proportional to the amount of juice currently in the mug. Let $J(t)$ be the amount of juice in the mug at time $t$.

   (b) Harry decides to put his Refilling Charm into action while Ron is drinking a mug of pumpkin juice. Ron drinks at a constant rate of 50 ml/min, while Harry uses his Refilling Charm as in the previous part. Let $J(t)$ be the amount of pumpkin juice in Ron's mug at time $t$.

The following problems are about solutions to differential equations.

10. Which of the following is a solution to $\frac{dy}{dx} = y$?

    i. $y = \frac{x^2}{2} + C$.

    ii. $y = e^x + C$.

    iii. $y = Ce^x$.

11. Give two solutions to $\frac{dy}{dx} = 5y$. What is the general solution?

12. Give two solutions to to $\frac{dy}{dx} = 5x$. What is the general solution?

13. Which of the following is the solution of $\dfrac{dM}{dt} = 0.06M + 600$ satisfying $M(0) = 1000$?

    i. $M(t) = 1000e^{0.06t} + 600t$

    ii. $M(t) = 11{,}000e^{0.06t} - 10{,}000$

    iii. $M(t) = (1000 + 600t)e^{0.06t}$

14. Show that, if $r$ is any constant, then $\lim_{n\to\infty} \left(1 + \dfrac{r}{n}\right)^n = e^r$.

# Slope Fields

1. (Problem Set 24, #8) Martha Stewart is mixing up a big batch of eggnog for the holidays. Suppose she has a big mixing vat containing 100 gallons of egg yolks. At time $t = 0$, her assistant starts pouring a milk and sugar mixture containing 1/2 lb of sugar per gallon into the vat at a rate of 4 gallons per minute. At the same time, mixed eggnog starts leaving the vat a rate of 4 gallons per minute.

   Write a differential equation for the amount $S(t)$ of sugar (measured in pounds) in the vat at time $t$. Can you write down an initial condition as well?

2. Suppose instead that, in the previous problem, mixed eggnog left the vat at a rate of 6 gallons per minute. How would this change the differential equation and initial condition?

3. Draw slope fields for the following differential equations:

   (a) $\dfrac{dy}{dt} = 1.$

   (b) $\dfrac{dy}{dt} = -t.$

(c) $\dfrac{dy}{dt} = -y.$

(d) $\dfrac{dy}{dt} = -\dfrac{t}{y}.$

Now, go back and try to figure out the general solution of each differential equation.

4. (a) If you have a differential equation of the form $\frac{dy}{dt} = f(t)$ (that is, $\frac{dy}{dt}$ depends only on $t$, not on $y$; an example is $\frac{dy}{dt} = t^2 + 1$), what can you say about its slope field? How do different solutions of the differential equation relate to each other?

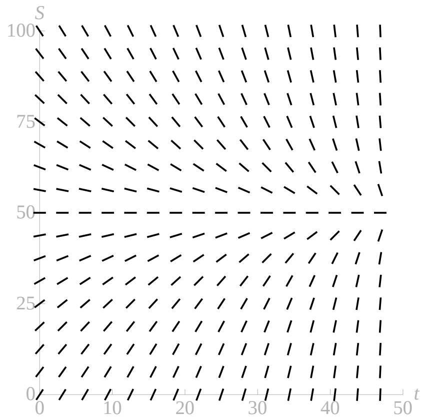
   (b) If you have a differential equation of the form $\frac{dy}{dt} = f(y)$ (that is, $\frac{dy}{dt}$ depends only on $y$, not on $t$), what can you say about its slope field? How do different solutions of the differential equation relate to each other?

5. (a) Draw the slope field for the differential equation $\frac{dy}{dt} = 1 - y$. Sketch two solutions to the equation.

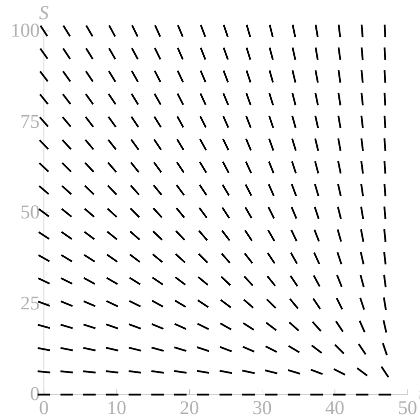
   (b) Which of the following is a solution to $\frac{dy}{dt} = 1 - y$?

      i. $y = Ce^{-t}$
      ii. $y = Ce^{t} - t$
      iii. $y = Ce^{-t} - 1$
      iv. $y = Ce^{-t} + 1$
      v. $y = Ce^{t} + 1$

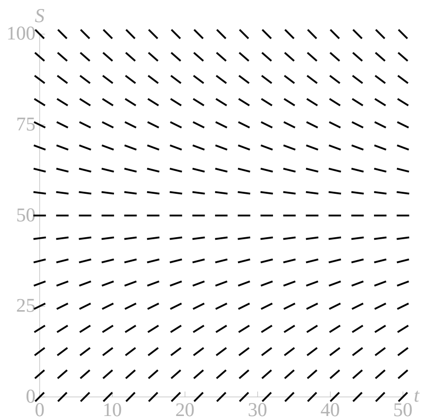
6. Here are four slope fields.



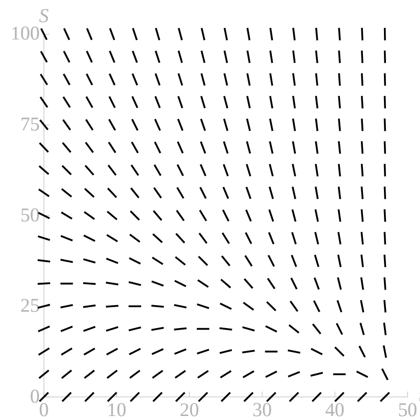
(A)



(B)



(C)



(D)

(a) Which is the slope field of the differential equation you found in #1? On that slope field, sketch the solution corresponding to your initial condition from #1. Over time, what happens to the amount of sugar in the vat?

(b) Which is the slope field of the differential equation in #2? On that slope field, sketch the solution corresponding to your initial condition. Over time, what happens to the amount of sugar in the vat?

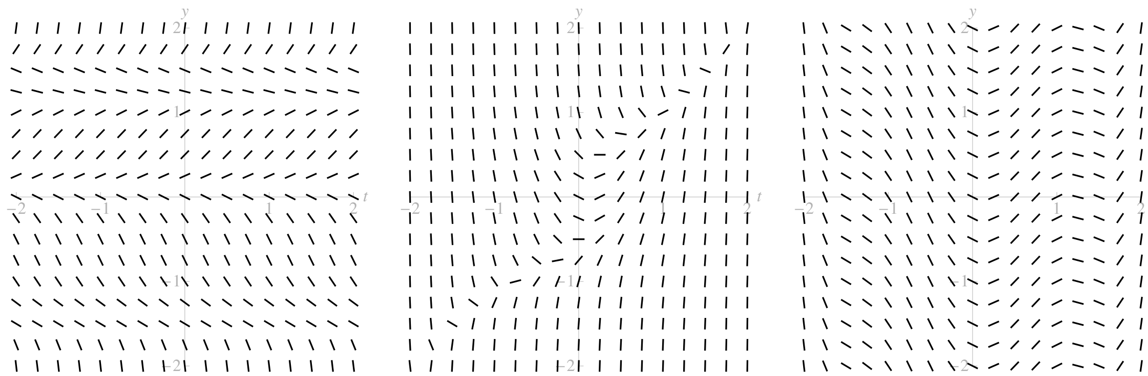
7. (a) Martha Stewart is still busy perfecting her 100 gallons of eggnog. Unfortunately, one of her assistants messed up and dumped too much sugar in the vat (Martha herself would never make a mistake), so the eggnog in the tank has a concentration of $\frac{3}{4}$ pounds of sugar per gallon. To fix the problem, Martha Stewart starts adding unsweetened eggnog to the vat at a rate of 1 gal/min. Unfortunately, another clumsy assistant punches a hole in the vat, so the mixed eggnog is dripping out of the vat at the rate of 3 gal/min.

   Let $t$ denote the time in minutes, with $t = 0$ the time that Martha Stewart starts pouring unsweetened eggnog into the vat. Let $S(t)$ be the amount of sugar in the vat at time $t$, measured in pounds. Write a differential equation and initial condition for $S(t)$.

   (b) Which of the four slope fields in #6 is the slope field of the differential equation you found? On that slope field, sketch the solution corresponding to your initial condition. Over time, what happens to the amount of sugar in the vat?

# Autonomous First-Order Differential Equations

1. Here are three slope fields.



(I) (II) (III)

For each of the following differential equations, find the picture above which shows its slope field.

(a) $\dfrac{dy}{dt} = t^5 - 4t^3 + 4t - \dfrac{1}{2}$.

(b) $\dfrac{dy}{dt} = y^5 - 4y^3 + 4y - \dfrac{1}{2}$.

2. Use qualitative analysis to sketch a family of representative solutions for each differential equation. Identify the equilibrium solutions of each differential equation, and say whether they are stable or unstable.

(a) $\dfrac{dy}{dt} = (y+1)(4-y)$.

(b) $\dfrac{dy}{dt} = (y+1)(y-2)^2$.

**Qualitative Analysis of $\boldsymbol{\dfrac{dy}{dt} = f(y)}$** or *How can we sketch solutions while being as lazy as possible?*

So far, when we've wanted to sketch solutions of a differential equation, we've sketched its slope field. Qualitative analysis gives us the same information when our differential equation has the form $\frac{dy}{dt} = f(y)$, with a lot less work! The basic method is:

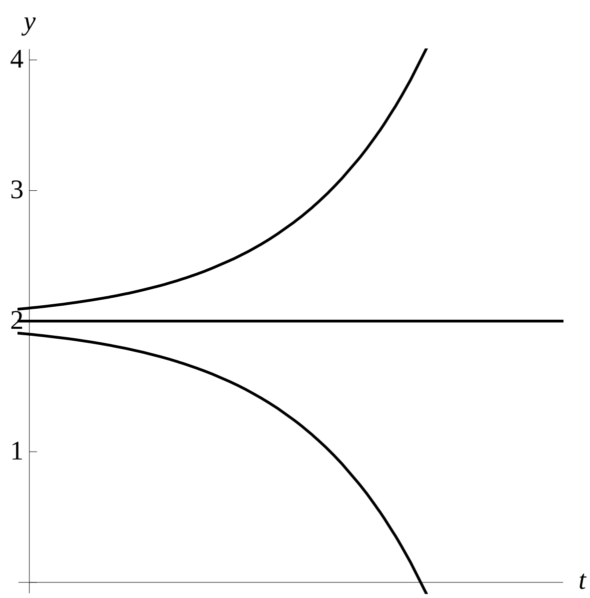
1. Find the equilibrium solutions and sketch these.

2. The equilibrium solutions divide the picture into horizontal strips. In each strip, test one point to determine whether $\frac{dy}{dt}$ is positive or negative. This tells you whether solutions in that strip increase or decrease, and then you can draw a representative solution in that strip.

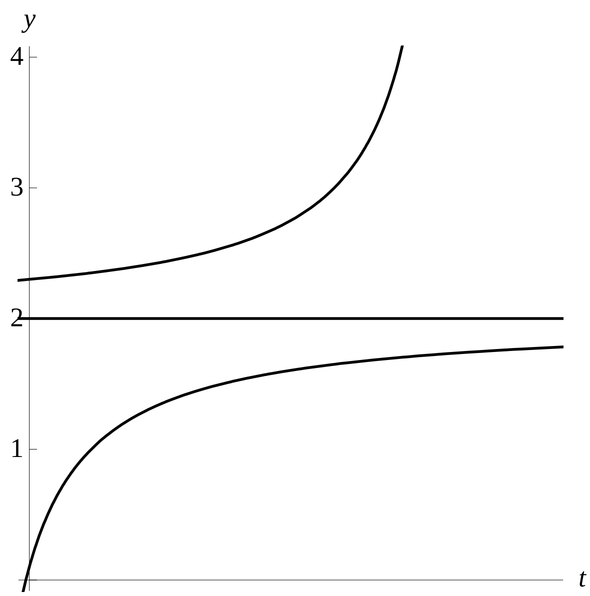
If you want to draw the solution satisfying an initial condition, then it is just a horizontal shift of one of your representative solutions.

3. For each picture, think of an autonomous first-order differential equation whose solution curves look like those in the picture.

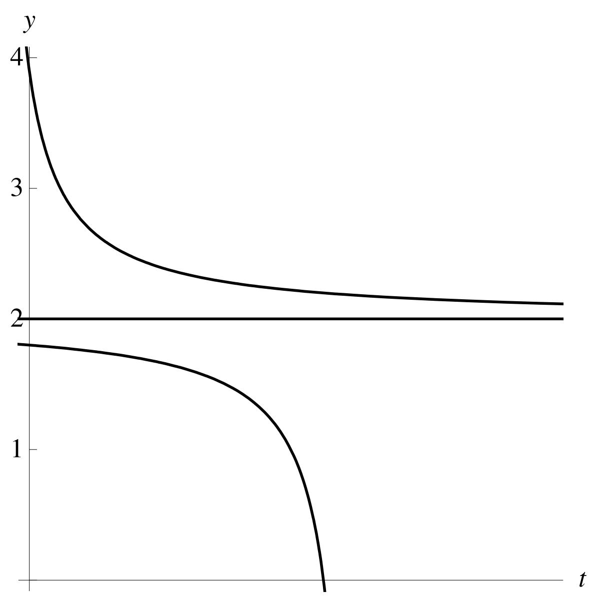
(a)



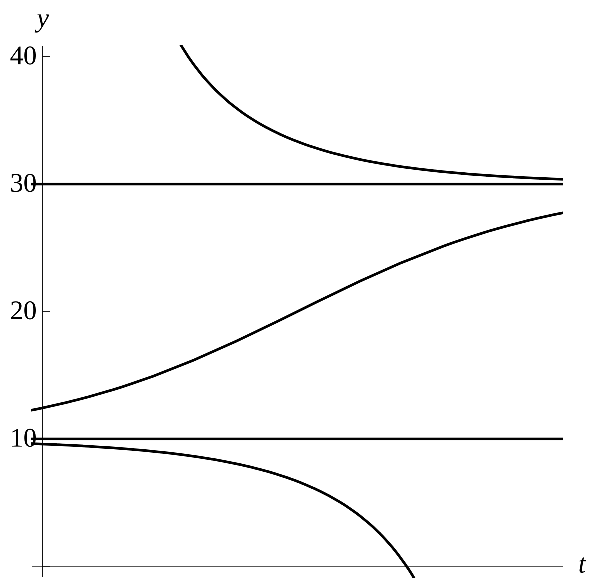
(b)



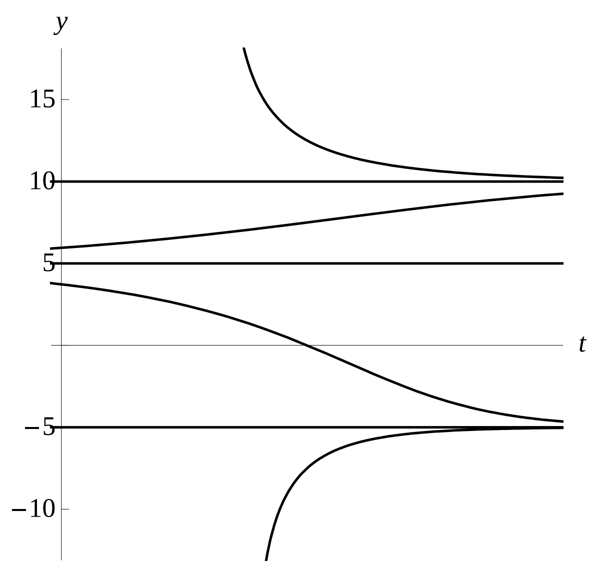

(c)



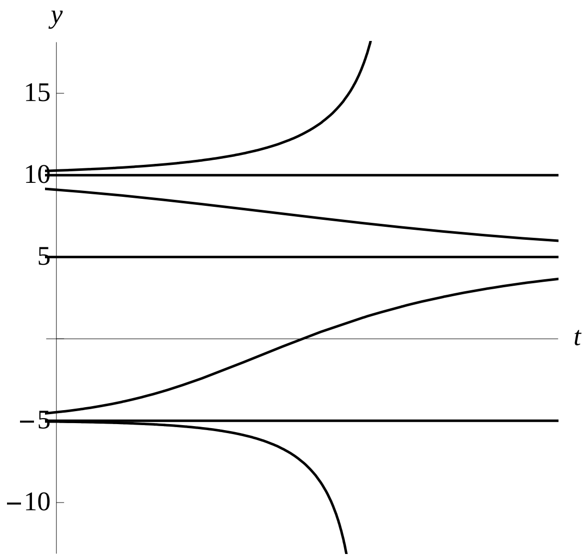
(d)



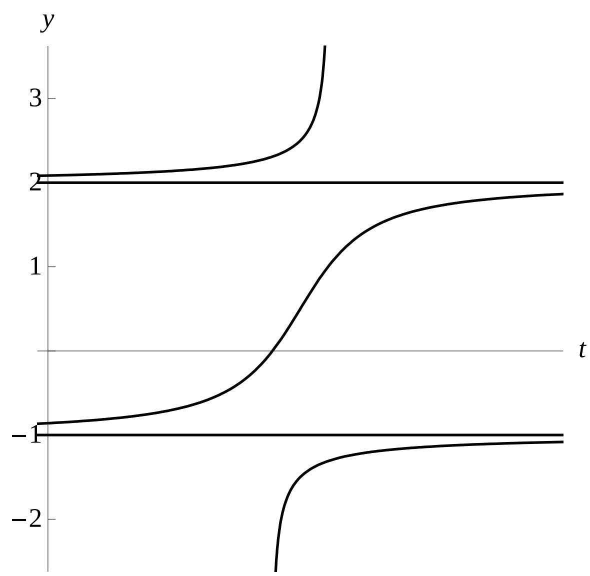

(e)



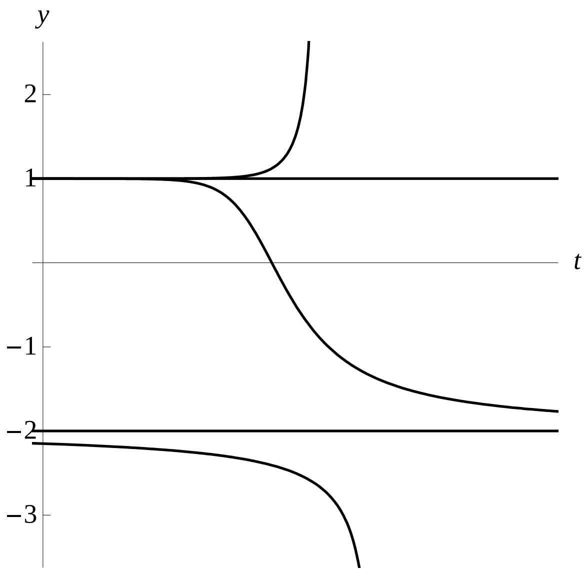
(f)




(g)



(h)



# Some Methods for Solving Differential Equations

1. Find the general solution of the differential equation $\dfrac{dM}{dt} = 2.4 - 0.2M$.

2. Solve the differential equation $\dfrac{dy}{dt} = -\dfrac{t}{y}$ using separation of variables. (Previously, we solved by drawing the slope field, guessing the solution, and checking that it worked by plugging into the differential equation.)

3. Decide whether each of the following differential equations is separable.

   (a) $\dfrac{dy}{dt} = t + y$.

   (b) $\dfrac{dy}{dt} = \dfrac{y}{\sin t}$.

   (c) $\dfrac{dy}{dt} = \dfrac{\sin t}{y} + t$.

4. In this problem, we'll solve the non-separable differential equation $y' + y = te^{-t}$ by using a substitution $u = e^t y$.

   (a) You are given a substitution $u(t) = e^t y(t)$; rewrite this equation to solve for $y(t)$.

   (b) Use your answer to (a) to rewrite the differential equation $y' + y = te^{-t}$ in terms of $u$ (and $t$).

   (c) Solve the differential equation you found in (b). (Find all solutions.)

   (d) Use your answer to the previous part to solve for $y(t)$.

   (e) Find the solution of $y' + y = te^{-t}$ satisfying the initial condition $y(0) = -1$.

5. Solve the differential equation $y' + 2xy^2 = 0$. (Find all solutions.)

6. Solve the differential equation $y' = \dfrac{y^2 + xy}{x^2}$ by using the substitution $u = \dfrac{y}{x}$.

7. Solve the differential equation $\dfrac{dy}{dt} = \dfrac{t^2}{y^2(1 + t^3)}$.

# Other Ways of Looking at Differential Equations

1. One way that psychologists study how people learn is by asking them to learn a list of nonsense words.[1] Suppose that $L(t)$ is the percent of such a list that a particular person has learned at time $t$, where $t$ is measured in days.

   (a) A simple model for learning is to assume that learning happens at a rate proportional to the percent of the list left to be learned. Write a differential equation that expresses this.

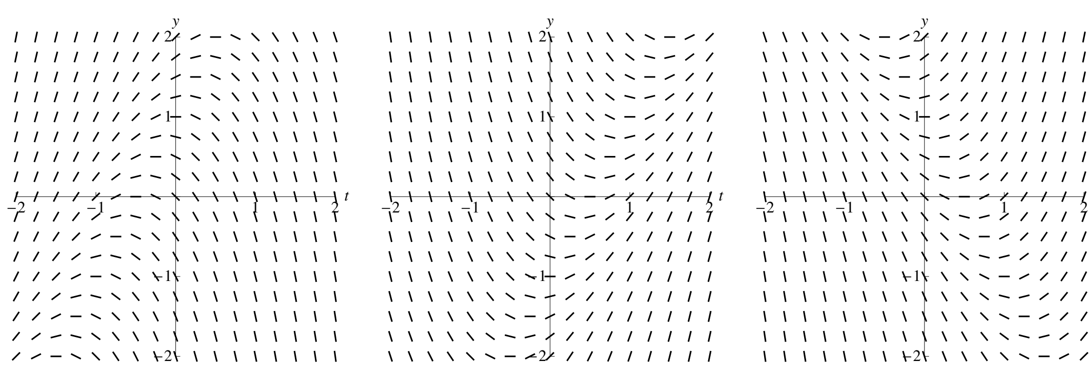
   (b) A student who has learned a fifth of the list already is currently learning at a rate of 10% of the list per day. Incorporate this into your model.

   (c) When this student just started learning the list (so knew none of it), at what rate was he learning?

   (d) How long does it take the student to go from knowing none of the list to knowing half of it?

2. In this problem, we'll look at the differential equation $\dfrac{dy}{dt} = 2t - y - 1$.

   (a) Which of the following is the slope field of $\dfrac{dy}{dt} = 2t - y - 1$?



(A) (B) (C)

[1] This is meant to be a "pure" experiment, in that the learners won't have any prior knowledge that would help them with the task.

(b) We're interested in the solution of $\dfrac{dy}{dt} = 2t - y - 1$ satisfying $y(0) = 1$. How might we approximate $y(2)$?

3. Let $f(x)$ be the solution of the initial value problem $y' = 2 - 8xy$, $y(0) = 1$. Use Euler's method with a step size of $\frac{1}{4}$ to approximate $f(1)$.

4. *Unfortunately, differential equations are often very difficult to solve; most methods only work for specific types of differential equations. Many differential equations end up being solved by a process of trial and error. One of the more common "methods" for solving a differential equation is to make a rough guess about the form of the solutions and then check that guess.*

   (a) Which of the following seems most likely to be a solution of the differential equation $y = y' + 3x^2$?

      i. Some sort of trigonometric function.

      ii. A polynomial of degree _______.

      iii. Some sort of exponential function.

   (b) Find all solutions of the form that you picked in (a).

5. According to some sources, the longest straight road in the U.S. is North Dakota Highway 46, which is 123 miles long. Imagine that a car is driving along ND 46; its acceleration at time $t$ (measured in hours) is given by $5 + 2t$ miles/hr$^2$.

   (a) If $s(t)$ gives the car's position along ND 46 at time $t$, write a differential equation for $s(t)$.

   (b) How many initial conditions do you think are needed to determine $s(t)$? What might they represent?

   (c) Find the general solution of the differential equation you wrote down in (a).

   (d) How many constants did your general solution have? How is this related to your answer to (b)?

6. *Getting information from a differential equation.* Suppose that $y = f(x)$ is a solution to the differential equation $y'' + y' = -x^2$. Why can't the the graph of $f(x)$ ever be both increasing and concave up?

7. In this problem, you'll find all solutions of the differential equation $y' = 4y - \sin t$ by using a combination of guesswork and substitution.

   (a) It seems reasonable to think that the differential equation might have a solution involving $\sin t$ and $\cos t$. Find all solutions of the differential equation of the form $a \sin t + b \cos t$. (There is one.)

   (b) Let $f(t)$ be the function you found in (a). Use the substitution $u = y - f(t)$ to solve the differential equation $y' = 4y - \sin t$.

# Second-Order Homogeneous Differential Equations with Constant Coefficients

1. Can you guess any solutions of $x'' - x = 0$? (It may help to think of the equation as $x'' = x$.) Guess as many solutions as you can, and check that your guesses are correct.

2. Can you guess any solutions of $x'' - 4x = 0$? (It may help to think of the equation as $x'' = 4x$.) Guess as many solutions as you can, and check that your guesses are correct.

3. In this problem, we'll look at the differential equation $x'' + 3x' + 2x = 0$.

   (a) Find all solutions of the differential equation of the form $x(t) = e^{rt}$. (There are two.)

   (b) Let $f(t)$ and $g(t)$ be the solutions you found in (a). Show that $C_1 f(t) + C_2 g(t)$ is also a solution of $x'' + 3x' + 2x = 0$.

   (c) The differential equation $x'' + 3x' + 2x = 0$ models the motion of a vibrating spring (why?). Explain, in terms of springs, why there are infinitely many solutions satisfying the initial condition $x(0) = 3$.

   (d) Explain, in terms of springs, why there is exactly one solution of $x'' + 3x' + 2x = 0$ satisfying the initial conditions $x(0) = 3$ and $x'(0) = -2$. What is this solution?

(e) Based on your reasoning above, do you think $C_1 f(t) + C_2 g(t)$ from (b) is the most general solution, or should there be even more solutions? Explain.

(f) In fact, we can use a clever substitution to solve the differential equation completely. Let $f(t)$ be one of the solutions you found in (a). First, use the substitution $u(t) = \frac{x(t)}{f(t)}$ to rewrite the differential equation $x'' + 3x' + 2x = 0$ as a differential equation about $u(t)$. Then, use the substitution $v(t) = u'(t)$ to solve the differential equation completely.

4. Solve the differential equation $y'' + 2y' - 15y = 0$.

5. Solve $x'' + 4x' + 4x = 0$.

6. Solve $x'' = 0$.

# $x'' + bx' + cx = 0$, Euler's Formula

1. Express each of the following complex numbers in the form $a+bi$, and show its position in the complex plane.

   (a) $e^{i\pi}$

   (b) $3e^{i\pi/4}$

   (c) $5e^{i\pi/3}$

   (d) $re^{i\theta}$ where $r > 0$

2. *Euler's formula* $e^{i\theta} = \cos\theta + i\sin\theta$ *is useful for many things besides solving differential equations. For example, this formula makes it very easy to derive some trigonometric identities.*

   If $A$ is any angle, then exponent rules tell us that

   $$e^{iA}e^{iA} = e^{2iA}.$$

   Use this, together with Euler's formula, to get formulas for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$.

3. In this problem, we'll look at the differential equation $x'' + x = 0$.

   (a) Does this model a spring? If so, does it model a spring with or without friction?

(b) Find the solution of $x'' + x = 0$ satisfying the initial conditions $x(0) = 1$, $x'(0) = 1$. (You may express your answer using complex numbers.)

(c) Should $x(t)$ be real or non-real?

(d) Use Euler's formula to rewrite $x(t)$.

(e) Does your solution seem consistent with your answer to (a)?

4. (a) Suppose a differential equation $x'' + bx' + cx = 0$ has general solution $C_1e^{2t} + C_2te^{2t}$. Could the differential equation model a spring?

(b) Think of a differential equation whose general solution is $C_1e^{2t} + C_2te^{2t}$.

# $x'' + bx' + cx = 0$, Concluded

1. Solve $x'' + 6x' + 13x = 0$ with initial conditions $x(0) = 7$ and $x'(0) = -1$.

2. Solve $x'' - 2x' + 5x = 0$. (Give the general solution.)

3. Solve $x'' + 9x = 0$.

4. (a) Solve $x'' + 8x' + 17x = 0$ with initial conditions $x(0) = 1$ and $x'(0) = 0$.

   (b) Interpret (a) in terms of a spring. What is happening to the spring as time goes on?

5. In each part, decide whether the differential equation has periodic solutions. If so, what is the period?

   (a) $x'' + 2x' - 3x = 0$.

   (b) $x'' + 2x' + 3x = 0$.

   (c) $x'' + 4 = 0$.

   (d) $x'' + 4x' = 0$.

   (e) $x'' + 4x = 0$.

   (f) $x'' + 4t = 0$.

   (g) $x'' - 4x = 0$.

   Does this agree with your interpretation of the differential equations in terms of springs? (Be careful! Not all of the differential equations are of the form we've been discussing, and even the ones that are cannot necessarily be interpreted in terms of springs.)

# Introduction to Systems of Differential Equations

1. Steve Strogatz, now a professor at Cornell, came up with the following system to model the relationship between Romeo and Juliet.

   Let $j(t)$ be a measure of Juliet's feelings for Romeo at time $t$, and let $r(t)$ be a measure of Romeo's feelings for Juliet, with $t = 0$ being the time of the Capulet ball where Romeo and Juliet met. Positive values indicate love, and negative values indicate hate. Then, (according to Steve Strogatz, at least), the following system models the relationship between Romeo and Juliet:

$$\begin{cases} \dfrac{dj}{dt} = -r \\ \dfrac{dr}{dt} = j \end{cases}$$

   (a) According to this model, which of the following statements are true?

      i. The more Romeo loves Juliet, the more strongly Juliet feels for him.

      ii. The more Juliet loves Romeo, the more strongly Romeo falls for her.

      iii. Juliet finds Romeo most attractive when he loves her least. When he loves her most, she turns away.

   (b) One of the following is the general solution of the system; which one?

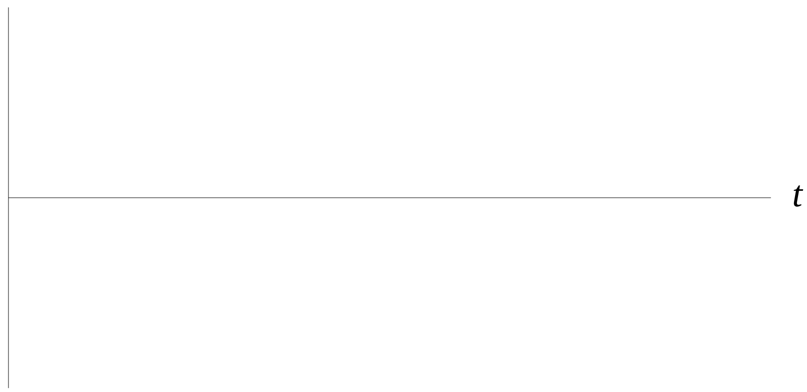
      i. $j(t) = -C_1 e^{-t} + C_2 e^{t}$ and $r(t) = C_1 e^{-t} + C_2 e^{t}$.

      ii. $j(t) = -C_1 \sin t + C_2 \cos t$ and $r(t) = C_1 \cos t + C_2 \sin t$.

      iii. $j(t) = C_1 \sin t + C_2 \cos t$ and $r(t) = C_1 \cos t + C_2 \sin t$.

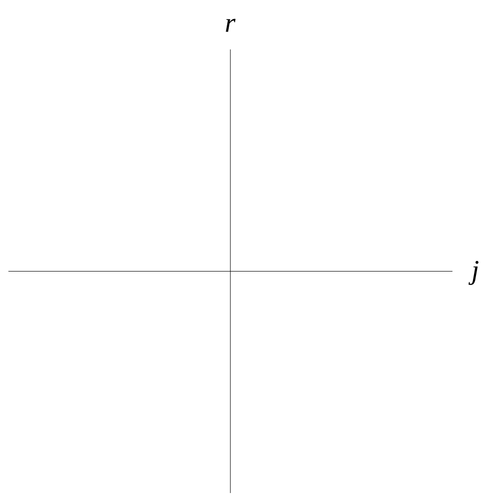
   (c) Find the particular solution satisfying $j(0) = 0$ and $r(0) = 1$.

(d) For the particular solution you found in (c), graph $j(t)$ and $r(t)$ on the following set of axes:




Describe Romeo and Juliet's relationship over time (using words, not math).

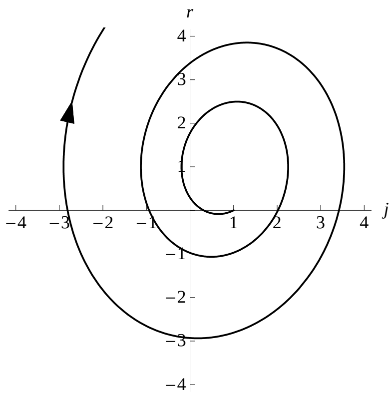
(e) Sketch the solution you found in (c) in the phase plane.




(f) Suppose that fate prevented Romeo from attending the Capulet ball. What initial condition could you then write for the system? Find the particular solution satisfying the initial condition. Graph it like you did in (d). Then, sketch the corresponding trajectory on your picture from (e).

(g) Others have suggested improvements to Strogatz's model. Suppose that the following is determined to be a more accurate solution trajectory. Sketch possible graphs of $r(t)$ and $j(t)$ on a single set of axes (like in (d)). (Label any times you think are important on both the trajectory and your graph.) What happens to Romeo and Juliet's relationship over time?




2. A large savannah is populated primarily by lions and gazelles. Let $L(t)$ be the population of lions at time $t$ and $G(t)$ be the population of gazelles at time $t$. Suppose that these satisfy the system

$$\begin{cases} \dfrac{dG}{dt} = 0.01G\left(1 - \dfrac{G}{6000}\right)\left(\dfrac{G}{2000} - 1\right) - 0.0001GL = -\dfrac{G\left[(G-2000)(G-6000) + 120000L\right]}{1200000000} \\ \dfrac{dL}{dt} = -0.1L + 0.00002GL = \dfrac{(G-5000)L}{50000} \end{cases}$$

(a) In the absence of gazelles, what does the model predict about the population of lions?

(b) In the absence of lions, what does the model predict about the population of gazelles?

(c) Find all equilibrium points of this system.

(d) In the phase plane, draw the equilibrium points and the solution trajectories corresponding to the situations in (a) and (b).

# Phase Plane Analysis

1. Mrs. Portwick moves from England to an uninhabited island. She plants a garden and decides that the two things missing in her garden are robins and worms. She decides to import some. Let $w(t)$ represent the number of hundreds of worms on the island at time $t$ and $r(t)$ be the number of hundreds of robins on the island at time $t$. She models their interaction as follows:

$$\frac{dr}{dt} = -0.4r + 0.1wr$$
$$\frac{dw}{dt} = 0.3w - 0.3wr$$

   Analyze the system.

2. On your phase plane analysis from #1, sketch trajectories with the following initial conditions:
   (a) $r(0) = 0$, $w(0) = 2$.
   (b) $r(0) = 2$, $w(0) = 0$.

3. Mrs. Portwick's nephew Edwin comes to visit, sees her work, and says that he believes that the second differential equation should be $\frac{dw}{dt} = 0.3w - 0.03w^2 - 0.3wr$.
   (a) What could his reasoning be?
   (b) Analyze the new system.

4. On your phase plane analysis from #3, sketch trajectories with the following initial conditions:
   (a) $r(0) = 0$, $w(0) = 2$.
   (b) $r(0) = 2$, $w(0) = 0$.
   (c) $r(0) = 0$, $w(0) = 12$.

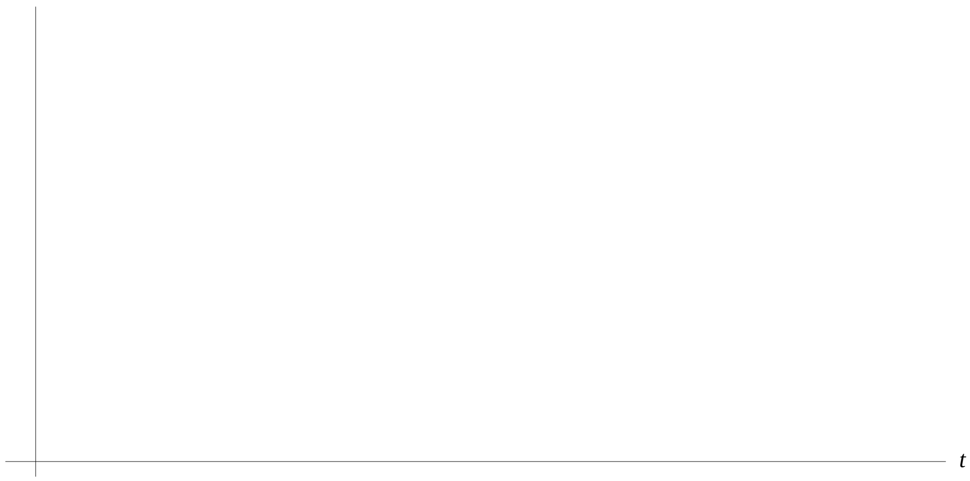
For the following questions, assume that we are modeling Mrs. Portwick's island using the model in #1.

5. *It turns out that the solution trajectories of the system in #1 are closed loops. (This is **NOT** something we can tell from the phase plane analysis.)*

   Suppose that, at some time that we'll call $t = 0$, Mrs. Portwick's garden has 200 robins ($r = 2$) and 400 worms ($w = 4$).

   (a) Describe in words what happens over time.

   (b) On the following set of axes, draw possible graphs of $r(t)$ and $w(t)$.




6. One day, Mrs. Portwick asks her butler Godfrey to do a quick survey of the garden. Godfrey finds that there are 150 robins and 300 worms. Mrs. Portwick feels that the garden has too many worms, so she asks Godfrey to take 50 of them to use as fishing bait. What effect does this have on the robin and worm populations in the long run?

# Systems and Shapes of Trajectories

1. (a) (Problem Set 33, #1) Let $x(t)$ be the number of hundreds of beasts of species X at time $t$ and $y(t)$ be the number of hundreds of beasts of species Y at time $t$. Do a phase plane analysis of the system

$$\begin{cases} \dfrac{dx}{dt} = 0.1x - 0.025xy & = 0.025(4x - xy) \\ \dfrac{dy}{dt} = 0.1y - 0.02xy - 0.01y^2 & = 0.01(10y - 2xy - y^2) \end{cases}$$

   (b) Using only your sketch from (a), which of the following questions can you answer?

      i. If there are currently 400 X beasts and 200 Y beasts, will the population of Y beasts ever reach 300?

      ii. If there are currently 500 X beasts and 600 Y beasts, will the population of X beasts ever fall below 300?

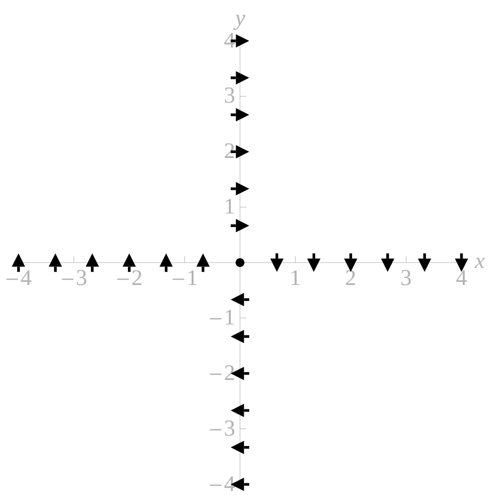
      iii. If $x(0) = 2$ and $y(0) = 5$, what are $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$?

      iv. If $x(0) = 1$ and $y(0) = 1$, what are $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$?

2. In this problem, we'll look at the system $\dfrac{dx}{dt} = y$, $\dfrac{dy}{dt} = -4x$.

   (a) By writing and solving a differential equation for $y$ in terms of $x$, determine the exact shape of the solution trajectories (in the phase plane) of this system.

   (b) The following diagram shows the sketch you get by doing a phase plane analysis of this system. Using this together with your answer to (a), sketch a phase portrait of the system. (This means: Sketch several representative solution trajectories in the phase plane.)




3. In this problem, we'll look at the system $\dfrac{dx}{dt} = y$ and $\dfrac{dy}{dt} = -3y$.

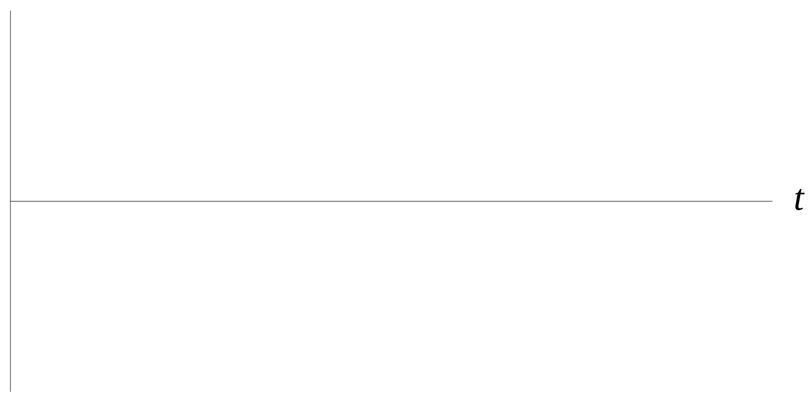
   (a) Find the shape of the solution trajectories.

   (b) Do a qualitative phase plane analysis of this system.

(c) Using your answers to (a) and (b), draw a phase portrait of the system $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -3y$.

(d) Let's focus on one particular solution, the one satisfying the initial conditions $x(0) = 0$ and $y(0) = 6$. What is the exact shape of the solution trajectory with these initial conditions? (Write an equation which describes the shape.)

(e) On the following set of axes, sketch the graphs of $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = 6$.




(f) In fact, you know enough to solve the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -3y$. (Hint: First solve for $y(t)$.) Find the general solution.

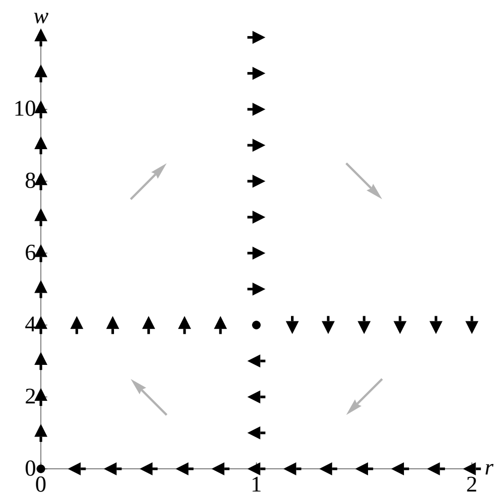
(g) Find the solution of the system satisfying $x(0) = 0$ and $y(0) = 6$. Does your answer match the graphs you sketched in (e)?

4. On the worksheet "Phase Plane Analysis", we used phase plane analysis to understand the system

$$\begin{cases} \dfrac{dr}{dt} = -0.1r(4-w) \\ \dfrac{dw}{dt} = 0.3w(1-r) \end{cases}$$

where $w(t)$ represented the number of hundreds of worms on an island at time $t$ and $r(t)$ represented the number of hundreds of robins on the island at time $t$. The sketch we got is below. The solution trajectories are closed curves (although we didn't show that).

Suppose that there are currently 100 robins and 200 worms on the island. What is the largest number of robins there will ever be on the island?



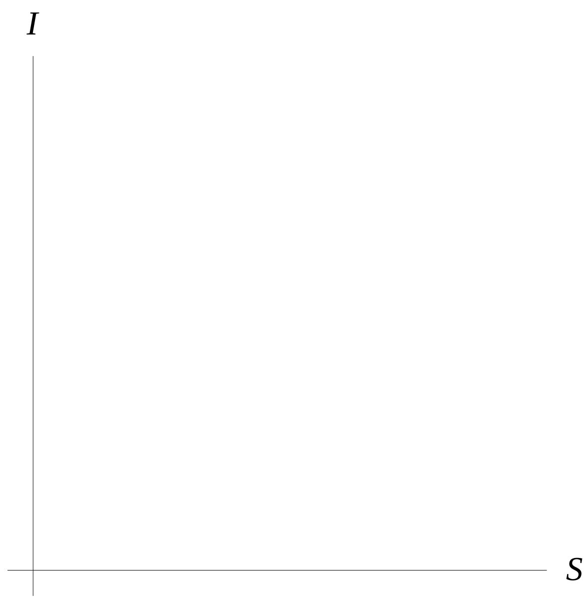
# Systems Wrap-Up

In Weekly Problem #32, you were given the following scenario. Suppose that there is a large population of people, and some of the people have a fatal disease. This disease is infectious, so anybody who doesn't have the disease is susceptible to getting it. Let $I(t)$ be the number of people infected at time $t$, and let $S(t)$ be the number of people who are susceptible at time $t$. You may ignore birth and death, except for death due to the disease, which you should include.

A reasonable system for the situation described is:

$$\begin{cases} \dfrac{dS}{dt} = -0.001IS \qquad\quad = -0.001IS \\ \dfrac{dI}{dt} = 0.001IS - 0.1I = 0.001I(S-100) \end{cases}$$

1. Do a phase plane analysis for this system.

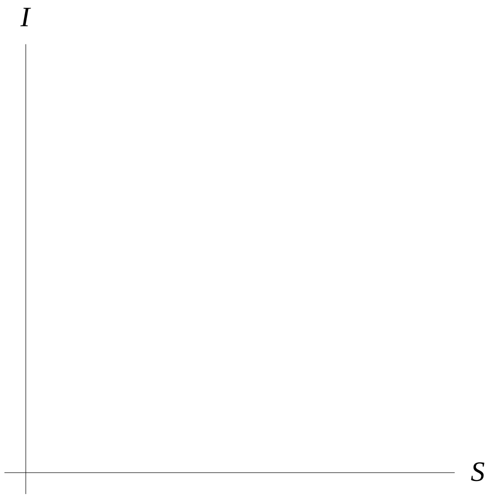



2. If the population starts with 50 infected people and 200 susceptible people, what will happen in the long run? Do $\lim_{t\to\infty} I(t)$ and $\lim_{t\to\infty} S(t)$ exist? If so, what are they?

3. Suppose you instead want to model an epidemic like measles, which is not fatal but confers immunity on people who've had it. How might you reasonably model such an epidemic?

In the next few problems, we'll look at the system $\dfrac{dS}{dt} = -IS + 50, \dfrac{dI}{dt} = IS - 10I$.

4. Do a qualitative phase plane analysis of this system.




5. Based on your phase plane analysis, what do you think the trajectories look like? Sketch a possible trajectory on your diagram if $S(0) = 5$ and $I(0) = 20$.

6. Using your trajectory, sketch a possible graph of $I(t)$ if $S(0) = 5$ and $I(0) = 20$.

7. This graph shows the number of cases of measles in 2 week periods in London from 1944 to 1966. (See `http://www.zoo.cam.ac.uk/zoostaff/grenfell/measles.htm` for the raw data and more information about it.) Does our system give the same qualitative behavior?