MATH 1060: Exam 3 Review Sheet

Useful Trig Identities

- $\csc(x) = \frac{1}{\sin(x)}$
- $\sec(x) = \frac{1}{\cos(x)}$
- $\cot(x) = \frac{1}{\tan(x)}$
- $\sin^2(x) + \cos^2(x) = 1$
- $\tan(x) = \frac{\sin(x)}{\cos(x)}$

Useful Trig Derivatives

- $\frac{d}{dx}(\sin(x)) = \cos(x)$
- $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
- $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
- $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
- $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

Useful Inverse Trig Derivatives

- $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left[\cos^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$
- $\frac{d}{dx}\left[\csc^{-1} x\right] = \frac{-1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}$

Note: Do not forget CHAIN RULE!! You must multiply by the derivative of the inside!
i.e.

$$\frac{d}{dx}\left[\sin^{-1}(g(x))\right] = \frac{1}{\sqrt{1-g(x)^2}} \cdot g'(x)$$

4.1: Maxima and Minima

abs max: at c if $f(c) \geq f(x)$ for all x in domain
abs min: at c if $f(x) \leq f(x)$ for all x in domain
local max: at c if $f(c) \geq f(x)$ when x is near c
local min: at c if $f(x) \leq f(x)$ when x is near c
Extreme Value Theorem: A continuous function on $[a,b]$ has an absolute max and an absolute min on $[a,b]$.
Closed Interval Method: (finding absolute extrema)

1. Find the critical points of f
2. Evaluate f at the critical points **AND** the endpoints
3. largest function value = abs max
   smallest function value = abs min

4.2: Mean Value Theorem

Rolle's Theorem: If $y = f(x)$ is

i) continuous on $[a,b]$

ii) differentiable on (a,b)

iii) $f(a) = f(b)$

then there is a number c in (a,b) such that $f'(c) = 0$.

Mean Value Theorem: If $y = f(x)$ is:

i) continuous on $[a,b]$

ii) differentiable on (a,b)

then there is a number c in (a,b) such that
$f'(c) = \frac{f(b)-f(a)}{b-a}$

4.3: 1st Derivatives and Shapes of a Graphs

Remember:

- If $f'(x) > 0$, then $f(x)$ is increasing.
- If $f'(x) < 0$, then $f(x)$ is decreasing

Increasing and Decreasing Test:

1. Find x values where either
   - $f'(x) = 0$
   - $f'(x)$ DNE
2. Make a sign chart!
3. Determine:
   - $f'(x) > 0 \implies f$ increasing
   - $f'(x) < 0 \implies f$ decreasing

local minimum: If $f'(x)$ changes from $-$ to $+$ at a critical point c, then $f(x)$ has a local minimum at $x = c$.
local maximum: If $f'(x)$ changes from $+$ to $-$ at a critical point c, then $f(x)$ has a local maximum at $x = c$
One Local $\implies$ Absolute: Suppose f is continuous one an interval that contains exactly one local extremum at c.

- If a local min occurs at c, then $f(c)$ is the absolute min of f on the interval.
- If a local max occurs at c, then $f(c)$ is the absolute max of f on the interval.

4.3: 2nd Derivatives and Shapes of a Graphs

Remember:

- If $f''(x) > 0$, then $f(x)$ is concave UP.
- If $f''(x) < 0$, then $f(x)$ is concave DOWN.

Concavity Test:

1. Find x values where
   - $f''(x) = 0$
   - $f''(x)$ DNE
2. Make a sign chart!
3. Determine:
   - $f''(x) > 0 \implies f(x)$ concave UP
   - $f''(x) < 0 \implies f(x)$ concave DOWN

Inflection Point: a point on the graph where the concavity CHANGES

Second Derivative Test: Suppose that $f''(x)$ is continuous near $x = c$ with $f'(c) = 0$. Then,

- $f''(c) > 0 \implies f(x)$ has a local min at $x = c$.
- $f''(c) < 0 \implies f(x)$ has a local max at $x = c$
- $f''(c) = 0 \implies$ inconclusive

4.4: Graphing Functions

1. Identify domain
2. Find intercepts
   - y-intercept: set $x = 0$ and solve for y
   - x-intercept: set $y = 0$ and solve for x
3. Check for Symmetry
   - y-axis: $f(-x) = f(x)$
   - origin: $f(-x) = -f(x)$
   - periodic: $\sin(x), \cos(x)$, etc.
4. Asymptotes
   - horizontal: if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$
   - vertical: points where denominator $= 0$.
   - slant: If degree of numerator is one more than degree of denominator, use long division to find quotient.
5. Increasing/Decreasing:
   - Find $f'(x)$ and make sign chart
6. Find Local Extrema
7. Find Concavity and Inflection Points
8. Sketch Graph!

4.5: Optimization

1. Define all variables.
2. Draw a picture!
3. State function to be optimized.
4. State constraints.
5. Write function as one variable (use constraints).
6. Find domain of function.
7. Find absolute extrema.
8. Verify absolute extrema.
9. Write a sentence!

4.6: Linearization and Differentials

Linearization: $L(x) = f(a) + f'(a)(x - a)$

1. Choose a value of a to produce a small error
2. Find $f(a)$, $f'(x)$, and $f'(a)$
3. Create $L(x)$ using formula
4. Plug in the quantity you want to calculate into $L(x)$

Differential: $dy = f'(a)dx$

1. Take derivative of given function
2. Plug in known values

4.7: L'Hopital's Rule

Indeterminate Forms for L'Hopital's: $\frac{0}{0}$, $\frac{\infty}{\infty}$

L'Hopital's Rule: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

- Apply when the limit gives you $\frac{0}{0}$ or $\frac{\infty}{\infty}$
- Take derivative of numerator and denominator SEPARATELY (no quotient rule!)

Related Indeterminate Forms: $\infty \cdot 0$, $\infty - \infty$

- For these indeterminate forms, try to use algebra to evaluate the limits.

Indeterminate Powers: 1^∞, 0^0, ∞^0

- Use e and $\ln(x)$ to change the form of the function:
$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln(f(x))}$$

1. Evaluate $\lim_{x\to a} \ln(f(x)) = L$
2. Exponentiate Step 1: $\lim_{x\to a} f(x) = e^L$

4.9: Antiderivatives

Antiderivative: F is the antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.
Theorem: If F is an antiderivative of f, then the most general antiderivative of f is $F(x) + C$
Indefinite Integral:

$$\int f(x)dx = F(x) + C$$

Power Rule: If exponent $\neq 1$:

1. Add 1 to exponent
2. Divide by new exponent

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Constant Multiple Rule:

$$\int af(x)dx = a\int f(x)dx$$

Sum Rule:

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

Integral of $\frac{1}{x}$:

$$\int \frac{1}{x}dx = \ln|x| + C; x \neq 0$$

Initial Value Problem:

1. Find antiderivative (don't forget $+C$!)
2. Plug in given x value in antiderivative and set equal to given function value
3. Solve for C.
4. Write final solution with the value found for C.

Rectilinear Motion:

- $v(t) = \int a(t)dt$
- $s(t) = \int v(t)dt$
- Initial Conditions: $s(0)$ and $v(0)$
- $g = 9.8m/s^2$ or $g = 32ft/s^2$

5.1: Areas, Distances, Riemann Sum

Displacement: displacement= velocity $\times$ time

- If velocity is alway positive, displacement is the distance traveled.
- Find displacement by finding the area under the curve of velocity function.

Approximating Areas by Riemann Sums

1. Divide interval $[a, b]$ into n subintervals of equal length.
   - $x_0 = a$ and $x_n = b$
   - Length of each subinterval: $\Delta x = \frac{b-a}{n}$
2. Choose a point in each subinterval, x_k^*, and make a rectangle whose height is the function evaluated at that point, $f(x_k^*)$.
   - We usually choose x_k^* as left endpoint, right endpoint, or midpoint.
     – Left: $x_k^* = a + (k-1)\Delta x$
     – Right: $x_k^* = a + k\Delta x$
     – Midpoint: $\overline{x_k} = a + (k - \frac{1}{2})\Delta x$
   - area of k^{th} rectangle: $f(x_k^*) \cdot \Delta x$
3. Add together all the areas of the n rectangles.
   - $\sum\limits_{k=1}^{n} f(x_k^*)\Delta x =$
     $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$

Common Sum Rules:

- $\sum\limits_{k=1}^{n} ca_k = c\sum\limits_{k=1}^{n} a_k$
- $\sum\limits_{k=1}^{n} (a_k + b_k) = \sum\limits_{k=1}^{n} a_k + \sum\limits_{k=1}^{n} b_k$
- $\sum\limits_{k=1}^{n} c = n \cdot c$

5.2: The Definite Integral

Net Area: the net area of the region bounded by a continuous function f and the x-axis between $x = a$ and $x = b$ is:

Area Above x-axis - Area Below x-axis

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Total Area: the total area of the region bounded by a continuous function f and the x-axis between $x = a$ and $x = b$ is:

Area Above x-axis + Area Below x-axis

$$\int_a^b |f(x)|dx$$

Properties of Definite Integrals:

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for a constant k
- $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b$
- $\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$

5.2: Definite Integrals with Riemann Sums

Calculate Definite Integral Using Riemann Sums:

1. Find $\Delta x = \frac{b-a}{n}$
2. Find an expression for the right endpoint /left endpoint/midpoint of the k^{th} subinterval
   - Left: $x_k = a + (k-1)\Delta x$
   - Right: $x_k = a + k\Delta x$
   - Midpoint: $x_k = a + (k - \frac{1}{2})\Delta x$
3. Find $f(x_k)$ by plugging in what you found for x_k everywhere there is an x in the function.
4. State the right/left/midpoint Riemann Sum, $\sum_{k=1}^{n} f(x_k)\Delta x$. This sum should be in terms of k and n.
5. Simplify the Riemann Sum using sum formulas given. The final answer should only be in terms of n.
6. Find the exact value of the definite integral by taking the limit of the simplified Riemann Sum

Sum Formulas: (will be given)

- $\sum\limits_{k=1}^{n} c = cn$
- $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

5.3: The Fundamental Theorem of Calculus

FTOC Part I: If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)dt$$

for $a \le x \le b$ is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$:

$$A'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Derivative of Integrals: If the lower limit is a constant and the upper limit is a function of x, use chain rule along with FTOC Part I:

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right) = f(g(x))g'(x)$$

FTOC Part II: If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Total Area Revisited:

1. Find x values where $f(x) = 0$ [x-intercepts].
2. Divide into subintervals using the x-intercepts.
3. Integrate f over each subinterval and add the absolute values.