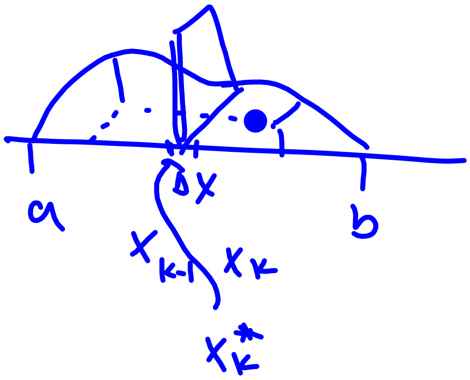
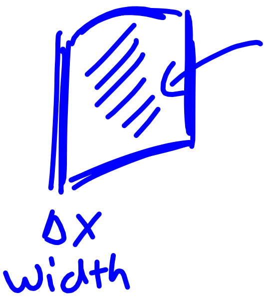
Section 6.2: Volumes

General slicing method



slice perp. to x-axis



some area $A(x_k^*)$

$$\text{Volume of a slice} = \Delta x \, A(x_k^*)$$

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} A(x_k^*) \Delta x$$

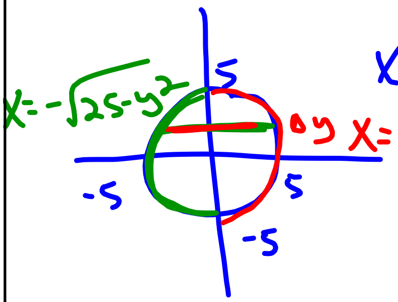
$$V = \int_a^b A(x)\,dx$$

cross sections perp. to x-axis of area $A(x)$

$$V = \int_c^d A(y)\,dy$$

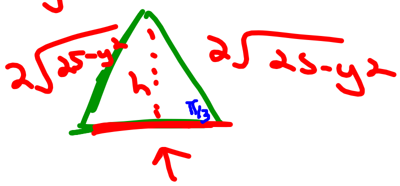
cross sections perp. to y-axis area $A(y)$

Ex. Find the vol. of the solid with circular base of radius 5 whose cross sections perp. to the base and parallel to x-axis are equilateral triangles.



$x^2+y^2=25$

$V=\int_{-5}^{5} A(y)\,dy$



$\sin\frac{\pi}{3} = \frac{h}{2\sqrt{25-y^2}}$

$\frac{\sqrt{3}}{2} = \frac{h}{2\sqrt{25-y^2}}$

$\sqrt{3}\sqrt{25-y^2} = h$

$x^2 = 25-y^2$

$x = \pm\sqrt{25-y^2}$

length $= \sqrt{25-y^2} - -\sqrt{25-y^2}$

$= 2\sqrt{25-y^2}$

base

$\tan \pi/3 = \frac{h}{\sqrt{25-y^2}}$

$h = \sqrt{3}\sqrt{25-y^2}$

$\sin \pi/3 = \frac{\sqrt{3}}{2}$

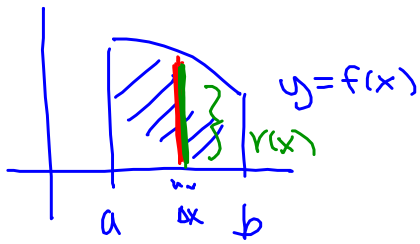
$\cos \pi/3 = \frac{1}{2}$

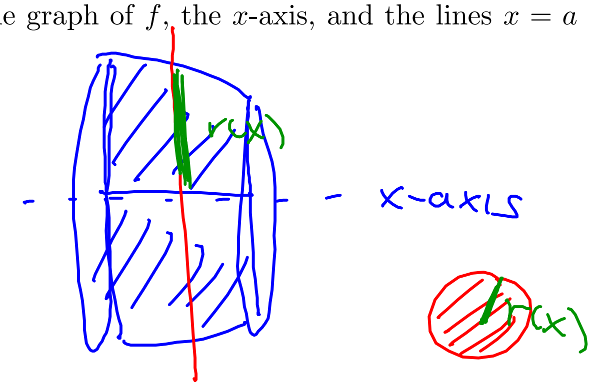
$$V = \int_{-5}^{5} \frac{1}{\cancel{2}} \overbrace{\cancel{2}\sqrt{25-y^2}}^{\text{base}}\ \overbrace{\sqrt{3}\sqrt{25-y^2}}^{\text{height}}\,dy$$

$$= 2\sqrt{3}\int_0^5 (25-y^2)\,dy = 2\sqrt{3}\left(25y - \frac{y^3}{3}\right)\Big|_0^5$$

$$= \cdot = \frac{500\sqrt{3}}{3}$$

A specific type of solid we will work with is a **solid of revolution.** Suppose f is a continuous function on $[a, b]$ and R is the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$.





Goal: Find the volume of the solid generated by revolving the region R about the x-axis.

Idea: Think about slices. cross sections are circular disks

→ slicing perpendicular to axis of rotation

$$V = \int_a^b A(x)\,dx$$

area of a circle $= \pi R^2$

$$V = \int_a^b \pi (r(x))^2\,dx$$

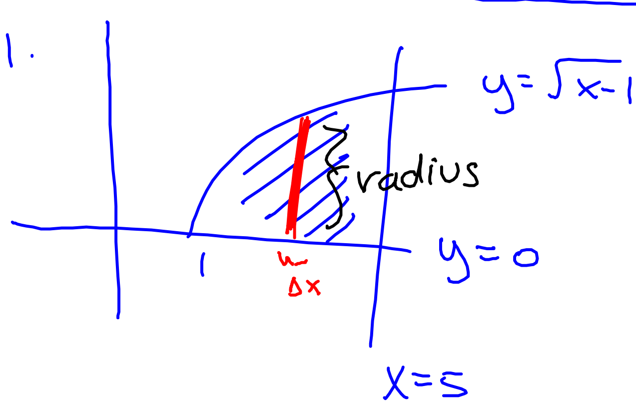
This method of finding vol. of a solid of rev is the disk method.

Example: Let R be the region bounded by $y = \sqrt{x-1}$, $y = 0$, and $x = 5$. Find the volume of the solid generated when R is rotated about the x-axis.

Disk method

Slice perp. to the x-axis

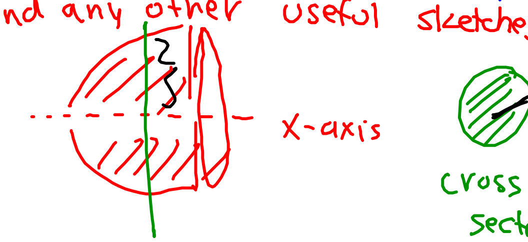
1.



2. Sketch a representative rectangle.

(don't skip this step!)

(and any other useful sketches)



3. identify radius $= \sqrt{x-1} - 0$

4, 5 set up integral and evaluate.

$$V = \int_1^5 \pi \left(\sqrt{x-1}\right)^2 dx = \pi \int_1^5 (x-1)\,dx$$

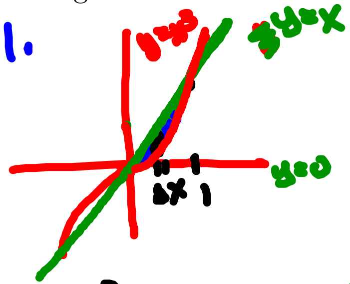
$$= \pi \left[\frac{x^2}{2} - x\right]\Big|_1^5 = \cdots = 8\pi$$

6. $\boxed{V = 8\pi}$

Example: Let R be the region bounded by $y = x^3$, $y = x$, and $x \geq 0$. Find the volume of the solid generated when R is rotated about the x-axis.

Disk Method

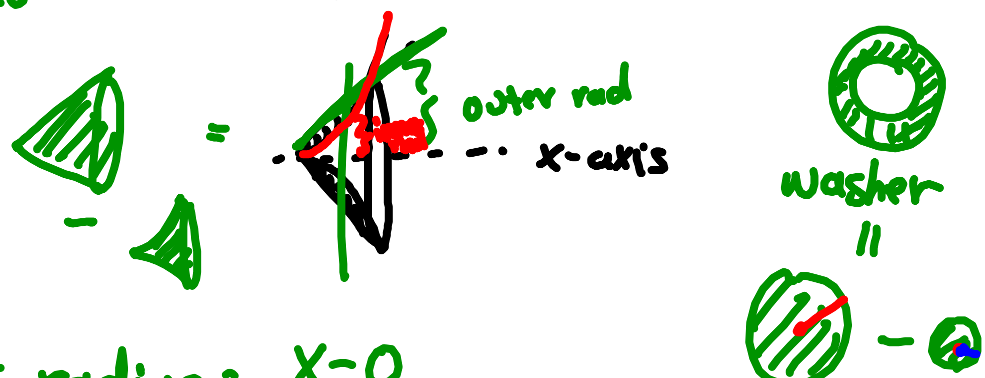
1.



2. Draw the rep. rect. perp. to the x-axis

$x^3 = x$

$x = 0, 1, -1$



3. Outer radius $= x - 0$

inner radius $= x^3 - 0$

4.,5. $V = \int_0^1 (\pi R^2 - \pi r^2)\,dx$

$= \int_0^1 \pi\left[(\text{outer rad})^2 - (\text{inner rad})^2\right]dx$

$= \int_0^1 \pi\left[x^2 - (x^3)^2\right]dx$

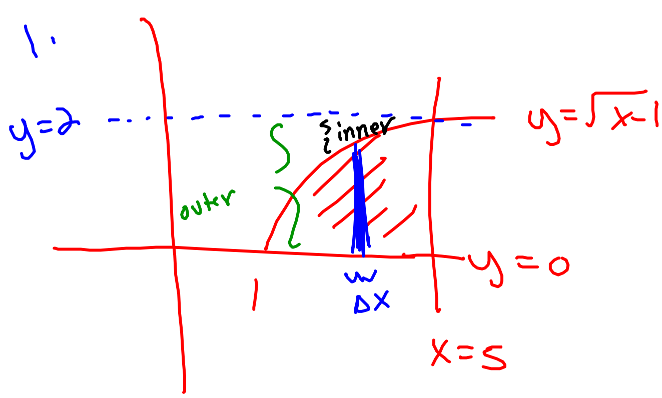
$= \pi\int_0^1 (x^2 - x^6)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^7}{7}\right]\Big|_0^1 \ldots = \frac{4\pi}{21}$

6. $V = \frac{4\pi}{21}$

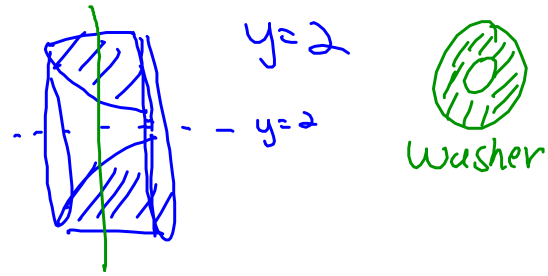
Example: Let R be the region bounded by $y = \sqrt{x-1}$, $y = 0$, and $x = 5$. Set up the integral(s) needed to find the volume of the solid generated when R is rotated about the line $y = 2$.

horizontal line

1.



2. slice perp. to $y=2$



3. outer rad $= 2 - 0$

inner rad $= 2 - \sqrt{x-1}$

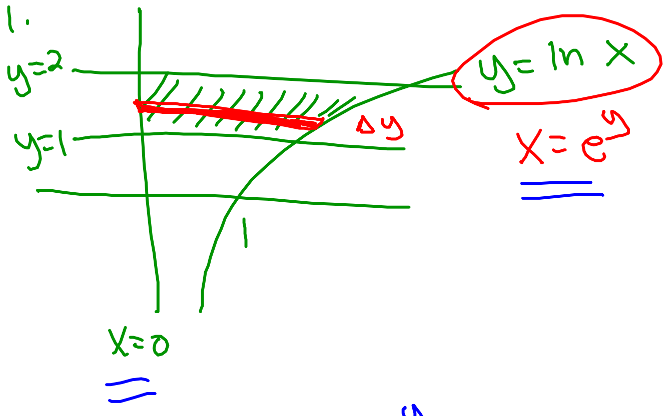
4. set up

$V = \int_1^5 \pi\left[2^2 - (2 - \sqrt{x-1})^2\right]dx$

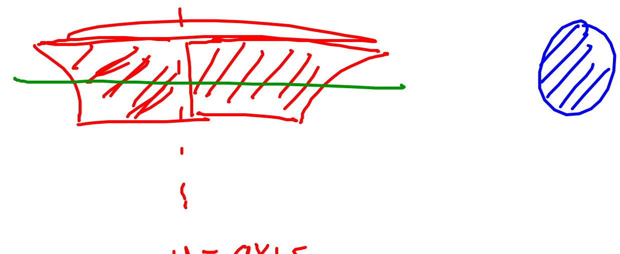
Example: Let R be the region bounded by $y = \ln x$, $y = 1$, $y = 2$, and $x = 0$.

(a) Find the volume of the solid generated when R is rotated about the y-axis. Disk method

1.



2. Draw a rep. rect. slice perp. to y-axis



3. radius $= e^y - 0$

4.,5. $V = \int_1^2 \pi (e^y)^2 \, dy$

$= \pi \int_1^2 e^{2y} \, dy = \frac{\pi}{2} e^{2y} \Big|_1^2 = \frac{\pi}{2}(e^4 - e^2)$

6. $V = \frac{\pi}{2}(e^4 - e^2)$

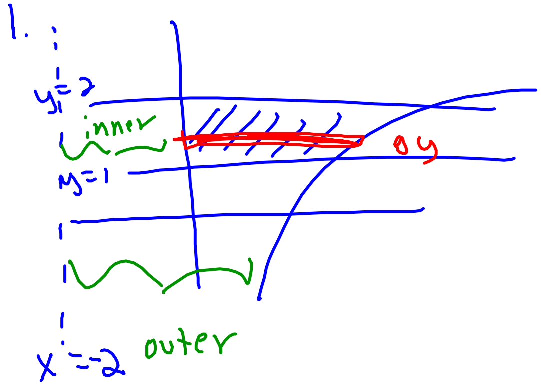
OR Let $u = 2y$
$du = 2dy$
$\frac{1}{2} du = dy$

$V = \pi \int_2^4 \frac{1}{2} e^u \, du$
$= \frac{\pi}{2} \int_2^4 e^u \, du = \frac{\pi}{2} e^u \Big|_2^4$
$= \frac{\pi}{2}(e^4 - e^2)$

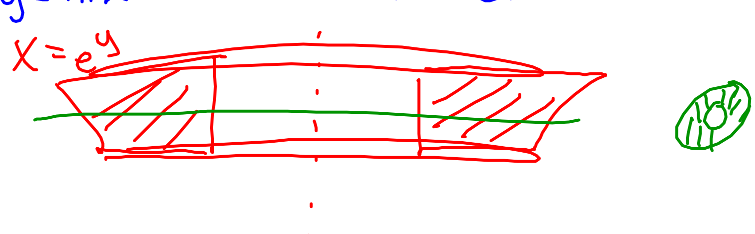
$\int e^{kx} \, dx = \frac{e^{kx}}{k} + C$

(b) Set up the integral(s) needed to find the volume of the solid generated when R is rotated about the line $x = -2$.

1.



2. Slice perp. to $X = -2$



3. outer rad $= e^y - -2 = e^y + 2$

inner rad $= 0 + 2 = 2$

4. $V = \int_1^2 \pi \left[(e^y + 2)^2 - 2^2 \right] dy$