3.4: The Product and Quotient Rules

Exponential Functions

$$\frac{d}{dx}[e^x] = e^x$$

This is because e is the number such that $\lim\limits_{h\to 0} \frac{e^h-1}{h} = 1$.

Example: Differentiate $h(t) = \sqrt[4]{t} - 4e^t$.
First, we rewrite $h(t)$ as $h(t) = t^{1/4} - 4e^t$.

Example: $k(r) = e^r + r^e$.
Don't freak out when you see e or π! They're just NUMBERS, so you should treat them as such.

Higher-Order Derivatives

If f is a differentiable function, then its derivative f' may also be differentiable.
The derivative of the function f' is called the **second derivative** and is denoted $(f')' = f''$.
Other notation: $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2 f(x)$

Note: We may also be also to continue taking higher derivatives.

$$f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

$$f^{(4)}(x) = y^{(4)} = \frac{d^4y}{dx^4}$$

Example: Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the following functions:

1. $f(x) = x^5 - 2x^3 + 10$

2. $f(x) = 2e^x$

The Product Rule

If f and g are both differentiable, then

$$\begin{aligned}\frac{d}{dx}\left[f(x)g(x)\right] &= \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right] \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

Proof. Let $F(x) = f(x)g(x)$ with both $f(x)$ and $g(x)$ differentiable. Then, by the limit definition of a derivative, we have that

$$\begin{aligned}\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-g(x+h)f(x)+g(x+h)f(x)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h)-g(x+h)f(x)}{h}+\frac{g(x+h)f(x)-f(x)g(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h)+f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right) \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h)+\lim_{h\to 0}f(x)\left[\frac{g(x+h)-g(x)}{h}\right] \\ &= f'(x)g(x)+f(x)g'(x)\end{aligned}$$

Therefore, $F'(x) = f'(x)g(x) + f(x)g'(x)$. □

Example: Find $f'(x)$ and $f''(x)$ given $f(x) = x^4e^x$.

Example: Find the derivative of $f(x) = 30x^6(3x + 2)$ by expanding the product and then using the power rule.

Example: Find the derivative of $f(x) = 30x^6(3x + 2)$ by using the product rule.

The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$
$$= \frac{\text{"ho dhi - hi dho"}}{\text{"ho squared"}}$$

Example: $\frac{d}{dx}\left(\frac{x+2}{3x^2-x}\right)$

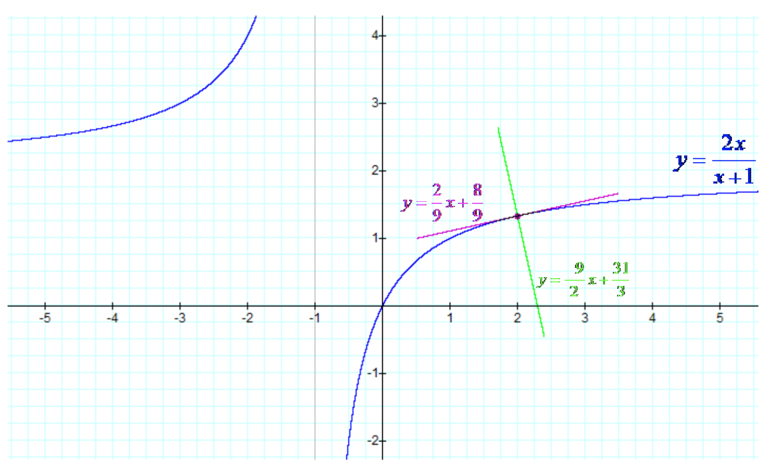
Example: Find the equation of the tangent line to the curve at the given point.

$$y = \frac{2x}{x+1} \text{ at } (2, \tfrac{4}{3})$$

First, we find the slope of the tangent line, which is just the derivative.

Then, we plug in our x value to find the slope of the tangent line at the point $(2, 4/3)$.

Then, we use point-slope formula to find the equation of the tangent line



Example: Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$:

- $h(x) = 5f(x) - 4g(x)$

- $h(x) = f(x)g(x)$

- $h(x) = \frac{g(x)}{1+f(x)}$

Example: If f is a differentiable function, find an expression for the derivative of each of the following functions.

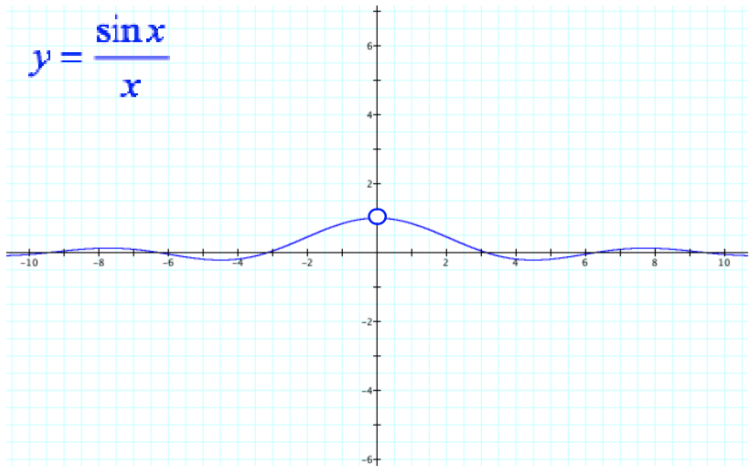
- $y = x^2 f(x)$

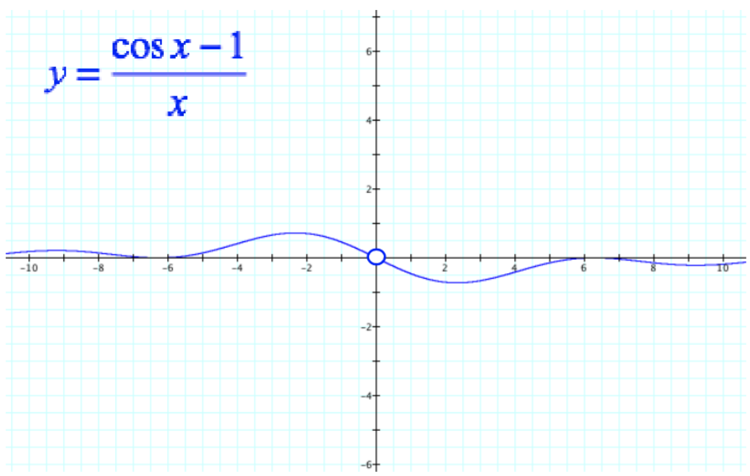
- $y = \frac{f(x)}{e^x}$

- $y = \frac{1+xf(x)}{\sqrt{x}}$

3.5: Derivatives of Trigonometric Functions

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$





Example: Find the limit of $\lim\limits_{h \to 0^-} \frac{h}{\sin 3h}$.

Example: Find the limit of $\lim\limits_{x \to 0} \frac{\sin 5x}{\sin 4x}$.

Example: Find the limit of $\lim\limits_{\theta \to 0} \frac{\sec \theta - 1}{\theta}$.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example: Find the derivative of the following function: $f(x) = \sin x \cos x$.

Example: Find the derivative of $y = \frac{1+\csc x}{1-\csc x}$.

Example: Find the equation of the tangent line to the curve at the given value of x.
$y = \frac{\cos x}{1-\cos x}$ at $x = \frac{\pi}{3}$
First, we find the derivative of the curve since the derivative is the slope of the tangent line.

This is the slope of the tangent line at any point on the curve. To find the slope at $x = \frac{\pi}{3}$, we plug in $\frac{\pi}{3}$ into our derivative.

Then, we find the y_1 value by plugging in $\frac{\pi}{3}$ into the ORIGINAL function.

Then, we plug everything we know into our point-slope formula and solve for y.

Example: Evaluate the following limits or state that they do not exists. Hint: Identify each limit as the derivative of a function at a point.

- $\lim\limits_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4}$.

  We know $f'(a) = \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$.

  So, comparing the two, we have that:
  - $f(x) = \tan x$
  - $a = \pi/4$

  So, we have that $f'(\pi/4) = \lim\limits_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4}$.

  Since we know $f(x) = \tan x$ and $a = \pi/4$, we can easily find $f'(\pi/4)$.

  So, $\lim\limits_{x \to \pi/4} \frac{\tan x - 1}{x - \pi/4} =$

- $\lim\limits_{h \to 0} \frac{\sin(\pi/6 + h) - 1/2}{h}$.

  We know $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

  So, comparing the two, we have that:
  - $f(x) = \sin x$
  - $a = \pi/6$

  So, we have that $f'(\pi/6) = \lim\limits_{h \to 0} \frac{\sin(\pi/6 + h) - 1/2}{h}$.

  Since we kmow $f(x) = \sin x$ and $a = \pi/6$, we can easily find $f'(\pi/6)$.

  So, $\lim\limits_{h \to 0} \frac{\sin(\pi/6 + h) - 1/2}{h} =$

3.6: Derivatives as Rates of Change

Instantaneous Rate of Change

The **instantaneous rate of change** is the derivative and is the limit of average rates of change. When we say "rate of change", we mean the instantaneous rate of change. The sign of the derivative indicates direction of motion.

Rectilinear Motion

The **displacement** of an object moving along a coordinate line with position $s = f(t)$ over a time interval from a to $a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a)$$

where Δt is the **elapsed time**.

The **average velocity** of the object over interval $[a, a + \Delta t]$ is

$$\frac{\text{displacement}}{\text{elapsed time}} = \frac{\Delta s}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

The **velocity** is the derivative of the position function with respect to time.

$$v(t) = \frac{ds}{dt} = s'(t) = f'(t)$$

So, the velocity at time a $(t = a)$ is $v(a) = s'(a) = f'(a)$.

The *sign* of the velocity indicates *direction*. If all we care about is rate of progess, we want the speed.
The **speed** is the absolute value of velocity

$$\text{speed} = |v(t)|$$

The **acceleration** is the derivative of the velocity with respect to time.

$$a(t) = \frac{dv}{dt} = v'(t) = \frac{d^2s}{dt^2} = s''(t)$$

Note: If an object is speeding up, we have that acceleration and velocity are in the same direction (i.e. they have the same sign). An object is slowing down when accelerating and velocity are in opposite directions (i.e. they have opposite signs).

Example: Suppose the position of an object moving horizontally along a line after t seconds is given by $s = f(t) = t^2 - 10t + 12$, $t \geq 0$, where s is measured in feet. Here, $s > 0$ corresponds to positions right of the origin.

- Find the velocity function.

- When is the object stationary, moving to the right, and moving to the left?

  The object is stationary (still), when it has zero velocity. So, we solve $v(t) = 0$.

  So, the object is stationary at $t = 5$ seconds.

  The object is moving to the right when velocity is positive. So, we want to solve $v(t) > 0$.

  So, the object is moving to the right after 5 seconds. So, we can write this as $t \in (5, \infty)$ s.

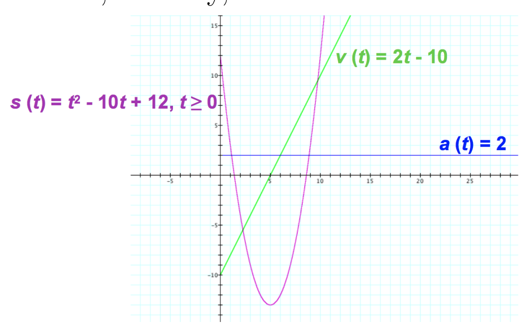
  The object is moving to the left when velocity is negative. So, we want to solve $v(t) < 0$.

  So, the object is moving to the right before 5 seconds. So, we can write this as $t \in [0, 5)$ s.

- Determine the velocity, speed, and acceleration after 3 seconds.

- Determine the acceleration of the object when its velocity is zero.

- Graph the position, velocity, and acceleration functions.



- On what intervals is the object speeding up?
  The object is speeding up when velocity and acceleration have the same sign.

- On what intervals is the object slowing down?
  The object is slowing down when velocity and acceleration have the opposite signs.

Free Fall

For object thrown vertically, the object reaches its **maximum height** when its *velocity is zero*. This is because for an object thrown vertically, it has an initial velocity going up, slowing to a zero velocity at its peak, then speeding up going down due to gravity.
When an object **hits the ground**, its *height is zero*.

Example: If a ball is thrown into the air with a velocity of 40 ft/s at an initial height of 0 ft, its height after t seconds is given by $s(t) = 40t - 16t^2$ ft.

- Find the velocity and acceleration as functions of t.

- What is the maximum height of the ball?
  The *time* the ball reaches its maximum height is when $v(t) = 0$.

  So, the *time* when the ball reaches its maximum height is $t = \frac{5}{4}$ s. To find the maximum height of the ball, we plug in $t = \frac{5}{4}$ s into our position function.

  So, the ball reaches a max height of 25 ft.

- With what velocity will the ball hit the ground?
  The ball hits the ground when $s(t) = 0$. So, we find this time.

  So, the ball hits the ground at $\frac{5}{2}$ seconds. So, we plug this time into our velocity function to find the velocity when the ball hits the ground.

  So, the ball hits the ground with a velocity of -40 ft/s.

Economics: Marginal Cost

Suppose $C(x)$ is the total cost to produce x units of a certain commodity. The **average cost** is $\frac{C(x)}{x}$ per item.
If the number of units produced is increased by Δx, then the **additional cost** is $\Delta C = C(x + \Delta x) - C(x)$.
So, the average cost per item of producing another Δx is $\frac{\Delta C}{\Delta x}$.
The **marginal cost** is the instantaneous rate of change of cost with respect to time and is a good estimate for the cost of producing one additional unit $C'(n) \approx \Delta C = C(n + 1) - C(n)$.

3.7: The Chain Rule

The Chain Rule: Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

We have two methods of computing the chain rule. Either method is valid.

Chain Rule: Method 1:
Let $y = f(g(x))$ be differentiable.

(1) Identify an outer function f and an inner function g. Let $u = g(x)$.
(2) Replace $g(x)$ with u to express y in terms of u
$$y = f(\underbrace{g(x)}_{u}) \implies y = f(u)$$
(3) Calculate the product $\frac{dy}{du} \cdot \frac{du}{dx}$
(4) Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$.

Example: find $\frac{dy}{dx}$ for $y = (2x^3 + x^2 + e)^{10}$ using Method 1 of Chain Rule.

Example: Find $\frac{dy}{dx}$ for $y = \sqrt{4 + 3x}$ using Method 1 of Chain Rule.

Note: Remember to replace u with $g(x)$! Your answer should be in terms of x only, not u!

Example: Find $\frac{dy}{dx}$ of $y = \cos(x^2 - 3)$ using Method 1 of Chain Rule.

Chain Rule: Method 2:
Let $y = f(g(x))$ be differentiable.

(1) Take the derivative of the outside function, leaving the inside UNCHANGED.
(2) Multiply by the derivative of the inside function.

Example: Find $\frac{dy}{dx}$ for $y = (2x^3 + x^2 + e)^{10}$ using Method 2 of the Chain Rule.

Example: Find the derivative of $f(x) = \left(\frac{1-x^3}{x+2}\right)^6$.

Example: Differentiate $g(v) = v^2(v^3+2)^4$.
Note we are going to have to use Product Rule AND Chain Rule!

Example: Differentiate $y = \sin^4 x$.

Example: Differentiate $y = \sin x^4$.

Example: Find $\frac{dy}{dx}$ for $y = \tan(\sin(2x))$.

Example: Find $\frac{dy}{dx}$ for $y = e^{k \tan \sqrt{x}}$.

Example: Differentiate $g(\theta) = \sec^3(3\theta)$.

Example: Let $h(x) = f(g(x))$ and $k(x) = g(g(x))$. Use the table to compute the following derivatives:

x	1	2	3	4	5
f (x)	0	3	-1	2	1
f '(x)	-6	-3	8	7	2
g (x)	4	1	5	2	3
g '(x)	9	7	3	-1	-5

- $h'(1)$

- $h'(2)$

- $k'(3)$

- $k'(1)$

Example: Find all values of x where the tangent line is horizontal.
$y = (3x-1)^2(5-x)^4$.

We know the derivative is the slope of the tangent line. For a line to be horizontal, we know its slope must be 0. So, we want to solve the derivative equal to 0.

Example: Suppose f is differentiable on $[-2, 2]$ with $f'(0) = 3$ and $f'(1) = 5$. Let $g(x) = f(\sin x)$. Evaluate the expressions.

- $g'(0)$

- $g'(\pi/2)$

Example: Assume f and g are differentiable on their domains with $h(x) = f(g(x))$. Suppose the equation of the tangent line to the graph of g at the point $(4, 7)$ is $y = 3x - 5$ and the equation of the line tangent to the graph of f at $(7, 9)$ is $y = -2x + 23$.

- Calculate $h(4)$ and $h'(4)$.

- Determine an equation of the line tangent to the graph of h at the point on the graph where $x = 4$.

3.8: Implicit Differentiation

Explicit Form: An equation is solved for one variable

$$y = x^2 + 2x + 1$$

Implicit Form: An equation that is *NOT* solved for one variable

$$y - x^2 - 2x = 1$$

Implicit Differentiation: The goal is to find $y' = \frac{dy}{dx}$ (derivative of y with respect to x) when the function is in implicit form.

Method of Implicit Differentiation:

1. Differentiate both sides with respect to x (independent variable).
2. Solve the resulting equation for $\frac{dy}{dx}$.

Example: $y = x^2 + 2x + 1$ is equivalent to $y - x^2 - 2x = 1$. Find $\frac{dy}{dx}$ for both equations and compare the results.
Explicit: $y = x^2 + 2x + 1$

Implicit: $y - x^2 - 2x = 1$

Note: This is a very simple example. We must apply the Chain Rule to terms involved y!

- $\frac{d}{dx}(y) = y' = \frac{dy}{dx}$
- $\frac{d}{dx}(y^2) = 2yy' = 2y\frac{dy}{dx}$
- $\frac{d}{dx}y^3 = 3y^2y' = 3y^2\frac{dy}{dx}$

When we evaluate a derivative $\frac{dy}{dx}$ that is in implicit form at a point (a, b), we use this following notation:

$$\frac{dy}{dx}\Big|_{(a,b)}$$

Example: Consider equation $y^2 + 3x = 2$.

- Use implicit differentiation to find $\frac{dy}{dx}$

- Verify that the point $(-1, \sqrt{5})$ lies on the curve.

- Find the slope of the curve at $(-1, \sqrt{5})$

Example: Find $\frac{dy}{dx}$ by implicit differentiation. Determine an equation of the tangent line and the normal line at the point $(1, 1)$.

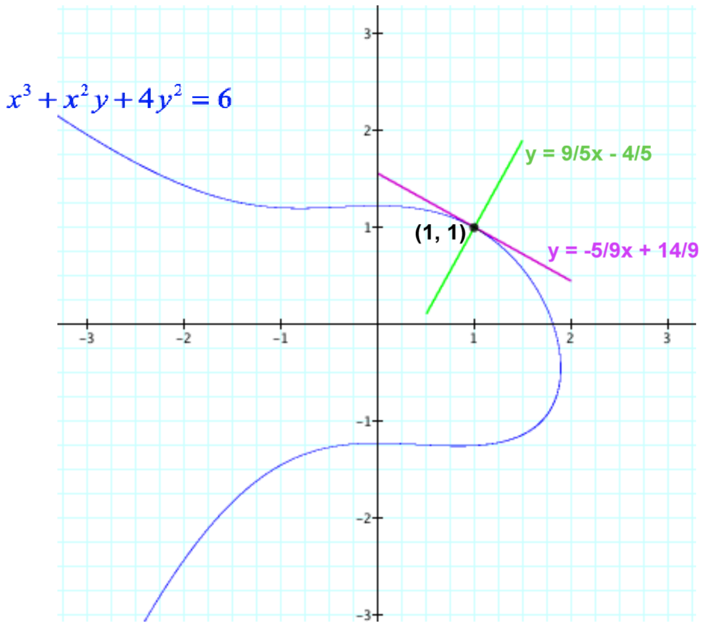
First, we find the derivative by implicit differentiation.

Now, we find the slope of the tangent line at $(1, 1)$.

Now, find the equation of the tangent line using point slope formula.

Now, find the slope of the normal line at $(1, 1)$

Lastly, find the equation of the normal line using point slope formula.



Example: Use implicit differentiation to find $\frac{dy}{dx}$ of $1 + x = \sin(xy^2)$

Example: Find y'' by implicit differentiation. $y^2 - 2x = 1 - 2y$
First, we find the first derivative:

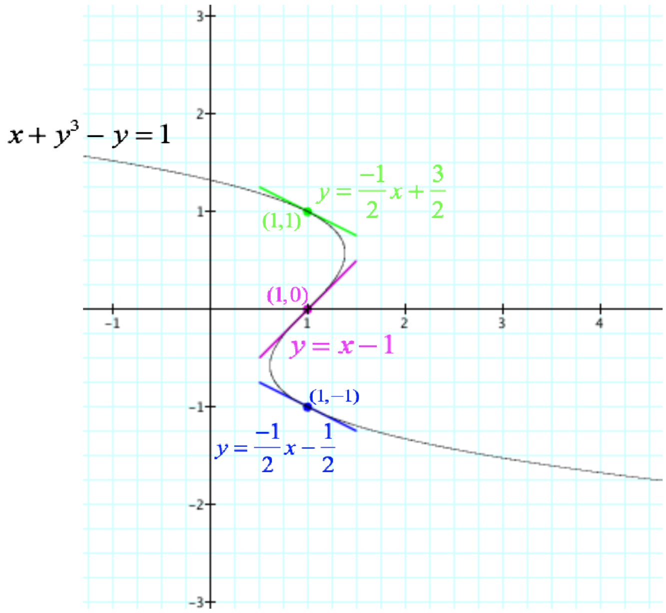
Then, we find the second derivative:

Example: Find the equations of all lines tangent to the curve, $x + y^3 - y = 1$, at $x = 1$.

We need to find the y values at $x = 1$.

So, we have that $y = 0, -1, 1$. So, our three slopes at $x = 1$ are

So, the equations of the lines tangent at $x = 1$ on the curve are



$x + y^3 - y = 1$
$y = \frac{-1}{2}x + \frac{3}{2}$
(1,1)
(1,0)
$y = x - 1$
(1,−1)
$y = \frac{-1}{2}x - \frac{1}{2}$

3.9: Derivatives of Logarithmic and Exponential Functions

Inverse Properties for e^x and $\ln x$

- $e^{\ln x} = x$ for $x > 0$
- $\ln(e^x) = x$ for all x
- $y = \ln x$ if and only if $x = e^y$
- For real numbers x and $b > 0$, $b^x = e^{\ln b^x} = e^{x \ln b}$

Derivative of Logarithms

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ and $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$

However, we have to apply the Chain RUle if you have the derivative of the natural log of a function $u = g(x)$

$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ aka $\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)}$

Example: Differentiate the function $f(x) = \ln(2x^8)$

Example: Differentiate the function $f(x) = \ln\left(\frac{x+1}{x-1}\right)$

Example: Find the derivative $h'(x)$, then evaluate $h'(\ln 2)$. $h(x) = \ln(e^{2x} + 1)$

Derivatives of Exponential Functions with Base a

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

Example: Find $\frac{dy}{dx}$ for $y = 10^{1-x^2}$

Logarithmic Differentiation

What if our base is a variable? Such as $y = x^{\tan x}$? We have to use Logarithmic Differentiation!

Method of Logarithmic Differentiation:

1. Take the natural log (ln) of BOTH SIDES.
2. Use the Laws of Logarithms to simplify the result.
3. Use implicit differentiation to take the derivative of both sides.
4. Isolate y' (or $\frac{dy}{dx}$)
5. Substitute y back into the result.

Example: Find the derivative of $y = x^{\tan x}$.

Example: Use logarithmic differentiation to find $f'(x)$ of $f(x) = \frac{\sin^{10} x}{(5x+3)^6}$

3.10: Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

IMPORTANT: Note that Chain Rule still applies! So, if your value of x inside the inverse trig function is not simply x, you must multiply by the derivative of the inside function. e.g:

$$\frac{d}{dx}(\sin^{-1} g(x)) = \frac{1}{\sqrt{1-g(x)^2}} g'(x)$$

Example: Find the derivative of $y = \tan^{-1}(x^2)$

Example: Find the derivative of $F(\theta) = \arcsin\sqrt{\sin\theta}$

Example: Find the derivative of $f(y) = \sec^{-1}(\ln y)$.

Example: Find an equation of the line tangent to the graph of f at the given value
$f(x) = \cos^{-1} x^2$ at $x = \frac{1}{\sqrt{2}}$

3.11: Related Rates

Related rates problem are about finding a rate of change given other rates of change. Usually, the rates are related by time. This is where we will use implicit differentiation as an application!

Method for Related Rates Problems:

1. Read the problem carefully, make a sketch, write down the given information, and identify the rate to find.
2. Write one or more equations that express the basic relationship among the variables (volume, area, etc.)
3. Introduce rates of change by differentiating the appropriate equations with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable; i.e. does your answer have the correct sign?

Example: The radius r and height h of a cone are related to the cone's volume by the equation: $V = \frac{1}{3}\pi r^2 h$.

- How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?

- How is $\frac{dV}{dt}$ realted to $\frac{dr}{dt}$ if h is constant?

- How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither r nor h is constant?

Example: The edges of a cube increase at a rate of 2 cm/s. How fast is the volume changing when the length of each edge is 50 cm?

Example: Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

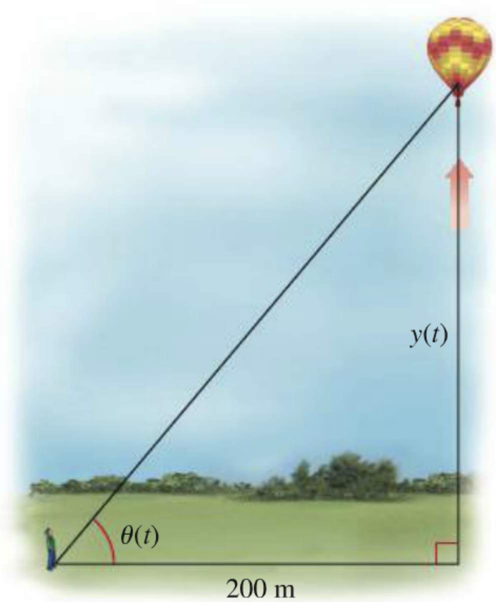
Let the distance the south traveling be denoted as y and the distance the car traveling west denoted as x. Call the distance between them z. We know by Pythagorean theorem: $x^2 + y^2 = z^2$. We know that $\frac{dy}{dt} = -60$ mi/h and $\frac{dx}{dt} = -25$ mi/h. We want to find $\frac{dz}{dt}$.

Before we do, we need to find the distance the south car has traveled after 2 hours (y) and the distance the west car has traveled after 2 hours (x). We also need to find the distance between them after 2 hours (z).

Example: A circle has an initial radius of 50 ft when the radius begins decreasing at a rate of 2 ft/min. What is the rate of change of the area at the instant the radius is 10 ft?

We know the area of a circle is $A = \pi r^2$. We are given that $\frac{dr}{dt} = -2$ ft/min and $r = 10$ ft.

Example: An observer stand 200 meters from the launch site of a hot-air balloon at an elevation equal to the elevation of the launch site. The balloon rises vertically at a constant rate of 4 m/s. How fast is the angle of elevation of the balloon increasing 30 seconds after the launch? (The angle of elevation is the angle between the ground and the observer's line of sight to the balloon.)



As the balloon rises, its distance from the ground y and its angle of elevation θ change simultaneously. We are given that $\frac{dy}{dt} = 4$ m/s. An equation expressing the relationship between these variables is

4.1: Maxima and Minima

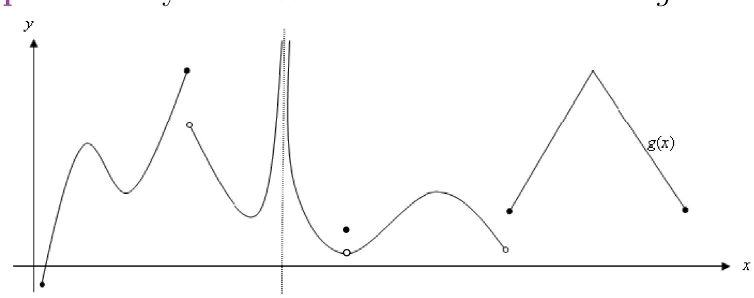
Absolute Extrema:

- A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f
- A function f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f.
- the maximum and minimum values of f are called the **extreme values** of f.
- Note that extreme values are the **function values**, NOT the x-coordinates

Local Extrema:

- A function f has a **local maximum/ relative maximum** at an interior point c of its domain if $f(c) \geq f(x)$ when x is near c, that is, for all x in some open interval containing c.
- A function f has a **local minimum/ relative minimum** at c if $f(c) \leq f(x)$ when x is near c.
- Local extrema do **NOT** occur at endpoints of a domain

Example: Identify the absolute and local extrema of g.



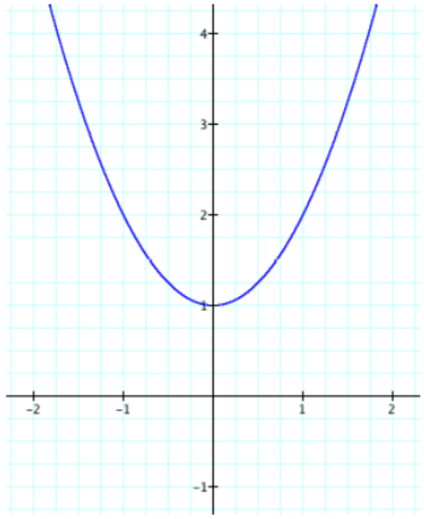
The Extreme Value Theorem:

The Extreme Value Theorem: A function that is continuous on a closed interval $[a, b]$ has an absolute maximum and an absolute minimum value of $[a, b]$.

Note:

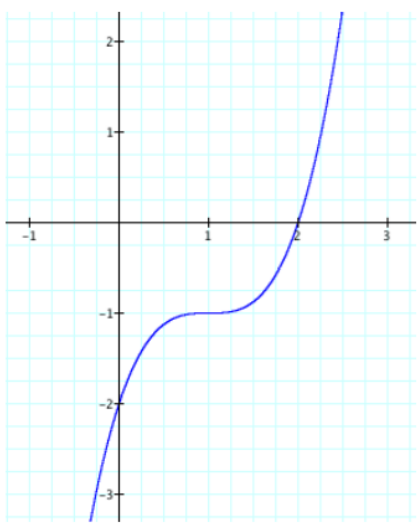
- The Extreme Value Theorem only guarantees extreme values on a closed interval on which a function is continuous. If a function is not continuous or the interval is open, extreme values are not guarenteed.
- A function doens't have to be continuous to have extreme values.

Example: $f(x) = x^2 + 1$



- Absolute max:
- Absolute min:

Example: $f(x) = (x - 1)^3 - 1$



- Absolute max:
- Absolute min:

Finding Absolute Extrema:

A **critical point** of a function f is an interior point c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Note: Local and absolute extrema only occur at critical points (or endpoints), but there may be neither at critical points.

We use The Closed Interval Method for locating absolute extrema of a continuous function f on a closed interval $[a, b]$.

The Closed Interval Method:

1. Find the critical points of f.
2. Evaluate f at the critical points **AND** the endpoints a and b
3. The largest function value found is the absolute maximum of f on $[a, b]$. The smallest function value found is the absolute minimum of f on $[a, b]$.

Example: $f(x) = 4 - x^2$ on $-3 \leq x \leq 1$

- Find the critcal points of f on the given interval.

- Determine the absolute extrema values of f on the given interval if they exist.

Example: Find the absolute extrema of $g(x) = \frac{4x}{x^3+1}$ on $[0, 2]$.

Example: Find the absolute extrema of $f(x) = \ln(x^3 + 1)$ on $[0, 2]$

4.2: The Mean Value Theorem

Rolle's Theorem: If $y = f(x)$ is

- continuous on $[a, b]$
- differentiable on (a, b)
- $f(a) = f(b)$

then here is at least one point c in (a, b) such that $f'(c) = 0$ (aka: there is a horizontal tangent line at some number between a and b).

Example: Consider the function $f(x) = \sin(2x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

(a) Verify that Rolle's Theorem applies to f on the given interval.

(b) Find all points c that are guaranteed to exist by Rolle's Theorem.

The Mean Value Theorem (MVT): If $y = f(x)$ is

- continuous on $[a, b]$
- differentiable on (a, b)

then there is at least one point c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ (aka: there is a tangent line at some number between a and b that has the same slope as the secant line between a and b).

Note: Remember in Rolle's Theorem how we had the extra condition $f(a) = f(b)$? So, the conclusion of Rolle's and MVT is the same, but in Rolle's we just have $f(a) = f(b)$, which is how we get $f'(c) = 0$:

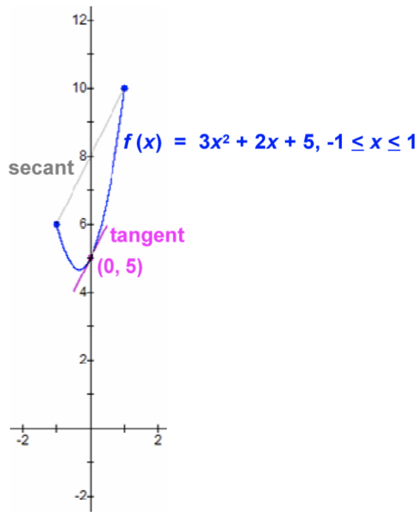
$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(a)-f(a)}{b-a} = \frac{0}{b-a} = 0$$

Application: If the position functoin $s(t)$ is continuous on the time interval $[a, b]$ and differentiable on (a, b), then there is a time $t = c$ between a and b such that instantaneous velocity at $t = c$ is the same as the average velocity over the entire interval.

Example: Consider the function $f(x) = 3x^2 + 2x + 5$ on the interval $[-1, 1]$.

(a) Verify that the function satisfies the conditions of the MVT on the given interval.

(b) Find all points c that are guarenteed to exist by MVT.



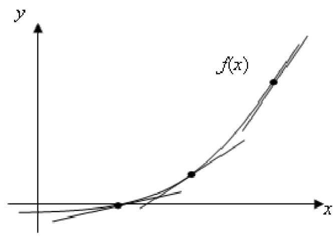
We will continue to use MVT in Section 4.9, but here is a glance at what that will look like:

Theorem: If $f'(x) = 0$ at each point x of an open interval, then $f(x)$ is a constant function on that interval.
This means that if the rate of change of f with respect to x is 0 for all x in an interval, then the function must be constant on that interval.

Theorem: If $f'(x) = g'(x)$ at each point x in an open interval, then there exists a constant C such that $f(x) = g(x) + C$ for all x in the interval.
This means that if the slopes of the tangent lines for two functions are the same for all x in an interval, then the functions are either the same or they are separated by a vertical shift.

4.3: What Derivatives Tell Us

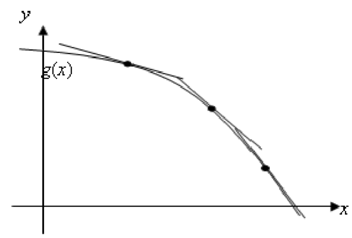
The First Derivative



f is an increasing function.

All of the tangent lines to f have positive slope.

So, if $f'(x) > 0$ on an interval, f is increasing on that interval.



g is a decreasing function.

All of the tangent lines to g have negative slope.

So, if $g'(x) < 0$ on an interval, g is decreasing on that interval.

These particular functions are said to be monotonic. Each function is either always increasing or always decreasing on its domain.

Increasing and Decreasing Test

1. Find all values of x for which $f'(x) = 0$ or $f'(x)$ does not exist. Note that we are looking not just in the domain, but over the entire function.
2. Make a sign chart to determine where $f'(x)$ is positive or negative
3. Derivative tells us:
   - $f'(x) > 0$: $f(x)$ is increasing
   - $f'(x) < 0$: $f(x)$ is decreasing

Example: Consider the function $h(x) = 2x^3 - 18x$.

(a) Find the intervals on which the function is increasing and decreasing.

The First Derivative Test for Local Extrema:

Suppose that c is a critical point of f.

- **local minimum at c:** f' changes from $-$ to $+$ at c
- **local maximum at c:** f' changes from $+$ to $-$ at c
- **no local min or max at c:** f' does not change sign at c

Example: Consider the same function $h(x) = 2x^3 - 18x$.

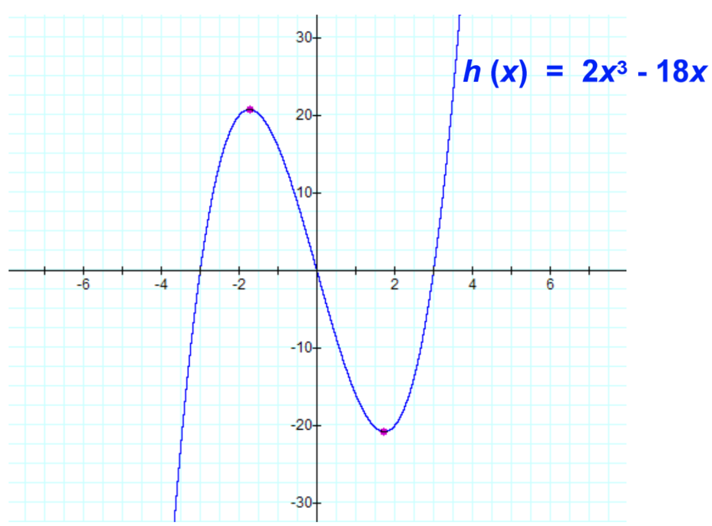
(b) Identify the function's local extrema values, if any, saying where they are located.

One Local Extremum Implies Absolute Extremum:
Suppose f is continuous on an interval I, possibly $(-\infty, \infty)$, that contains exactly one local extremum at c.

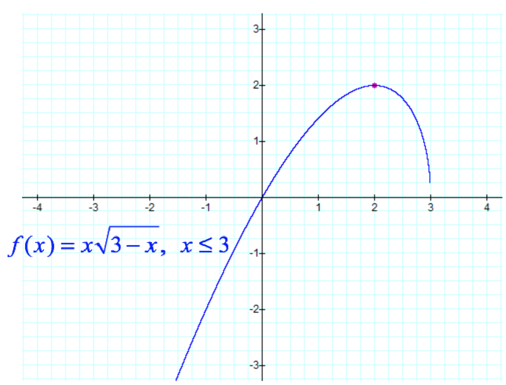
- If a local min occurs at c, then $f(c)$ is the **absolute minimum** of f on I.
- If a local max occurs at c, then $f(c)$ is the **absolute maximum** of f on I.

Example: Consider the same function $h(x) = 2x^3 - 18x$.

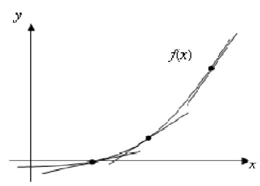
(c) Which, if any, of the extreme values are absolute?



Example: Find the location and value of the one absolute extreme of the function on the given interval.
$f(x) = x\sqrt{3-x}$; $x \leq 3$.



The Second Derivative

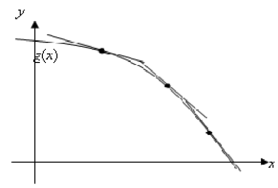


f **is concave up on its domain.**

All of the tangent lines to *f* are below the graph.

The slopes of the tangents are increasing.
=> *f* '(*x*) increasing
=> *f* "(*x*) positive

So, if *f* "(*x*) > 0 on an interval, *f* is concave up on that interval.



g **is concave down on its domain.**

All of the tangent lines to *g* are above the graph.

The slopes of the tangents are decreasing.
=> *g* '(*x*) decreasing
=> *g* "(*x*) negative

So, if *g* "(*x*) < 0 on an interval, *g* is concave down on the interval.

Determining Concavity:

1. Find all values of x for which $f''(x) = 0$ or $f''(x)$ DNE.
2. Make a sign chart with the numbers from Step 1 to determine where f'' is positive and negative.
3. Second Derivative tells us:
   - $f''(x) > 0$: f is concave up (CCU)
   - $f''(x) < 0$: f is concave down (CCD)

Example: Consider the function $2x^3 - 3x^2 + 12$.

(a) Find the intervals on which the function is concave up or down.

An **inflection point** is a point on the graph where the *concavity changes*.

Concavity Test to Find Inflection Points:
Suppose that f is defined at c. If f'' changes signs at c, f has an inflection point at c.

Example: Again consider the function $f(x) - 2x^3 - 3x^2 + 12$.

(b) Identify the function's inflection points, if any.

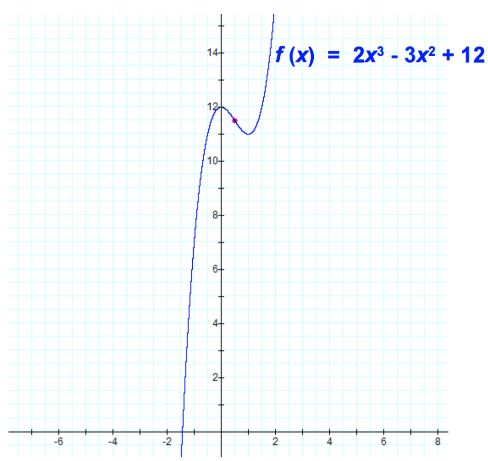
The Second Derivative Test for Local Extrema:

Suppose that f'' is continuous near c with $f'(x) = 0$.

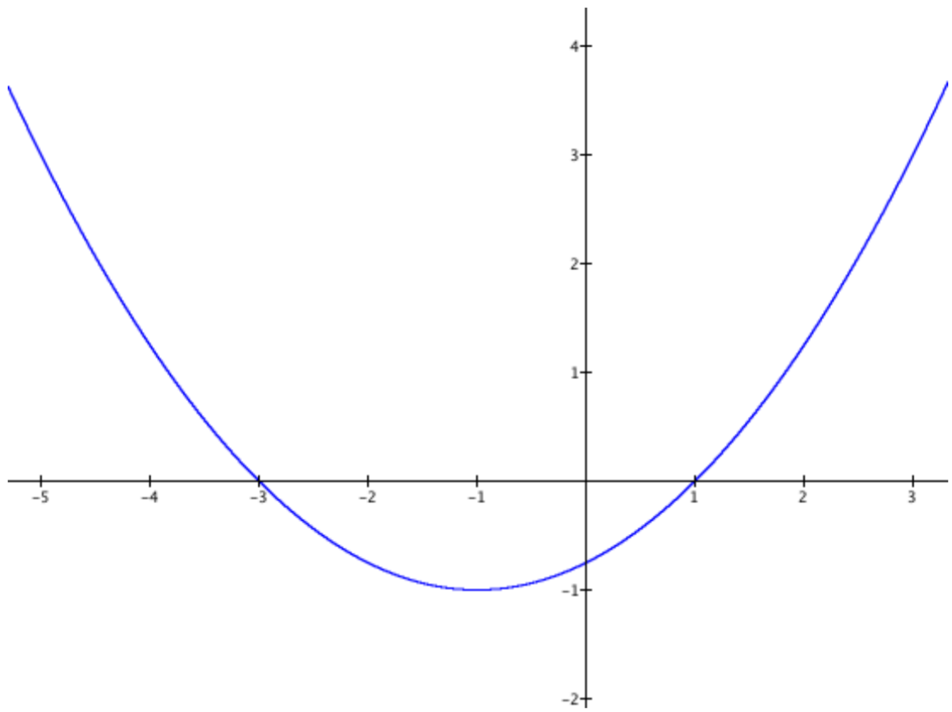
- $f''(c) > 0$: f has a local minimum at c.
- $f''(c) < 0$: f has a local maximum at c.
- $f''(c) = 0$: this test is inconclusive, use the first derivative text or another method.

Example: Again consider the function $f(x) - 2x^3 - 3x^2 + 12$.

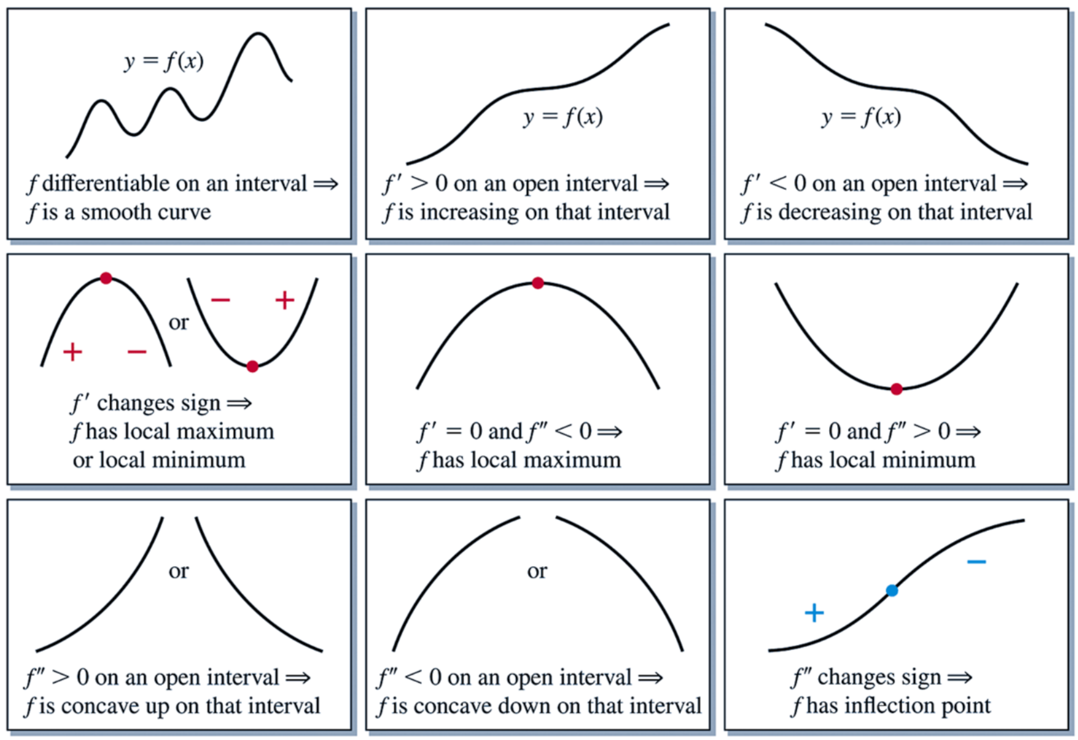
(c) Use the Second Derivative Test to determine whether the critical numbers of f correspond to local extrema or whether the test is inconclusive.



Example: The following figure gives teh graph of the derivative of a continuous function f that passes through the origin. Sketch a possible graph of f on the same set of axes. Note that the graphs of f are not unique.



Summary



$y = f(x)$
f differentiable on an interval $\Rightarrow$
f is a smooth curve
$y = f(x)$
$f' > 0$ on an open interval $\Rightarrow$
f is increasing on that interval
$y = f(x)$
$f' < 0$ on an open interval $\Rightarrow$
f is decreasing on that interval
+ − or − +
f' changes sign $\Rightarrow$
f has local maximum
or local minimum
$f' = 0$ and $f'' < 0 \Rightarrow$
f has local maximum
$f' = 0$ and $f'' > 0 \Rightarrow$
f has local minimum
or
$f'' > 0$ on an open interval $\Rightarrow$
f is concave up on that interval
or
$f'' < 0$ on an open interval $\Rightarrow$
f is concave down on that interval
+ −
f'' changes sign $\Rightarrow$
f has inflection point

4.4: Graphing Functions

Strategy for Graphing $y = f(x)$:

1. **Domain**
   Identify the domain of the function on the given interval. (Look for values that must be excluded)
2. **Intercepts**
   - y-intercept: Set $x = 0$ and solve for y
   - x-intercept: Set $y = 0$ and solve for x
3. **Symmetry**
   - Evaluate $f(-x)$
     - **EVEN:** symmetric about y-axis; $f(-x) = f(x)$
     - **ODD:** symmetric about origin; $f(-x) = -f(x)$
   - Look for periodic functions too (i.e. $\sin x$, $\cos x$)
4. **Asymptotes**
   - **Horizontal Asymptotes:** $\lim\limits_{x\to\pm\infty} f(x)$. If an answer is a number L, there is a horizontal asymptote, $y = L$
   - **Vertical Asymptotes:** Solve denominator $= 0$. $x = a$ is a vertical asymptotes

     - Verify Vertical Asymptote: $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$. If answer is ∞ or $-\infty$, then vertical asymptotes is $x = a$.
   - **Slant Asymptote:** If the degree of the numerator is *one more* than the degree of denominator, use long division to find the quotient.
5. **Intervals of Increase and Decrease**
   - find y'
   - Make a sign chart for y'
   - $y' < 0$: y is decreasing
   - $y' > 0$: y is increasing
6. **Local Extrema**
   - local min location: y' changes from $-$ to $+$
   - local max location: y' changes from $+$ to $-$
   - local extrema value: evaluate y at these points

7. **Concavity and Points of Inflection**
   - find y''
   - make a sign chart for y''
   - $y'' < 0$: y concave down (CCD)
   - $y'' > 0$: y concave up (CCU)
   - inflection points: where the sign changes
   - find the y values at these points for graphing purposes
8. **Sketch**
   - draw all asymptotes as dotted lines
   - plot and label all intercepts, maximum and minimum points, and inflection points
   - sketch a smooth curve through points following concavity and increasing/decreasing

Example: Graph the function $y = f(x) = \frac{(x+1)^3}{(x-1)^2}$

1. **Domain**

2. **Intercepts**

3. **Symmetry**

4. Asymptotes

- horizontal?

- vertical?

- slant?

5. Increase/Decrease

$f'(x) = \frac{(x-5)(x+1)^2}{(x-1)^3}$

6. Local Extrema

7. Concavity and Inflection Points

$f''(x) = \frac{24(x+1)}{(x-1)^4}$

8. Graph