MATH 1080
Calculus of One Variable II

KEY

Exam 2 – Version A
Second Summer 2020

Instructions: You are not permitted to use a calculator of any kind on any portion of this exam. You are not allowed to use a textbook, notes, cell phone, computer, or any other technology on any portion of this exam. All devices must be turned off and stored away while you are in the examination room.

During this exam, any form of communication with any person other than the instructor or designated proctor is understood to be a violation of academic integrity.

No part of this exam may be removed from the examination room.

Read each question carefully. To receive full credit in the free response, you must:

- ☐ show legible, logical, and relevant justification which supports your final answer,
- ☐ use complete and correct mathematical notation,
- ☐ include proper units, where necessary, and
- ☐ give answers as exact values whenever possible.

You have ninety (90) minutes to complete this entire examination. *Good luck!*

Some helpful information:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin 2x = 2 \sin x \cos x$$

A series of the form $\displaystyle\sum_{k=0}^{\infty} cr^k$ converges if $|r| < 1$ and diverges otherwise.

$$\int u\,dv = uv - \int v\,du$$

The use of **ANY additional resource beyond (1) this PDF file, (2) something to write with, and (3) blank paper is cheating.**

The penalty for proven cheating is an automatic F in the course.

Free Response. Answer all questions clearly and legibly. *If your work is unintelligible, the grader has no obligation to attempt to understand it and it may be marked incorrect.* Additionally, correct answers without appropriate work shown will receive little or no credit. For each question, show specific mathematical notation, work, and any appropriate units. Before you begin, read the directions at the beginning of the exam. This exam is worth eighty-five (85) points, five (5) of which are allotted for appropriate mathematical notation.

1. **(10 pts)** Apply **integration by parts** in the evaluation of the integral: $\int (\arcsin x)^2 \, dx$.

① $u = (\arcsin x)^2 \qquad du = \frac{2\arcsin x}{\sqrt{1-x^2}}$

$dv = dx \qquad v = x$

$$= x(\arcsin x)^2 - \int \frac{2x\arcsin x}{\sqrt{1-x^2}}\,dx$$

② $u = \arcsin x \qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dv = \frac{2x}{\sqrt{1-x^2}}\,dx \qquad v = -2\sqrt{1-x^2}$

$$= x(\arcsin x)^2 - \left(-2\sqrt{1-x^2}\arcsin x + \int \frac{-2\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}}}\,dx\right)$$

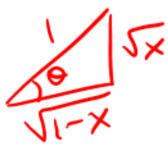
$$= x(\arcsin x)^2 + 2\arcsin x\sqrt{1-x^2} - 2x + C$$

2. **(10 pts)** Apply the **trig substitution method** in the evaluation of the integral: $\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$.

Hint: If $x > 0$, then $x = \sqrt{x}^2$.

Note that $1 - x = 1^2 - \sqrt{x}^2$, hence

we let $\sqrt{x} = \sin\theta$, $\theta = \arcsin\sqrt{x}$



$\cos\theta = \sqrt{1-x}$

$x = \sin^2\theta$

$\frac{1}{2\sqrt{x}}\,dx = \cos\theta\,d\theta$

TRIG SUB

$\frac{1}{\sqrt{x}}\,dx = 2\cos\theta\,d\theta$

$$\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx = \int \sqrt{1-\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= 2\int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= 2\int \cos^2\theta\,d\theta$$

$$= \cancel{2}\int \left(\frac{1+\cos 2\theta}{\cancel{2}}\right)d\theta$$

$$= \theta + \frac{\boxed{\sin 2\theta}}{2} + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \arcsin\sqrt{x} + \sqrt{x - x^2} + C$$

$x^5 + x^3 = x^3(x^2+1)$

3. **(10 pts)** Find the **partial fraction decomposition** for $f(x) = \dfrac{7x^4+5x^3+8x^2+4x+3}{x^5+x^3}$.

$$\frac{7x^4+5x^3+8x^2+4x+3}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}$$

$$7x^4+5x^3+8x^2+4x+3 = Ax^2(x^2+1) + Bx(x^2+1) + C(x^2+1) + (Dx+E)x^3$$

$$= Ax^4 + Ax^2 + Bx^3 + Bx + Cx^2 + C + Dx^4 + Ex^3$$

Equating Coefficients

$\boxed{C=3}$

$A + C = 8$

$A + 3 = 8$

$\boxed{A=5}$

$\boxed{B=4}$

$B + E = 5$

$4 + E = 5$

$\boxed{E=1}$

$A + D = 7$

$5 + D = 7$

$\boxed{D=2}$

4. **(10 pts)** Suppose $a \in \mathbb{R}$ and $k > 0$. Solve the following equation for a: $\displaystyle\int_a^\infty e^{-kx}\,dx = 1$.

In other words, find the interval $[a, \infty)$ so that the area between $y = e^{-kx}$ and the x-axis is one.

$$\int_a^\infty e^{-kx}\,dx = \lim_{N\to\infty} \int_a^N e^{-kx}\,dx$$

$$= \lim_{N\to\infty} \left. \frac{e^{-kx}}{-k} \right|_a^N$$

$$= \lim_{N\to\infty} \left(\cancel{\frac{e^{-kN}}{-k}}^{\to 0} - \frac{e^{-ka}}{-k} \right)$$

$$= \frac{1}{k} e^{-ka}$$

We want: $\quad \dfrac{1}{k} e^{-ka} = 1$

$$k = e^{-ka}$$

$$\ln k = \ln(e^{-ka})$$

$$\ln k = -ka$$

$$\boxed{a = \frac{-1}{k} \ln k}$$

$$a = \frac{1}{k} \ln\left(\frac{1}{k}\right)$$

equivalent

5. **(10 pts)** Suppose $|j| > 1$. Find the sum of the following series, or otherwise conclude it diverges.

$$\frac{2}{3} + \frac{1}{3j} + \frac{1}{6j^2} + \frac{1}{12j^3} + \cdots \qquad \text{ratio} = \frac{1}{2j}$$

Any claim of convergence must be justified by referencing a specific test and showing that the conditions of that test have been met.

$$= \frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{1}{2j}\right)^k$$

Since $|j| > 1$, then $\left|\frac{1}{2j}\right| < 1$ and so the series converges by the GST.

$$= \frac{2}{3}\cdot\frac{1}{1-\frac{1}{2j}}$$

$$= \frac{4j}{3(2j-1)}$$

6. **(10 pts)** Evaluate the integral: $\displaystyle\int \cos^2 x \sin^2 x \, dx$

The use of a reduction formula or an identity other than the ones on the front of this exam will result in zero points awarded.

$$= \int \cos^2 x \cdot (1 - \cos^2 x)\, dx$$

$$= \int \cos^2 x \, dx - \int \cos^4 x \, dx$$

$$= \int \left(\frac{1+\cos 2x}{2}\right) dx - \int (\cos^2 x)^2 \, dx$$

$$= \frac{1}{2}x + \frac{\sin 2x}{4} - \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{2}x + \frac{\sin 2x}{4} - \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{2}x + \cancel{\frac{\sin 2x}{4}} - \frac{1}{4}x - \cancel{\frac{\sin 2x}{4}} - \frac{1}{4}\int \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{\sin 4x}{32} + C$$

$$= \frac{1}{8}x - \frac{\sin 4x}{32} + C$$

An alternate solution exists.

7. **(10 pts)** Respond to the following statements as being true or false.

You do not need to justify your selection.

a. Suppose that $0 < j < 1$. The geometric series $\displaystyle\sum_{k=2}^{\infty}\left(\frac{1}{j}\right)^k$ converges. False

b. If $\{a_n\}_{n=0}^{\infty}$ diverges, then $\displaystyle\left\{\frac{1}{a_n}\right\}_{n=0}^{\infty}$ converges. False

c. Every divergent sequence is unbounded. False

d. Suppose that $0 < \theta < \frac{\pi}{2}$. The geometric series $\displaystyle\sum_{k=0}^{\infty}\left(\sin\theta\right)^k$ diverges. False

e. If $a_k > 0$ for all $k \in \mathbb{N}$ and $\displaystyle\sum_{k=0}^{\infty} a_k$ converges, then $\displaystyle\sum_{k=10}^{\infty} a_k > \sum_{k=5}^{\infty} a_k$. False

8. **(10 pts)** Consider the sequence $\left\{ \frac{1}{n} - \frac{1}{n-2} \right\}_{n=3}^{\infty}$.

a. **(2 pts)** Write the first two (2) terms of the sequence as simplified fractions.

$$a_1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$a_2 = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$$

b. **(2 pts)** Determine if the sequence converges or diverges.

$$\lim_{n\to\infty} \left(\cancel{\frac{1}{n}}^{\,0} - \cancel{\frac{1}{n-2}}^{\,0} \right) = 0 \quad \therefore \text{ converges}$$

c. **(6 pts)** Determine if the series $\sum_{k=3}^{\infty} \left(\frac{1}{k} - \frac{1}{k-2} \right)$ converges or diverges.

$$S_n = \left(\cancel{\frac{1}{3}} - 1\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{2}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-2}} + \cancel{\frac{1}{n-4}}\right) + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n-3}}\right) + \left(\frac{1}{n} - \cancel{\frac{1}{n-2}}\right)$$

$$S_n = -\frac{3}{2} + \frac{1}{n}$$

$$\lim_{n\to\infty} S_n = -\frac{3}{2}$$

$\downarrow$

series converges to $-3/2$

9. **(5 pts)** Notation. *Please mark this question on your last page submitted to Gradescope.*

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