Math 1060 Class notes
Fall 2019

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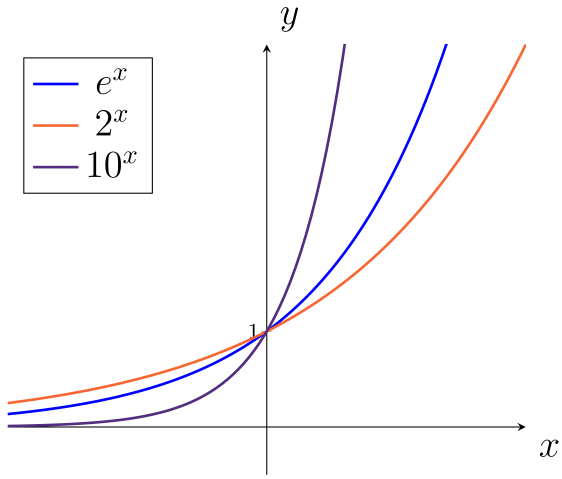
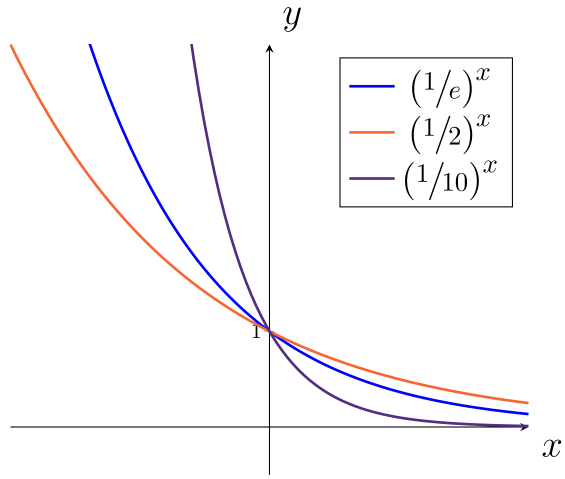
1.3 Inverse, Exponential and Logarithmic Functions

Definition.
An **exponential function** has the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. Exponential functions have a horizontal asymptote of $y =$ ___ and y-intercept of $(0, \_)$. When b is such that $0 < b < 1$, then $f(x)$ is ______________________ and when $b > 1$, then $f(x)$ is ______________________.
Exponential functions have domain _____ and range _____.





Definition.
The **natural exponential function** is

$$f(x) = e^x.$$

where e is the irrational constant $e \approx 2.718281828459045\ldots$.

Laws of Exponents: For $a > 0$, we have the following laws:

a) $a^{x+y} = a^x a^y$

b) $a^{x-y} = \frac{a^x}{a^y}$

c) $(a^x)^y = a^{xy}$

d) $(ab)^x = a^x b^x$

Example. For the following expressions, use the Laws of Exponents to simplify:

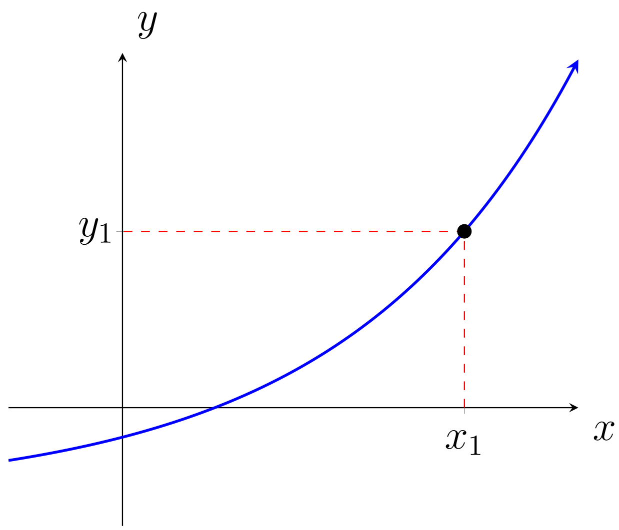
a) $\left(x^2y^3\right)^5$

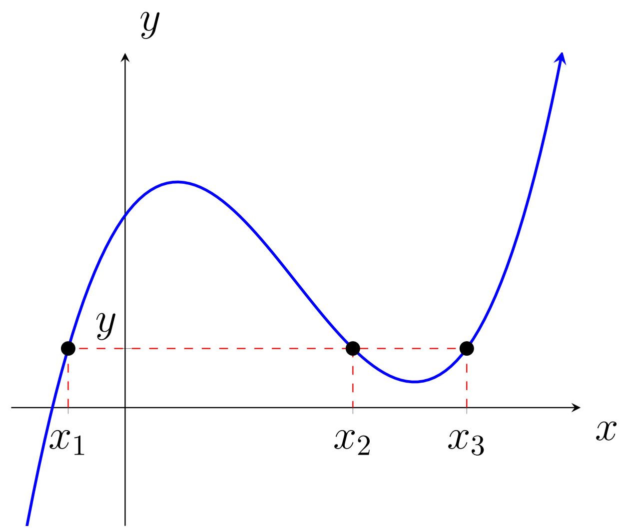
b) $\left(\sqrt{3}\right)^{1/2} \cdot \left(\sqrt{12}\right)^{1/2}$

c) $\left(\dfrac{x^{-2}}{x^8}\right)^{-2}$

d) $\left(\dfrac{-1}{27}\right)^{4/3}$

Definition. (One-to-One Functions and the Horizontal Line Test)
A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on d if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D.
The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once.





Finding an Inverse Function Suppose f is one-to-one on an interval I. To find f^{-1}, use the following steps:

1. Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
2. Interchange x and y and write $y = f^{-1}(x)$.

Example. Find $f^{-1}(x)$:

$$f(x) = x^2 - 2x + 1, \ x \geq 1$$

$$g(x) = \frac{x}{2} - \frac{7}{2}$$

$$h(x) = \sqrt[3]{5x+1}$$

$$j(x) = \frac{2x}{1-x}$$

$$k(x) = e^x$$

Existence of Inverse Functions

Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

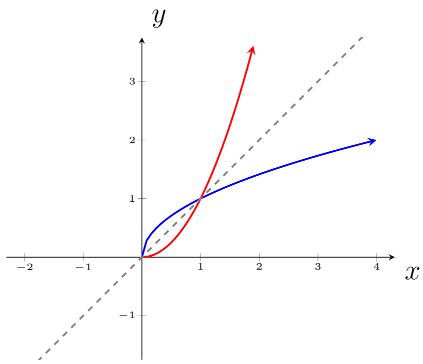
$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f\big(f^{-1}(y)\big) = y$$

where x is in D and y is in R.

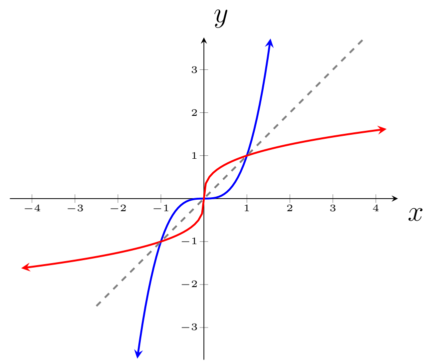
Example. For $f(x) = \sqrt[3]{4x-1} + 2$, show that $f^{-1}(f(x)) = f\big(f^{-1}(x)\big) = x$

Note: A function is symmetric with it's inverse with respect to $y = x$.

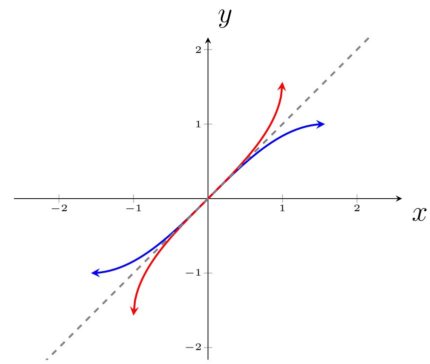
$$f(x) = \sqrt{x}$$
$$f^{-1}(x) = x^2,\ x > 0$$



$$f(x) = x^3$$
$$f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$$

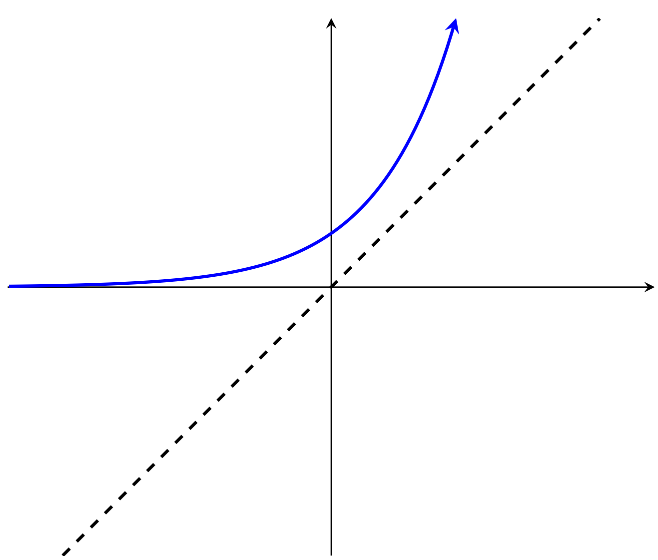


$$f(x) = \sin x \text{ on } [-\pi/2, \pi/2]$$
$$f^{-1}(x) = \sin^{-1} x$$

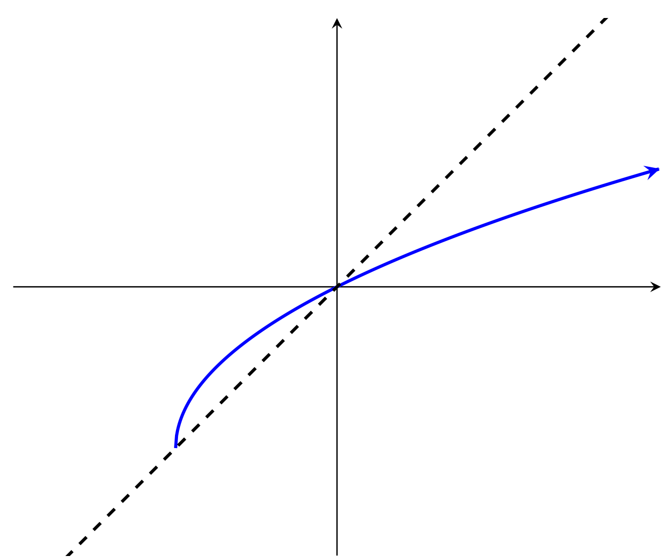


Example. Draw the function inverses:

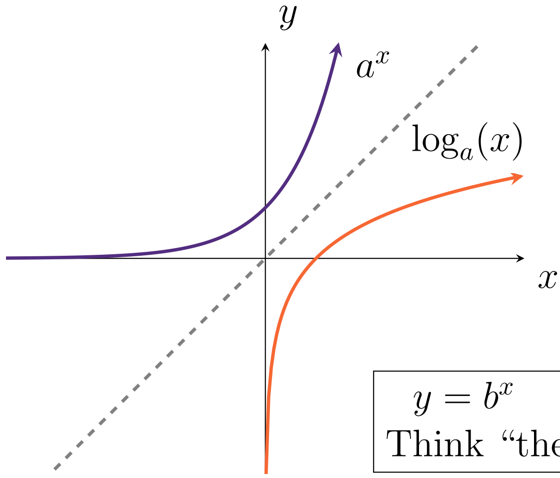
$$f(x) = 2^x$$



$$f(x) = \sqrt{x+1} - 2$$



Definition. (Logarithmic Function Base b)
For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $y = \log_b(x)$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln(x)$.



Note:

	a^x	$\log_a(x)$
Domain:	$(-\infty, \infty)$	$(0, \infty)$
Range:	$(0, \infty)$	$(-\infty, \infty)$

$$y = b^x \iff \log_b(y) = x$$
Think "the base stays the base"

Example. Evaluate:

$\log_9(81)$ $\qquad$ $\log_3(\sqrt{3})$

$\log_{\frac{1}{2}}(8)$ $\qquad$ $\left(\log_5(5^{-3})\right)^2$

Note: In this course, the **common logarithm** is $\log_{10}(x)$ and is denoted by $\log(x)$.
– Sometimes other disciplines use $\log(x)$ to represent other bases.

Example. Evaluate:

$\log 100000$ $\qquad$ $\log \frac{1}{1000}$

Recall that for a function f and its inverse g:

- $f(g(x)) = x$
- $g(f(x)) = x$
- Domain of f=Range of g
- Domain of g=Range of f

Inverse Relations for Exponential and Logarithmic Functions
For any base $b > 0$, with $b \neq 1$, the following inverse relations hold:

$$b^{\log_b x} = x \qquad \log_b(b^x) = x, \text{ for all real values of } x$$

Example. Evaluate:

$2^{\log_2 8}$ $\qquad$ $\log_b b^\pi$ $\qquad$ $\log 10^3$

Example. Write each expression in terms of one logarithm:

$\log_2 6 - \log_2 15 + \log_2 20$

$\log_3 100 - (\log_3 18 + \log_3 50)$

Laws of Logarithms
For $x, y > 0$:

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(x)$
3. $\log_a(x^r) = r \log_a(x)$
4. $\log_a(1) = 0$
5. $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Example. Solve each equation checking for extraneous solutions:

$\log_{64} x^2 = \frac{1}{3}$

$\log(3x + 2) + \log(x - 1) = 2$

$\log_2 x^2 - \log_2(3x - 8) = 2$

$\log_4 x - \log_4(x - 1) = \frac{1}{2}$

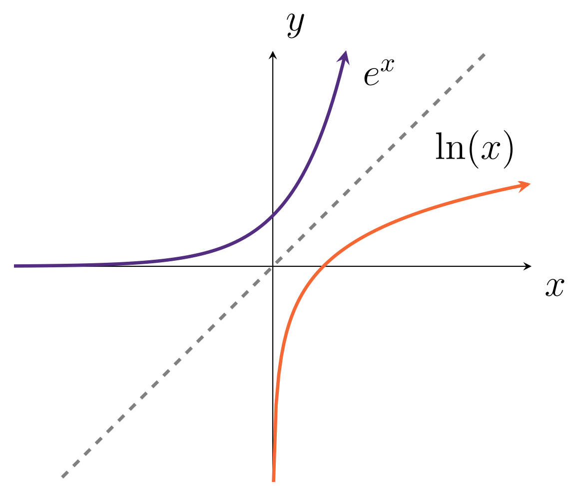
$\log_3(x + 6) - \log_3(x - 6) = 2$

$\log_3(x^2 - 5) = 2$

Definition.
The **Natural Logarithmic Function** uses base e and is denoted $\ln(x) = \log_e x$.

Note: The natural log is the inverse of e^x:

$$\ln(x) = y \quad \Longleftrightarrow \quad e^y = x$$



Inverse Properties for a^x and $\log_a x$

Base a: $\quad a^{\log_a x} = x, \qquad \log_a a^x = x, \qquad a > 0,\ a \neq 1,\ x > 0$

Base e: $\quad e^{\ln x} = x, \qquad \ln e^x = x, \qquad x > 0$

Example. Simplify

$e^{-\ln 0.3}$

$e^{\ln \pi x - \ln 2}$

$\ln\left(\frac{1}{e}\right)$

$e^{4\ln x}$

Example. Write each expression in terms of one logarithm:

$\ln(a+b) + \ln(a-b) - 2\ln c$

Laws of the Natural Logarithm
For $x, y > 0$:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(x)$
3. $\ln(x^r) = r\ln(x)$
4. $\ln(1) = 0$
5. $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$

$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right]$

Note: Many common mistakes come from using the logarithm rules incorrectly:

$$\ln A - \ln B \neq \frac{\ln A}{\ln B} \qquad \ln(A+B) \neq \ln(A)\ln(B)$$

Example. Solve:

$2^x = 55$

$5^{3x} = 29$

$e^{2x} - 5e^x - 14 = 0$

$4e^{2x} - 7e^x = 15$

Example. Solve for y in terms of x:

$\ln(y - 40) = 5x$

$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

Example. Solve:

$\ln(t) + \ln(t^2) = 6$

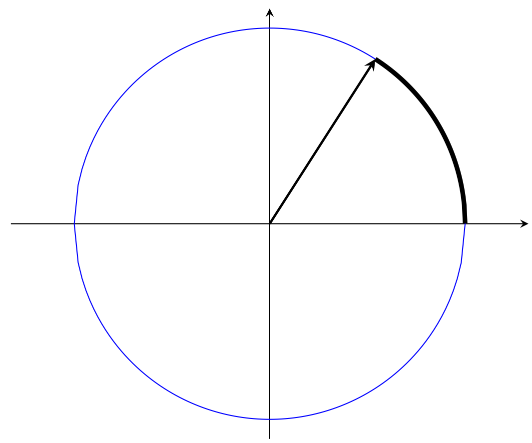
$e^{x^2+2x-3} = 1$

$\ln x = -1$

$e^{-0.3t} = 27$

1.4 Trigonometric Functions and Their Inverses

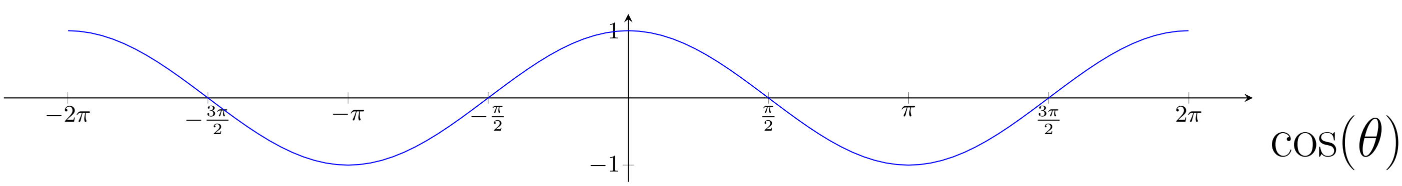
Definition.
The **unit circle** is the circle of radius 1 that is centered at the origin.
The angle corresponding to an arc length of 1 on a unit circle is called a **radian**.

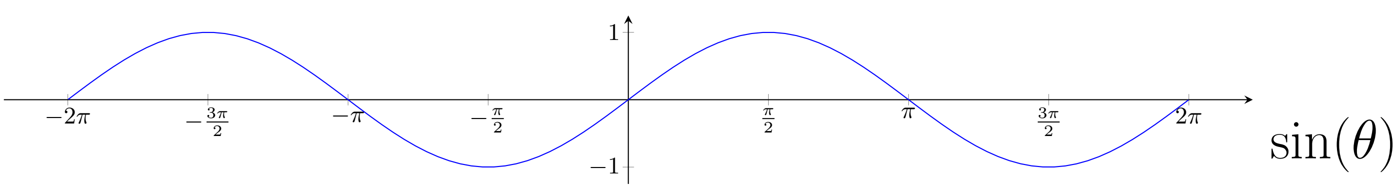


A circle is 2π radians or 360°. Thus:

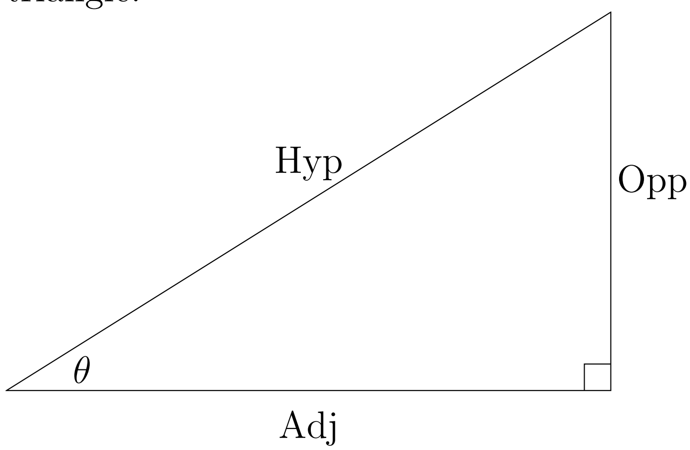
$$2\pi = 360^\circ \Longrightarrow 1 = \frac{180^\circ}{\pi} = \frac{\pi}{180^\circ}$$

Definition.
The coordinates of a unit circle are given by $(\cos(\theta), \sin(\theta))$ for each θ.
The $\sin(\theta)$ and $\cos(\theta)$ functions are **periodic** since these functions repeat themselves over a fixed interval.





Definition.
Alternatively, $\cos(\theta)$ and $\sin(\theta)$ can be consider the ratio of the sides of a right angle triangle.

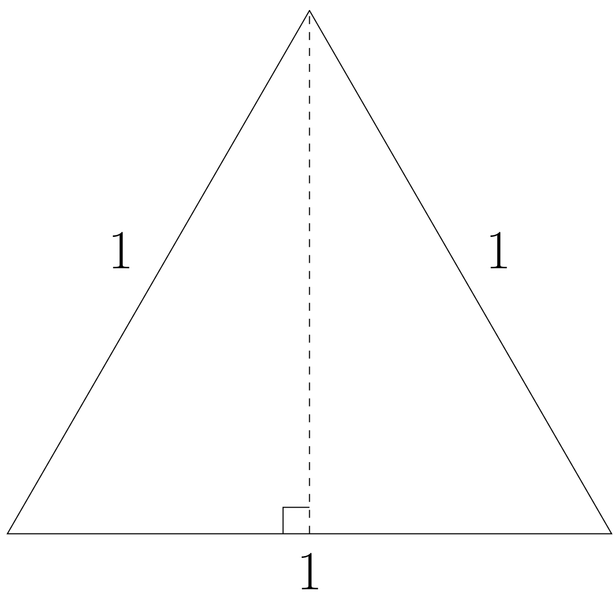


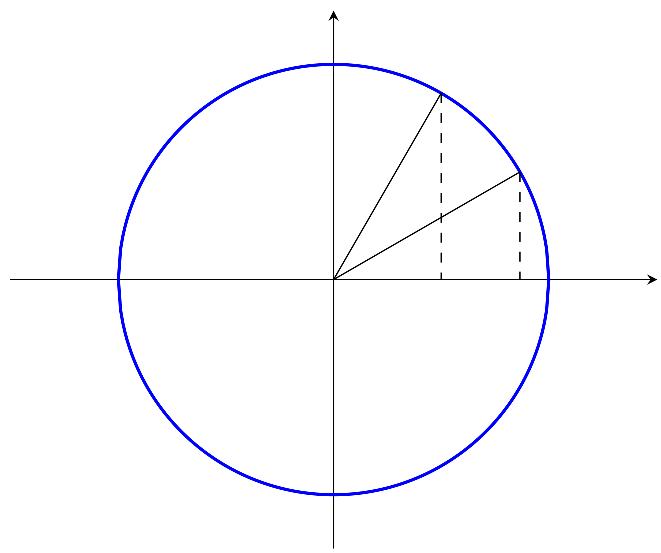
$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}}$$

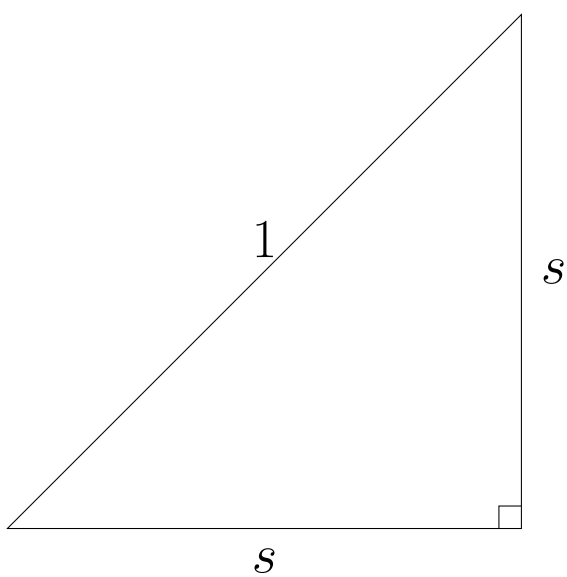
$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}}$$

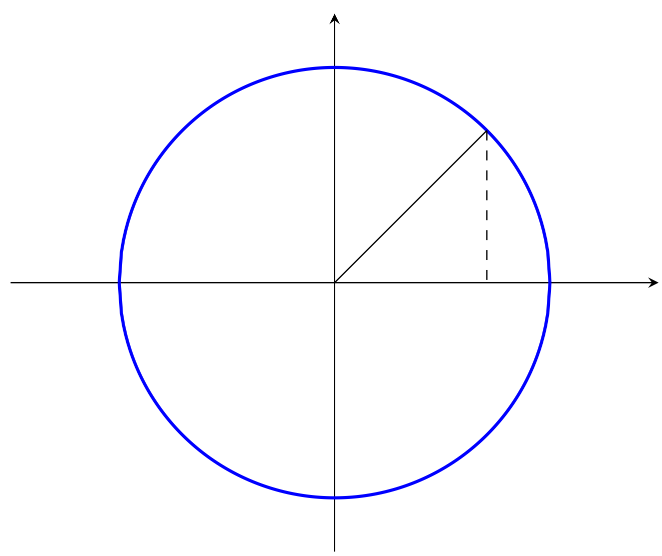
$$\tan\theta = \frac{\text{Opp}}{\text{Adj}}$$

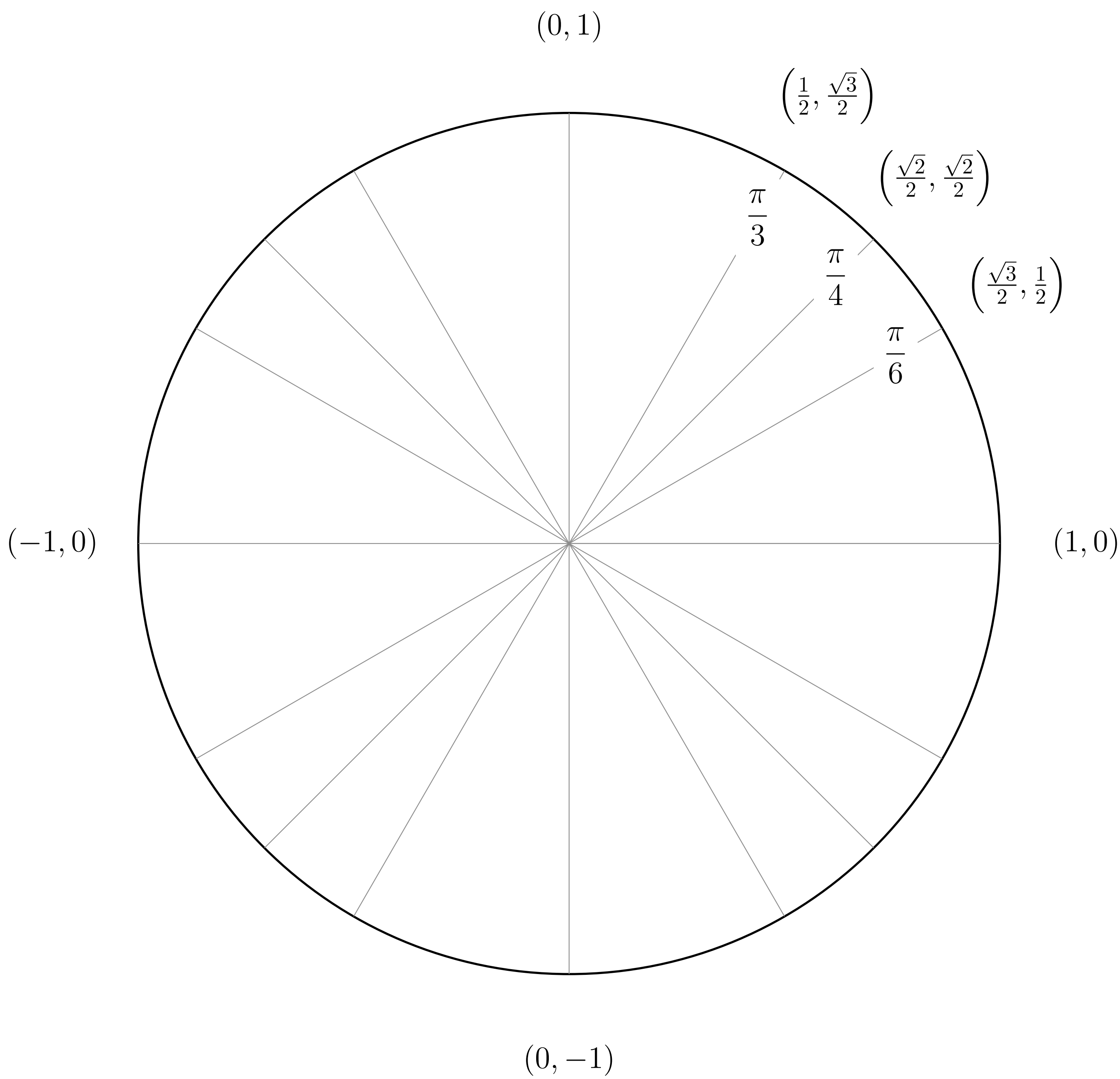
Example. Find $\cos(\theta)$ and $\tan(\theta)$ given that $\sin(\theta) = \frac{3}{5}$ and $\pi/2 \leq \theta \leq \pi$ (2nd quadrant).


1
1
1




1
s
s



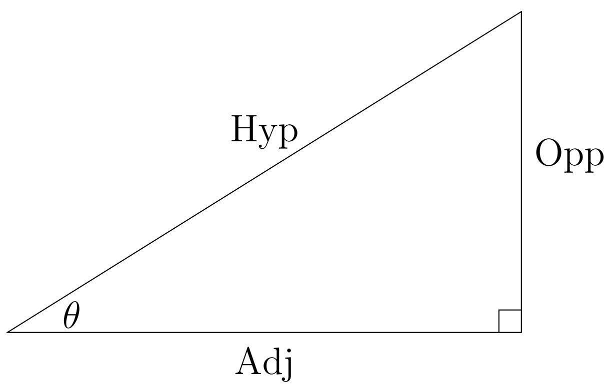

$(0,1)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{3}$
$\frac{\pi}{4}$
$\frac{\pi}{6}$
$(-1,0)$
$(1,0)$
$(0,-1)$

There are 6 trig functions, all of which can be written in terms of $\sin(\theta)$ and $\cos(\theta)$:

$$\tan\theta = \qquad\qquad\qquad \cot\theta =$$

$$\sec\theta = \qquad\qquad\qquad \csc\theta =$$

These functions can also be represented with the sides of a right triangle:



$$\tan\theta = \qquad\qquad \cot\theta =$$

$$\sec\theta = \qquad\qquad \csc\theta =$$

Using these functions, we have the following Pythagorean Identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\tan^2(\theta) + 1 = \underline{\qquad\qquad\qquad}$$

$$1 + \cot^2(\theta) = \underline{\qquad\qquad\qquad}$$

Example. Show the identity: $\tan\theta + \cot\theta = \sec\theta\csc\theta$.

Definition.
The **Angle Sum Formulas** are

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Note: Since $\cos(\theta)$ is even and $\sin(\theta)$ is odd, we can derive the difference formula from the sum formula.

Definition.
The **double-angle formulas** are a special case of the angle-sum formulas:

$$\begin{aligned}\sin(2\theta) &= \sin(\theta + \theta) \\ &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\ &= \boxed{2\sin(\theta)\cos(\theta)}\end{aligned}$$

$$\begin{aligned}\cos(2\theta) &= \cos(\theta + \theta) \\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\ &= \boxed{\cos^2(\theta) - \sin^2(\theta)}\end{aligned}$$

Note: Using the Pythagorean Identity, we have 2 additional representations of $\cos(2\theta)$.

Example. Evaluate $\sin\left(\frac{2\pi}{3}\right)$ using the double-angle formula.

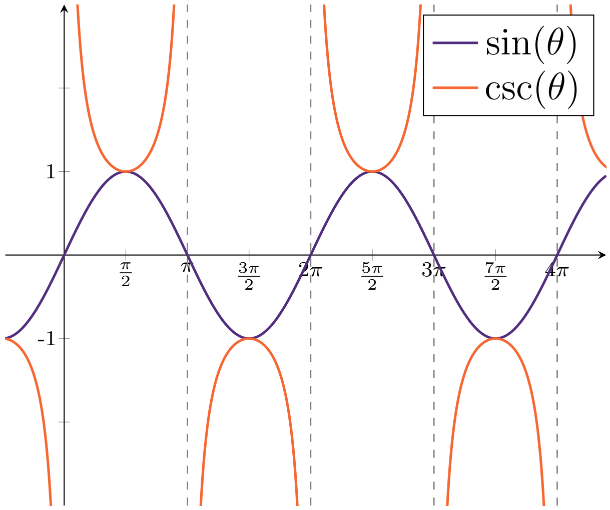
Definition.
The **half-angle formulas** are derived from the double angle formula:

$$\sin(\theta) = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$
$$\cos(\theta) = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

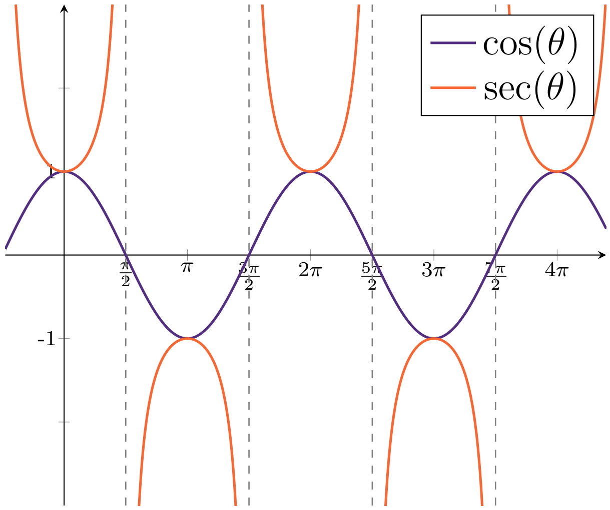
Example. Evaluate $\cos\left(\frac{\pi}{12}\right)$ using the half-angle formula.

Example. Solve the following equation:

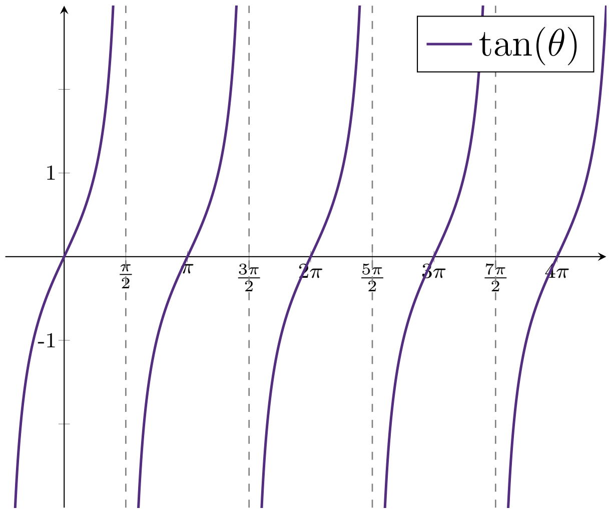
$$\cos(3\theta) = \sin(3\theta), \quad 0 \leq \theta < 2\pi$$



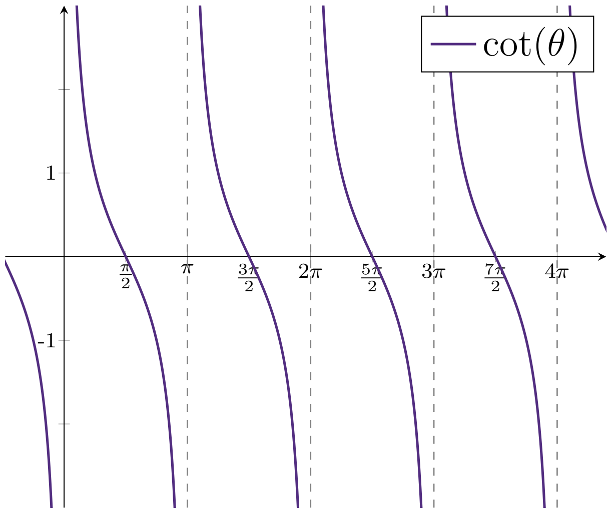
sin(θ)
csc(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π



cos(θ)
sec(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

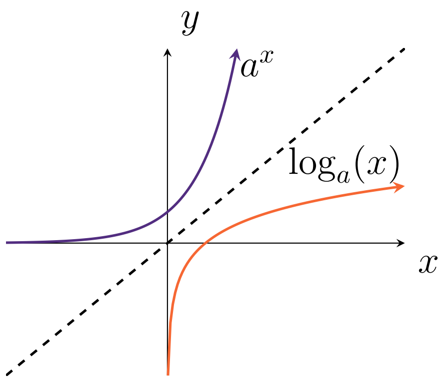


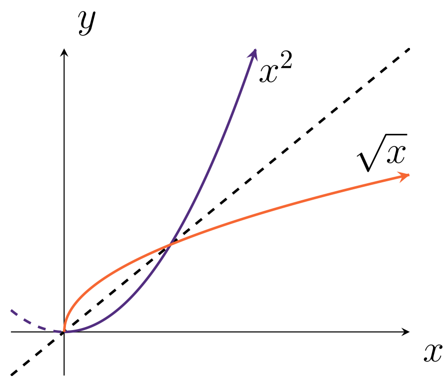
tan(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

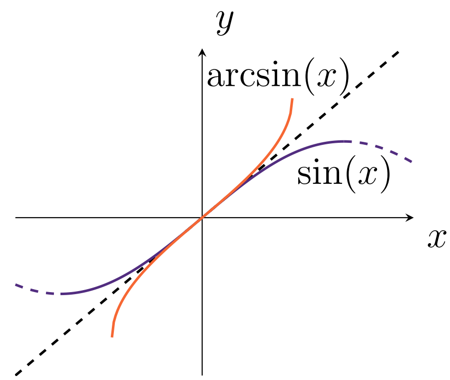


cot(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

Recall that a function has an inverse if it is 1-to-1 (e.g. it passes the horizontal line test).

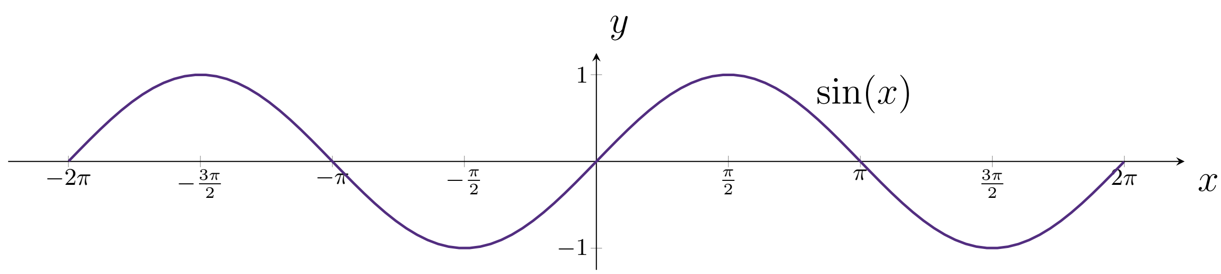




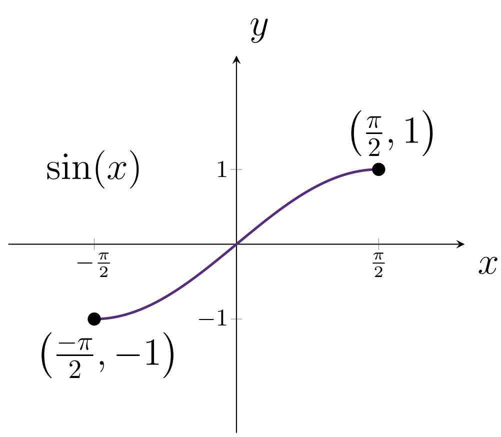


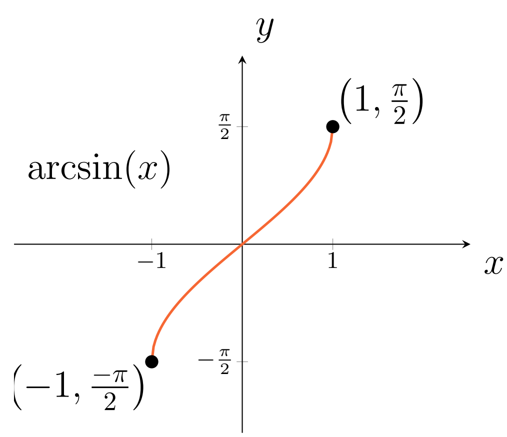
Notice that x^2 and $\sin(x)$ are on restricted domains.

Without restriction on its domain, $\sin(x)$ is NOT 1-to-1:



The range of $\sin(x)$ is $[-1, 1]$ and all of these values are attained on a restricted domain of $[-\pi/2, \pi/2]$:



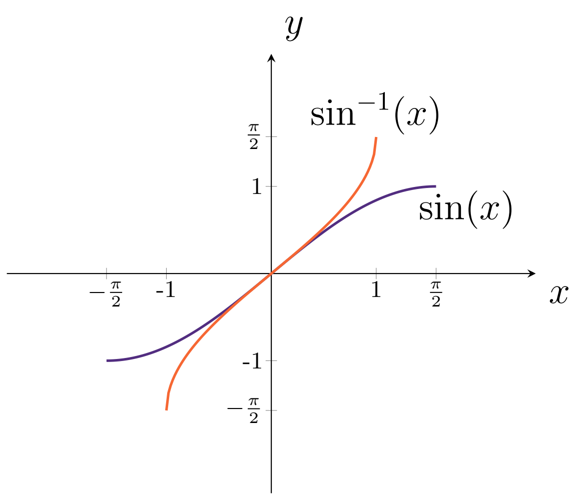


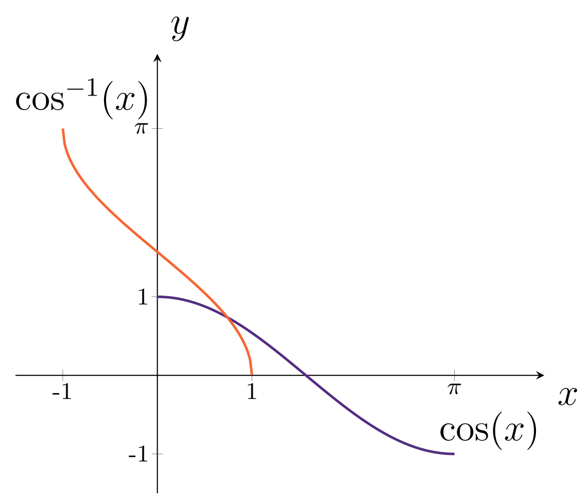
Definition. (Inverse Sine and Cosine)
$y = \sin^{-1}(x)$ is the value of y such that $x = \sin(y)$, where $-\pi/2 \le y \le \pi/2$.

$y = \cos^{-1}(x)$ is the value of y such that $x = \cos(y)$, where $0 \le y \le \pi$.

The domain of both $\sin^{-1}(x)$ and $\cos^{-1}(x)$ is $\{x \mid -1 \le x \le 1\}$.

Note: The inverse sine function can be denoted as $\arcsin(x)$ or $\sin^{-1}(x)$.
This means that $\sin^{-1}(x) \neq \dfrac{1}{\sin(x)}$.
Similarly, $\arccos(x)$ and $\cos^{-1}(x)$ denote the inverse cosine functions.





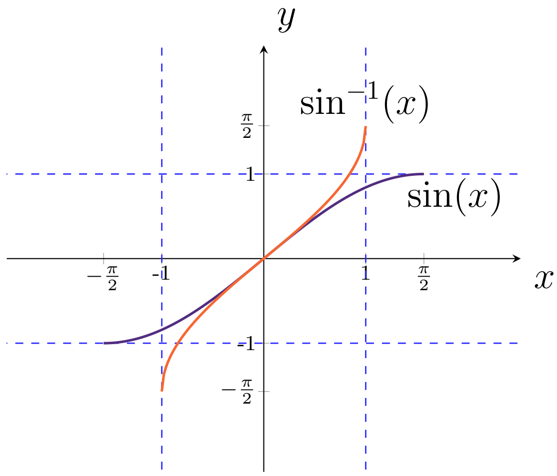
Example. Solve the following:

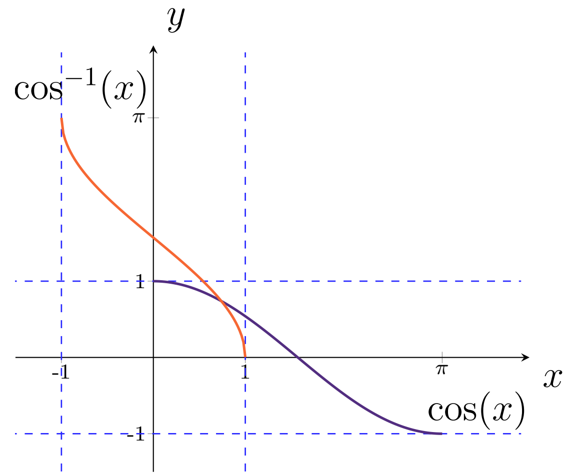
$\sin^{-1}(0)$	$\arcsin(1)$	$\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$
$\cos^{-1}(-1)$	$\cos^{-1}\left(-\dfrac{1}{2}\right)$	$\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$

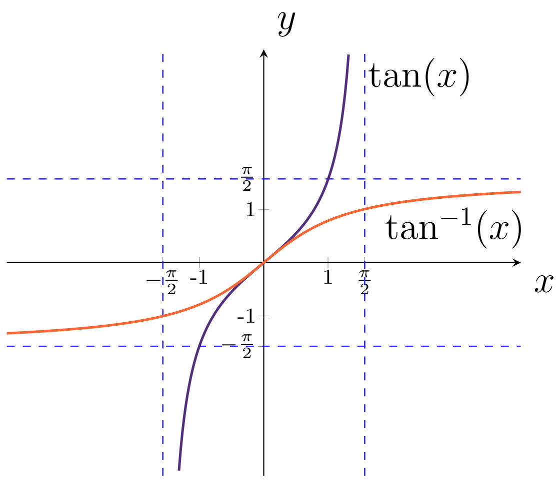
Definition. (Inverse Tangent and Secant)
$y = \tan^{-1}(x)$ is the value of y such that $x = \tan(y)$, where ${-\pi}/2 < y < \pi/2$.
The domain of $\tan^{-1}(x)$ is $\{x \mid -\infty < x < \infty\}$.
$y = \sec^{-1}(x)$ is the value of y such that $x = \sec(y)$, where $0 \leq y \leq \pi,\ y \neq \pi/2$.
The domain of $\sec^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$

Function	Restricted Domain	Range
$\sin(x)$	$[-\pi/2, \pi/2]$	$[-1, 1]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$
$\tan(x)$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
$\cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$\sec(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$

Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$


y
$\sin^{-1}(x)$
$\sin(x)$
x
$\frac{\pi}{2}$
1
-1
$-\frac{\pi}{2}$


y
$\cos^{-1}(x)$
π
1
-1
$\cos(x)$
x


y
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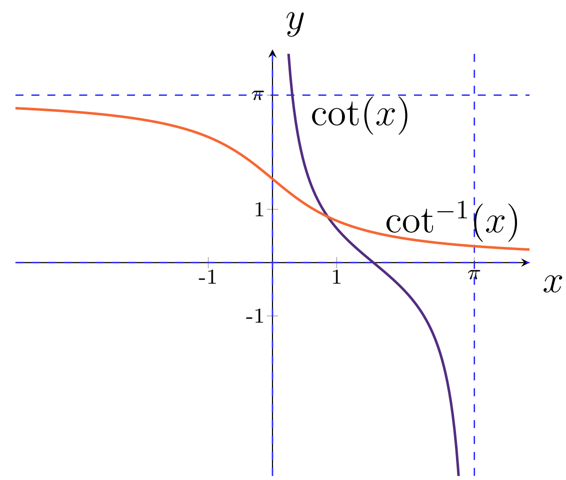
x


y
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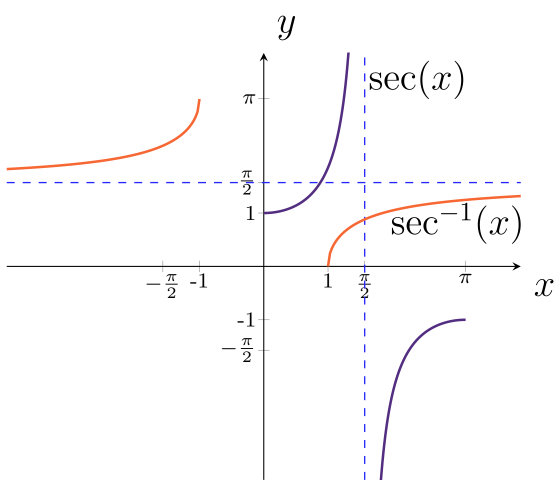
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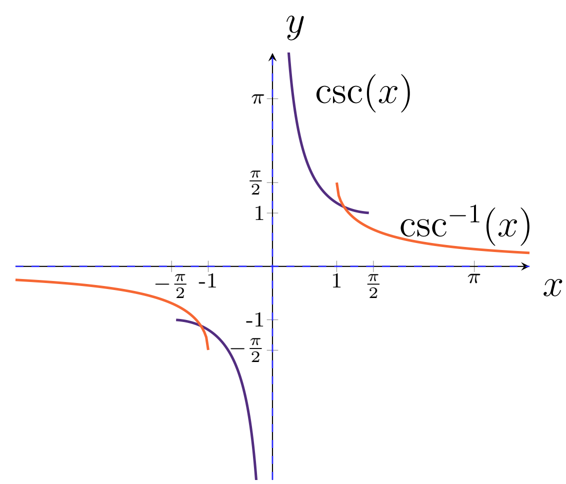

y
$\csc(x)$
$\csc^{-1}(x)$
π
$\frac{\pi}{2}$
1
-1
$-\frac{\pi}{2}$
x

Example. Solve the following:

$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ $\qquad$ $\sec^{-1}(2)$ $\qquad$ $\cot^{-1}\left(-\sqrt{3}\right)$

While $\sin(x)$ and $\sin^{-1}(x)$ are inverse functions, the inverse relationship only holds when working in the correct domains:

$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0 \neq \pi$ $\qquad$ $\sin\left(\sin^{-1}(-1)\right) = \sin(-\pi/2) = -1$

Example. Solve the following:

$\tan(\tan^{-1}(5))$ $\qquad$ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$\cos\left(\arcsin\frac{1}{2}\right)$ $\qquad$ $\cos^{-1}(\cos(5\pi))$

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right) \qquad \tan\left(\sec^{-1}(10)\right) \qquad \sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$$

Example. Simplify the following using triangles.

$\cos\left(\tan^{-1}(x)\right)$

$\sec\left(\sin^{-1}(x)\right)$

$\cos\left(2\sin^{-1}(x)\right)$

$\sin\left(2\tan^{-1}(x)\right)$

2.1 The Idea of Limits

Definition.
The **average velocity** is the distance traveled over some time period.
The **instantaneous velocity** is the limit of the average velocities as the length of the time period goes to zero.

Example. An unladen swallow is flying from Camelot to the Castle Anthrax and back. It's current position, in miles, is given by

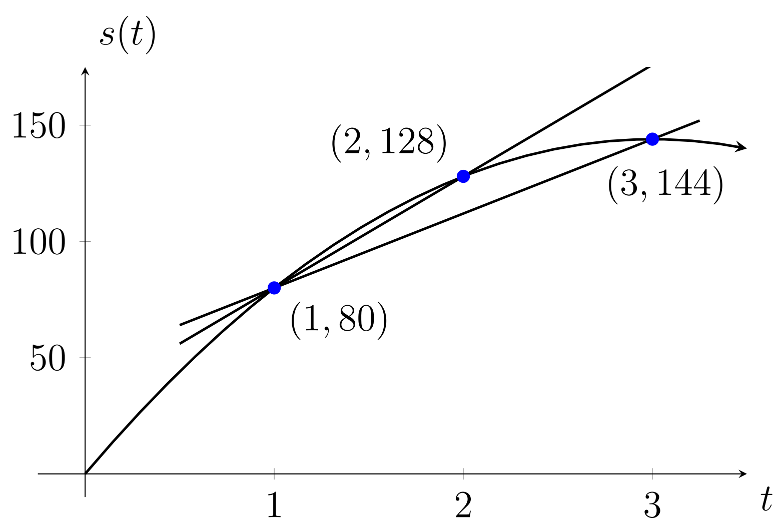
$$s(t) = -16t^2 + 96t$$

where t is given in hours. Find the average velocity between:

a) $t = 1$ and $t = 3$,

b) $t = 1$ and $t = 2$,

Example. Find the instantaneous velocity using $s(t)$ by computing the average velocity between $t = 1$ and $t = h$:



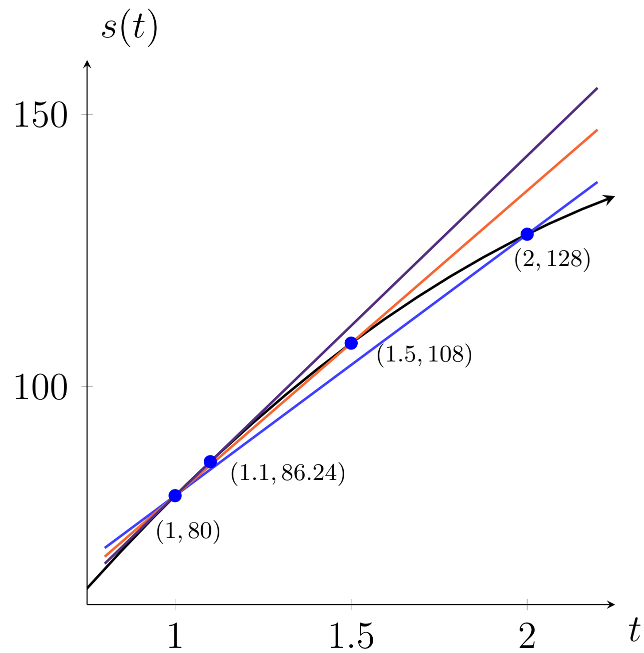
Definition.
The **secant line** is the line that intersects the function in two places.
The **tangent line** is the line that intersects the function in exactly one place (locally).

Note– The average velocity is the slope of the secant line. The instantaneous velocity is the slope of the tangent line.

Example. Using the average velocity between $t = 1$ and $t = h$:

$$v_{avg} = -16(h - 5)$$

compute the instantaneous velocity at $h = 1$.



Example. Find the instantaneous velocity of

$$f(x) = 2x^2 - 4x + 1$$

for any value of x.

2.2 Definitions of Limits

Definition.
(Briggs) Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

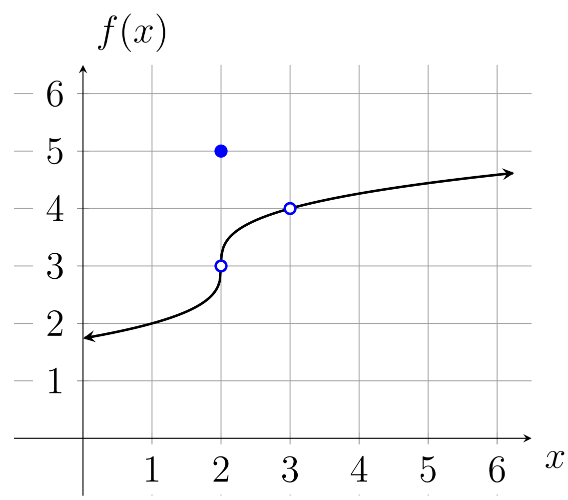
$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

Note– Most of the time, we can think of the limit as the value of the function if it could be evaluated at a specific point.

Example. Using the graph of f, determine the following values:

- $f(1)$ and $\lim_{x \to 1} f(x)$



- $f(2)$ and $\lim_{x \to 2} f(x)$

- $f(3)$ and $\lim_{x \to 3} f(x)$

Definition.
(Briggs)

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

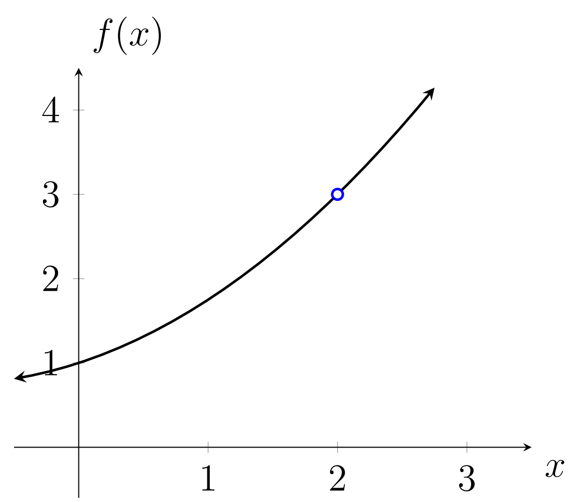
$$\lim_{x \to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

Example. For $f(x) = \dfrac{x^3 - 8}{4(x-2)}$, find

- $\lim\limits_{x \to 2^+} f(x)$

- $\lim\limits_{x \to 2^-} f(x)$



Definition.
(Briggs) **Relationship Between One-Sided and Two-Sided Limits**
Assume f is defined for all x near a except possibly at a. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

Example. For $f(x)$ above, is $\lim_{x\to 2} f(x)$ defined? If so, what is it? What is $f(2)$?

Example. Consider the graph of

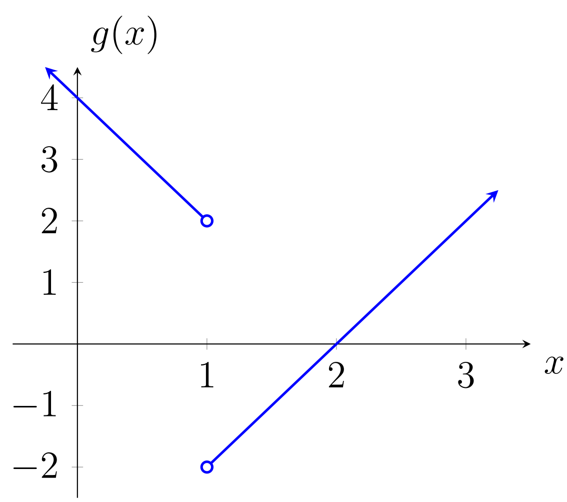
$$g(x) = \frac{2x^2 - 6x + 4}{|x-1|} = \begin{cases} -2(x-2) & x < 1 \\ 2(x-2) & x > 1 \end{cases}$$

Find

- $\lim_{x\to 1^-} g(x)$

- $\lim_{x\to 1^+} g(x)$

- $\lim_{x\to 1} g(x)$



Example. Consider the function

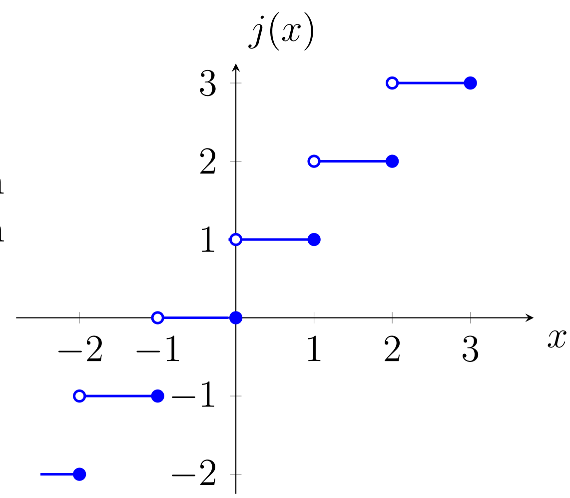
$$h(x) = \frac{x^2 - 81}{2x + 18}$$

What does this function look like? What is $h(-9)$? What is $\lim_{x \to -9} h(x)$?

Example. The ceiling function is

$$j(x) = \lceil x \rceil$$

where $\lceil x \rceil$ returns the smallest integer greater than or equal to x. In other words, the ceiling function always rounds up. Find the following:



$\lim_{x \to 1^-} j(x)$

$\lim_{x \to 1^+} j(x)$

$\lim_{x \to 1} j(x)$

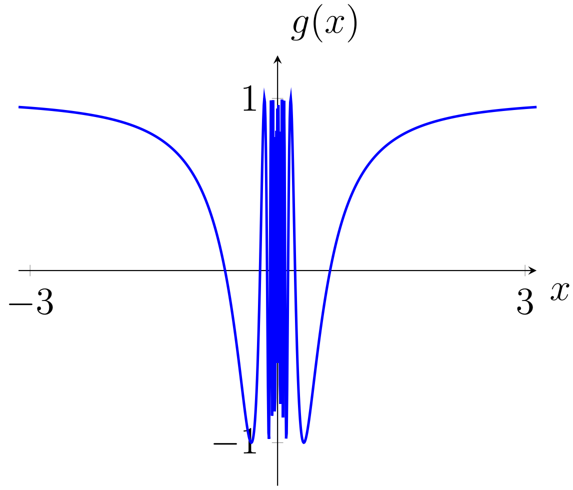
$j(1)$

$\lim_{x \to 1.5^-} j(x)$ $\lim_{x \to 1.5^+} j(x)$ $\lim_{x \to 1.5} j(x)$ $j(1.5)$

Example. Consider the function

$$h(x) = \cos\left(\frac{1}{x}\right)$$

What is $\lim\limits_{x\to 0} h(x)$?



Consider $x = 1/(n\pi)$. As $n \to \infty, x \to 0$, then,

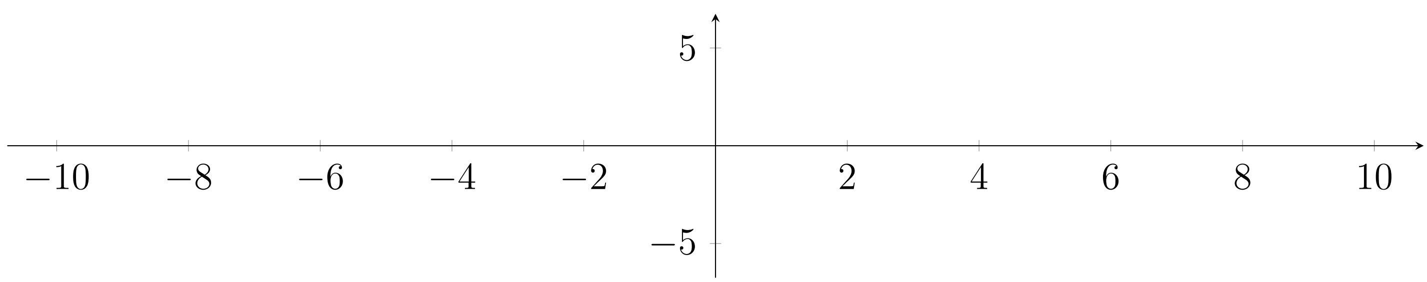
$$\cos\left(\frac{1}{x}\right) = \cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

Example. Graph an example with the following characteristics:

$$\lim_{x\to -2^-} f(x) = -4 \qquad \lim_{x\to -2^+} f(x) = 2 \qquad f(-2) = 0$$

$$\lim_{x\to 4} f(x) = 2 \qquad f(4)\ \textbf{DNE}$$

$$\lim_{x\to 8} f(x) = -2 \qquad f(8) = -2$$



2.3 Techniques for Computing Limits

Example.

a) $\lim_{x\to 3} \frac{1}{2}x - 7$

b) $\lim_{x\to 2} 6$

Definition. (Briggs)
Limit Laws: Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following properties hold, where c is a real number, and $n > 0$ is an integer.

1. **Sum:** $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. **Difference:** $\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. **Constant multiple:** $\lim_{x\to a} (cf(x)) = c \lim_{x\to a} f(x)$
4. **Product:** $\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
5. **Quotient:** $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
6. **Power:** $\lim_{x\to a} (f(x))^n = (\lim_{x\to a} f(x))^n$
7. **Root:** $\lim_{x\to a} (f(x))^{1/n} = (\lim_{x\to a} f(x))^{1/n}$

Example. Suppose $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$ and $\lim_{x\to 2} h(x) = 8$.

a) $\lim_{x\to 2} \dfrac{f(x) - g(x)}{h(x)}$

b) $\lim_{x\to 2} (6f(x)g(x) + h(x))$

c) $\lim_{x\to 2} (g(x))^3$

Example. For $g(x) = \dfrac{x+6}{x^2 - 36}$, find

1. $\lim_{x\to 0} g(x)$

2. $\lim_{x\to -6} g(x)$

Example. $\lim_{x \to 2} \dfrac{\sqrt{2x^3+9}+3x-1}{4x+1}$

Example. $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} -2x+4 & \text{if } x \leq 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$

Example. $\lim\limits_{x \to 2} \dfrac{x^2 - 6x + 8}{x^2 - 4}$

Example. $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

Example. $\lim\limits_{x \to -4} \sqrt{16 - x^2}$

Example. $\lim_{x\to 2} \dfrac{x^3 - 6x^2 + 8x}{\sqrt{x-2}}$

Example. $\lim_{y\to a} \dfrac{(y-a)^{12} + 6y - 6a}{y-a}$

The Squeeze Theorem: Assume the functions f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Example. Consider the function $f(x) = x^2 \sin({}^1\!/_x)$. What is $\lim\limits_{x\to 0} f(x)$?

Example. Use the squeeze theorem on $-|x| \leq x \sin \frac{1}{x} \leq |x|$.

Example. $\lim_{x \to 0} \dfrac{\sin^2 x}{1 - \cos x}$

Example. $\lim_{x \to 0} \dfrac{1 - \cos 2x}{\sin x}$

2.4 Infinite Limits

An infinite limit occurs when function values increase or decrease without bound near a point.

Limits which have an infinite value are called **infinite limits**. They are a special case of limits that do not exist, but we indicate that they approach infinity.

Example. Consider the function
$f(x) = \frac{1}{x}$

$\lim_{x \to 0^+} f(x)$

$\lim_{x \to 0^-} f(x)$

$\lim_{x \to \infty} f(x)$

$\lim_{x \to -\infty} f(x)$

Consider $f(x) = \dfrac{1}{(x-2)^2}$.
Find $\lim_{x \to 2} f(x)$.

Consider $g(x) = \dfrac{1}{x+1}$.
Find $\lim_{x \to -1} g(x)$.

Consider $h(x) = -\dfrac{1}{(x+3)^4}$.
Find $\lim_{x \to -3} h(x)$.

Definition.
Infinite Limits
Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

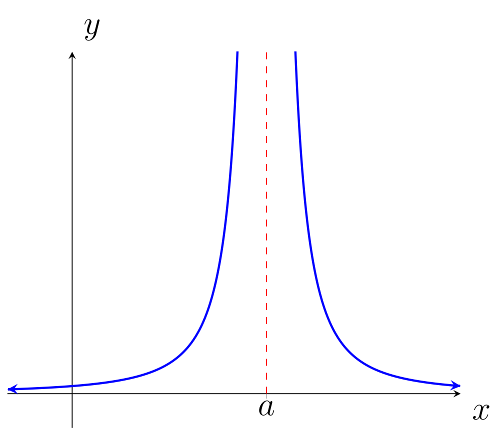
$$\lim_{x \to a} f(x) = \infty$$

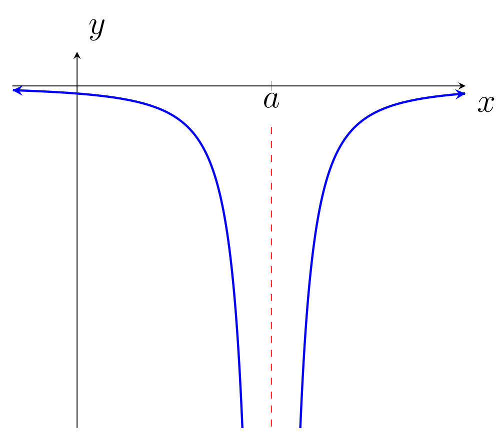
and say the limit of $f(x)$ as x approaches a is infinity.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty$$

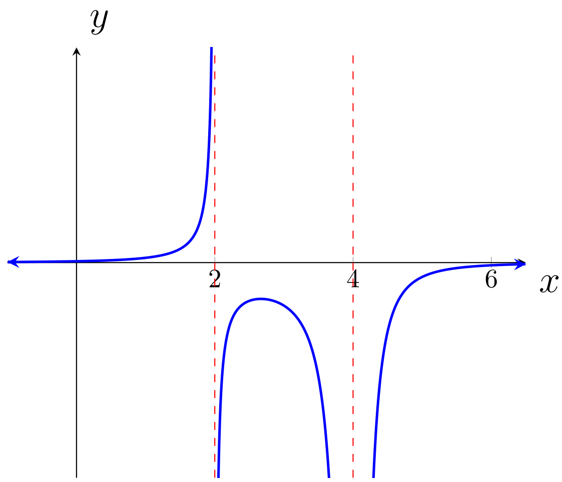
and say the limit of $f(x)$ as x approaches a is negative infinity.
In both cases, the limit does not exist.





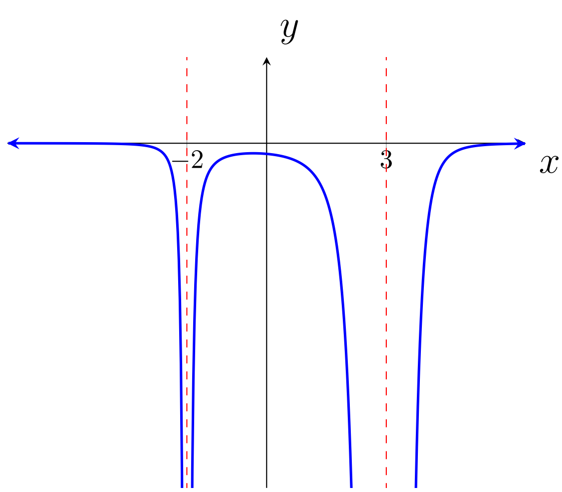
Definition.
Vertical Asymptote
If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

Example. The graph of $\ell(x)$ has vertical asymptotes $x = 2$ and $x = 4$. Find the following limits:



1. $\lim_{x \to 2^-} \ell(x)$
2. $\lim_{x \to 2^+} \ell(x)$
3. $\lim_{x \to 2} \ell(x)$
4. $\lim_{x \to 4^-} \ell(x)$
5. $\lim_{x \to 4^-} \ell(x)$
6. $\lim_{x \to 4} \ell(x)$

Example. The graph of $p(x)$ has vertical asymptotes $x = -2$ and $x = 3$. Find the following limits:



1. $\lim_{x \to -2^-} p(x)$
2. $\lim_{x \to -2^+} p(x)$
3. $\lim_{x \to -2} p(x)$
4. $\lim_{x \to 3^-} p(x)$
5. $\lim_{x \to 3^+} p(x)$
6. $\lim_{x \to 3} p(x)$

Note: When computing the limit, $\lim_{x \to a} f(x)$ we can try to evaluate $f(a)$.

If $f(a)$ is of the form $\frac{0}{0}$, try factoring, conjugates, etc. (Section 2.3)

If $f(a)$ is of the form $\frac{c}{0}$ where $c \neq 0$, the limit is infinite. Here, we must consider the signs of the numerator and the denominator.

$$\lim_{x \to 3^+} \frac{\overbrace{2-5x}^{-13}}{\underbrace{x-3}_{\text{small pos}}} = -\infty \qquad\qquad \lim_{x \to 3^-} \frac{\overbrace{2-5x}^{-13}}{\underbrace{x-3}_{\text{small neg}}} = \infty$$

Example. Evaluate:

a) $\lim_{x \to 3^-} \frac{2}{(x-3)^3}$ b) $\lim_{x \to 3^+} \frac{2}{(x-3)^3}$ c) $\lim_{x \to 3} \frac{2}{(x-3)^3}$

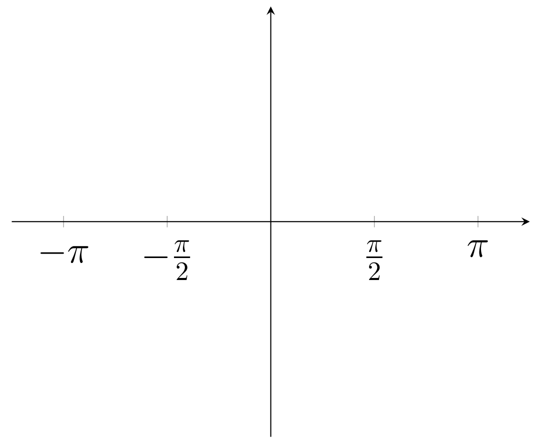
Example. For $h(t) = \frac{t^2 - 4t + 3}{t^2 - 1}$, find $\lim_{t \to 1} h(t)$ and $\lim_{t \to -1} h(t)$.

Are these infinite limits or limits at infinity?

Example. Evaluate $\lim\limits_{\nu \to 7} \dfrac{4}{(\nu - 7)^2}$.

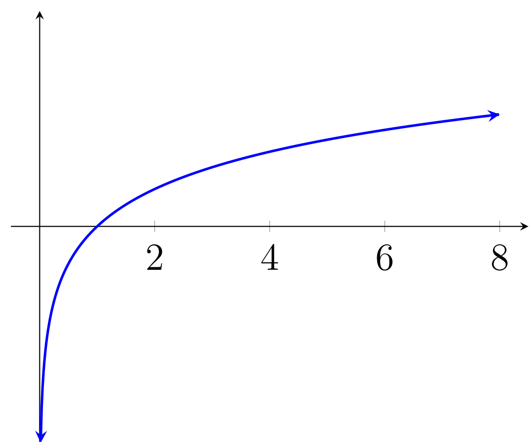
Example. Evaluate $\lim\limits_{r \to 1} \dfrac{r}{|r - 1|}$.

Example. Evaluate



- $\lim\limits_{x \to \pi/2^-} \tan x$
- $\lim\limits_{x \to \pi/2^+} \tan x$
- $\lim\limits_{x \to -\pi/2^-} \tan x$
- $\lim\limits_{x \to -\pi/2^+} \tan x$

Example. Below is the graph of $\ln(x)$. Use this to evaluate the following limits:



- $\lim\limits_{x \to 0^+} \ln(x)$
- $\lim\limits_{x \to \infty} \ln(x)$

Example. Find all vertical asymptotes, $x = a$, for $f(x) = \dfrac{\cos x}{x^2 + 2x}$.

2.5 Limits at Infinity

Definition.
Limits at Infinity and Horizontal Asymptotes
If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case, the line $y = L$ is a **horizontal asymptote** of f. The limit at negative infinity,

$$\lim_{x\to-\infty} f(x) = M$$

is defined analogously. When this limit exists, $y = M$ is a horizontal asymptote.

Note: The function *can* cross it's horizontal asymptote (consider $\frac{\sin x}{x}$).

Note: A function can have 0, 1 or 2 horizontal asymptotes.

Example. For each of the following functions, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

a) $f(x) = \dfrac{1}{x^2}$

b) $f(x) = \dfrac{1}{x^3}$

c) $f(x) = 2 + \dfrac{10}{x^2}$

d) $f(x) = 5 + \dfrac{\sin x}{\sqrt{x}}$

e) $f(x) = \left(5 + \dfrac{1}{x} + \dfrac{10}{x^2}\right)$

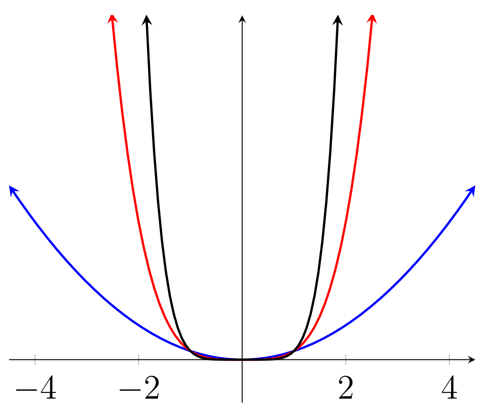
f) $f(x) = \left(3x^{12} - 9x^7\right)$

g) $f(x) = \sin(x)$

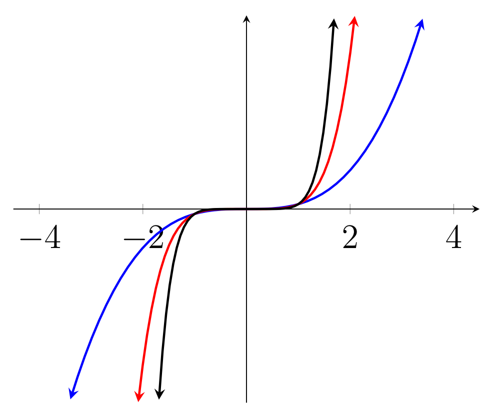
h) $f(x) = \dfrac{\sin x}{x}$

> **Definition.**
> **Infinite Limits at Infinity** If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write
> $$\lim_{x\to\infty} f(x) = \infty$$
> The limits $\displaystyle\lim_{x\to\infty} = -\infty$, $\displaystyle\lim_{x\to-\infty} = \infty$ and $\displaystyle\lim_{x\to-\infty} = -\infty$ are defined similarly.

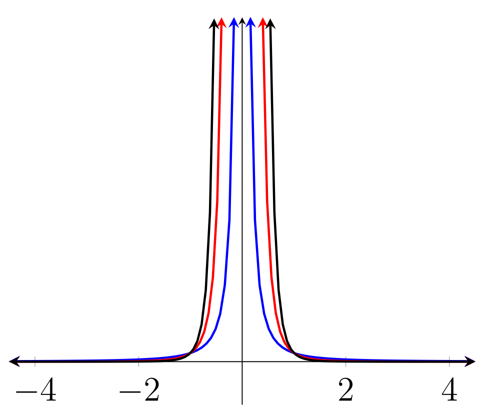
Even functions



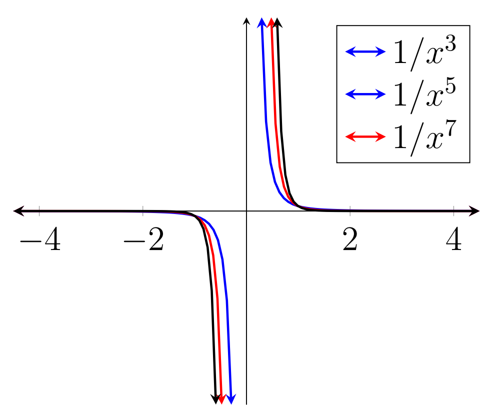
Odd functions



$1/x^n$: n Even



$1/x^n$: n Odd



Theorem. Limits at Infinity of Powers and Polynomials
Let n be a positive integer and let p be the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \text{ where } a_n \neq 0.$$

1. $\lim\limits_{x \to \pm\infty} x^n = \infty$ when n is even.
2. $\lim\limits_{x \to \infty} x^n = \infty$ and $\lim\limits_{x \to -\infty} x^n = -\infty$ when n is odd.
3. $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^n} = \lim\limits_{x \to \pm\infty} x^{-n} = 0$.
4. $\lim\limits_{x \to \pm\infty} p(x) = \lim\limits_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Note: All previous limit laws still apply (e.g. constant multiplier rule)

Note: This theorem *ONLY* applies for $x \to \pm\infty$. When $x \to a, |a| < \infty$, we compute the left and right limits and use sm+/sm- (as done in section 2.4).

Example. For the following, find the limits as $x \to -\infty$ and $x \to \infty$:

$f(x) = 2x^{-8}$

$g(x) = -12x^{-5}$

$h(x) = 3x^{12} - 9x^7$

$\ell(x) = 2x^{-8} + 4x^3$

When finding the limit as $x \to \pm\infty$ of a rational function, $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, we multiply the function by $\dfrac{1/x^n}{1/x^n}$, where n is the highest degree in the denominator $q(x)$.

Note: To receive full credit for questions of this type, you must show all the fractions in your intermediate steps.

Example.

a) $\displaystyle\lim_{x\to\infty} \frac{1-x}{2x}$

b) $\displaystyle\lim_{x\to\infty} \frac{1-x}{x^2}$

c) $\displaystyle\lim_{x\to\infty} \frac{1-x^2}{2x}$

Theorem. End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

with $a_m \neq 0$ and $b_n \neq 0$.

1. **Degree of numerator less than degree of denominator**

   If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of f.

2. **Degree of numerator equals degree of denominator**

   If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m/b_n$ and $y = a_m/b_n$ is a horizontal asymptote of f.

3. **Degree of numerator greater than degree of denominator**

   If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$ and f has no horizontal asymptote.

4. **Slant Asymptote**

   If $m = n+1$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote, but f has a slant asymptote.

5. **Vertical asymptotes**

   Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Example. Evaluate the limits of the following as $x \to -\infty$ and $x \to \infty$. State the equation of the horizontal asymptote.

1. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

2. $g(x) = \dfrac{1}{x^3 - 4x + 1}$

3. $h(x) = \dfrac{3x^5 + 2x^2 - 2}{4x^4 - 3x}$

4. $j(x) = \dfrac{4x^2 - 2x + 3}{7x^2 - 1}$

5. $\ell(x) = \dfrac{1 - x^2}{3 + 2x - x^3}$

Definition.
When the degree of the numerator, m, is greater than the degree of the denominator, n, the function has an oblique asymptote:

$$f(x) = \frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)}$$

where $a(x)$ is the resulting polynomial that we get from polynomial long division and $r(x)$ is the remainder. We are interested in the special case where $m = n + 1$, and $f(x)$ has a **slant asymptote**.

Example. For the following functions, find the vertical asymptotes and the slant asymptotes:

1. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$

2. $f(x) = \dfrac{x^2 - 1}{x + 2}$

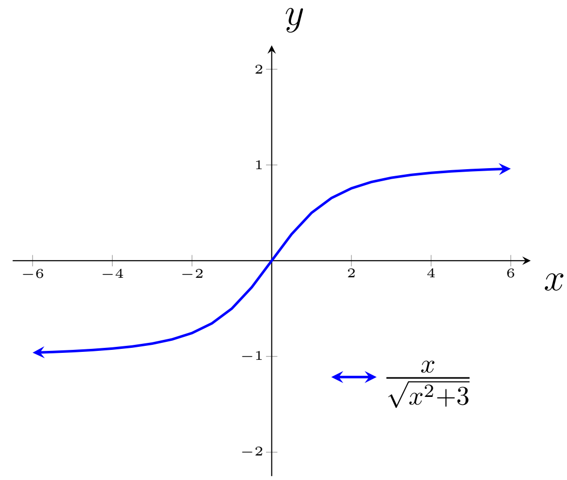
3. $g(t) = \dfrac{t^2 - 1}{2t + 4}$

4. $h(u) = \dfrac{u^2}{u - 1}$

If the denominator has a square root, we need to change our work depending on if $x \to -\infty$ or $x \to \infty$:

Example. For the following, find the equation of the horizontal asymptotes:

a) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+3}}$



b) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+3}}$

c) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}}$

d) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}}$

e) $\dfrac{7x^3 - 2}{-x^3 + \sqrt{25x^6 + 4}}$

f) $\dfrac{\sqrt[3]{x^6 + 8}}{4x^2 + \sqrt{3x^4 + 1}}$

g) $\dfrac{2x}{\sqrt{x^2 - x - 2}}$

Example. For the following, sketch a graph with the following properties:

1. $\lim_{x \to 0} f(x) = -\infty$

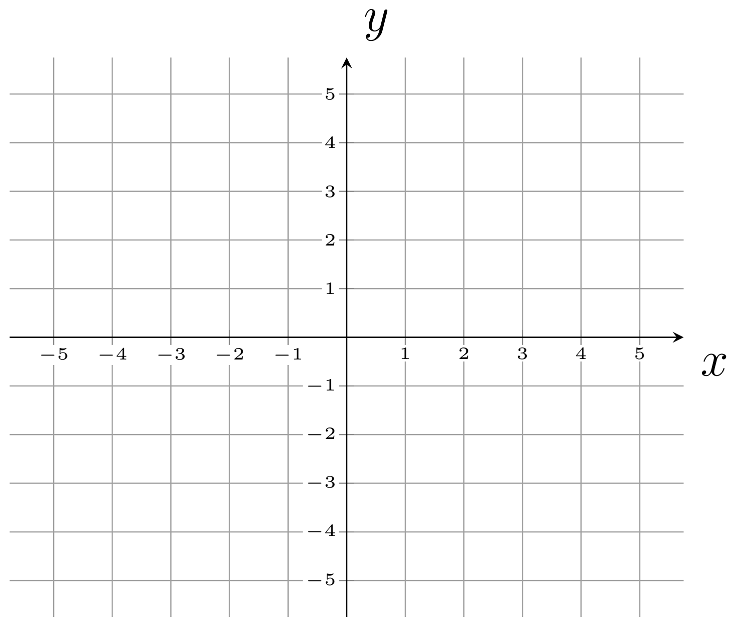
   $\lim_{x \to 2} f(x) = \frac{5}{4}$

   $\lim_{x \to \pm\infty} f(x) = 1$

   $f(2)$ DNE

   $f(1) = 1$

   $f(-1) = -1$



2. $\lim_{x \to -1^-} f(x) = \infty$

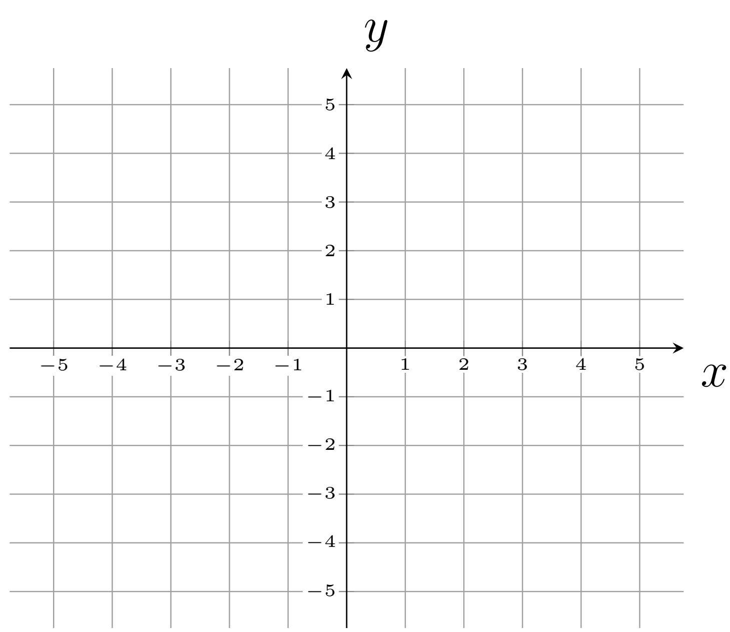
   $\lim_{x \to -1^+} f(x) = -\infty$

   $\lim_{x \to \pm\infty} f(x) = 2$

   $f(0) = -2$

   $f(1) = 1$

   $f(-2) = 4$

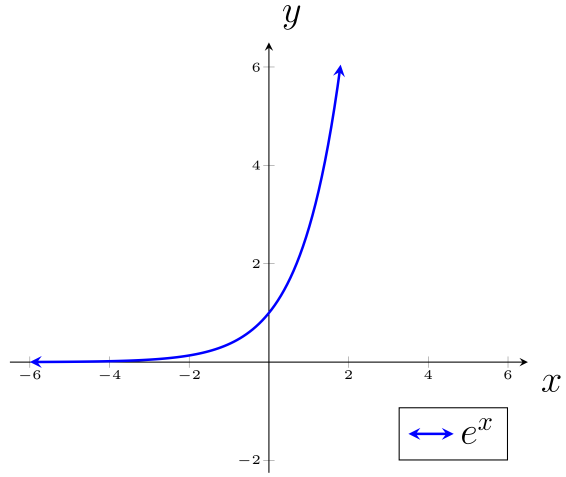


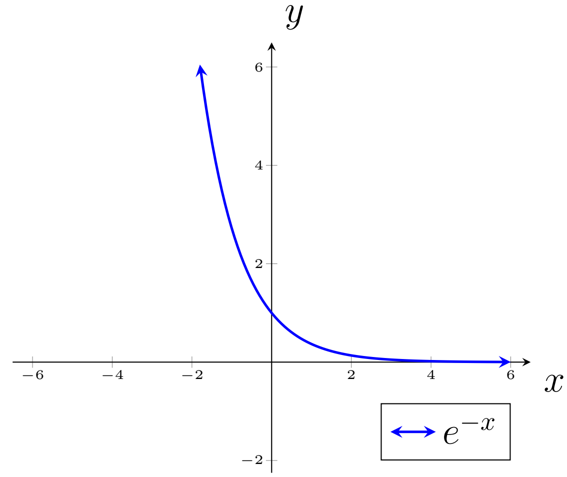
Example. Find *all* asymptotes (vertical, horizontal, slant)

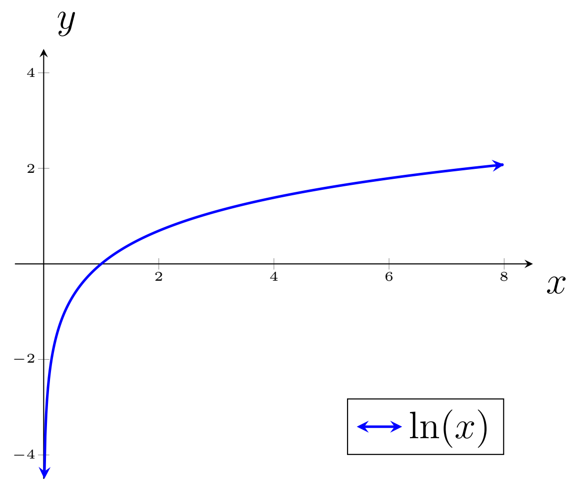
1. $\dfrac{x^3 - 10x^2 + 16x}{x^2 - 8x}$

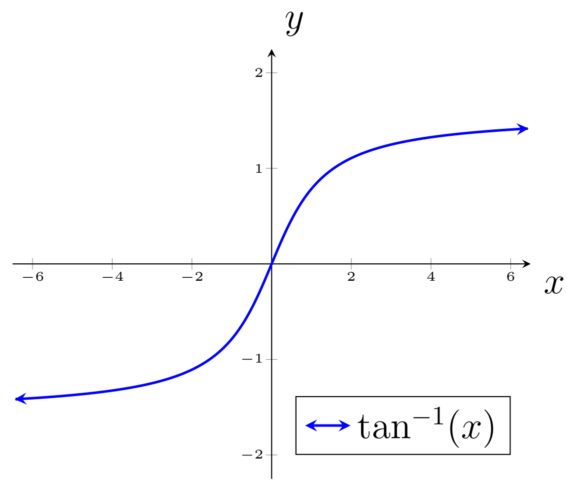
2. $\dfrac{\cos x + 2\sqrt{x}}{\sqrt{x}}$

Other function end behavior to consider include $e^x, e^{-x}, \ln(x)$ and $\tan^{-1}(x)$:









a) $\lim\limits_{x \to -\infty} \sin x$

b) $\lim\limits_{x \to \infty} \sin x$

c) $\lim\limits_{x \to -\infty} \cos x$

d) $\lim\limits_{x \to \infty} \cos x$

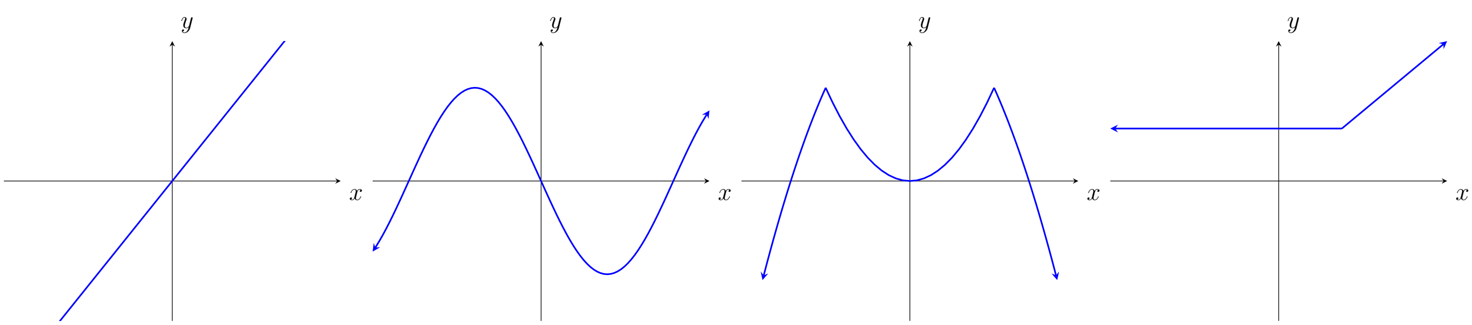
2.6 Continuity

Definition. (Continuity at a point)
A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

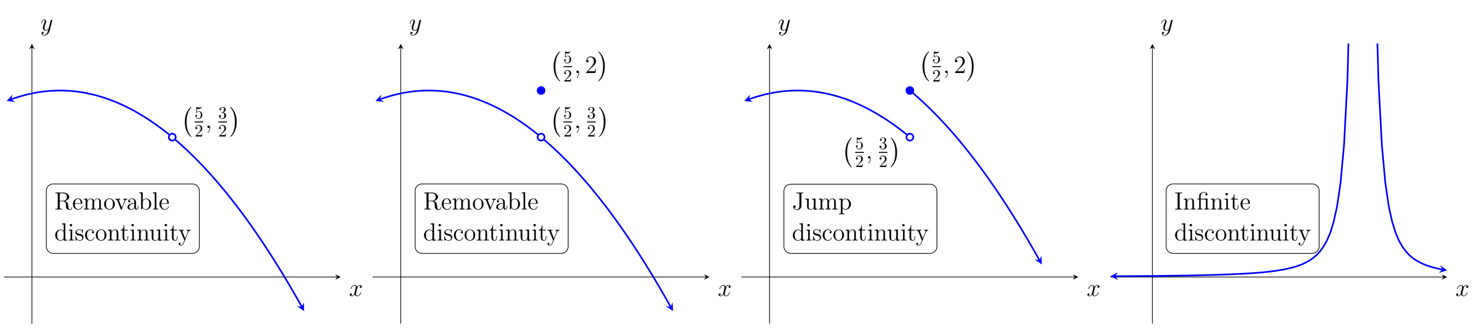
Continuity Checklist:
In order for f to be continuous at a, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f),
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Graphically:



Types of discontinuity:

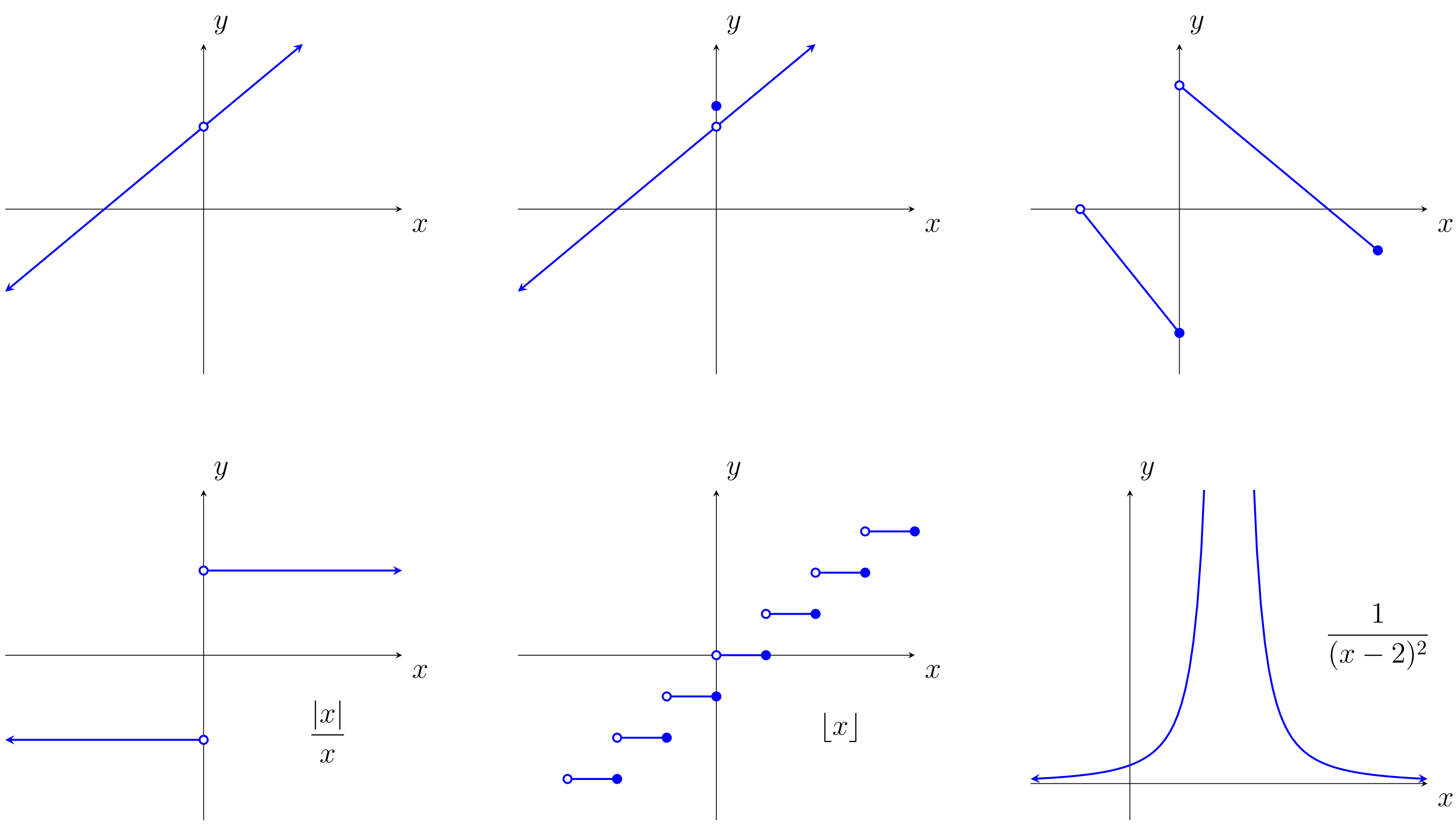


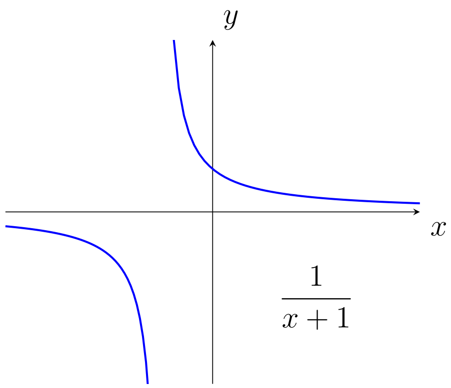
Definition.

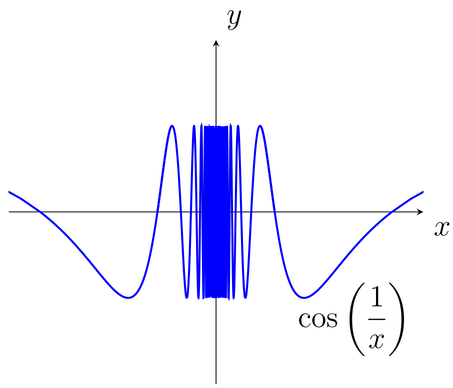
A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

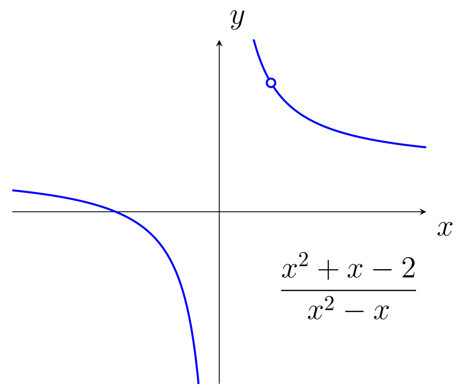
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

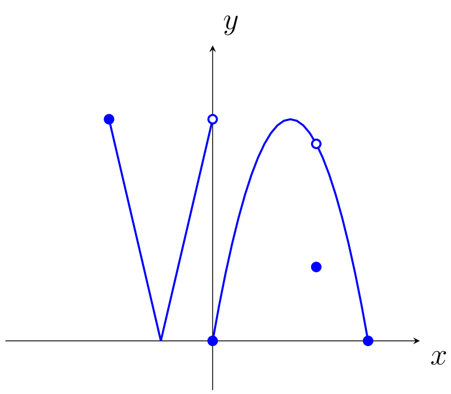
A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

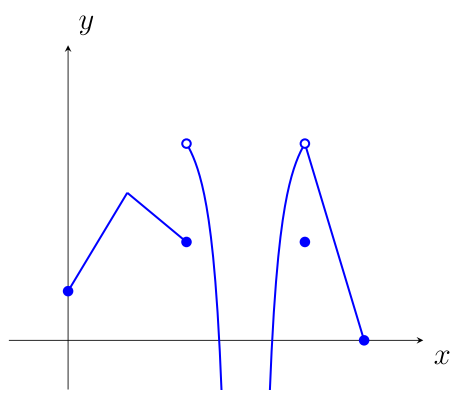


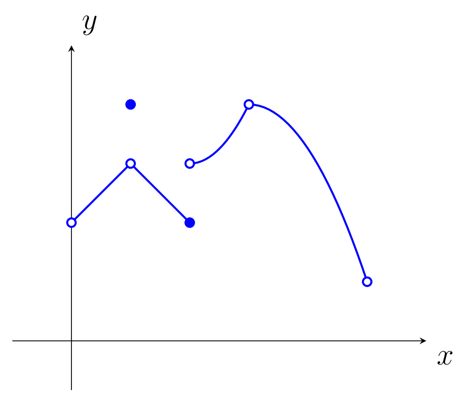






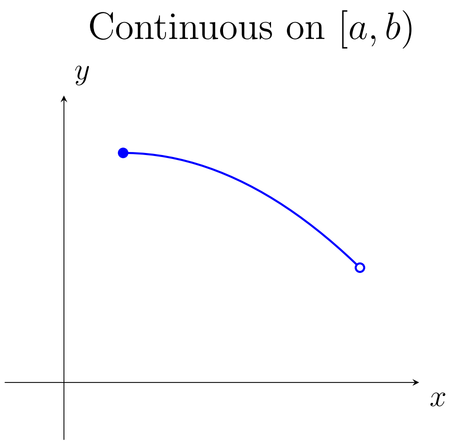






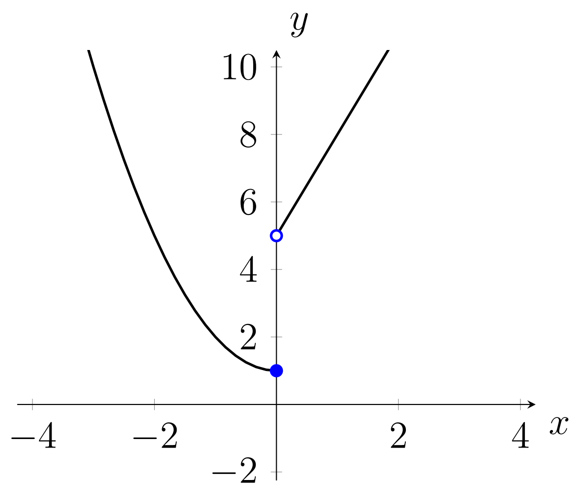
Definition. (Continuity at Endpoints)
A function f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x\to a^+} f(x) = f(a)$, and f is **continuous from the left** (or **left-continuous**) at b if $\lim_{x\to b^-} f(x) = f(b)$.

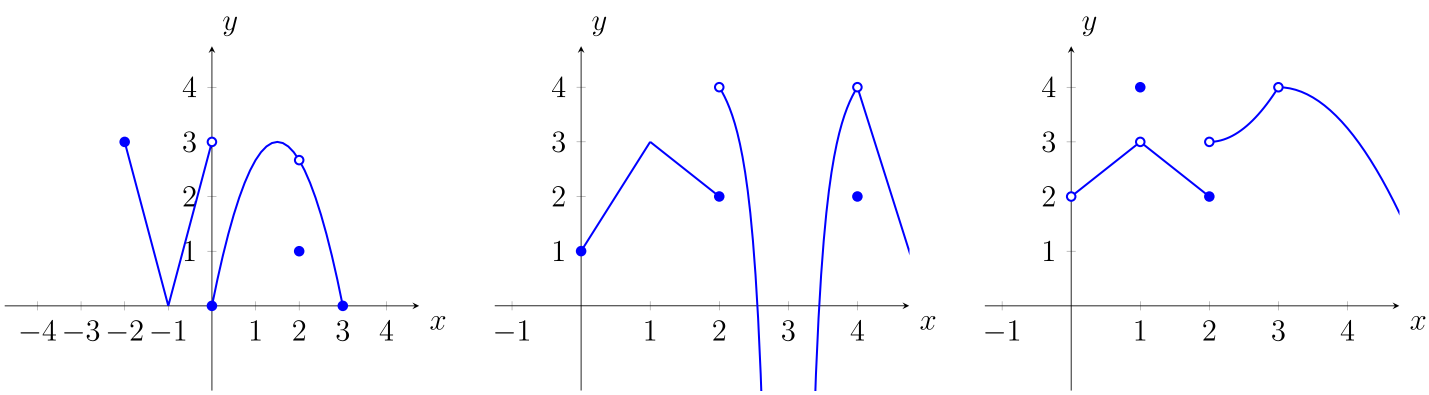
Definition. (Continuity on an Interval)
A function f is **continuous on an interval I** if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.



Example. Determine the interval of continuity for the following:

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ 3x + 5, & x > 0 \end{cases}$$





Example. Determine whether the following are continuous at a:

$$f(x) = x^2 + \sqrt{7-x},\ a = 4$$

$$g(x) = \frac{1}{x-3},\ a = 3$$

$$h(x) = \begin{cases} \dfrac{x^2+x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases},\ a = -1$$

$$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases},\ a = 0$$

$$k(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases},\ a = 2$$

Theorem 2.9: Continuity Rules
If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a) $f + g$

b) $f - g$

c) cf

d) fg

e) f/g, provided that $g(a) \neq 0$.

f) $(f(x))^n$

Theorem 2.10: Polynomial and Rational Functions

a) A polynomial function is continuous for all x.

b) A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Theorem 2.11: Continuity of Composite Functions at a Point
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Theorem 2.12: Limits of Composite Functions

1. If g is continuous at a and f is continuous at $g(a)$, then
$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$
2. If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then
$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$

Theorem 2.13: Continuity of Functions with Roots
Assume n is a positive integer. If n is an odd integer, then $(f(x))^{1/n}$ is continuous at all points at which f is continuous.
If n is even, then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous at $f(a) > 0$.

Theorem 2.14: Continuity of Inverse Functions
If a function f is continuous on an interval I and has an inverse on I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

Theorem 2.15: Continuity of Transcendental Functions
The following functions are continuous at all points of their domains.

Trigonometric		**Inverse Trigonometric**		**Exponential**	
$\sin x$	$\cos x$	$\sin^{-1} x$	$\cos^{-1} x$	b^x	e^x
$\tan x$	$\cot x$	$\tan^{-1} x$	$\cot^{-1} x$	**Logarithmic**	
$\sec x$	$\csc x$	$\sec^{-1} x$	$\csc^{-1} x$	$\log_b x$	$\ln x$

Example. Determine the intervals of continuity for the following functions:

a) $g(x) = \dfrac{3x^2 - 6x + 7}{x^2 + x + 1}$

b) $h(x) = \dfrac{3x^2 - 6x + 7}{x^2 - x - 1}$

c) $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

d) $t(x) = \dfrac{x^2 - 4x + 3}{x^2 + 1}$

e) $q(x) = \sqrt[3]{x^2 - 2x - 3}$

f) $r(x) = \sqrt{x^2 - 2x - 3}$

g) $a(x) = \sec x$

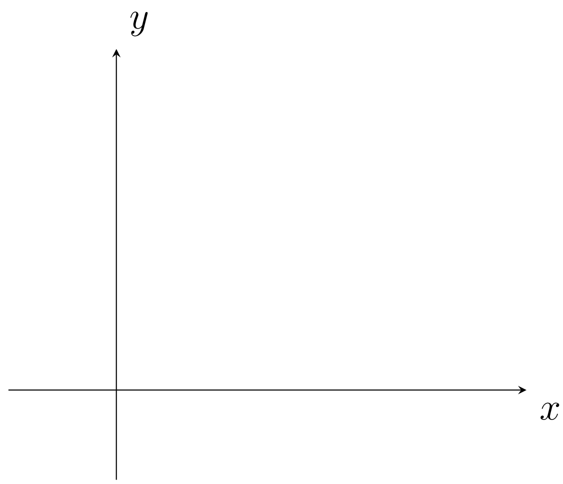
h) $b(x) = \sqrt{\sin x}$

i) $\ell(x) = \begin{cases} x^3 + 4x + 1, & x \leq 0 \\ 2x^3, & x > 0 \end{cases}$

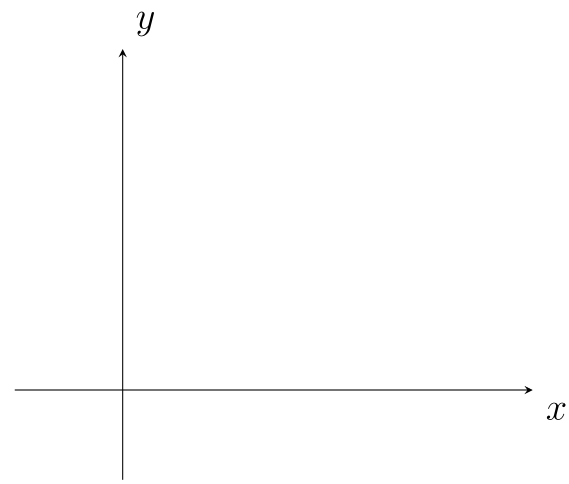
j) $m(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x \geq \frac{\pi}{4} \end{cases}$

Example. Sketch a function that:

Is defined, but not continuous at $x = 1$,



Has a limit, but not continuous at $x = 1$.



Example. Determine the value of a for which $f(x)$ is continuous:

1. $f(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ a, & x = 1 \end{cases}$

2. $f(t) = \begin{cases} \dfrac{t^2 + 3t - 10}{t - 2}, & t \neq 2 \\ a, & t = 2 \end{cases}$

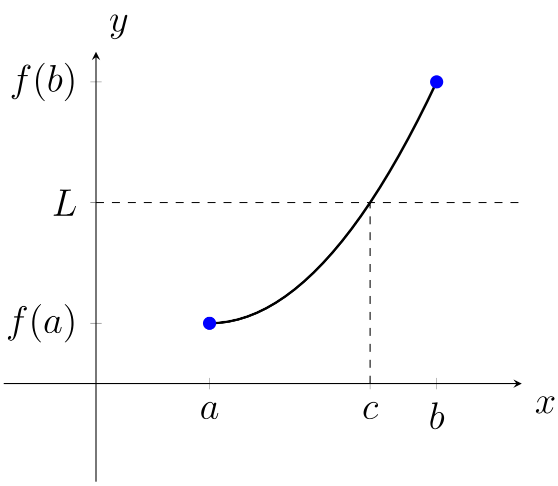
3. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}$

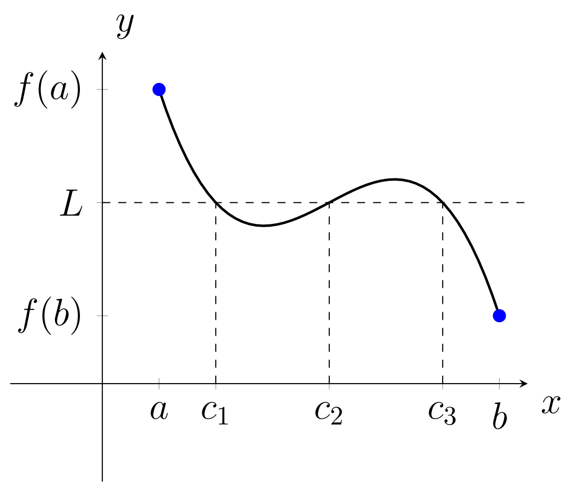
Example. Redefine the following functions so that they are continuous everywhere:

1. $g(x) = \dfrac{x^3 - x^2 - 2x}{x-2}$

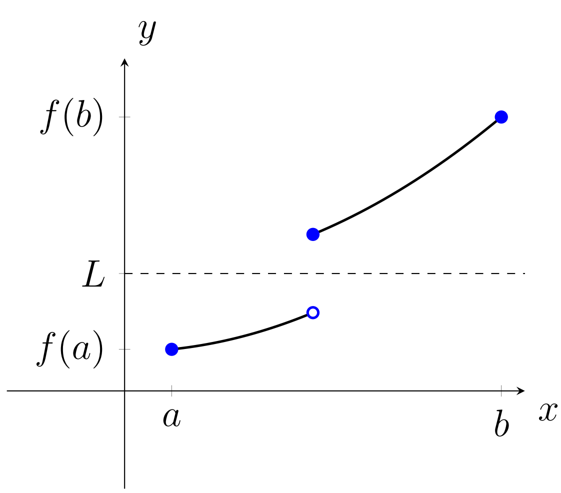
2. $g(x) = \dfrac{x^2 + x - 6}{x-2}$

Theorem 2.16: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Example. Show that $f(x)$ has a root using the IVT: $f(x) = x^3 + 4x + 4$

Example. Show that $\sqrt{x^4 + 25x^3 + 10} = 5$ on the interval $(0, 1)$.

Example. Show that $-x^5 - 4x^2 + 2\sqrt{x} + 5 = 0$ on $(0, 3)$.

2.7 Precise Definition of Limits

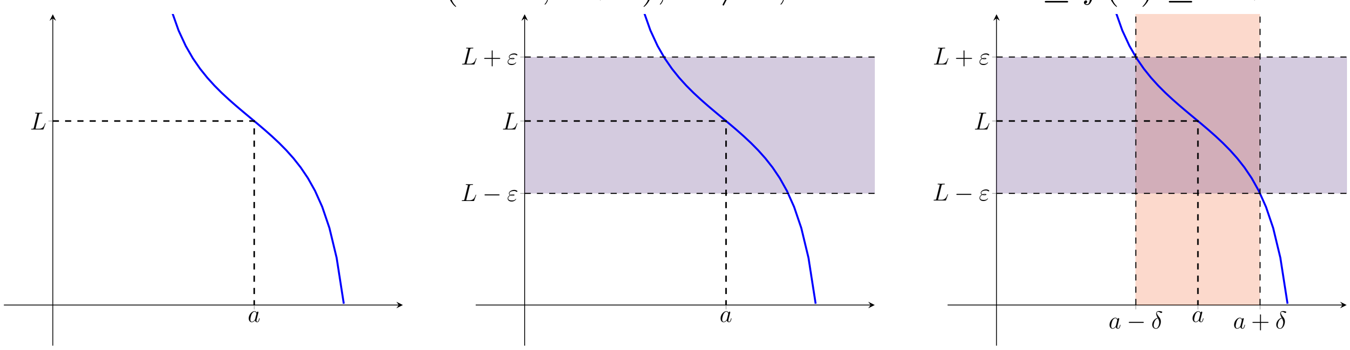
Definition. (Limit of a Function)
Assume $f(x)$ is defined for all x in some open interval containing a, except possibly at a. We say **the limit of $\boldsymbol{f(x)}$ as $\boldsymbol{x}$ approaches $\boldsymbol{a}$ is $\boldsymbol{L}$**, written

$$\lim_{x \to a} f(x) = L$$

if for *any* number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

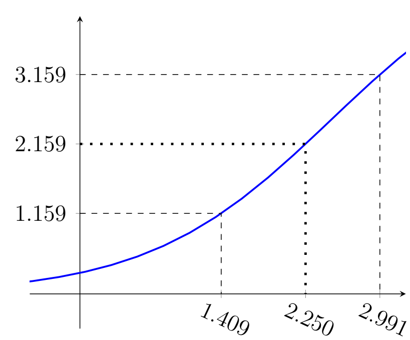
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

If we know L and $\varepsilon > 0$ is given, we can draw horizontal lines $L - \varepsilon$ and $L + \varepsilon$. Using the intersections of the graph and the horizontal lines, we can solve for $\delta > 0$ such that for values of x in the interval $(a - \delta, a + \delta)$, $x \neq a$, we have $L - \varepsilon \leq f(x) \leq L + \varepsilon$.

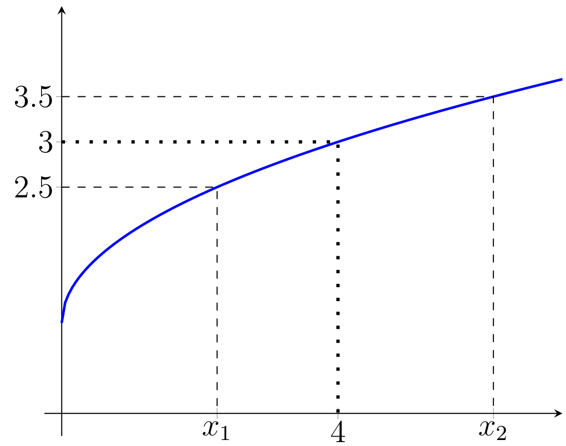


Note: As ε becomes smaller, δ will become smaller as well.

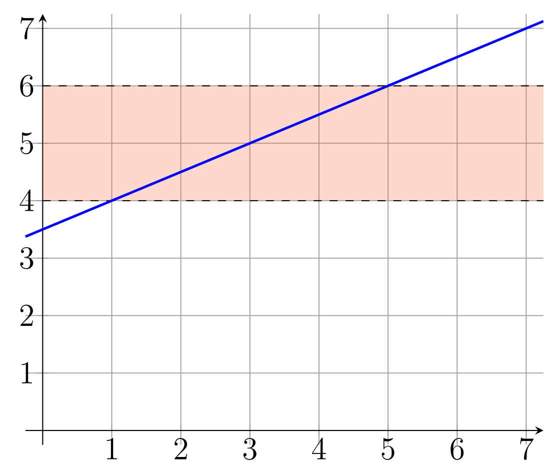
Example. Use the graph of f below to find a number δ such that if $0 < |x - 2.25| < \delta$ then $|f(x) - 2.159| < 1$.



Example. Use the graph of $g(x) = \sqrt{x}+1$ to help find a number δ such that if $|x-4| < \delta$ then $\left|\left(\sqrt{x}+1\right)-3\right| < \dfrac{1}{2}$.



Example. Use the graph of the following linear function where $\lim_{x\to 3} h(x) = 5$ to find $\delta > 0$ such that $|h(x)-5| < 1$ whenever $0 < |x-3| < \delta$.



Steps for proving that $\lim\limits_{x \to a} f(x) = L$

1. **Find $\boldsymbol{\delta}$.** Let ε be an arbitrary positive number. Use the inequality $|f(x) - L| < \varepsilon$ to find a condition of the form $|x - a| < \delta$, where δ depends only on the value of ε.
2. **Write a proof.** For any $\varepsilon > 0$, assume $0 < |x - a| < \delta$ and use the relationship between ε and δ found in Step 1 to prove that $|f(x) - L| < \varepsilon$.

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim\limits_{x \to 4}(2x - 5) = 3$.

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim\limits_{x \to 2} \dfrac{x}{5} = \dfrac{2}{5}$.

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2} = 5$.

Example. Use the $\varepsilon - \delta$ definition of a limit to prove $\lim_{x\to 3} \dfrac{x^2+2x-15}{2x-6} = 4$.

3.1 Introducing the Derivative:

Recall that when given a distance function $s(t)$, the average velocity over the interval $[a, t]$ is

$$v_{\text{avg}} = \frac{s(t) - s(a)}{t - a}$$

and the instantaneous velocity is the limit of the average velocities as $t \to a$:

$$v_{\text{inst}} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

Furthermore, the average velocity is the slope of the secant line through the points $(a, s(a))$ and $(t, s(t))$ and the instantaneous velocity is the slope of the tangent line at the point $(a, s(a))$. `https://www.desmos.com/calculator/08syaijrdo`

Definition. (Rate of Change and the Slope of the Tangent Line)

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. This **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is

$$y - f(a) = m_{\text{tan}}(x - a)$$

Example. Find an equation of the line tangent to the graph of $f(x) = \dfrac{3}{x}$ at $\left(2, \dfrac{3}{2}\right)$.

Definition. (Rate of Change and the Slope of the Tangent Line)

The **average rate of change** in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

Example. Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.

Definition. (The Derivative of a Function at a Point)

The **derivative of f at a**, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1) \quad \text{or } f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable at** a.

Example. Find an equation of the line tangent to the graph of $f(x) = \dfrac{8}{x^2}$ at $(2, 2)$.

Example. An equation of the line tangent to the graph of f at the $(2, 7)$ is $y = 4x - 1$. Find $f(2)$ and $f'(2)$.

Example. An equation of the line tangent to the graph of g at $x = 3$ is $y = 5x + 4$. Find $g(3)$ and $g'(3)$.

Example. If $h(1) = 2$ and $h'(1) = 3$, find an equation of the line tangent to the graph of h at $x = 1$.

Example. If $f'(-2) = 7$, find an equation of the line tangent to the graph of f at the point $(-2, 4)$.

3.2 The Derivative as a Function

Definition. (The Derivative Function)

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is **differentiable** at x. If f is differentiable at every point on an open interval I, we say that f is differentiable on I.

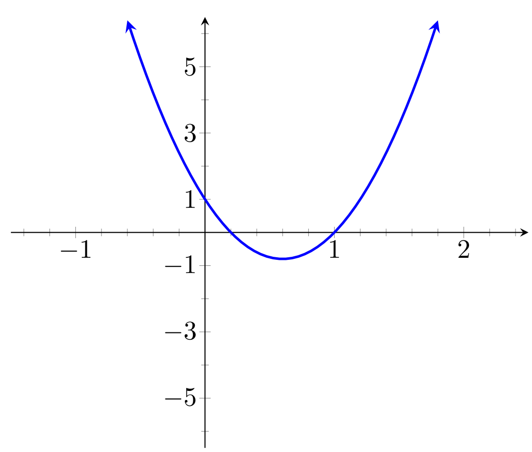
Note: The derivative of f has several notations:

$$f'(x) \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

Note: The derivative of f evaluated at a has several notations:

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Example. Use the limit definition of a derivative to find the derivative function $f'(x)$ for the function $f(x) = 5x^2 - 6x + 1$.



Example. Find the derivative of the following functions. If a point is specified, find the tangent line at that point.

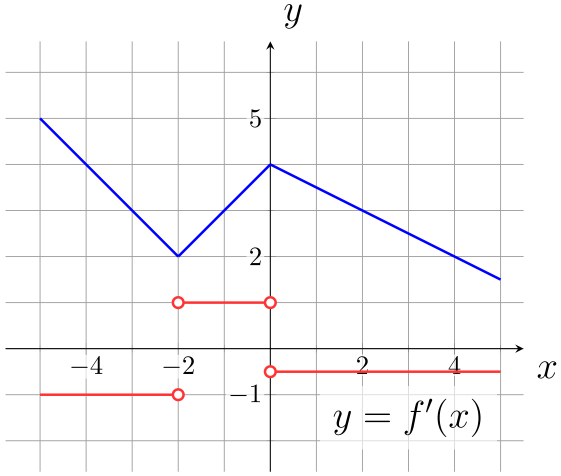
$f(w) = \sqrt{4w - 3},\ w = 3$

$g(v) = \dfrac{v}{v+2}, v = 0$

$h(m) = 1 + \sqrt{m},\ m = 1/4,\ m = 1$

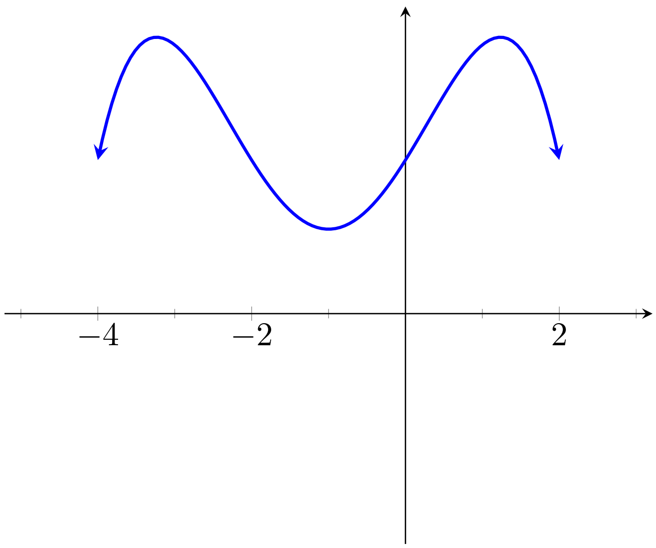
$\dfrac{d}{dx}\big(\sqrt{ax+b}\big).$ Then find $\dfrac{d}{dx}(f(x))$ where $f(x) = \sqrt{5x+9}$ and find $f'(-1)$.

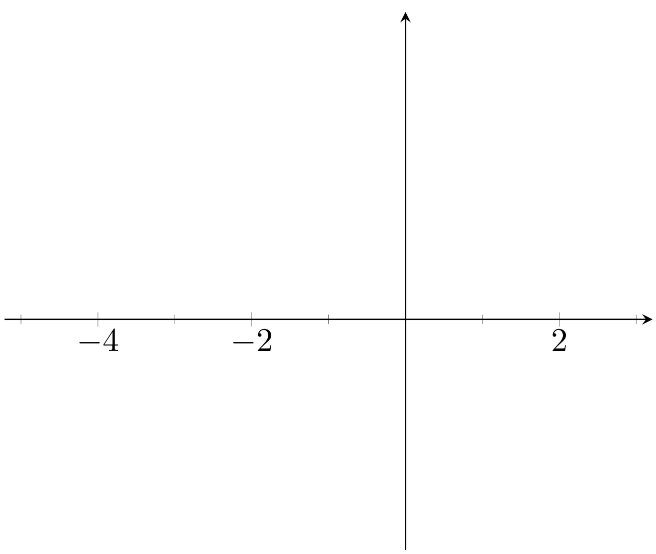
$$\frac{d}{dx}\left(ax^2 + bx + c\right)$$



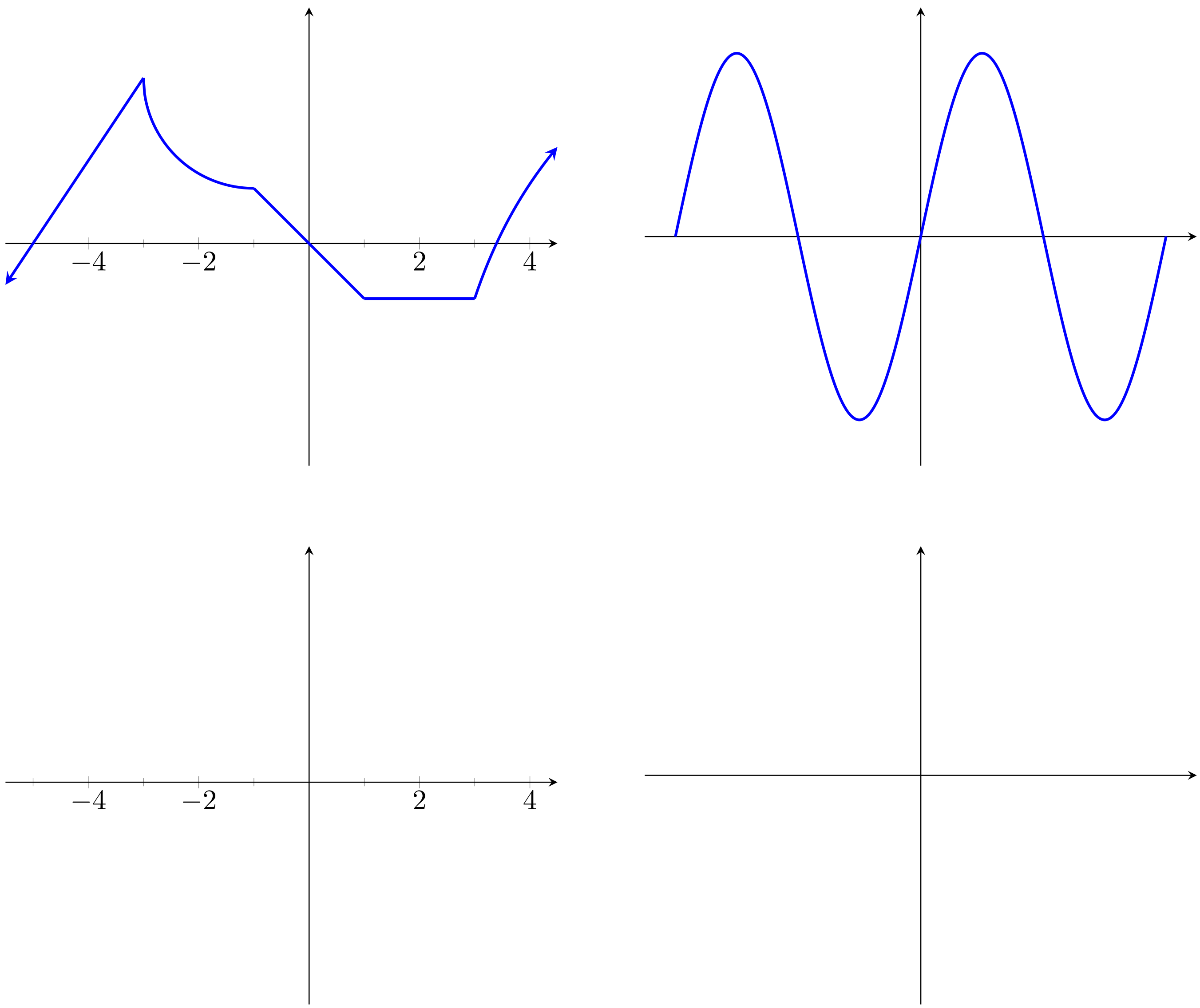
Function	Derivative
Increasing	
Decreasing	
Smooth Min/Max	
Constant	
Linear	
Quadratic	

Example. Graph the slope graph of the following function



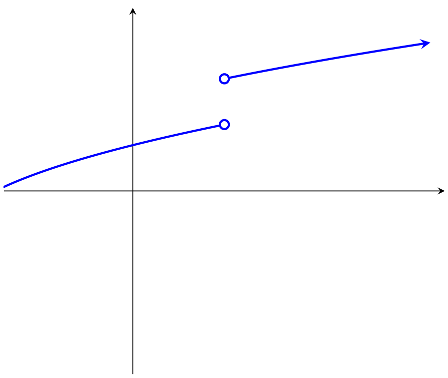
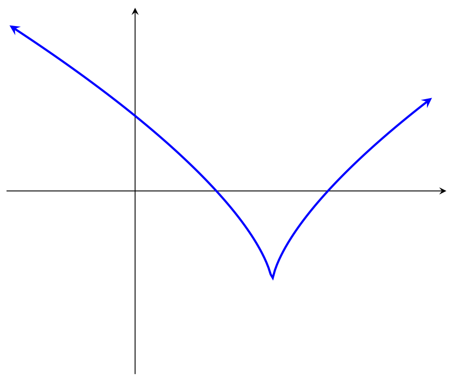
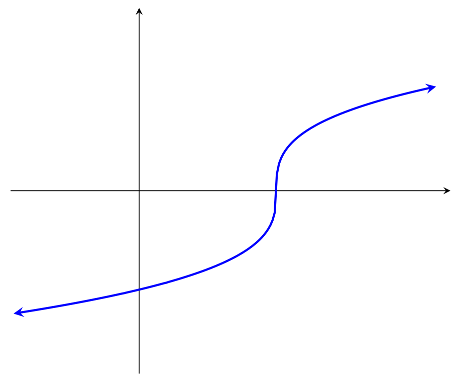


Example. Graph the slope graph of the following functions



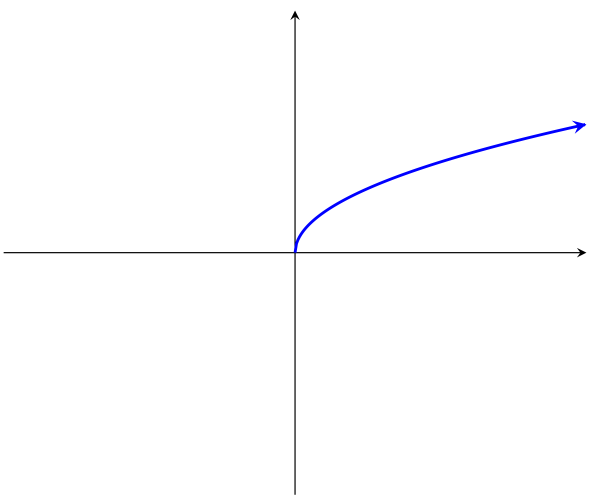
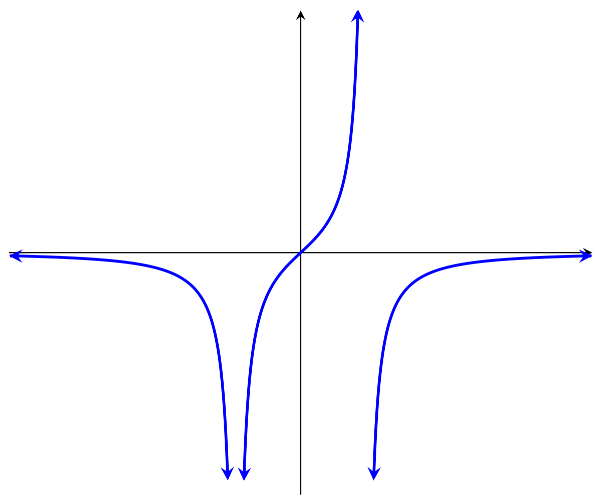
When is a Function Not Differentiable at a Point?
A function f is *not* differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a
2. f has a corner at a
3. f has a vertical tangent at a

Theorem: Differentiable Implies Continuous
If f is differentiable at a, then f is continuous at a.

Theorem: Not Continuous Implies Not Differentiable
If f is not continuous at a, then f is not differentiable at a.

Definition.
The **normal** line at $(a, f(a))$ is the line perpendicular to the tangent line that crosses the point $(a, f(a))$.

Example. Find the derivative of $g(x) = \sqrt{x-2}$. Use your result to find the tangent line and the normal line at $x = 11$.

Example. Find the tangent line and normal line of $h(x) = \dfrac{2}{\sqrt{x^2 + x - 2}}$ at $x = 4$.

3.3 Rules of Differentiation

> **Theorem 3.2** Constant Rule
> If c is a real number, then $\dfrac{d}{dx}(c) = 0$.

Example. Find the derivatives of

$f(x) = 3$ $\qquad$ $g(x) = \pi$ $\qquad$ $h(x) = e^{\pi}$

> **Theorem 3.3** Power Rule
> If n is a nonnegative integer, then $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Example. Find the derivative of

$j(x) = x^3$ $\qquad$ $\ell(x) = x^{\pi}$ $\qquad$ $m(x) = \pi^{42\cos(e)}$

Proof. (Briggs, p153)

Let $f(x) = x^n$ and use the definition of the derivative in the form

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

With $n = 1$ and $f(x) = x$, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1$$

as given by the Power Rule.

With $n \geq 2$ and $f(x) = x^n$, note that $f(x) - f(a) = x^n - a^n$. A factoring formula gives

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Therefore,

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= \underbrace{a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}}_{n \text{ terms}} = na^{n-1}
\end{aligned}$$

$\square$

Theorem 3.4 Constant Multiple Rule
If f is differentiable at x and c is a constant, then

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

Example.

$$\frac{d}{dx}\left(-4x^9\right) \qquad \frac{d}{dx}\left(-\frac{7x^{11}}{8}\right) \qquad \frac{d}{dx}\left(\tfrac{1}{3}x^3\right)$$

Theorem 3.5 Sum Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Example. Find the derivative of the following:

$$p(x) = 3x^{100} + 4x^e - 17x + 24 - \pi^{\cos(e)} \qquad t(w) = 2w^3 + 9w^2 - 6w + 4$$

Definition.
(The Number e)
The number $e = 2.718281828459\ldots$ satisfies

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

It is the base of the natural exponential function $f(x) = e^x$

Note: One way to show the above result is to recall that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Theorem 3.6 The Derivative of e^x
The function $f(x) = e^x$ is differentiable for all real numbers x, and

$$\frac{d}{dx}(e^x) = e^x$$

Proof.

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

□

Example. Find the derivatives of the following

e^x $\qquad$ $42e^x$ $\qquad$ $7e^x - 14x^e$

Example. *Note:* Simplify the expression before taking the derivative

a) $\dfrac{d}{ds}\left(\dfrac{12s^3 - 8s^2 + 12s}{4s}\right)$

b) $h(x) = \dfrac{x^3 - 6x^2 + 8x}{x^2 - 2x}$

c) $\dfrac{d}{dx}\left(\dfrac{x - a}{\sqrt{x} - \sqrt{a}}\right)$

d) $g(w) = \begin{cases} w + 5e^w, & \text{if } w \leq 1 \\ 2w^3 + 4w + 5, & \text{if } w > 1 \end{cases}$

Example. Use the table to find the following derivatives:

x	1	2	3	4	5
$f'(x)$	3	4	2	1	4
$g'(x)$	2	4	3	1	5

a) $\left.\dfrac{d}{dx}[f(x)+g(x)]\right|_{x=1}$ b) $\left.\dfrac{d}{dx}[1.5f(x)]\right|_{x=2}$ c) $\left.\dfrac{d}{dx}[2x-3g(x)]\right|_{x=4}$

Example. Find the equation of the tangent line to $y = x^3 - 4x^2 + 2x - 1$ at $a = 2$

Example. Find the equation of the tangent line to $y = \dfrac{e^x}{4} - x$ at $a = 0$.

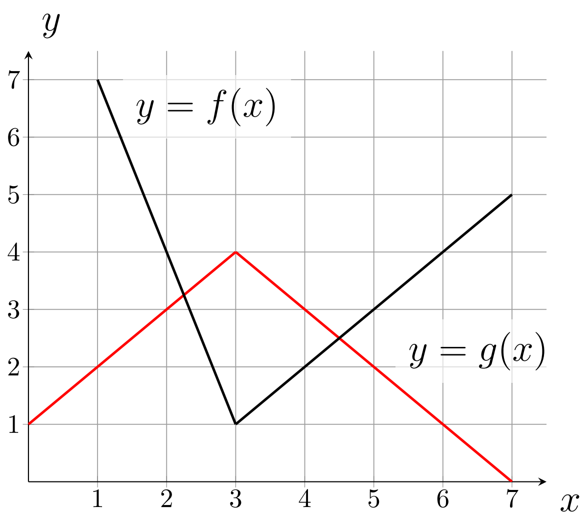
Example. Find the equation of the normal line to $f(x) = 1 - x^2$ at $x = 2$.

Example. Find the equations of the tangent line and normal line to $y = \frac{1}{2}x^4$ at $a = 2$.

Example. At what x-values does $f(x) = x - 2x^2$ have horizontal tangents?

Example. Find an equation of the line having slope $\frac{1}{4}$ that is tangent to the curve $y = \sqrt{x}$.

Example.



a) $f'(2)$

b) $g'(2)$

c) $f'(5)$

d) $g'(5)$

Example. The line tangent to the graph of f at $x = 5$ is $y = \frac{1}{10}x - 2$. Find $\left.\frac{d}{dx}(4f(x))\right|_{x=5}$

Example. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$.

Example. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and parallel to the line $12x - y = 1$.

Definition.
Higher-Order Derivatives
Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x))$$

For integers $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}\Big(f^{(n-1)}(x)\Big)$$

Example. Find all the derivatives of $y = \dfrac{x^5}{120}$

Example. Find the first, second and third derivatives of $f(x) = 5x^4 + 10x^3 + 3x + 6$

Example. Find the first, second and third derivatives of $f(x) = x^2\big(2 + x^{-3}\big)$.

3.4 The Product and Quotient Rule

Theorem 3.7: Product Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Note: This can also be denoted

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)].$$

Example. For $f(x) = \left(3x^2\right)(2x)$, find $f'(x)$ by using the product rule and by

Example. For $g(x) = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$, find $g'(x)$.

Example. For $h(x) = (x-1)\left(x^2 + x + 1\right)$, find $h'(x)$.

Example. Use the product rule to find the derivative of $1 - e^{2t}$.

Theorem 3.8 Quotient Rule
If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of f/g at x exists and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Note: A common phrase for the quotient rule is

"Lo De Hi minus Hi De Lo over Lo squared"

Example. Find the derivative of $y = \dfrac{t^2 + 1}{3t^2 - 2t + 1}$.

Example. Find the derivatives of the following functions:

$$f(t) = \frac{2t}{4+t^2} \qquad\qquad w = (2x-7)^{-1}(x+5)$$

$$y = \frac{e^x}{1-e^x} \qquad\qquad h(w) = \frac{w^2-1}{w^2+1}$$

Example. Find the derivative of the following functions. Is using the quotient rule recommended here?

$$w(z) = \frac{4}{z^3} \qquad\qquad f(x) = \frac{x^2 - 2ax + a^2}{x - a}$$

Example. Find the second derivative of the following functions.

$$f(x) = x^{\frac{5}{2}} e^x \qquad\qquad y(t) = \frac{t}{t+2}$$

Example. Use the table below to evaluate the following

x	1	2	3	4	5
$f(x)$	5	4	3	2	1
$f'(x)$	3	5	2	1	4
$g(x)$	4	2	5	3	1
$g'(x)$	2	4	3	1	5

$$\frac{d}{dx}[f(x) \cdot g(x)]\Big|_{x=5}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\Bigg|_{x=3}$$

$$\frac{d}{dx}[x \cdot g(x)]\Big|_{x=2}$$

$$\frac{d}{dx}\left[\frac{x \cdot f(x)}{g(x)}\right]\Bigg|_{x=4}$$

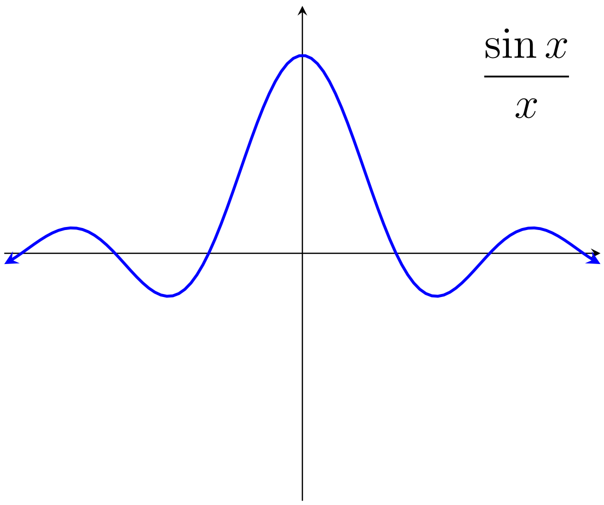
$h(x) = \left(-2x^3\right) \cdot f(x)$, find $h'(4)$.

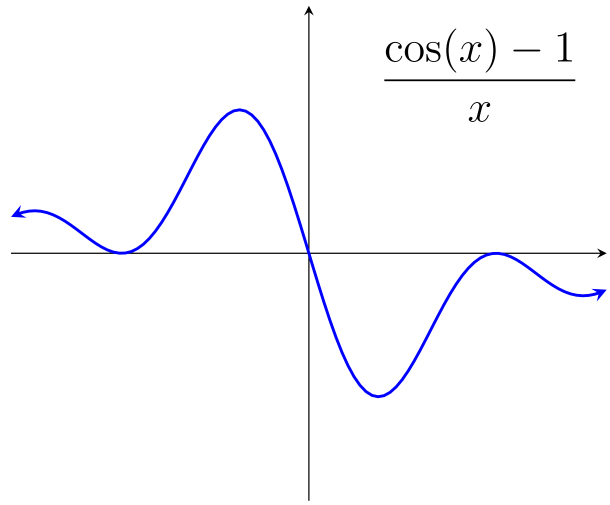
$r(x) = \dfrac{2g(x)}{-3\sqrt[4]{x}}$, find $r'(1)$.

3.5 Derivatives of Trigonometric Functions

Theorem 3.10 Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$





Example. Evaluate the following limits:

$$\lim_{x \to 0} \frac{\sin(3x)}{3x} \qquad \lim_{x \to 0} \frac{\sin(4x)}{x} \qquad \lim_{h \to 0} \frac{5h}{\sin(3h)}$$

$$\lim_{t \to \frac{\pi}{2}} \frac{\sin\left(t - \frac{\pi}{2}\right)}{t - \frac{\pi}{2}} \qquad \lim_{x \to 0} \frac{\tan(2x)}{x} \qquad \lim_{x \to 0} \frac{\sin(7x)}{\sin(5x)}$$

Theorem 3.11 Derivatives of Sine and Cosine

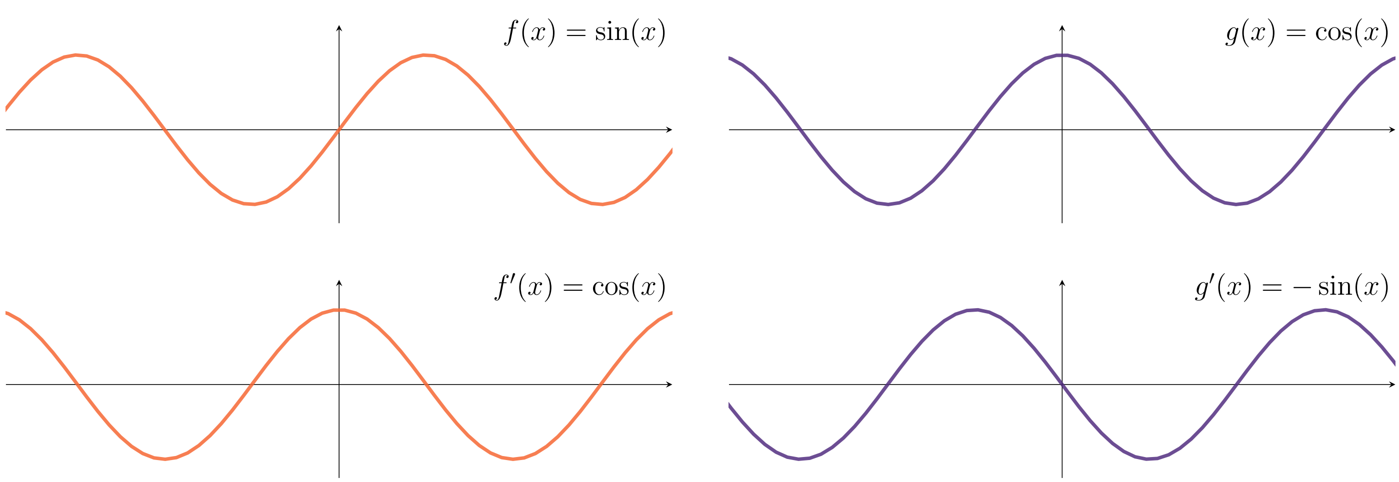
$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad\qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} = \cos(x)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} = -\sin(x)
\end{aligned}$$

□



Example. Find the derivative of the following functions:

$$y = 3\cos(x) - 2x^{\frac{3}{2}} \qquad z = \frac{\sin(x)}{x} \qquad w = \frac{x}{\cos(x)}$$

$$\ell = e^x \cos(x) \qquad m = \frac{\cos(x)}{\sin(x)} \qquad n = \sin^2(x) + \cos^2(x)$$

Example. Find the equation of the line tangent to $y = \cos(x)$ at $x = \frac{\pi}{4}$.

Example. Find the derivative of $y = \dfrac{x\cos(x)}{1+\sin(x)}$ and simplify.

Theorem 3.12 Derivatives of the Trigonometric Functions

$$\begin{aligned} \frac{d}{dx}\sin(x) &= \cos(x) & \frac{d}{dx}\cos(x) &= -\sin(x) \\ \frac{d}{dx}\tan(x) &= \sec^2(x) & \frac{d}{dx}\cot(x) &= -\csc^2(x) \\ \frac{d}{dx}\sec(x) &= \sec(x)\tan(x) & \frac{d}{dx}\csc(x) &= -\csc(x)\cot(x) \end{aligned}$$

Example. Find the derivatives of the following:

$y = \frac{4}{x} - \frac{9}{13}\tan(x)$

$f(x) = -4x^3\cot(x)$

$$g(\theta) = \frac{\sec(\theta)}{1+\sec(\theta)}$$

$h(w) = e^w \csc(w)$

Example. Evaluate

$$\frac{d}{dx}[\tan(x)]\Big|_{x=\frac{\pi}{4}}$$

$$\frac{d}{dx}[(\sin(x)+\cos(x))\csc(x)]$$

$$\frac{d}{d\theta}\left[\theta^2\sin(\theta)\tan(\theta)\right]$$

Example. Find the following higher order derivatives:

y'' when $y = \cos(x)$

$f''(x)$ when $f(x) = \sin(x)$

$y^{(42)}$ when $y = \cos(x)$

$$\frac{d^2}{d\theta^2}[\sin(\theta)\cos(\theta)]$$

$$\frac{d^2}{dx^2}\left[\tfrac{1}{2}e^x\cos(x)\right]$$

$$\frac{d^2}{dx^2}[\cot x]$$

Example. For

$$f = \begin{cases} \frac{3\sin(x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

Find a such that f is continuous.

Example. Find the equation of the line tangent to $y = \dfrac{\cos(x)}{1 - \cos(x)}$ at $x = \frac{\pi}{3}$.

Example. For what values of x does $x - \sin(x)$ have a horizontal tangent line?

Example. Evaluate the following limits

$$\lim_{x \to \pi/4} \frac{\tan(x) - 1}{x - \pi/4}$$

$$\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h}$$

$$\lim_{x \to \pi/4} \frac{\cot x - 1}{x - \frac{\pi}{4}}$$

3.6 Rate of Change Applications

Definition. (Average and Instantaneous Velocity)
Let $s = f(t)$ be the position function (sometimes referred to as the **displacement** function) of an object moving along a line. The **average velocity** of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$:

$$v_{avg} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

The **instantaneous velocity** at a is the slope of the line tangent to the position curve, which is the derivative of the position function:

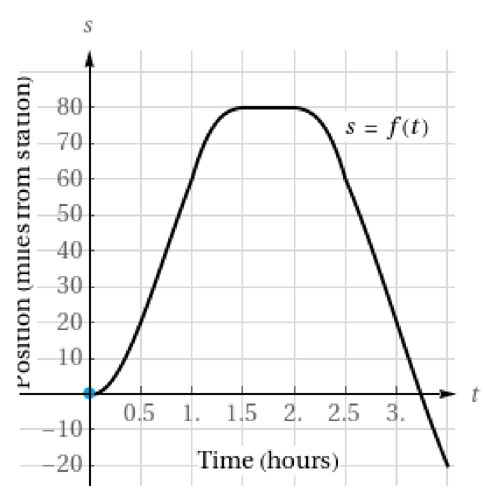
$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Example (Position and velocity of a patrol car)**.**
Assume a police station is located along a straight east-west freeway. At noon ($t = 0$), a patrol car leaves the station heading east. The position function of the car $s = f(t)$ gives the location of the car in miles east ($s > 0$) or west ($s < 0$) of the station t hours after noon.

a) Describe the location of the patrol car during the first 3.5hr of the trip.

b) Calculate the displacement and average velocity of the car between 2:00 P.M. and 3:30 P.M. ($2 \le t \le 3.5$).

c) At what time(s) is the instantaneous velocity greatest *as the car travels east*?



Definition. (Velocity, Speed, and Acceleration)
Suppose an object moves along a line with position $s = f(t)$. Then

$$\text{the } \textbf{velocity} \text{ at time } t \text{ is} \qquad v = \frac{ds}{dt} = f'(t)$$

$$\text{the } \textbf{speed} \text{ at time } t \text{ is} \qquad |v| = |f'(t)|, \text{ and}$$

$$\text{the } \textbf{acceleration} \text{ at time } t \text{ is} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t).$$

- Velocity indicates direction:

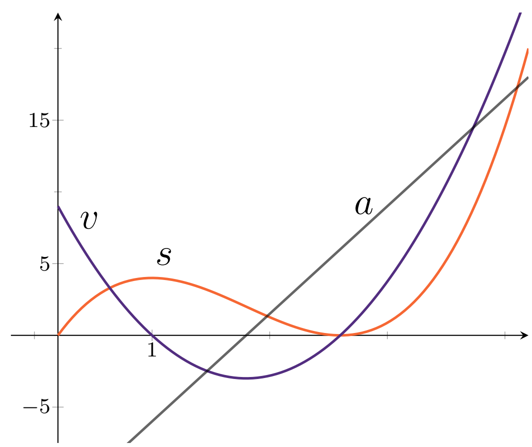
  forward is positive, backward is negative

- Speed is direction independent:

  $$v(t) = -30m/s \Rightarrow s(t) = 30m/s.$$

- If velocity changes signs, then velocity was zero.

  A velocity of zero does not indicate a change in direction.



Example. $s = -t^3 + 3t^2 - 3t,\ 0 \leq t \leq 3$ gives the position $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

1. Find the body's displacement and average velocity for the given time interval.
2. Find the body's speed and acceleration at the endpoints of the interval.
3. When, if ever, during the interval does the body change direction?

For vertical motion (e.g. an object thrown up in the air), an object's maximum height occurs when velocity is zero and hits the ground at height zero.

Example. A rock is thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t sec.

1. Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
2. How long does it take for the rock to reach it's highest point?
3. How high does the rock go?
4. When does the rock hit the ground?
5. What is the velocity at that instant?

Example. Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 32 ft/s from a height of 48 ft above the ground. The height (in feet) of the stone above the ground t seconds after it is thrown is $s(t) = -16t^2 + 32t + 48$.

1. Determine the velocity v of the stone after t seconds.
2. When does the stone reach its highest point?
3. What is the height of the stone at the highest point?
4. When does the stone strike the ground?
5. With what velocity does the stone strike the ground?
6. On what intervals is the speed increasing?

Example (Velocity of a bullet). A bullet is fired vertically into the air at an initial velocity of 1200 ft/s. On Mars, the height s (in feet) of the bullet above the ground after t seconds is $1200t - 6t^2$ and on Earth, $s = 1200t - 16t^2$. How much higher will the bullet travel on Mars than on Earth?

Definition. (Average and Marginal Cost)
The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process. The **average cost** to produce x items is $\bar{C}(x) = C(x)/x$. The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items.

Example. Suppose $C(x) = 10,000+5x+0.01x^2$ dollars is the estimated cost of producing x items. The marginal cost at the production level of 500 items is:

Example. The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

1. Find and interpret $C'(100)$.
2. Compare $C'(100)$ with the cost of producing the 101st item.

Example. For the following cost functions,

a) Find the average cost and marginal cost functions.

b) Determine the average cost and the marginal cost when $x = a$.

c) Interpret the values obtained in part (b)

1. $C(x) = 500 + 0.02x,\ 0 \le x \le 2000,\ a = 1000.$

2. $C(x) = -0.01x^2 + 40x + 100,\ 0 \le x \le 1500,\ a = 1000.$

3.7 The Chain Rule

Theorem 3.13 The Chain Rule
Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{2}$$

Example. Take the derivatives of the following functions

a) $y = \left(3x^3 + 1\right)^2$

b) $y = \left(3x^3 + 1\right)^7$

c) $y = 6\cos^2(x)$

d) $y = \sin\left(x + \cot(x)\right)$

To use the chain rule,

- Identify the inner and outer function
- Take the derivative of the outside, leaving the original inner function
- Multiply by the derivative of the inner function

e) $y(x) = e^{-4x}$

f) $y(x) = \sin\left(x + \cot(x)\right)$

g) $y(x) = \sqrt{\sec(x)}$

h) $y(x) = 2(8x-1)^3$

i) $y(x) = \left(\frac{x}{2} - 1\right)^{-10}$

j) $y(x) = e^{\sin(t)} + \sin(e^t)$

k) $y(x) = x^2 e^{x^2}$

l) $\dfrac{f(x)}{g(x)} = f(x) \cdot [g(x)]^{-1}$

m) $y(x) = f(g(h(x)))$

n) $y(x) = -12e^{3x^7}$

o) $y(x) = \dfrac{\cos^2(x)}{e^x(x^2+4)}$

JIT 13.1: Solving Linear Equations Involving Derivatives

Recall that for a function $f(x)$, we can denote the derivative as $\dfrac{df}{dx}$.

Example. Solve the following for $\dfrac{dy}{dx}$:

$$2 + 3\frac{dy}{dx} = 1$$

$$2x + 3y' = 3x - 5y'$$

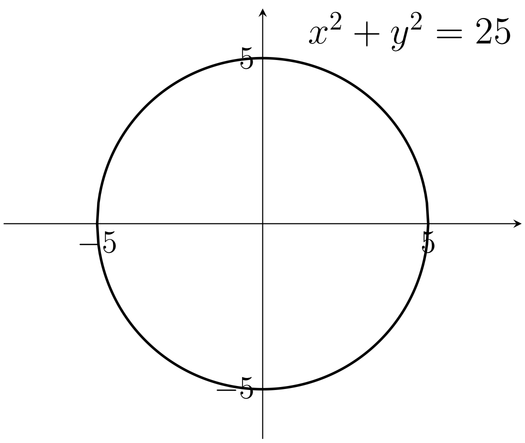
$$x + 2y\frac{dy}{dx} = -\frac{dy}{dx} + y$$

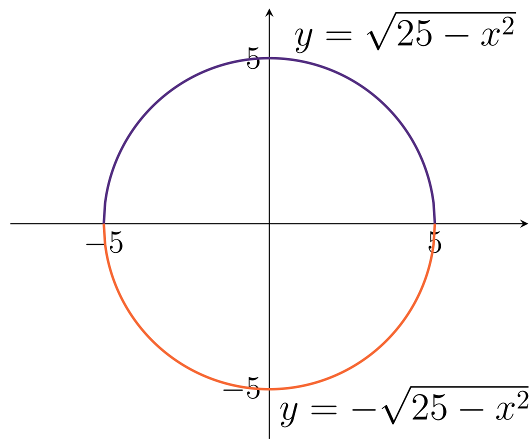
$$5xy + 4\frac{dy}{dx} = 3x^2 - 2xy^2\frac{dy}{dx}$$

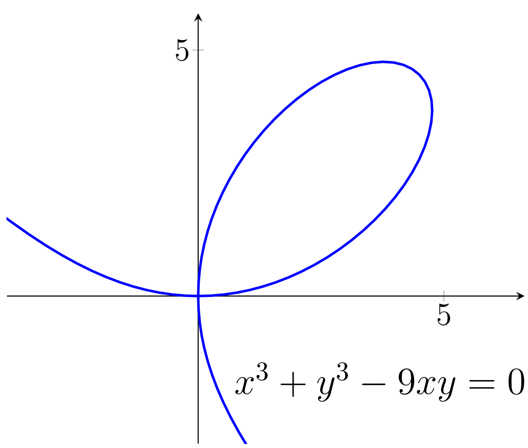
3.8: Implicit Differentiation

Up until now, we have only taken the derivatives of *explicitly* defined functions (functions defined in terms of only x).
An *implicitly* defined function will be written in terms of both x and y:

$$x^2 + y^2 = 25$$







Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with ${}^{dy}\!/_{dx}$ on one side of the equation.
3. Solve for ${}^{dy}\!/_{dx}$.

Example. Find the derivatives of the following by rewriting each function explicitly before taking the derivative, and by using implicit differentiation. Compare the results.

$y^2 = x$

$\sqrt{x} + \sqrt{y} = 4$

Example. Find the derivatives of the following equations:

$x^2 + y^2 = 25$

$x^3 + y^3 - 9xy = 0$

$2y = x^2 + \sin y$

$x^2y^2 + x \sin y = 4$

$y^5 + x^2y^3 = 1 + x^4y$

$1 + x = \sin\left(xy^2\right)$

Example. Find the derivatives of the following equations:

$$x^3 - xy + y^3 = 1$$

$$xe^y = x - y$$

$$\frac{1}{x} + \frac{1}{y} = 1$$

$$x^2 - 2x^3y^4 + y^2 = 30y$$

$$\tan(xy) = x + y$$

$$x^2 = \frac{x-y}{x+y}$$

Example. Find the second derivative implicitly for the following equations:

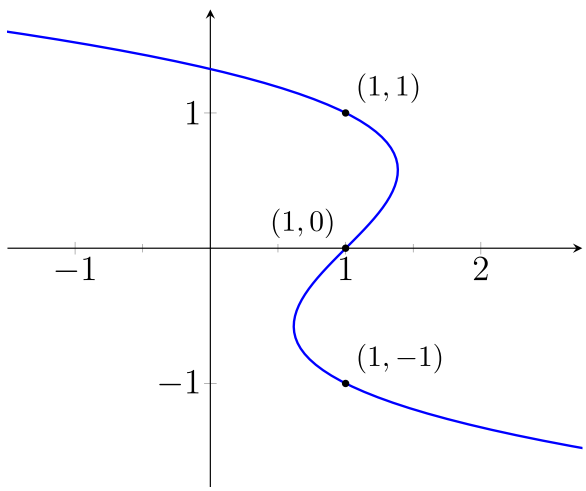
$y^2 - 2x = 1 - 2y$

$xy = \cot(xy)$

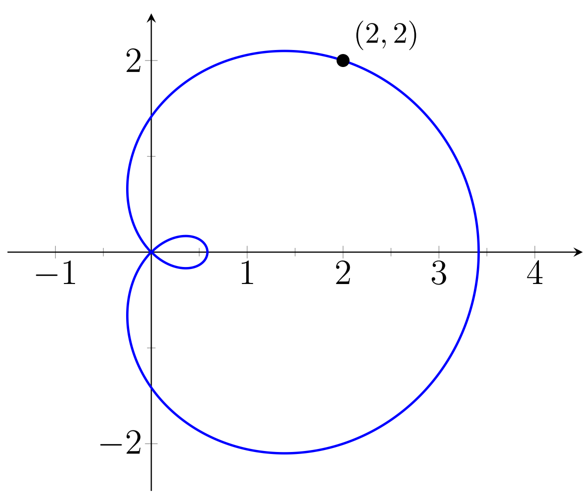
$x^3 + y^3 = 1$

$x = e^y$

Example. Find the equation of all lines tangent to the curve $x + y^3 - y = 1$ at $x = 1$.



Example. Find the equation of the tangent line and normal line for $\left(x^2 + y^2 - 2x\right)^2 = 2\left(x^2 + y^2\right)$ at $(x, y) = (2, 2)$.

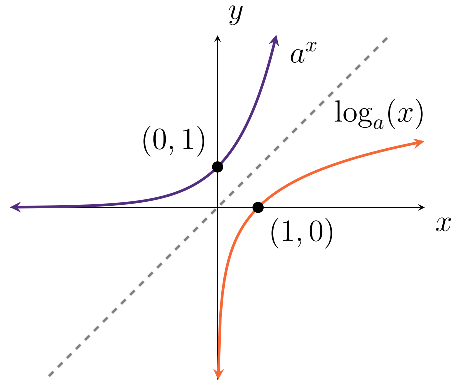


3.9: Derivatives of Logarithmic and Exponential Functions

Recall that $y = \log_a(x)$ and $y = a^x$ are inverse functions:

Inverse Properites of a^x and $\log_a(x)$

1. $a^{\log_a(x)} = x$, for $x > 0$, and $\log_a(a^x) = x$, for all x.
2. $y = \log_a(x)$ if and only if $x = a^y$.
3. For real numbers x and $b > 0$, $b^x = a^{\log_a(b^x)} = a^{x \log_a(b)}$.

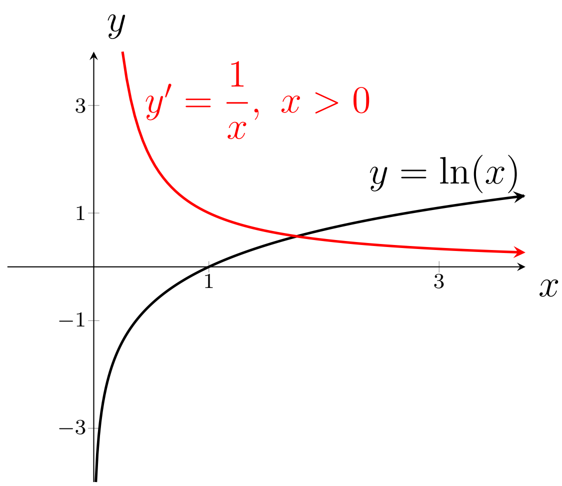


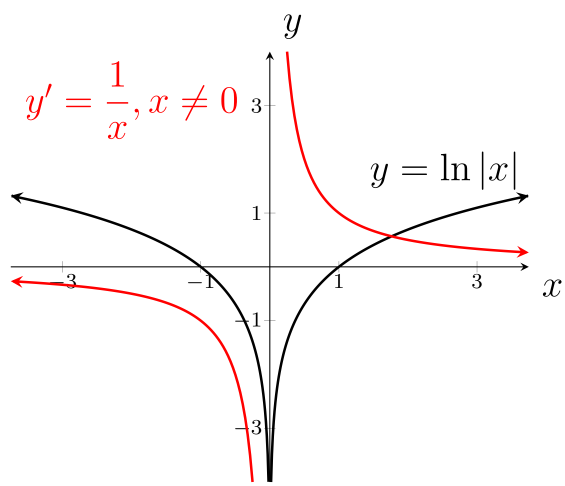
Theorem 3.15: Derivative of $\ln(x)$.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \text{ for } x > 0 \qquad \frac{d}{dx}(\ln|x|) = \frac{1}{x}, \text{ for } x \neq 0$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}$$





Example. Use implicit differentiation to prove $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.
Then, use the piecewise definition of $|x|$ to prove that $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$.

Example. Find the derivatives of the following functions:

$y = \ln(x)$ $\qquad$ $y = \ln(4x)$ $\qquad$ $y = \ln\left(4x^2 + 2\right)$

$f(x) = \sqrt{x}\ln(x^2)$ $\qquad$ $f(x) = \ln\left(\frac{10}{x}\right)$ $\qquad$ $f(x) = \dfrac{\ln(x)}{1 + \ln(x)}$

$f(x) = \sqrt[5]{\ln(3x^4)}$

$f(x) = \ln \sqrt[5]{3x}$

$f(x) = \ln(\ln(\ln(4x)))$

$f(x) = \ln \left| x^2 - 1 \right|$

$y = \ln\left(\sec^2 \theta\right)$

$y = (\ln(\sin(3x)))^2$

Theorem 3.16: Derivative of b^x.
If $b > 0$ and $b \neq 1$, then for all x.

$$\frac{d}{dx}[b^x] = b^x \ln(b).$$

Example. Using the properties of exponents and logarithms, prove the above theorem. Extend this theorem by stating the derivative of $y = b^{f(x)}$.

Example. Find the derivatives of the following functions:

$y = 5^{3x}$

$s(t) = \cos(2^t)$

$g(v) = 10^v(\ln(10^v) - 1)$

$y = 6^{x \ln(x)}$

Theorem 3.18: Derivative of $\log_b(x)$.
If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln b}, \text{ for } x > \text{ and } \frac{d}{dx}[\log_b |x|] = \frac{1}{x \ln b}, \text{ for } x \neq 0.$$

Example. Using the properties of exponents and logarithms, prove the above theorem. Extend this theorem by stating the derivative of $y = \log_b(g(x))$.

Example. Find the derivatives of the following functions:

$f(x) = \log_4\left(4x^2 + 3x\right)$

$f(x) = \log_5(xe^x)$

$y = 2x \log_{10} \sqrt{x}$

$$y = \frac{\log_3\left(\tan\left(e^2 x\right)\right)}{\pi \cdot e^{-4x}}$$

Derivative rules for exponential functions:

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}[b^x] = \ln(b) \cdot b^x \qquad \frac{d}{dx}\left[b^{g(x)}\right] = \ln(b) \cdot b^{g(x)} \cdot g'(x)$$

Derivative rules for logarithmic functions:

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad \frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{\ln(b)x} \qquad \frac{d}{dx}[\log_b(g(x))] = \frac{g'(x)}{\ln(b)g(x)}$$

Laws of Logarithms

For $x, y > 0$:

$$\log_a(xy) = \log_a(x) + \log_a(y) \qquad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(x)$$

$$\log_a(x^r) = r\log_a(x) \qquad \log_a(1) = 0$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \qquad \log_a(a) = 1$$

Example. For the following functions, use the laws of logarithms to rewrite the function before taking the derivative:

$$F(t) = \ln\left(\frac{(2t+1)^3}{(3t+1)^4}\right)$$

$$y = \ln\sqrt[3]{\frac{1+x}{1-x}}$$

$$y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$$

Logarithmic Differentiation:

1. Take the natural logarithm of both sides of the equation.
2. Use logarithm laws to simplify.
3. Use implicit differentiation to take the derivative of both sides.
4. Solve for $\frac{dy}{dx}$.

Example. Find the derivatives of the following functions:

$$y = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}$$

$$h(\theta) = \frac{\theta\sin(\theta)}{\sqrt{\sec(\theta)}}$$

$$g(x) = \sqrt[10]{\frac{3x+4}{2x-4}}$$

$$y = \frac{e^{-x}\cos^2(x)}{x^2+x+1}$$

Note: Whenever the function is of the form $f(x)^{g(x)}$, then *Logarithmic Differentiation* is the only option!

$y = x^x$

$y = (\ln(x))^x$

$y = (\tan x)^{\frac{1}{x}}$

$y = (2x)^{3x}$

Example. Use the definition of the derivative to evaluate the following limits:

$$\lim_{x \to e} \frac{\ln(x) - 1}{x - e}$$

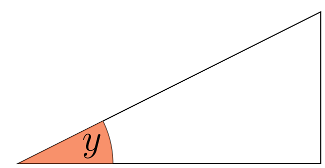
$$\lim_{h \to 0} \frac{\ln(e^8 + h) - 8}{h}$$

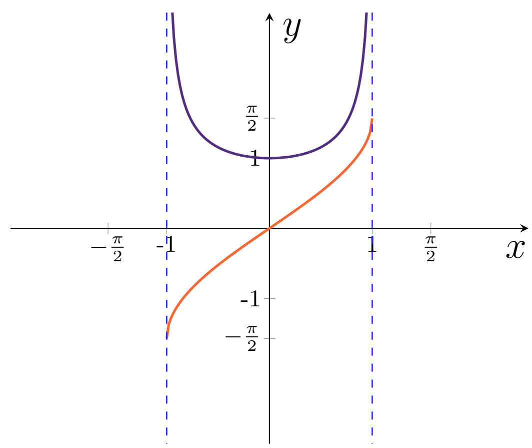
$$\lim_{x \to 2} \frac{5^x - 25}{x - 2}$$

$$\lim_{h \to 0} \frac{(3 + h)^{3+h} - 27}{h}$$

3.10: Derivatives of Inverse Trigonometric Functions

Example. Recall that $y = \sin^{-1}(x) \iff \sin(y) = x$. Use this fact and implicit differentiation to derive the derivative of $\sin^{-1}(x)$.



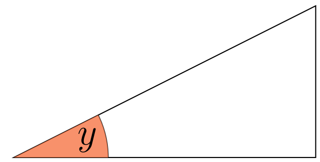


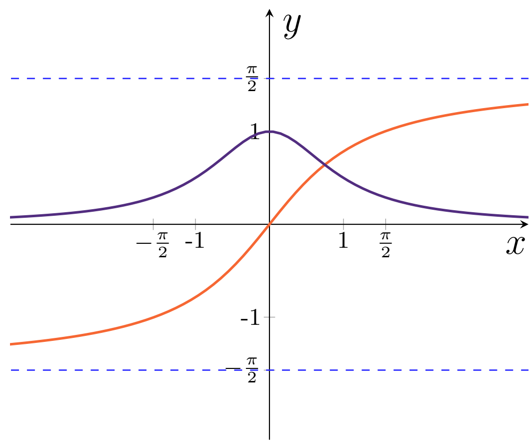
Next, extend this definition for the derivative of $\sin^{-1}(f(x))$.

Example. Find the derivative of the following

$$f(x) = \sqrt{1 - x^2}\,\arcsin(x) \qquad\qquad y = \sin^{-1}\left(\sqrt{2}t\right)$$

Example. Recall that $y = \tan^{-1}(x) \iff \tan(y) = x$. Use this fact and implicit differentiation to derive the derivative of $\tan^{-1}(x)$.





Next, extend this definition for the derivative of $\tan^{-1}(f(x))$.

Example. Find the derivative of the following

$y = \sqrt{\tan^{-1}(x)}$

$y = \tan^{-1}\left(\sqrt{x}\right)$

Derivatives of Inverse Trigonometric Functions

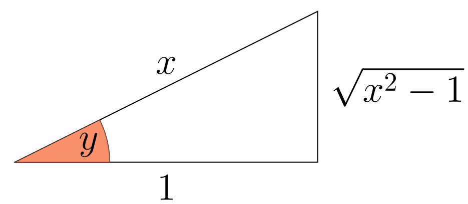
$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\csc^{-1}(x)\right] = -\frac{1}{|x|\sqrt{x^2-1}}$$

Derivative of $y = \sec^{-1}(x)$

$$\begin{aligned} y &= \sec^{-1}(x) \\ \sec(y) &= x \\ \sec(y)\tan(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec(y)\tan(y)} \end{aligned}$$



$$\sin^2(x) + \cos^2(x) = 1$$
$$\tan^2(x) + 1 = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

Now we rewrite $\sec(y)\tan(y)$ in terms of x. Note that the domain of $\sec(y)$ is restricted to $[0, \pi/2) \cup (\pi/2, \pi]$. If we look at these quadrants of the unit circle, we see that $\sec(y)\tan(y)$ is always positive, so the resulting derivative is always positive:

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

Example. Find the derivatives of the following functions:

$h(t) = e^{\sec^{-1}(t)}$

$y = \arccos\left(e^{2x}\right)$

$y = \sin^{-1}(2x+1)$

$y = \sec^{-1}(5r)$

$f(x) = \csc^{-1}\left(\tan(e^x)\right)$

$f(x) = \tan^{-1}(10x)$

$y = x\sin^{-1}(x) + \sqrt{1-x^2}$

$h(t) = \cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right)$

$f(x) = 2x\tan^{-1}(x) - \ln\left(1+x^2\right)$

$f(t) = \ln\left(\tan^{-1}(t)\right)$

Example. Find the equation of the tangent line to $f(x) = \cos^{-1}(x^2)$ at $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$.

Example. Find the equation of the tangent line to $f(x) = \sec^{-1}(e^x)$ at $\left(\ln(2), \frac{\pi}{3}\right)$.

3.11: Related Rates

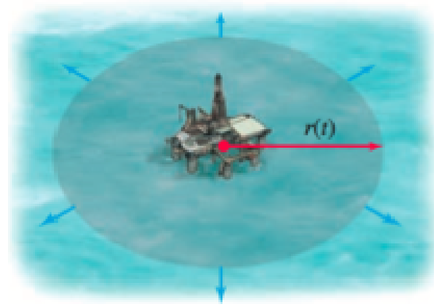
Related rates are problems that use a mathematical relationship between two or more objects under specific constraints. From this, we can differentiate this relationship and examine how each variable changes with respect to time.

Example. An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of $30m/hr$, how fast is the area of the patch increasing when the patch has a radius of $100m$?

Solution:

We are given the radius $r = 100\,m$ which is increasing at a rate of $\frac{dr}{dt} = 30\,m/hr$. First, we note the formula for the area of a circle:

$$A = \pi r^2$$



Now, we differentiate *with respect to time* t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

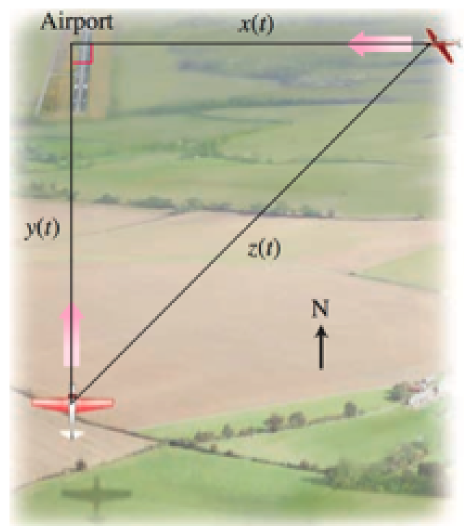
Note that we want to solve for $\dfrac{dA}{dt}$ — the rate of change of the area of the oil patch:

$$\frac{dA}{dt} = 2\pi(100\,m)(30\,m/hr) = 6000\pi\,m^2/hr$$

Example. Two small planes approach an airport, one flying due west at $120\,mi/hr$ and the other flying due north at $150\,mi/hr$. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is $180\,mi$ from the airport and the northbound plane is $225\,mi$ from the airport?

Solution:

From the problem statement, we know that



$$x = 180\,mi \qquad y = 225\,mi$$
$$\frac{dx}{dt} = -120\,mi/hr \qquad \frac{dy}{dt} = -150\,mi/hr$$

Now we use the Pythagorean theorem to relate the three sides:

$$x^2 + y^2 = z^2$$

Next, we differentiate and get the resulting relationship between the related rates:

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 2z\,\frac{dz}{dt}$$

Before we solve for $\frac{dz}{dt}$, we must find z:

$$z = \sqrt{180^2 + 225^2} = 45\sqrt{4^2 + 5^2} = 45\sqrt{41} \approx 288\,mi$$

This gives us

$$\frac{dz}{dt} = \frac{x\,\frac{dx}{dt} + y\,\frac{dy}{dt}}{z} = \frac{(180\,mi)(-120\,mi/hr) + (225\,mi)(-150\,mi/hr)}{45\sqrt{41}\,mi}$$
$$= \frac{1230}{\sqrt{41}}\,mi/hr \approx -192\,mi/hr$$

Steps for Related-Rate Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

Note: The *Just-In-Time* book has some examples in chapter 14 that are helpful in setting up the relationships outlined in these types of word problems.

Example. A ladder 13 feet long rests against a vertical wall and is sliding down the wall at a constant rate of $3\,ft/s$. How fast is the foot of the ladder moving away from the wall when the foot of the ladder is $5\,ft$ from the base of the wall?

Example. The volume of a cube decreases at a rate of $0.5\, ft^3/min$. What is the rate of change of the side length when the side lengths are $12\, ft$?

Example. The length of a rectangle is increasing at a rate of $8\,cm/s$ and its width is increasing at a rate of $3\,cm/s$. When the length is $20\,cm$ and the width is $10\,cm$, how fast is the area of the rectangle increasing?

Example. A coffee mug has the shape of a right circular cylinder with inner diameter 4 inches and height 5 inches. If the mug is filled with hot chocolate at a constant rate of $2\,in^3/sec$, how fast is the level of the liquid rising?

Example. A rectangular swimming pool 10 ft wide by 20 ft long and of uniform depth is being filled with water.

a) At what rate is the volume of the water increasing if the water level is rising at $\frac{1}{4}\,ft/min$?

b) At what rate is the water level rising if the pool is filled at a rate of 10 ft^3/min?

Example. At all times, the length of a rectangle is twice the width w of the rectangle as the area of the rectangle changes with respect to time t.

a) Find the equation that relates A to w.

b) Find the equation that relates dA/dt to dw/dt.

Example. Assume x, y and z are functions of t with $z = x + y^3$. Find dz/dt when $dx/dt = -1,\ dy/dt = 5$ and $y = 2$.

Example. Assume $w = x^2y^4$, where x and y are functions of t. Find dw/dt when $x = 3$, $dx/dt = 2$, $dy/dt = 4$, and $y = 1$.

Example. The sides of a square decrease in length at a rate of $1\,m/s$.

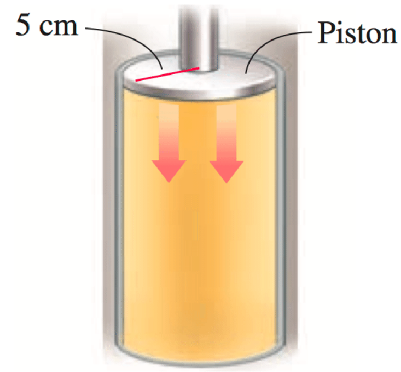
a) At what rate is the area of the square changing when the sides are $5\,m$ long?

b) At what rate are the lengths of the diagonals of the square changing?

Example. At noon bicyclist A is 25 miles south of an intersection and bicyclist B is 8 miles west of the same intersection. Bicyclist A is traveling north at 11 miles per hour and bicyclist B is traveling 6 miles per hour west of the intersection. How is the distance between riders changing at 2pm?

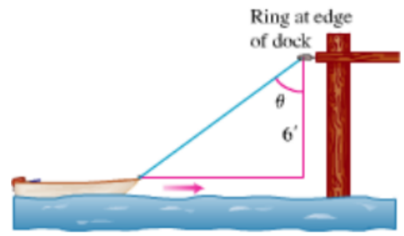
Example. A ladder 13 feet long rests against a vertical wall and is sliding down the wall at a constant rate of $3\,ft/s$. How fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ radians?

Example. Piston compression A piston is seated at the top of a cylindrical chamber with radius $5\,cm$ when it starts moving into the chamber at a constant speed of $3\,cm/s$. What is the rate of change of the volume of the cylinder when the piston is $2\,cm$ from the base of the chamber?



Example. Suppose we have a snowball that is a perfect sphere. If the snowball is melting at a rate of $5\,in^3/min$, how fast is the radius changing when the radius is 10 inches? How fast is the surface area changing?

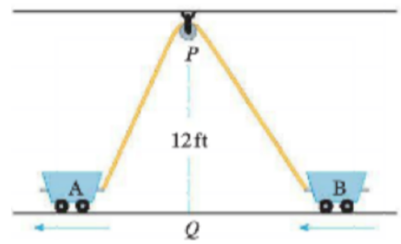
Example. A dingy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at a rate of 2 ft/sec. At what rate is the angle θ changing when 10 ft of rope is out?



Example. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4pm?

Example. Once Kate's kite reaches a height of 50 ft (above her hands), it rises no higher but drifts due east in a wind blowing 5 ft/s. How fast is the string running through Kate's hands at the moment that she has released 120 ft of string?

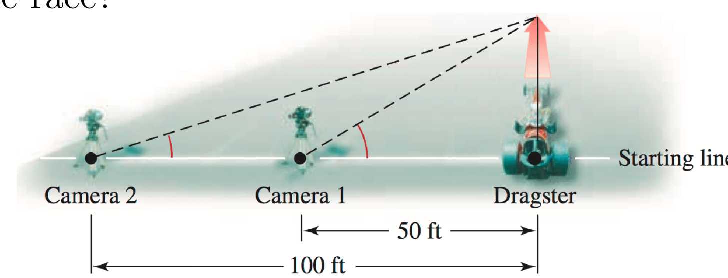
Example. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



Example. A camera is set up at the starting line of a drag race 50ft from a dragster at the starting line (camera 1 in the figure). Two seconds after the start of the race, the dragster has traveled 100ft and the camera is turning at 0.75rad/s while filming the dragster.

a) What is the speed of the dragster at this point?

b) A second camera (camera 2 in the figure) filming the dragster is located on the starting line 100 ft away from the dragster at the start of the race. How fast is this camera turning 2 s after the start of the race?



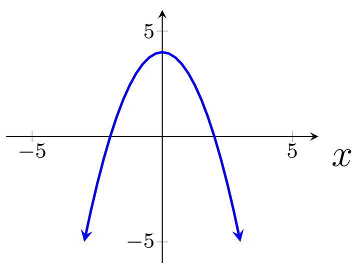
Example. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft. How fast is the distance from the television camera to the rocket changing at that moment? Also, if the television camera is always kept aimed at the rocket, how fast is the cameras angle of elevation changing at that same moment?

4.1: Maxima and Minima

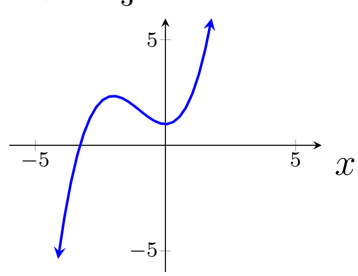
Definition. (Absolute Maximum and Minimum)
Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

Example. Determine whether the function has any absolute extreme values

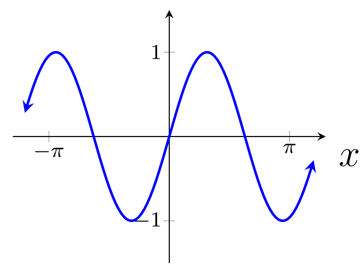
$y = -x^2 + 4$



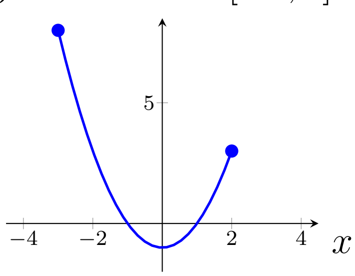
$y = \frac{x^3}{3} + x^2 + 1$



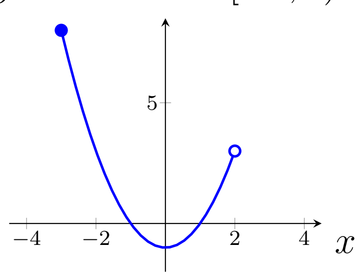
$y = \sin(x)$



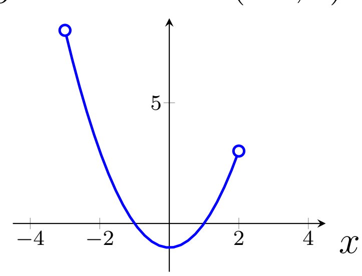
$y = x^2 - 1$ on $[-3, 2]$



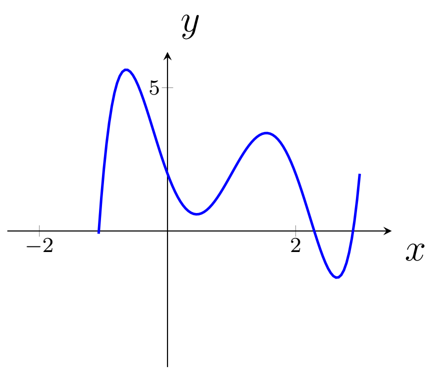
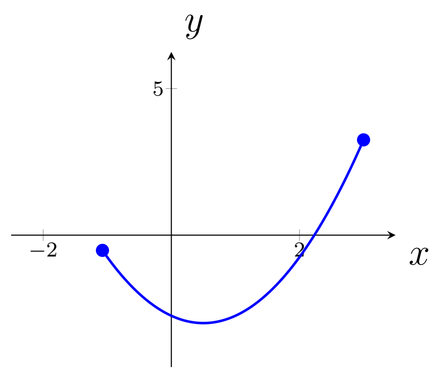
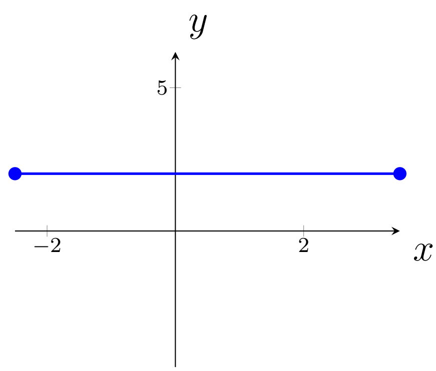
$y = x^2 - 1$ on $[-3, 2)$



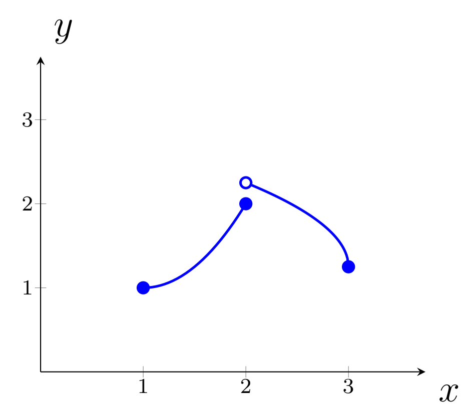
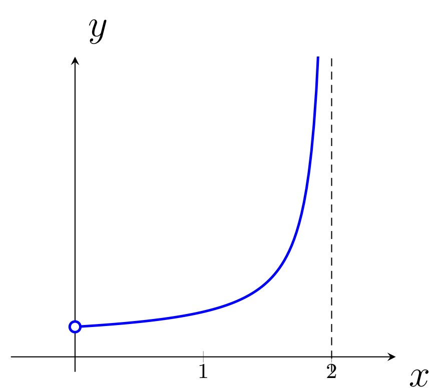
$y = x^2 - 1$ on $(-3, 2)$



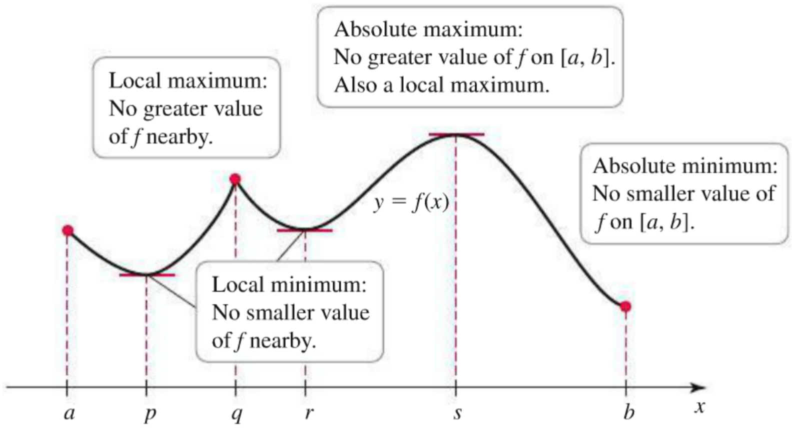
Theorem 4.1: Extreme Value Theorem
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

Note: It is important that the function is both continuous *and* the interval is closed:

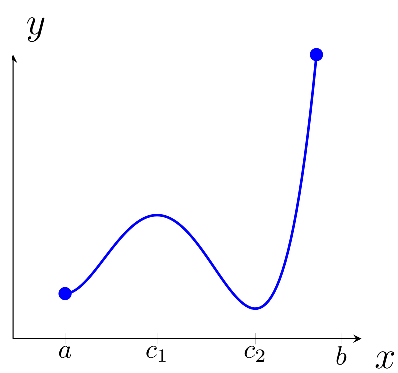



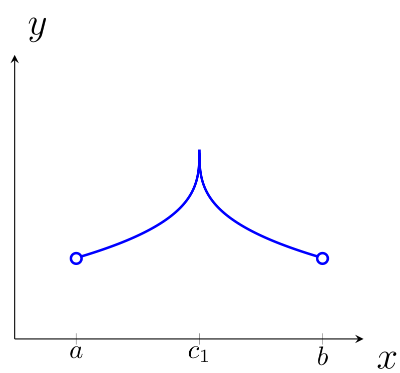
Definition. (Local Maximum and Minimum Values)
Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum value of f.**

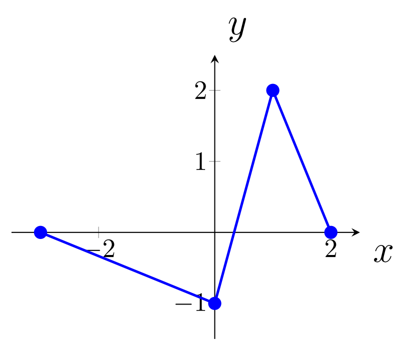


Note: Local extrema **CANNOT** occur at endpoints.

Example. State the absolute extrema and the local extrema:







Example. Sketch the graph of a continuous function f on $[0, 4]$ satisfying the given properties:

1. $f'(x) = 0$ for $x = 1, 2$ and 3; f has an absolute minimum at $x = 1$; f has no local extremum at $x = 2$; and f has an absolute maximum at $x = 3$.

2. $f'(1)$ and $f'(3)$ are undefined; $f'(2) = 0$; f has a local maximum at $x = 1$; f has a local minimum at $x = 2$; f has an absolute maximum at $x = 3$; and f has an absolute minimum at $x = 4$.

3. $f'(x) = 0$ at $x = 1$ and 3; $f'(2)$ is undefined; f has an absolute maximum at $x = 2$; f has neither a local maximum nor a local minimum at $x = 1$; and f has an absolute minimum at $x = 3$.

Theorem 4.2: Local Extreme Value Theorem
If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

Note: If the derivative is zero, then the function **MIGHT** have a max/min.

Example. Sketch a graph of a function $f(x)$ that has a local maximum value at a point c where $f'(c)$ is defined.

Example. Sketch a graph of a function $f(x)$ that has a local maximum value at a point c where $f'(c)$ is undefined.

Example. Graph

a) $f(x) = x^3$

b) $f(x) = |x|$

Definition. (Critical Point)
An interior point c of the domain of f at which $f'(c) = 0$ *or* $f'(c)$ fails to exist is called a **critical point** of f.

Example. Find the critical points of

$$f(x) = x^3 + 3x^2 - 24x$$

$$g(x) = \sqrt{4 - x^2}$$

$$h(t) = 3t - \sin^{-1}(t)$$

Example. Find the critical points of

$$f(x) = \sin(x)\cos(x) \text{ on } [0, 2\pi].$$

$$f(t) = t^2 - 2\ln(t^2 + 1)$$

$$f(x) = x\sqrt{x - a}$$

$$f(x) = \sin^{-1}(x)\cos^{-1}(x)$$

How to find the absolute max/min of $f(x)$ on $[a, b]$:

1. Find $f'(x)$
2. Find all critical points ($f'(x) = 0$ or $f'(x)$ **DNE**)
3. Evaluate $f(x)$ at the critical points within $[a, b]$ and the end points.
4. Identify the absolute max and absolute min using the values found. Include value and location (e.g. ordered pair (x, y))

Example. Find the absolute max and min of $f(x) = 2x - 2x^{\frac{2}{3}}$ on $[-1, 3]$.

Example. Find the absolute max and min of $f(x) = \frac{x}{x^2+1}$ on $[0, 2]$.

Example. Find the absolute max and min of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

Example. Find the absolute max and min of $f(x) = \sin(3x)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

Example. Find the absolute max and min of $f(x) = xe^{1-\frac{x}{2}}$ on $[0, 5]$.

Example. Find the absolute max and min of $f(x) = e^x - 2x$ on $[0, 2]$.

Example. Find the absolute max and min of $f(x) = x^{\frac{1}{3}}(x+4)$ on $[-27, 27]$.

Example. Find the absolute max and min of $y = \sqrt{x^2 - 1}$.

Example. Find the absolute/local max and min of $f(x) = x^2\big(x^2 + 4x - 8\big)$ on $[-5, 2]$.

Example. Minimum-surface-area box

All boxes with a square base of length x and a volume V have a surface area given by $S(x) = x^2 + \frac{4V}{x}$. Find x such that the box has volume $50\, ft^3$ with minimal surface area.

Example. Trajectory high point

A stone is launched vertically upward from a cliff $192\, ft$ above the ground at a speed of $64\, ft/s$. Its height above the ground t seconds after the launch is given by $s = -16t^2 + 64t + 192$, for $0 \leq t \leq 6$. When does the stone reach its maximum height?

Example. Maximizing Revenue

A sales analyst determines that the revenue from sales of fruit smoothies is given by $R(x) = -60x^2 + 300x$, where x is the price in dollars charged per item, for $0 \leq x \leq 5$.

a) Find the critical points of the revenue function.

b) Determine the absolute maximum value of the revenue function and the price that maximizes the revenue.

Example. Find the absolute and local extreme values of the following

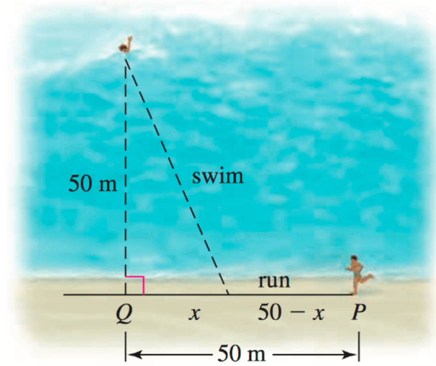
1. $f(x) = |x - 3| + |x + 2|$ on $[-4, 4]$,

2. $g(x) = |x - 3| - 2|x + 1|$ on $[-2, 3]$.

Example. Every second counts (4.01: Q87)

You must get from a point P on the straight shore of a lake to a stranded swimmer who is $50\,m$ from a point Q on the shore that is $50\,m$ from you. Assuming that you can swim at a speed of $2\,m/s$ and run at a speed of $4\,m/s$, the goal of this exercise is to determine the point along the shore, x meters from Q, where you should you stop running and start swimming to reach the swimmer in the minimum time.

a) Find the function T that gives the travel time as a function of x, where $0 \le x \le 50$.



b) Find the critical point of T on $(0, 50)$.

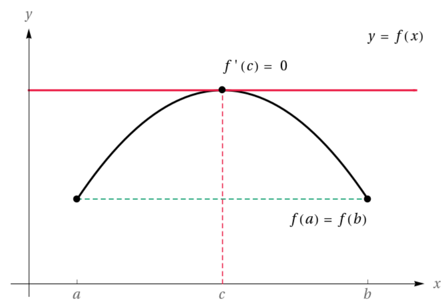
c) Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?

d) Graph the function T to check your work.

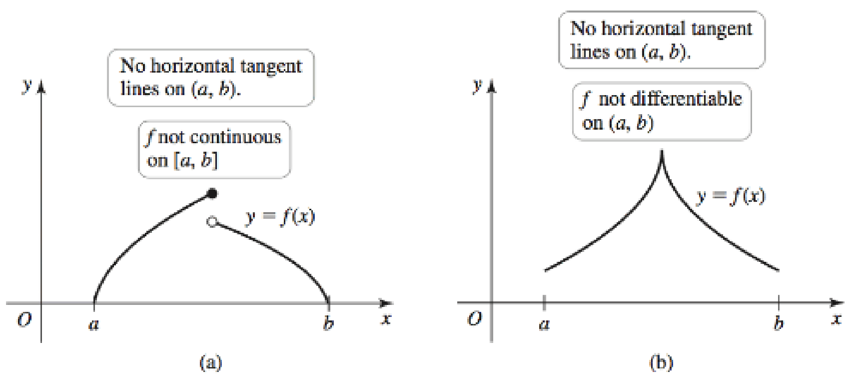
4.2: Mean Value Theorem

Before studying the *Mean Value Theorem*, we must first learn *Rolle's Theorem*:

Theorem 4.3: Rolle's Theorem
Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.



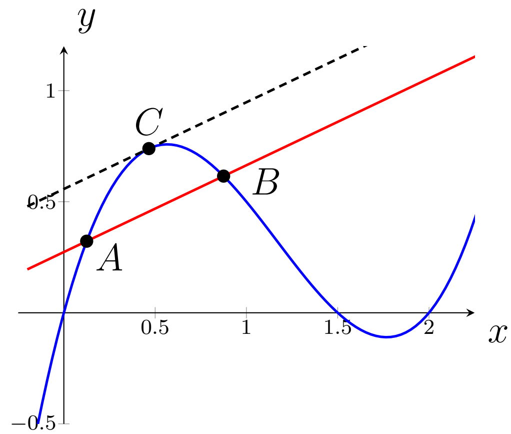
Note: Rolle's Theorem requires f be both *continuous* and *differentiable*:



Example. Determine whether Rolle's Theorem applies for the following. If it applies, find the point(s) c such that $f'(c) = 0$.

$f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$

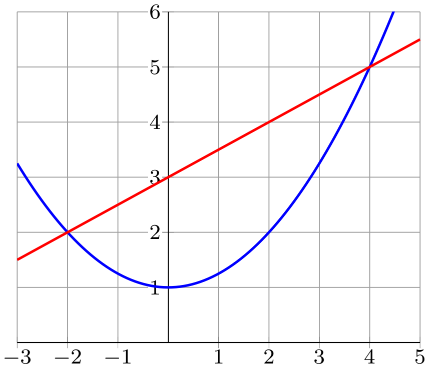
Theorem 4.4: Mean Value Theorem
If f is continuous on a closed interval $[a,b]$ and differentiable on (a,b), then there is at least one point c in (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.



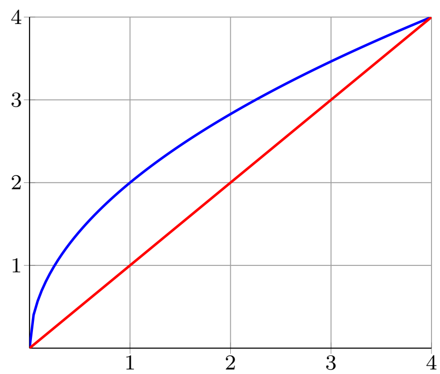
Example. For $f(x) = x^{\frac{2}{3}}$ on the interval $[0,1]$, does the Mean Value Theorem apply? If so, find the point(s) c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Example. For each function, associated interval and graph determine if the conditions for the Mean Value Theorem are met and find the value(s) of c such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

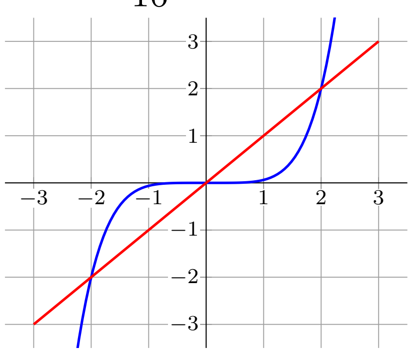
1. $f(x) = \dfrac{x^2}{4} + 1$ on $[-2, 4]$



2. $f(x) = 2\sqrt{x}$ on $[0, 4]$



3. $f(x) = \dfrac{x^5}{16}$ on $[-2, 2]$



Example. Determine whether Rolle's Theorem applies to the following functions and find the point(s) c if applicable.

1. $f(x) = x(x-1)^2$ on $[0, 1]$.

2. $f(x) = x^3 - x^2 - 5x - 3$ on $[-1, 3]$.

Example. Determine whether the Mean Value Theorem applies to the following functions and find the point(s) c if applicable.

1. $f(x) = 3x^2 + 2x + 5$ on $[-1, 1]$.

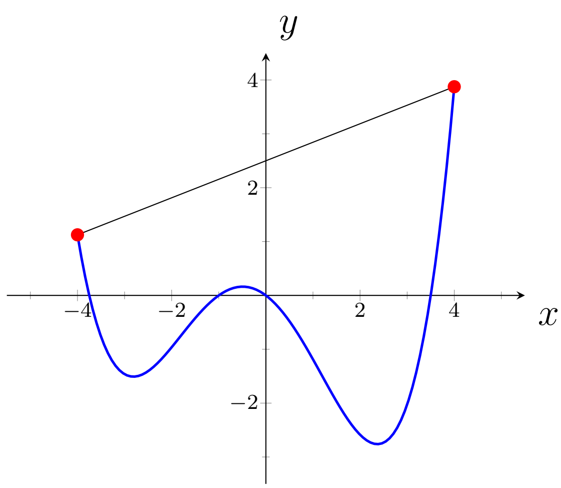
2. $f(x) = x^{-\frac{1}{3}}$ on $\left[\frac{1}{8}, 8\right]$.

3. $f(x) = \begin{cases} \frac{\sin(x)}{x}, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}.$

4. $f(x) = |x-1|$ on $[-1, 4]$.

Example. Mean Value Theorem and graphs

Locate all points on the graph at which the slope of the tangent line equals the secant line on the interval $[-4, 4]$.



Example. Find the number that satisfies the hypotheses of the Mean Value Theorem for $f(x) = \sqrt{x}$ on $[0, 4]$. Graph the function, the secant line through the endpoints, and the tangent line at c, visually verify that the secant and tangent are parallel.

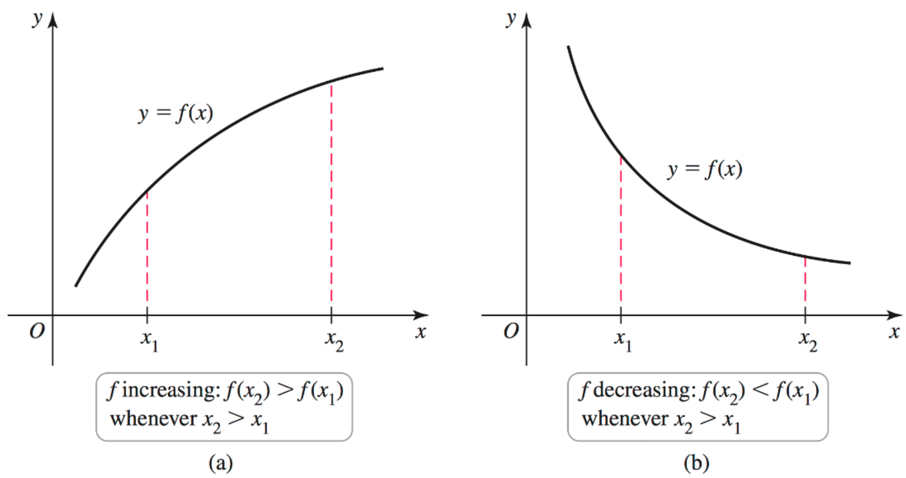
Example. Drag racer acceleration

The fastest drag racers can reach a speed of $330\, mi/hr$ over a quarter-mile strip in 4.45 (from a standing start). Complete the following sentence about such a drag racer: At some point during the race, the maximum acceleration of the drag racer is at least _______ $mi/hr/s$.

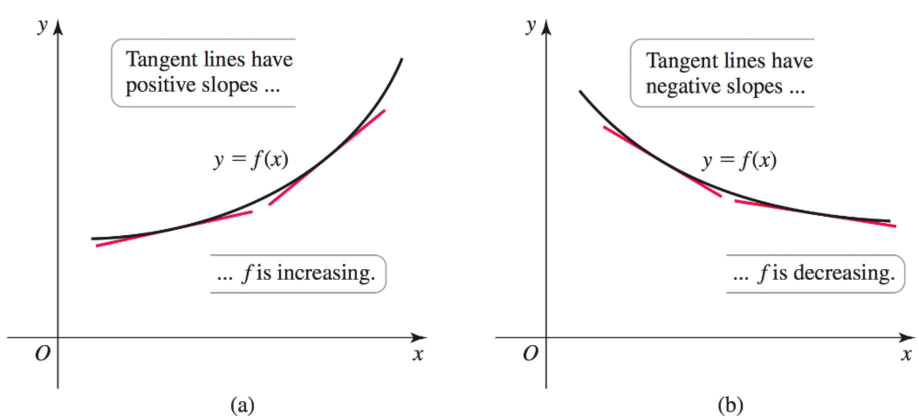
Example. A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to another officer 35 miles from the on-ramp. When the car reached the location of the second officer, 30 minutes later, it was clocked going $60\, mph$. The driver of the car was given a ticket for exceeding the $65 -\ mph$ speed limit. Why can the officer conclude that the driver exceeded the speed limit?

4.3: What Derivatives Tell Us

Definition. (Increasing and Decreasing Functions)
Suppose a function f is defined on an interval I. We say that f is increasing on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is decreasing on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.



Theorem 4.7: Test for Intervals of Increase and Decrease
Suppose f is continuous on an interval I and differentiable at every interior point of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.



Proof. **(Theorem 4.7: Test for Intervals of Increase and Decrease) p258**
Let a and b be any two distinct points in the interval I with $b > a$. By the Mean Value Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

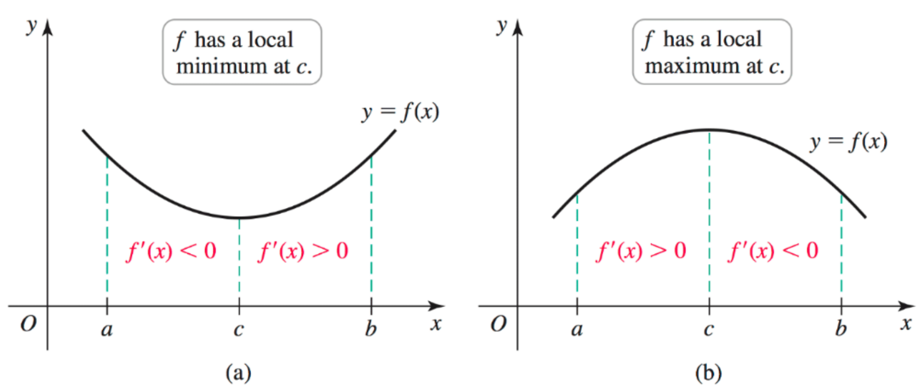
for some c between a and b. Equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Notice that $b - a > 0$ by assumption. So if $f'(c) > 0$, then $f(b) - f(a) > 0$. Therefore, for all a and b in I with $b > a$, we have $f(b) > f(a)$, which implies that f is increasing on I. Similarly, if $f'(c) < 0$, then $f(b) - f(a) < 0$ or $f(b) < f(a)$. It follows that f is decreasing on I. □

Theorem 4.8: First Derivative Test
Assume that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a local maximum at c.
- If f' changes sign from negative to positive as x increases through c, then f has a local minimum at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has no local extreme value at c.



Example. Consider the function $f(x) = 6x - x^3$.

a) $f'(x) =$

b) Find the intervals on which the function is increasing and decreasing.

c) Identify the function's local extreme values, if any. (e.g. "local max of __ at $x =$ __")

d) Which, if any, of the extreme values are absolute?

Example. Consider the function $f(t) = 12t - t^3$ on $-3 \leq t < \infty$.

a) $f'(t) =$

b) Find the intervals on which the function is increasing and decreasing.

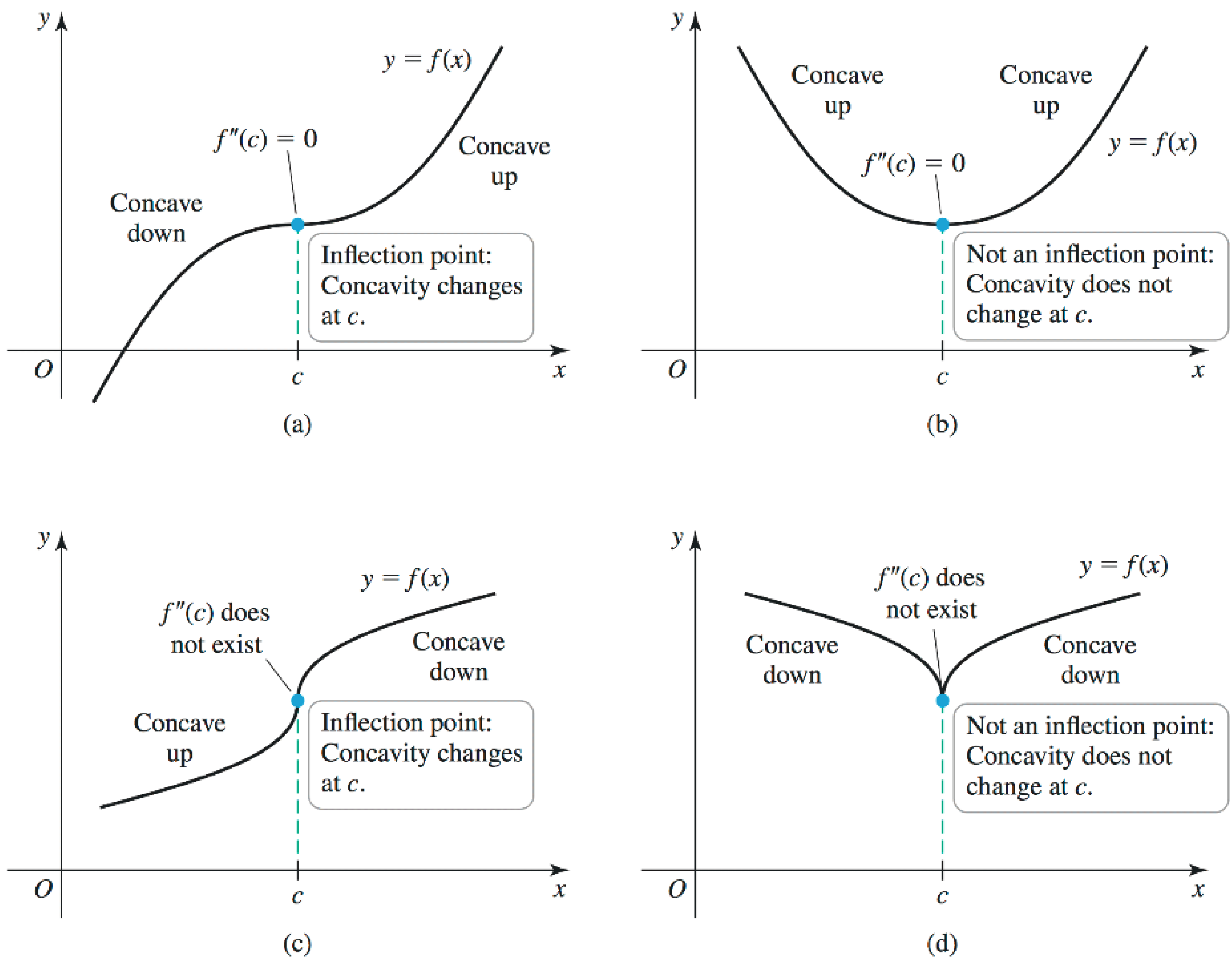
c) Identify the function's local extreme values, if any. (e.g. "local max of __ at $x =$ __")

d) Which, if any, of the extreme values are absolute?

Example. Consider the function $f(x) = \cos^2(x)$ on $[-\pi, \pi]$. Find the intervals on which f is increasing and the intervals on which it is decreasing.

Definition. (Concavity and Inflection Point)
Let f be differentiable on an open interval I.

- If f' is increasing on I, then f is *concave up* on I.
- If f' is decreasing on I, then f is *concave down* on I.
- If f is continuous at c and f changes concavity at c, then f has an *inflection point* at c.



Theorem 4.10: Test for Concavity
Suppose f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c, then f has an inflection point at c.

Example. Consider $f(x) = 5 - 3x^2 + x^3$

a) Find the intervals of increasing/decreasing and local max/min.

b) Find intervals of concavity and inflection points.

c) Draw a rough sketch of the function.

Example. Consider $f(x) = xe^{-x^2/2}$

a) Find the intervals of increasing/decreasing and local max/min.

b) Find intervals of concavity and inflection points.

c) Draw a rough sketch of the function.

Example. Consider $f(x) = x\left(\sqrt[3]{(x-3)^2}\right)$

a) Find the intervals of increasing/decreasing and local max/min.

b) Find intervals of concavity and inflection points.

c) Draw a rough sketch of the function.

Example. Consider $f(x) = x^2 - x - \ln(x)$

a) Find the intervals of increasing/decreasing and local max/min.

b) Find intervals of concavity and inflection points.

c) Draw a rough sketch of the function.

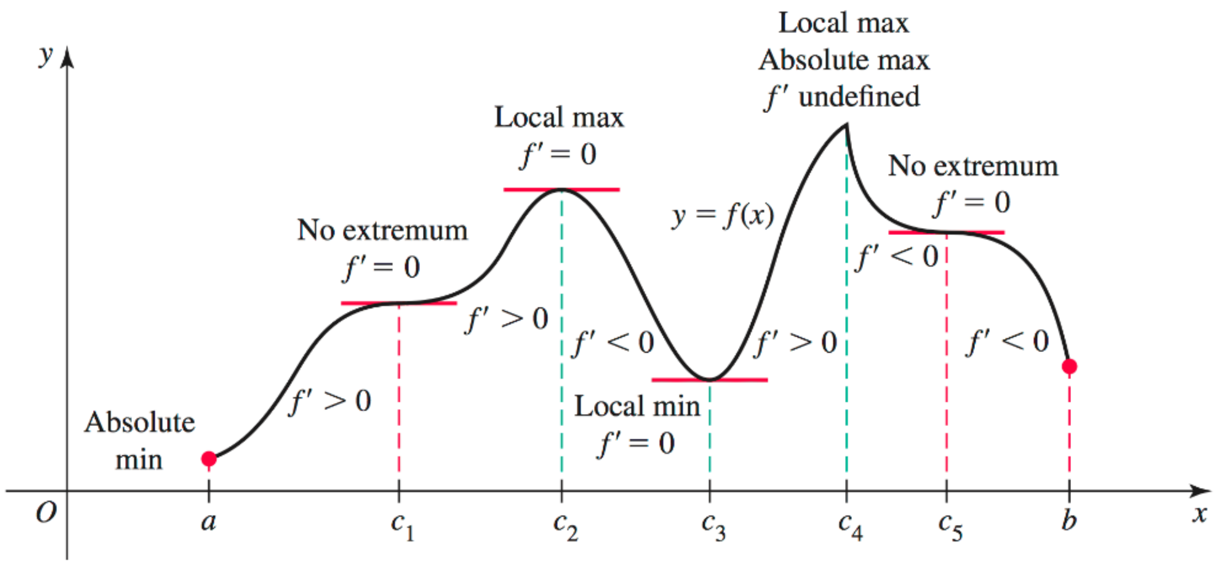
Example. Given the first derivative $y' = (x-2)^{-1/3}$

a) Find the intervals of increasing/decreasing and local max/min.

b) Find intervals of concavity and inflection points.

c) Draw a rough sketch of the function.

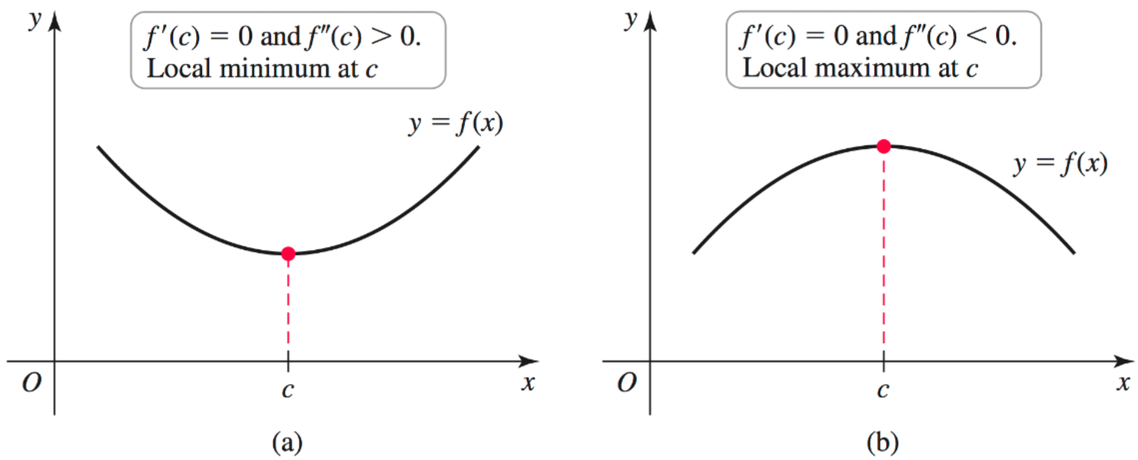
As a visual summary, here is figure 4.27 from Briggs:



$f(x)$	$f'(x)$	$f''(x)$
increasing	positive	—
decreasing	negative	—
max	zero (pos to neg)	—
min	zero (neg to pos)	—
concave up	increasing	positive
concave down	decreasing	negative
Inflection point	max/min	changes sign

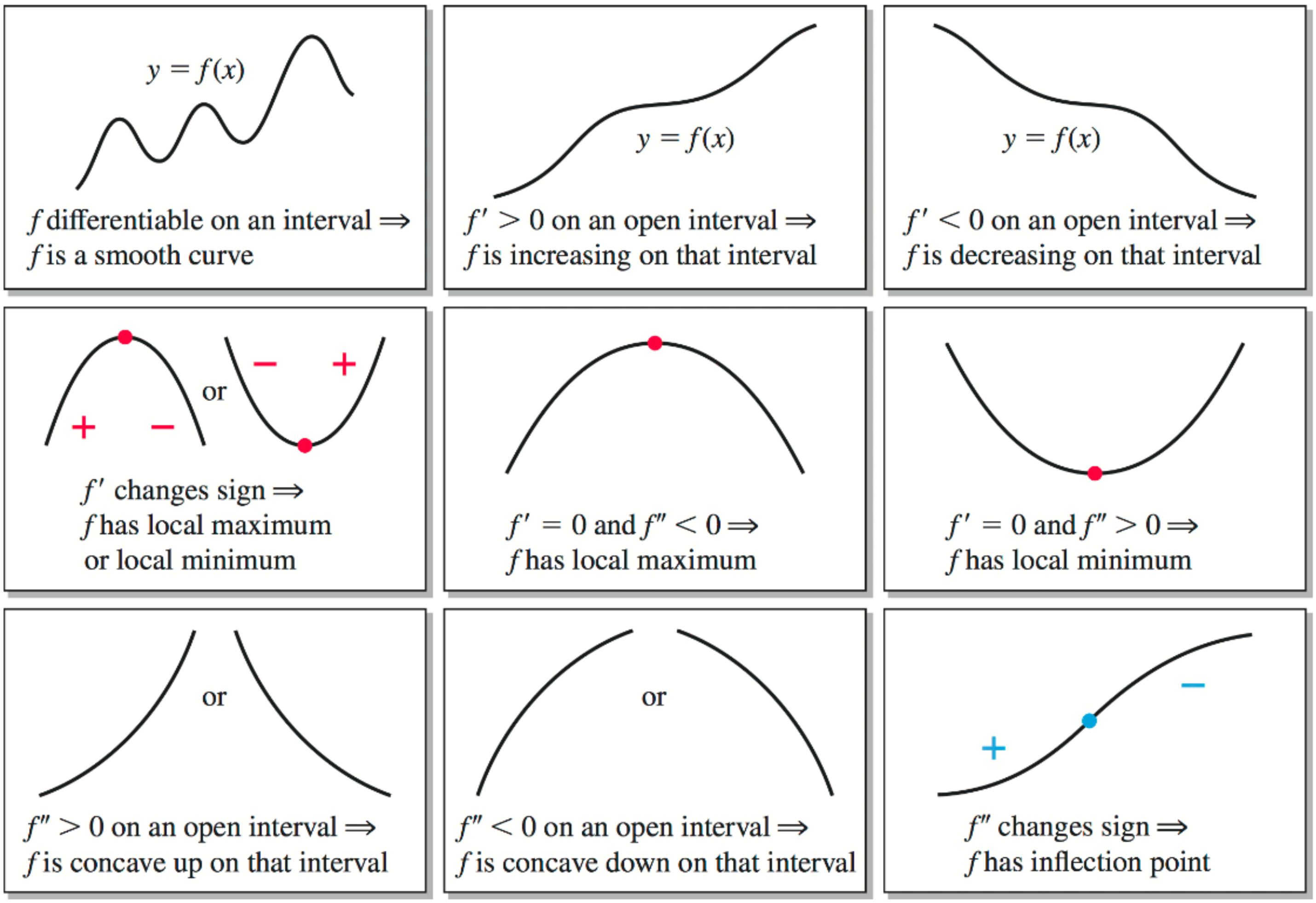
Theorem 4.11: Second Derivative Test for Local Extrema
Suppose f" is continuous on an open interval containing c with f'(c)=0.

- If $f''(c) > 0$, then f has a local minimum at c (Figure 4.40a).
- If $f''(c) < 0$, then f has a local maximum at c (Figure 4.40b).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.



Example. For each of the following, use the Second Derivative Test for Extrema to determine local max/mins:

a) $f(x) = x^5 - 5x + 3$ b) $f(x) = 2x^3 - 3x^2 + 12$ c) $f(x) = x^3 - 6$


$y = f(x)$
f differentiable on an interval $\Rightarrow$
f is a smooth curve
$y = f(x)$
$f' > 0$ on an open interval $\Rightarrow$
f is increasing on that interval
$y = f(x)$
$f' < 0$ on an open interval $\Rightarrow$
f is decreasing on that interval
$+$ $-$ or $-$ $+$
f' changes sign $\Rightarrow$
f has local maximum
or local minimum
$f' = 0$ and $f'' < 0 \Rightarrow$
f has local maximum
$f' = 0$ and $f'' > 0 \Rightarrow$
f has local minimum
or
$f'' > 0$ on an open interval $\Rightarrow$
f is concave up on that interval
or
$f'' < 0$ on an open interval $\Rightarrow$
f is concave down on that interval
$+$ $-$
f'' changes sign $\Rightarrow$
f has inflection point

4.4: Graphing Functions

Graphing Guidelines for $y = f(x)$

1. Identify the domain or interval of interest.
2. Determine if the function is symmetric.
3. Use $f'(x)$ and $f''(x)$ to determine
   - intervals of increasing/decreasing ..... $f'(x) > 0$ or $f'(x) < 0$
   - relative max/mins .... 1st or 2nd derivative test
   - concave up/concave down .... $f''(x) > 0$ or $f''(x) < 0$
   - inflection points .... $f''(x)$ changes signs
4. Find asymptotes
   - vertical .... $\lim_{x\to a} f(x) = \pm\infty$
   - horizontal .... $\lim_{x\to\pm\infty} f(x) = c$
   - slant .... $y = ax + b$
5. Intercepts
   - y-intercept .... evaluate $f(0)$
   - x-intercept .... solve $f(x) = 0$
6. Sketch using the characteristics found above

Example. Sketch the following functions using derivatives:

1. $f(x) = x^4 - 6x^2$

2. $g(x) = 200 + 8x^3 + x^4$

3. $y = \dfrac{x}{(x-1)^2}$

4. $y = \dfrac{x^2 - 4}{x^2 - 2x}$

5. $f(x) = \dfrac{(x+1)^3}{(x-1)^2}$

6. $y = \dfrac{x^2}{x^2+3}$

7. $f(x) = \dfrac{x^3}{(x+1)^2}$

8. $y = x^{1/5}$

9. $y = 2\sqrt{x} - x$

10. $y = \dfrac{1}{1+e^{-x}}$

11. $f(x) = \dfrac{x^2}{x^2 - 4}$

Example. Using the following derivatives, sketch a possible graph of the original function.

1. $f'(x) = \dfrac{1}{6}(x+1)(x-2)^2(x-3)$

2. $g'(x) = x^2(x+2)(x-1)$

4.5: Optimization Problems

Note: Please note that these ‘word problems’ are different from the related rates problems that we did in section 3.11.

Guidelines for Optimization Problems

1. Read the problem and identify variables with a diagram. Only put numbers on the diagram if they are constant.
2. Express the function that will be optimized.
3. Identify the constraint(s). Use the constraint(s) to eliminate/rewrite all but the single independent variable of the objective function.
4. Use derivatives to find the absolute max/min of the objective function.

   Make sure you are finding the correct extrema!

   - First derivative test ($f'(x)$ changes signs at $x = c$)
   - Second derivative test ($f'(c) < 0$ (neg) means $f(c)$ is a max).
   - Evaluate at endpoints and critical points.
5. Summarize your result in a sentence.

1. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He decides not to fence along the river. What are the dimensions of the field that has the largest area?

   Draw a diagram. Define your variables.

   Express your constraints and the variable that needs optimizing.

   Find the local maximum.

   Verify the critical point is the absolute maximum.

   State your answer ensuring that you are answering the question asked.

2. Show among all rectangles with an 8 meter perimeter, the one with the largest area is a square.

3. Find x and y such that $xy = 50$ but $x + y$ is minimal.

4. A box with a square base and an open top must have a volume of $32,000\,cm^3$. Find the dimensions of the box that minimizes the amount of material used.

5. If $y = 2x - 89$, what is the minimum value of the product xy?

6. A farmer has 900 meters of fencing. The fencing is to be used to enclose a rectangular field and to divide it in half. Find the dimensions of the field that has maximum area.

7. A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 48 - x^2$. What are the dimensions of the rectangle with the maximum area? What is the area?

8. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

9. A cylindrical can is to be made to hold $1\,L$ $(1000\,cm^3)$ of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.

10. A man launches his boat from point A on a bank of a straight river, $3\,km$ wide, and wants to reach point B, $8\,km$ downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point between point C and point B and then run to B. If he can row $6\,km/h$ and run $8\,km/h$, where should he land to reach B as soon as possible? (*Assume that the speed of the water is negligible compared with the speed at which the man rows.*)

11. The top and bottom margins of a poster are each $6\,cm$ and the side margins are each $4\,cm$. If the area of printed material on the poster is fixed at $384\,cm^2$, find the dimensions of the poster with the smallest area.

12. A cabin is located $2\,km$ directly into the woods from a mailbox on a straight road. A store is located on the road $5\,km$ from the mailbox. A woman wished to walk from the cabin to the store. She can walk $3\,km/h$ through the woods and $4\,km/h$ along the road. Find the point on the road toward which she would walk in order to minimize the total time for her walk.

13. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $\$10/ft^2$ and the material used to build the sides costs $\$6/ft^2$. If the box must have a volume of $50\,ft^3$ determine the dimensions that will minimize the cost to build the box.

14. An open-top rectangular box is constructed from a $10\,in.$ by $16\,in.$ piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find the dimensions of the box of largest volume and the maximum volume.

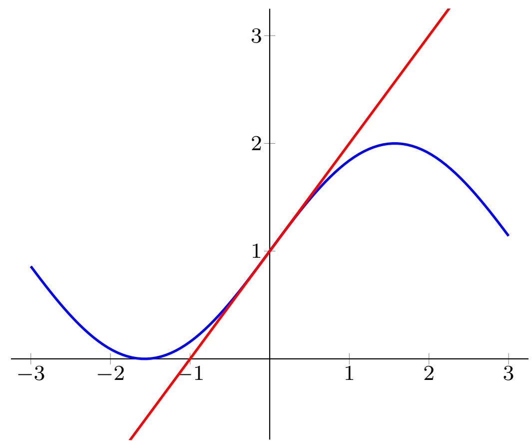
15. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius $\sqrt{2}\,cm$.

16. A rectangle is constructed with one side on the positive x-axis, one side on the positive y-axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?

17. A Norman window has the shape of a rectangle surmounted by a semi-circle. (Thus, the diameter of the semi-circle is equal to the width of the rectangle.) If the perimeter of the window is 20 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

4.6: Linear Approximation and Differentials

Example. Find the equation of the line tangent to $y = 1 + \sin(x)$ at $a = 0$. State the line in the slope–intercept form.



Definition.
Linear Approximation of f at a.
Suppose f if differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function.

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

Example. Consider the function $f(x) = \dfrac{x}{x+1}$. Find the linearization at $a = 1$. Use the linearization to estimate $f(1.1)$ and compare with the true value of $f(1.1)$.

Example. Find the linearization $L(x)$ of $f(x) = e^{3x-6}$ at $a = 2$.

Example. Find the linearization $L(x)$ of $f(x) = 9(4x + 11)^{\frac{2}{3}}$ at $a = 4$.

Example. Find a linearization over an interval which will contain the given point x_0. Choose your center at a point *near* x_0 but not at x_0 so that the given function and its derivative are easy to evaluate. Lastly, use the linearization to approximate $f(x_0)$.

a) $f(x) = x^2 + 2x,\ x_0 = 0.1$

b) $f(x) = \sqrt[3]{x},\ x_0 = 8.5.$

Example. Use a linear approximation to estimate $\sqrt{146}$.

Example. Use a linear approximation to estimate $(1.999)^4$.

Example. Use a linear approximation to estimate $\sqrt{\frac{5}{29}}$.

Example. Find the linearization $L(x)$ of $f(x) = \sqrt{x^2 + 9}$ at $a = -4$. Use $L(x)$ to estimate $f(-4.1)$ and $\sqrt{23.44}$.

Example. Use a linearization to show that 0.05 is a good approximation for $\ln(1.05)$.

Example. Find the linearization of the following functions at the given point and use concavity to identify if the linearization is an overestimate or an underestimate.

a) $f(x) = \frac{2}{x}$; $a = 1$

b) $f(x) = e^{-x}$; $a = \ln(2)$

Summary: Uses of Linear Approximation

1. To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

2. To approximate the change Δy in the dependent variable when the independent variable x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a)\Delta x.$$

Definition. (Differentials)
Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)\,dx.$$

Example. Find the differential dy.

$y = \cos(x^2)$

$y = \sqrt{1 - x^2}$

$y = 4x^2 - 3x + 2$

$y = x\tan(x^3)$

$y = \cos^5(x)$

$f(x) = \sin^{-1}(x)$

Example. Let $y = x^2$

a) Find dy

b) If $x = 1$ and $dx = 0.01$, find dy.

c) Compare dy and Δy at this point.

Example. Let $y = \sqrt{3 + x^2}$

a) Find dy

b) If $x = 1$ and $dx = -0.1$, find dy.

c) Compare dy and Δy at this point.

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.01) - f(5) = 0.25$. Use linear approximation to estimate the value of $f'(5)$.

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.99) = 7$ and $f(6) = 7.002$. Use linear approximation to estimate the value of $f'(6)$.

Example. Compute dy and Δy for $y = e^x$ when $x = 0$ and $\Delta x = 0.5$.

Example. Approximate the change in the area of a circle when its radius increases from 2.00 to 2.02 m.

Example. Approximate the change in the magnitude of the electrostatic force between two charges when the distance between them increases from $r = 20\,m$ to $r = 21\,m$ ($F(r) = 0.01/r^2$).

Example. Approximate the change in the volume of a right circular cylinder of fixed radius $r = 20\,cm$ when its height decreases from $h = 12\,cm$ to $h = 11.9\,cm$ ($V(h) = \pi r^2 h$).

Example. Approximate the change in the volume of a right circular cylinder of a right circular cone of fixed height $h = 4\,m$ when its radius increases from $r = 3\,m$ to $r = 3.05\,m$ ($V(r) = \frac{1}{3}\pi r^2 h$).

Example. The radius of a sphere is measured and found to be 0.7 inches with a possible error in measurement of at most 0.01 inches.

a) What is the maximum error in using the value of the radius to compute the volume of the sphere?

b) Find the relative error of the volume: *relative error* $\dfrac{dV}{V}$

What is the percentage error?

Example. The radius of a circular disk is given as $24\,cm$ with a maximum error in measurement of $0.2\,cm$.

a) Use differentials to estimate the maximum error in the calculated area of the disk.

b) What is the relative error? What is the percentage error?

Example. Use differentials to estimate the amount of paint needed to apply a coat of paint $0.05\,cm$ thick to a hemispherical dome with diameter $50\,m$.

4.7: L'Hôpital's Rule

Theorem 4.12: L'Hôpital's Rule
Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Theorem 4.13: L'Hôpital's Rule (∞/∞)
Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Note: Limits of the form $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are called *indeterminant forms.*

Notes on grading:

1. Unless specifically told to use L'Hôpital's Rule, you may use any valid method to evaluate limits.
2. Remember to
   a) keep your limit notation until the direct substitution step
   b) connect each step with equal signs
   c) notate the equal signs where L'Hôpital is used
3. L'Hôpital does NOT replace the quotient rule!

Example. Find the following limits with and without L'Hôpital's Rule:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} \qquad\qquad \lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1}$$

Note: L'Hôpital's Rule only works for indeterminant forms!

Example. Find the following limit with and without L'Hôpital's Rule:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin(x)}{1-\cos(x)}$$

Example. Find the following limits:

$$\lim_{t\to 1} \frac{t^3 - 1}{4t^3 - t - 3}$$

$$\lim_{z\to 0} \frac{\tan(4z)}{\tan(7z)}$$

Example. Find the following limits. Repeat L'Hôpital's Rule each time you get an indeterminant form:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

$$\lim_{t\to 0} \frac{t\sin(t)}{1 - \cos(t)}$$

Example. Evaluate:

$$\lim_{x\to 0^+} \frac{\ln(x)}{x}$$

$$\lim_{x\to 3} \frac{2x^2 - 5x + 1}{x^2 + x - 6}$$

$$\lim_{x\to \frac{1}{2}} \frac{6x^2 + 5x - 4}{4x^2 + 16x - 9}$$

$$\lim_{x\to \infty} \frac{x - 8x^2}{12x^2 + 5x}$$

$$\lim_{t\to 0} \frac{e^{2t} - 1}{\sin(t)}$$

$$\lim_{t\to 0} \frac{8^t - 5^t}{t}$$

Note: $0 \cdot \infty$ and $\infty - \infty$ are also indeterminant forms. L'Hôpital's Rule can be used after these functions are converted into rational functions of indeterminant form.

Example. Find the following limits. Convert into indeterminant form as needed:

$$\lim_{x \to 1^-} (1 - x) \tan\left(\frac{\pi x}{2}\right)$$

$$\lim_{x \to \infty} x^2 \sin\left(\frac{1}{4x^2}\right)$$

$$\lim_{x \to 0^+} (\csc(x) - \cot(x) + \cos(x))$$

Indeterminant forms 1^∞, 0^0, and ∞^0.
Assume $\lim\limits_{x\to a} f(x)^{g(x)}$ has the indeterminant form 1^∞, 0^0, or ∞^0.

1. Analyze $L = \lim\limits_{x\to a} g(x)\ln\left(f(x)\right)$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by L'Hôpital's Rule.
2. When L is finite, $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$. If $L = \infty$ or $L = -\infty$, then $\lim\limits_{x\to a} f(x)^{g(x)} = \infty$ or $\lim\limits_{x\to a} f(x)^{g(x)} = 0$, respectively.

Note: 0^∞ and ∞^∞ are NOT indeterminant forms.

$$\lim_{x\to 0^+} x^{-1/\ln(x)}$$

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln(x))}$$

$$\lim_{x\to 0^+} x^{x^2}$$

$$\lim_{x\to 0^+} x^{\sqrt{x}}$$

Note: L'Hôpital does not always work!

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin(x)}} \qquad \lim_{x\to 0^+} \frac{\cot(x)}{\csc(x)} \qquad \lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$$

Example. Find the following limits:

$$\lim_{x\to 2\pi} \frac{x\sin(x)+x^2-4\pi^2}{x-2\pi}$$

$$\lim_{x\to \frac{\pi}{2}} \frac{2\tan(x)}{\sec^2(x)}$$

$$\lim_{x\to -1} \frac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2}$$

$$\lim_{x\to\infty} \frac{27x^2+3x}{3x^2+x+1}$$

$$\lim_{x\to 0} \frac{x+\sin(x)}{x+\cos(x)}$$

$$\lim_{t\to 0} \frac{2t}{\tan(t)}$$

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$$

$$\lim_{x \to 0} \frac{x^2 - 2x}{x^2 - \sin(x)}$$

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin(\theta)}{\csc(\theta)}$$

$$\lim_{x \to 2} \frac{\sqrt[3]{3x + 2} - 2}{x - 2}$$

$$\lim_{x \to \infty} \frac{100x^3 - 3}{x^4 - 2}$$

$$\lim_{x \to 0} \frac{\sin^2(3x)}{x^2}$$

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin(\theta)}{\csc(\theta)}$$

$$\lim_{x \to 0} \cot(2x) \sin(6x)$$

$$\lim_{x \to 0} \left(\cot(x) - \frac{1}{x} \right)$$

$$\lim_{\theta\to 0}\left(\frac{1}{1-\cos(\theta)} - \frac{2}{\sin^2(\theta)}\right)$$

$$\lim_{x\to 0}(1-2x)^{\frac{1}{x}}$$

$$\lim_{\theta\to 0^+}(\sin(\theta))^{\tan(\theta)}$$

$$\lim_{x\to 0^+}(\tan x)^{x}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}$$

$$\lim_{x\to 0}\left(e^{ax}+x\right)^{\frac{1}{x}}$$

$$\lim_{x\to 0^+}\frac{x^x-1}{\ln(x)+x-1}$$

Definition. (Growth Rates of Functions (as $x \to \infty$))
Suppose f and g are functions with $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$. Then f *grows faster than* g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, equivalently, if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

The functions f and g have *comparable growth rates* if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

where $0 < M < \infty$ (M is positive and finite)

Theorem 4.14: Ranking Growth Rates as $x \to \infty$
Let $f \ll g$ mean that g grows faster than f as $x \to \infty$. With positive real numbers p, q, r, and s and with $b > 1$,

$$(\ln(x))^q \ll x^p \ll x^p(\ln(x))^r \ll x^{p+s} \ll b^x \ll x^x$$

Example. Rank the functions in order of increasing growth rates as $x \to \infty$:

x^3, $\ln(x)$, x^x, and 2^x

x^{100}, $\ln\left(x^{10}\right)$, x^x, and 10^x

Example. Use limits to compare and rank growth ranks of the following functions:

$\ln\left(x^{20}\right)$; $\ln(x)$

$\ln(x)$; $\ln(\ln(x))$

100^x; x^x

e^{x^2}; $x^{x/10}$

4.9: Antiderivatives

Definition. (Antiderivative)
A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for x in I.

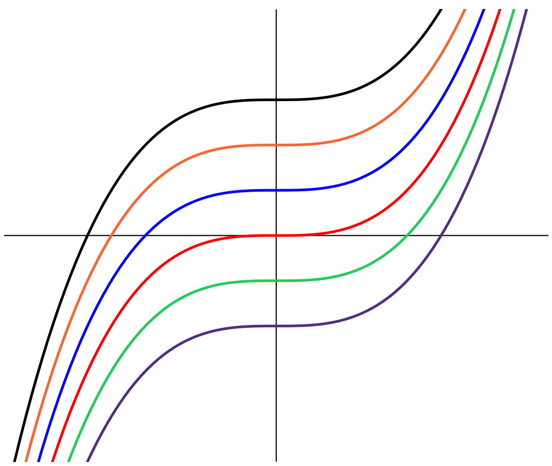
Note: we will denote this relationship in the following way:

Function	Anti-derivative
$f'(x)$	$f(x)$
$f(x)$	$F(x)$

Example. If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$. In this case, $\tan(x)$ is the *antiderivative* of $\sec^2(x)$.

Theorem 4.15: The Family of Antiderivatives
Let F be any antiderivative of f on an interval I. Then *all* antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

Example. If $f'(x) = x^2$, then $f(x) = \dfrac{x^3}{3} + C$ is the family of antiderivatives of $f'(x)$.



Example. Find the most general antiderivative of the following functions

$f(x) = \sin(x)$

$k(x) = \dfrac{1}{1+x^2}$

$g(x) = x^n,\ n \neq -1$

$h(x) = {}^1\!/\!_x$

$j(x) = 3x^2$

$n(x) = 6x^5$

$\ell(x) = \dfrac{41}{x} + 4e^x$

$m(x) = \dfrac{1}{2x^3}$

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2(x)$	$\tan(x)$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec(x)\tan(x)$	$\sec(x)$
$x^n \ (n \neq 1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)$
$\cos(x)$	$\sin(x)$	$\sin(x)$	$-\cos(x)$
e^x	e^x		

Note: There are some more 'complicated' antiderivatives as well:

$$
\begin{aligned}
f(x) = e^{g(x)} \quad &\Rightarrow \quad F(x) = \frac{e^{g(x)}}{g'(x)} \\
f(x) = k\sec^2(kx) \quad &\Rightarrow \quad F(x) = \tan(kx)
\end{aligned}
$$

Focus more on "What can I take the derivative of to get ..." rather than memorizing formulas.

Definition.
Recall that $\dfrac{d}{dx}[f(x)]$ represents taking the derivative of $f(x)$ with respect to x.

- Finding the antiderivative of f with respect to x is the **indefinite integral** $\displaystyle\int f(x)\,dx$.
- The **integrand** is the function $f(x)$ we are integrating.
- The **variable of integration**, dx, indicates which variable we are integrating with respect to.

Theorem 4.16: Power Rule for Indefinite Integrals

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Theorem 4.17: Constant Multiple and Sum Rules
Constant Multiple Rule: $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$, for real numbers c.
Sum Rule: $\displaystyle\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

Table 4.9: Indefinite Integrals of Trigonometric Functions

$\dfrac{d}{dx}[\sin(x)] = \cos(x)$	$\Longrightarrow$	$\int \cos(x)\,dx = \sin(x) + C$
$\dfrac{d}{dx}[\cos(x)] = -\sin(x)$	$\Longrightarrow$	$\int \sin(x)\,dx = -\cos(x) + C$
$\dfrac{d}{dx}[\tan(x)] = \sec^2(x)$	$\Longrightarrow$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\dfrac{d}{dx}[\cot(x)] = -\csc^2(x)$	$\Longrightarrow$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\dfrac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\Longrightarrow$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\dfrac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	$\Longrightarrow$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

Table 4.10: Other Indefinite Integrals

$\dfrac{d}{dx}[e^x]$	$= e^x$	$\Longrightarrow$	$\displaystyle\int e^x\,dx = e^x + C$
$\dfrac{d}{dx}[\ln\lvert x\rvert]$	$= \dfrac{1}{x}$	$\Longrightarrow$	$\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\dfrac{d}{dx}\left[\sin^{-1}(x)\right]$	$= \dfrac{1}{\sqrt{1-x^2}}$	$\Longrightarrow$	$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$
$\dfrac{d}{dx}\left[\tan^{-1}(x)\right]$	$= \dfrac{1}{1+x^2}$	$\Longrightarrow$	$\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$
$\dfrac{d}{dx}\left[\sec^{-1}\lvert x\rvert\right]$	$= \dfrac{1}{x\sqrt{x^2-1}}$	$\Longrightarrow$	$\displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}\lvert x\rvert + C$

Example. Verify the following integration formulas using differentiation.

$$\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$$

$$\int \sec^2(5x-1)\,dx = \frac{1}{5}\tan(5x-1) + C$$

$$\int \cos^3(x)\,dx = \sin(x) - \frac{1}{3}\sin^3(x) + C$$

$$\int \frac{x}{\sqrt{x^2+1}} = \sqrt{x^2+1} + C$$

Example. Find the most general anti-derivative or indefinite integral

$$\int \left(\frac{t^2}{2} + 4t^3 \right) dt \qquad \int \left(\frac{1}{5} - \frac{2}{x^3} + 2x \right) dx$$

$$\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) dx \qquad \int \frac{\csc\theta \cot\theta}{2} \, d\theta$$

$$\int \left(1 + \frac{1}{4u^3} - (3u)^2\right) du$$

$$\int \left(\frac{7}{\sqrt{1-x^2}} - \frac{3}{\cos^2(x)}\right) dx$$

$$\int \left(\frac{1}{4e^x} - \frac{4}{x} + 4^x\right) dx$$

$$\int \frac{x^2 - 36}{x - 6}\, dx$$

$$\int \frac{2+3\cos(y)}{\sin^2(y)}\,dy$$

$$\int (u+4)(2u+1)\,du$$

$$\int e^{x+2}\,dx$$

$$\int \left(\sqrt[4]{x^3}+\sqrt{x^5}\right)dx$$

$$\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx$$

$$\int \left(\sin(4x) - \frac{3}{\sin^2(x)}\right) dx$$

$$\int \left(\csc^2(2t) - 2e^t\right) dt$$

$$\int x\left(1 + 2x^4\right) dx$$

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\int \frac{\sin(2x)}{\sin(x)}\,dx$$

$$\int \left(\pi + \frac{2}{yt}\right) dt$$

$$\int \frac{t^2 - e^{2t}}{t + e^t}\,dt$$

Definition.

- An equation involving an unknown function and its derivative is called a **differential equation**.
- An **initial condition** allows us to determine the arbitrary constant C.
- A differential equation coupled with an initial condition is called an **initial value problem**.

$f'(x) = g(x)$,	where g is given, and	Differential equation
$f(a) = b$,	where a and b are given.	Initial condition

Example. Solve the initial value problem:

$$\frac{dy}{dx} = 9x^2 - 4x + 5,\ y(-1) = 0$$

$$f'(x) = 8x^3 + 12x + 3,\ f(1) = 6$$

$$f'(x) = 1 + 3\sqrt{x},\ f(4) = 25$$

$$\frac{dr}{d\theta} = \cos(\pi\theta),\ r(0) = 1$$

$$f'(t) = 2\cos(t) + \sec^2(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}, \quad f\left(\frac{\pi}{3}\right) = 4$$

$$g'(x) = 7x\left(x^6 - \tfrac{1}{7}\right); \quad g(1) = 2$$

$$f'''(x) = \sin(x), \quad f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1$$

$$f'(x) = \frac{4}{\sqrt{1-x^2}}, \ f\left(\frac{1}{2}\right) = 1$$

$$\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left.\frac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$$

$$f''(x) = 4 + 6x + 4x^2, \ f(0) = 3, \ f(1) = 10$$

Initial Value Problems for Velocity and Position
Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem.

$$s'(t) = v(t),\ s(0) = s_0,\ \text{where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t),\ v(0) = v_0,\ \text{where } v_0 \text{ is the (known) initial velocity.}$$

Recall:

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

Example. Solve the following velocity and position initial value problems

$v(t) = \sin(t) + 3\cos(t); \quad s(0) = 4$

$a(t) = 2e^t - 12, \quad v(0) = 1, \quad s(0) = 0$

Example. The acceleration of gravity near the surface of Mars is $3.72 m/s^2$. A rock is thrown straight up from the surface with an initial velocity of 23 m/s. How high does the rock go?

a) Write the initial value problem

b) Find the velocity and position functions.

c) Maximum height is reached when velocity is 0. Find the time when this happens and the maximum height.

Example. A ball is thrown vertically upward from a height of 48 feet above ground at a speed of $32ft/s$. Assume the acceleration due to gravity is 32 m/s^2.

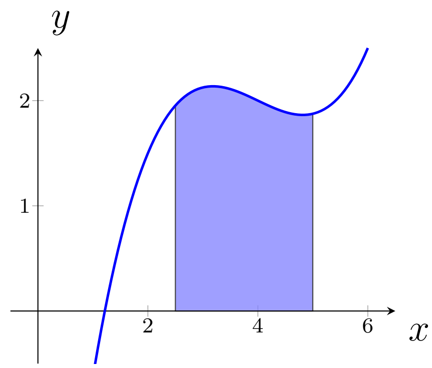
a) How high above the ground will it get?

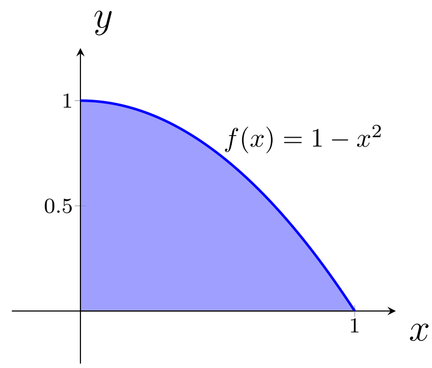
b) How long after it is thrown will it hit the ground?

Example. A stone was dropped off a cliff and hits the ground with a speed of $120 ft/s$. What is the height of the cliff (assuming $a(t) = -32 ft/s^2$).

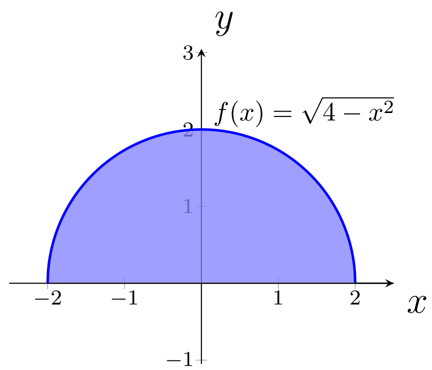
Example. Find the antiderivative of $f(x) = \dfrac{2 + x^2}{1 + x^2}$

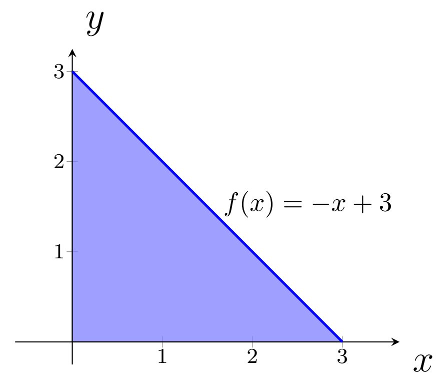
5.1: Approximating Areas under Curves

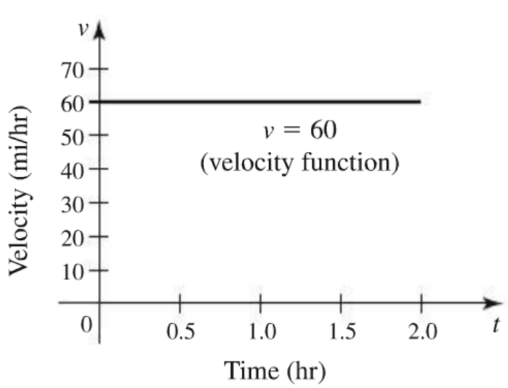


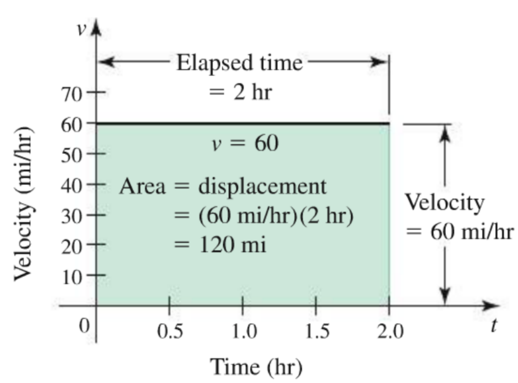


Finding the area under the curve is simple in some cases:

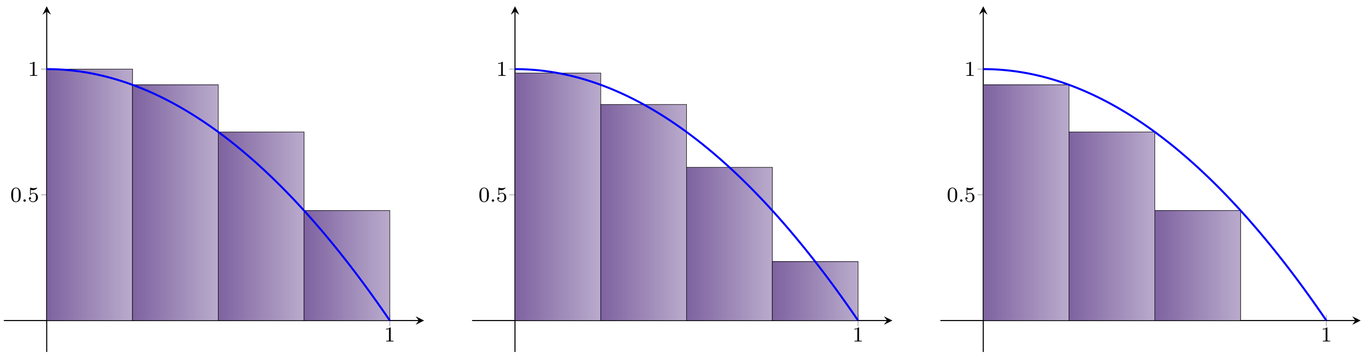






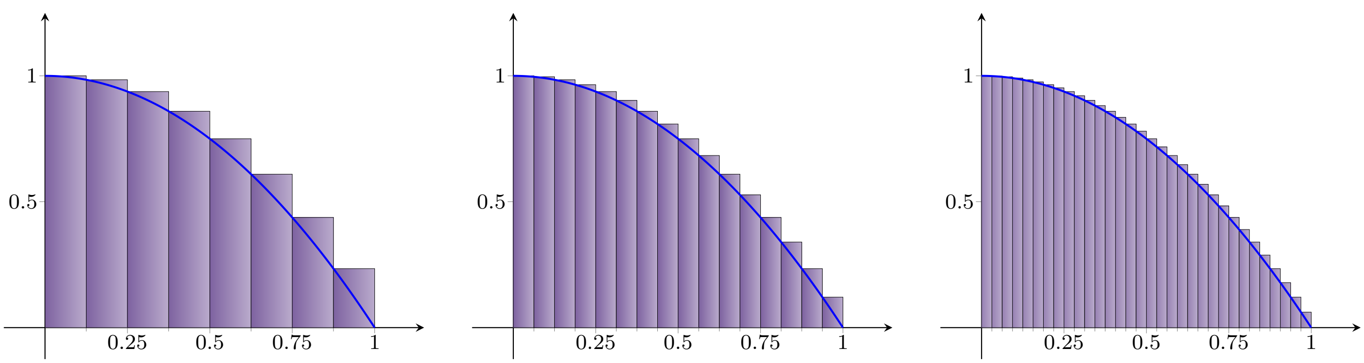


For functions whose curves are irregular shapes, we can approximate the area using rectangles:



Left rectangles   Midpoint rectangles   Right rectangles

These approximations are much more accurate when more rectangles are used:



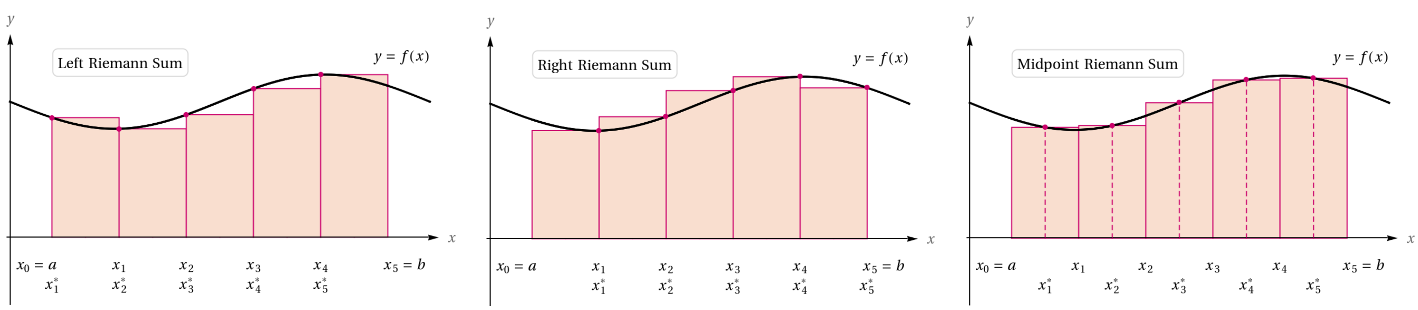
Definition. (Riemann Sum)
Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the k-th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$,
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$, and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

In general, midpoint rectangles give better approximations then left or right rectangles.



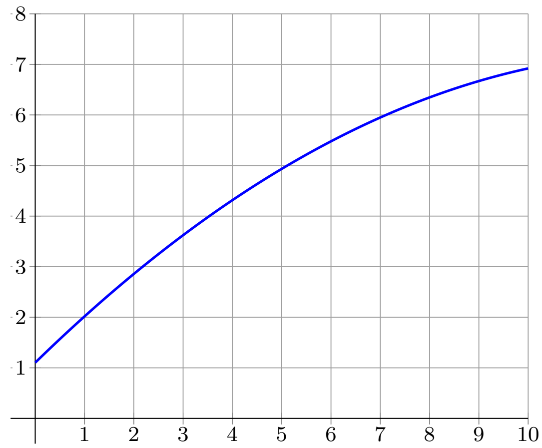
When estimating the area under the graph on the interval $[a, b]$, we define

$$\Delta x = \frac{b-a}{n}$$

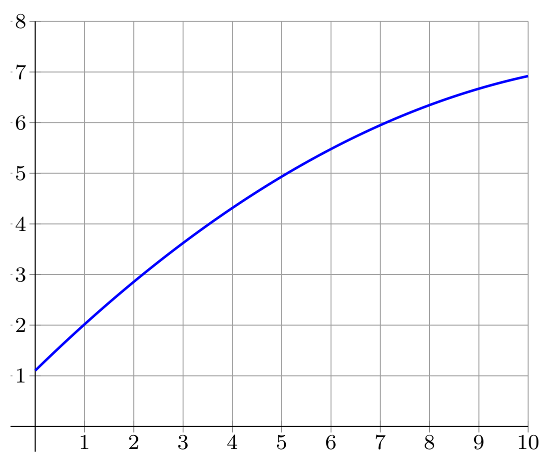
to be the width of the rectangles. The height of the rectangles is given by $f(x_i)$, where the x_i's are Δx apart.

Example. Use the plots below to estimate the area under the given graph of $f(x)$:

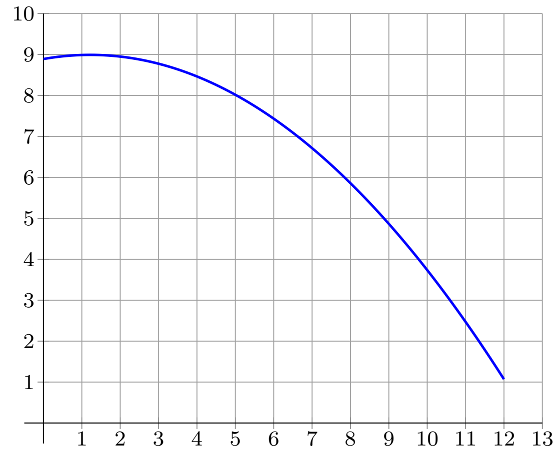
Sketch five rectangles and use them to find a lower estimate for the area under the given graph of $f(x)$ from $x = 0$ to $x = 10$.



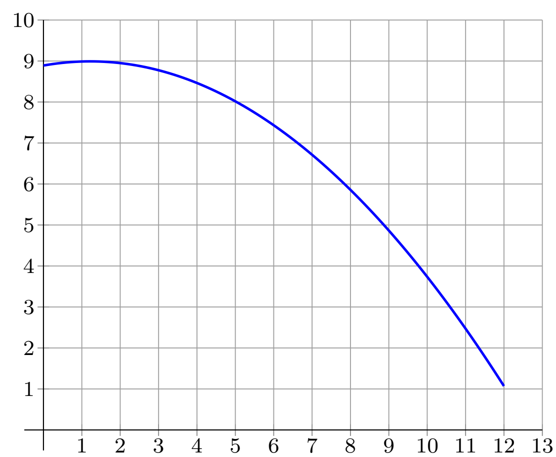
Sketch five rectangles and use them to find a upper estimate for the area under the given graph of $f(x)$ from $x = 0$ to $x = 10$.



Use six right rectangles to estimate the area under the given graph of $f(x)$ from $x = 0$ to $x = 12$. Is the estimate an under-approximation or an over-approximation? Why?



Use six midpoint rectangles to estimate the area under the given graph of $f(x)$ from $x = 0$ to $x = 12$. Talk about the quality of this estimate.



Example. Consider $f(x) = x^2$ on the interval $[0, 1]$.

a) Use two rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.

b) Use four rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.

c) Use four midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.

Example. Consider $f(x) = \dfrac{1}{x}$ on the interval $[1, 5]$.

a) Use two rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.

b) Use four rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.

c) Use two rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.

d) Use four rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.

e) Use two midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.

f) Use four midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.

Example. Use the tabulated values of f to evaluate both the left and right Reimann sums. ($n = 8$, $[1, 5]$)

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	0	2	3	2	2	1	0	2	3

When velocity is a continuous function, the area between the curve and the x-axis gives the displacement.

Example. The velocities (in m/s) of an automobile moving along a straight freeway over a four-second period are given in the following table. Find the midpoint Riemann sum approximation to the displacement on $[0, 4]$ with $n = 2$ and $n = 4$ subintervals.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	20	25	30	35	30	30	35	40	40

Example. Use the following table of recorded velocities answer the following questions:

Time (min)	**Velocity (m/sec)**	**Time (min)**	**Velocity (m/sec)**
0	1.0	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

a) Estimate the total displacement using 12 subintervals of length 5 with left-endpoint values.

b) Estimate the total displacement using 12 subintervals of length 5 with right-endpoint values.

Definition.
If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

Example. Rewrite the sums without sigma notation and evaluate:

a) $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

b) $\displaystyle\sum_{k=1}^{4} (-1)^k \cos(k\pi)$

Example. Rewrite the following sum without sigma notation. Note that n denotes the number of rectangles and the letters i, j, and k are typically used for indexing.

$$\sum_{k=1}^{n} \frac{1}{n}(k^2 + 1)$$

Example. Which of the following summations represent the sum $1+2+4+8+16+32$?

a) $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ b) $\displaystyle\sum_{k=0}^{5} 2^{k}$ c) $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

Example. Which of the following summations represent the sum $1-2+4-8+16-32$?

a) $\displaystyle\sum_{k=1}^{6}(-2)^{k-1}$ b) $\displaystyle\sum_{k=0}^{5}(-1)^{k}2^{k}$ c) $\displaystyle\sum_{k=-2}^{3}(-1)^{k+1}2^{k+2}$

Example. Express the following sums in sigma notation:

a) $-1+4-9+16-25$

b) $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}$

c) $\dfrac{5}{2}+\dfrac{10}{3}+\dfrac{15}{4}+\dfrac{20}{5}+\dfrac{25}{6}+\dfrac{30}{7}$

d) $4+9+14+\cdots+44$

Example. Suppose that $\displaystyle\sum_{k=1}^{n} a_k = 0$ and $\displaystyle\sum_{k=1}^{n} b_k = 1$, evaluate the following

a) $\displaystyle\sum_{k=1}^{n} 8a_k$

b) $\displaystyle\sum_{k=1}^{n} 250b_k$

c) $\displaystyle\sum_{k=1}^{n} (a_k+1)$

d) $\displaystyle\sum_{k=1}^{n} (b_k-1)$

Theorem 5.1: Sums of Powers of Integers
Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Example. Prove that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Example. Evaluate the following sums

a) $\displaystyle\sum_{k=1}^{10} k$ b) $\displaystyle\sum_{k=1}^{10} \left(1+k^2\right)$ c) $\displaystyle\sum_{k=1}^{10} k^3$

d) $\displaystyle\sum_{k=1}^{7} (-2k-4)$ e) $\displaystyle\sum_{k=1}^{6} (3-k^2)$ f) $\displaystyle\sum_{m=1}^{3} \frac{2m+2}{3}$

Definition. (Left, Right and Midpoint Riemann Sums in Sigma Notation) Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** for f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- $\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **midpoint Riemann sum** if $x_k^* = a + (k - 1/2)\Delta x$.

Example. For the function $f(x) = 3x^2$, find a formula for the upper sum obtained by dividing the interval $[0, 1]$ into n equal subintervals.

Now take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 3x^2$ over $[0, 1]$.

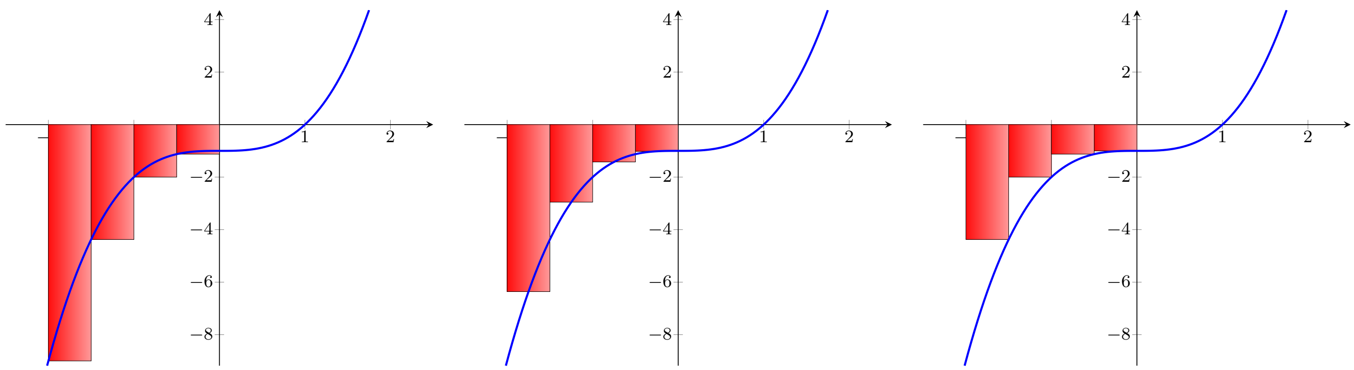
Example. For the function $f(x) = 2x$, find a formula for the upper sum obtained by dividing the interval $[0, 3]$ into n equal subintervals.

Now take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 2x$ over $[0, 3]$.

5.2: Definite Integrals

In section 5.1, we assumed f was nonnegative on the interval $[a, b]$.
When $f(x) \leq 0$ on the interval $[a, b]$, then the area between $f(x)$ and the x-axis is negative.

Example. Consider the function $x^3 - 1$ on $[-2, 0]$. Using Riemann sums, we can approximate the area between the curve and the x-axis:



$$L_4 = f(-2)\frac{1}{2} + f(-1.5)\frac{1}{2} + f(-1)\frac{1}{2} + f(-0.5)\frac{1}{2} \qquad = -8.25$$

$$M_4 = f(-1.75)\frac{1}{2} + f(-1.25)\frac{1}{2} + f(-0.75)\frac{1}{2} + f(-0.25)\frac{1}{2} \qquad = -5.875$$

$$R_4 = f(-1.5)\frac{1}{2} + f(-1)\frac{1}{2} + f(-0.5)\frac{1}{2} + f(0)\frac{1}{2} \qquad = -4.25$$

Actual area: $\boxed{-6}$

Definition. (Net Area)
Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

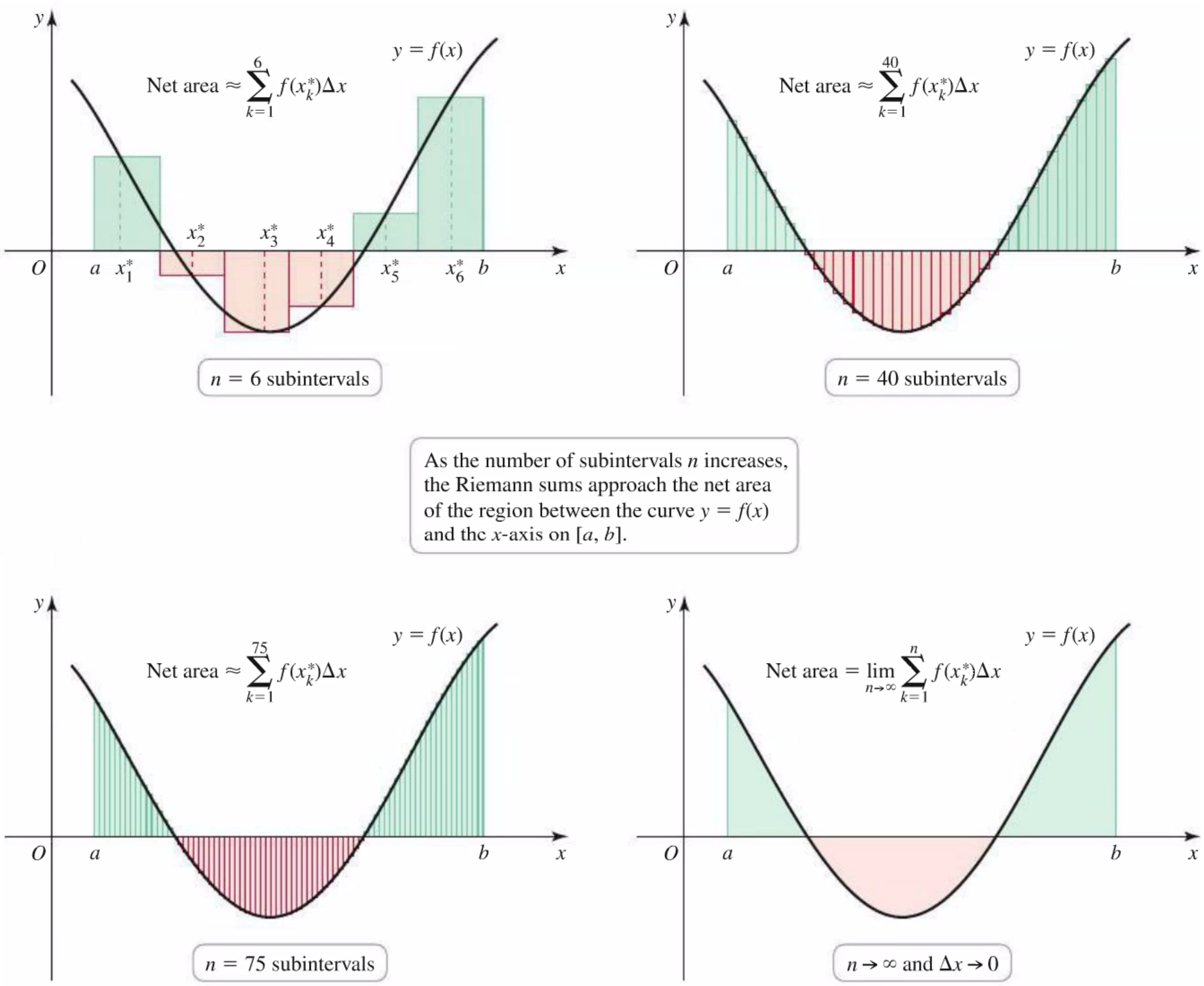
Example. If $f(x) = x^2 - 2x,\ 0 \le x \le 3$, evaluate the Riemann sum with $n = 6$, taking the sample points to be right endpoints.

Example. Find the Riemann sum for $f(x) = \sin(x),\ 0 \le x \le \frac{3\pi}{2}$, with six terms, taking the sample points to be right endpoints.

Definition. (Definite Integral)

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$



Example.

a) For the function $f(x) = 3x + 2x^2$, find a formula for the upper sum obtained by dividing the interval $[0, 1]$ into n equal subintervals.

b) Take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 3x + 2x^2$ over $[0, 1]$.

Example. Use the limit of the Riemann sum notation to evaluate $\int_0^2 (2 - x^2)\, dx$.

Example. Use the definition of the definite integral to evaluate $\int_1^4 \left(x^2 - 1\right) dx$.

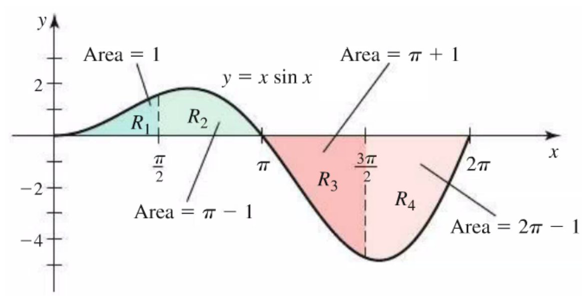
Example. Use the definition of the definite integral to evaluate $\displaystyle\int_1^4 \left(x^2 - 4x + 2\right) dx$.

Example. Use the definition of the definite integral to evaluate $\displaystyle\int_0^2 4x^3\, dx$.

Example. Use a sketch and geometry to evaluate the following integral

$$\int_1^{10} g(x)\,dx, \text{ where } g(x) = \begin{cases} 4x, & \text{if } 0 \le x \le 2 \\ -8x+16, & \text{if } 2 < x \le 3 \\ -8, & \text{if } x > 3 \end{cases}$$

Example. Use the following figure to evaluate the integrals below:



a) $\displaystyle\int_0^{\pi} x\sin(x)\,dx$

b) $\displaystyle\int_0^{3\pi/2} x\sin(x)\,dx$

c) $\displaystyle\int_0^{2\pi} x\sin(x)\,dx$

d) $\displaystyle\int_{\pi/2}^{2\pi} x\sin(x)\,dx$

Example. Graph the following integrands and compute the areas to evaluate the integrals.

a) $\displaystyle\int_{1/2}^{3/2} (-2x+4)\,dx$

b) $\displaystyle\int_{-2}^{4} \left(\frac{x}{2}+3\right) dx$

c) $\displaystyle\int_{0}^{3} \left(\frac{1}{2}x-1\right) dx$

d) $\displaystyle\int_{-1}^{3} (3-2x)\,dx$

e) $\displaystyle\int_{-4}^{0} \sqrt{16-x^2}\,dx$

f) $\displaystyle\int_{-1}^{1} \left(1+\sqrt{1-x^2}\right) dx$

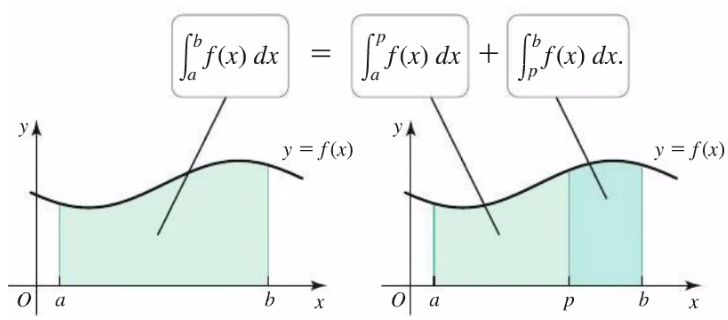
g) $\displaystyle\int_{-2}^{1} |x|\,dx$

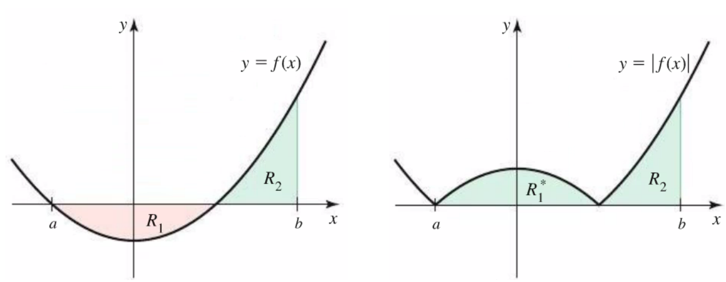
h) $\displaystyle\int_{-1}^{1} (2-|x|)\,dx$

Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
5. $\displaystyle\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.





Example. Suppose that $\displaystyle\int_{-3}^{0} g(t)\,dt = \sqrt{2}$. Evaluate the following:

a) $\displaystyle\int_{-3}^{0} g(u)\,du$

b) $\displaystyle\int_{0}^{-3} g(t)\,dt$

c) $\displaystyle\int_{0}^{-3} [-g(x)]\,dx$

d) $\displaystyle\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\,dr$

Example. Suppose that $\displaystyle\int_{0}^{3} f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Evaluate the following:

a) $\displaystyle\int_{3}^{4} f(z)\,dz$

b) $\displaystyle\int_{4}^{3} f(z)\,dz$

Example. Use the fact that $\displaystyle\int_0^{\pi/2} (\cos(\theta) - 2\sin(\theta))\, d\theta = -1$ to evaluate the following

a) $\displaystyle\int_0^{\pi/2} (2\sin(\theta) - \cos(\theta))\, d\theta$

b) $\displaystyle\int_{\pi/2}^{0} (4\cos(\theta) - 8\sin(\theta))\, d\theta$

Example. Suppose that $\displaystyle\int_1^9 f(x)\, dx = -1$, $\displaystyle\int_7^9 f(x)\, dx = 5$ and $\displaystyle\int_7^9 h(x)\, dx = 4$. Evaluate the following:

a) $\displaystyle\int_1^9 -2f(x)\, dx)$

b) $\displaystyle\int_7^9 [f(x) + h(x)]\, dx$

c) $\displaystyle\int_7^9 [2f(x) - 3h(x)]\, dx$

d) $\displaystyle\int_9^1 f(x)\, dx$

e) $\displaystyle\int_1^7 f(x)\, dx$

f) $\displaystyle\int_7^9 [h(x) - f(x)]\, dx$

Example. Given $\displaystyle\int_1^3 e^x\,dx = e^3 - e$, find $\int_1^3 (2e^x - 1)\,dx$

Example. Suppose that $f(x) \geq 0$ on $[0, 2]$ and $f(x) \leq 0$ on $[2, 5]$ where $\displaystyle\int_0^2 f(x)\,dx = 6$ and $\displaystyle\int_2^5 f(x)\,dx = -8$. Evaluate the following:

a) $\displaystyle\int_0^5 f(x)\,dx$

b) $\displaystyle\int_0^5 |f(x)|\,dx$

c) $\displaystyle\int_2^5 4|f(x)|\,dx$

d) $\displaystyle\int_0^5 (f(x) + |f(x)|)\,dx$

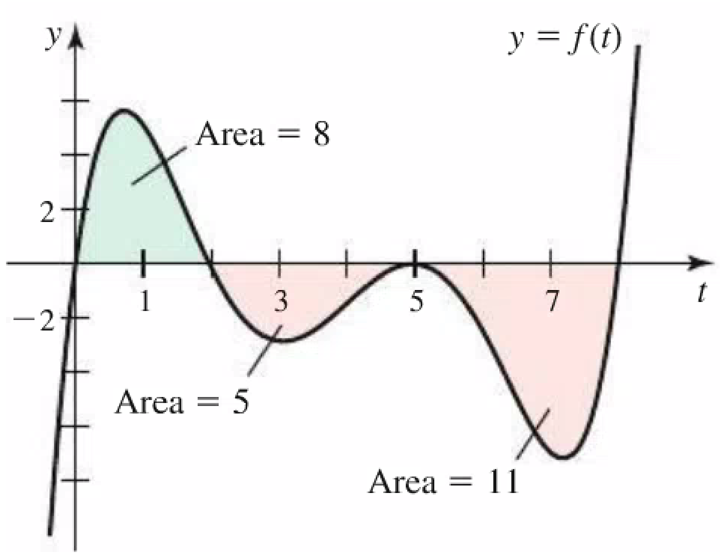
5.3: Fundamental Theorem of Calculus

Definition. (Area Function)
Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a is**

$$A(x) = \int_a^x f(t)\,dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Example. The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions:



a) $A(2)$ b) $F(5)$ c) $A(0)$ d) $F(8)$

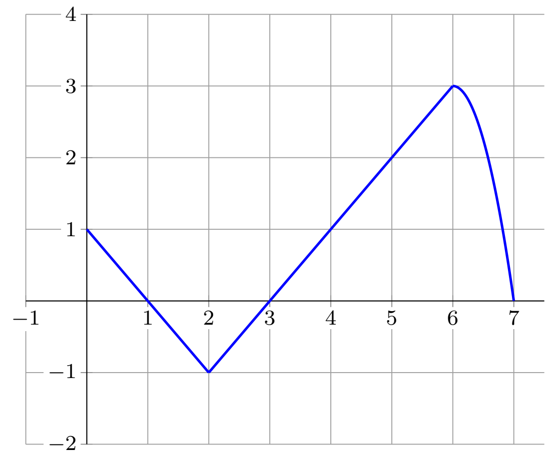
e) $A(8)$ f) $A(5)$ g) $F(2)$

Example. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$ and 6.

b) Estimate $g(7)$.

c) Where does g have a maximum value? Where does it have a minimum value?



Example. For the area function $A(x) = \int_a^x f(t)\,dt$, graph the area function and then verify that $A'(x) = f(x)$.

a) $f(t) = 10,\ a = 4$

b) $f(t) = 2,\ a = -3$

c) $f(t) = 2t + 5,\ a = 0$

d) $f(t) = 4t + 2,\ a = 0$

Example. Let $f(x) = c$, where c is a positive constant. Explain why an area function of f is an increasing function.

Example. The linear function $f(x) = 3 - x$ is decreasing on the interval $[0, 3]$. Is its area function for f (with left endpoint 0) increasing or decreasing on the interval $[0, 3]$?

Theorem 5.3 (Part I) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example. For the following functions, find the derivatives

a) $g(x) = \displaystyle\int_0^x \sqrt{1-2t}\,dt$

b) $g(x) = \displaystyle\int_3^x e^{t^2-t}\,dt$

c) $g(y) = \displaystyle\int_2^y t^2\sin(t)\,dt$

d) $y = \displaystyle\int_x^2 \cos(t^2)\,dt$

e) $y = \displaystyle\int_1^{\cos(x)} (t+\sin(t))\,dt$

f) $y = \displaystyle\int_{-3}^{3x^4} \frac{t}{t^2-4t}\,dt$

g) $y = \int_1^{e^x} \ln(t)\, dt$

h) $y = \int_0^{x^4} \cos^2(\theta)\, d\theta$

i) $y = \int_{\tan(x)}^0 \frac{dt}{1+t^2}$

j) $y = \int_{\sin(x)}^1 \sqrt{1+t^2}\, dt$

Theorem 5.3 (Part II) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Example. Evaluate the following integrals using graphs and the Fundamental Theorem of Calculus.

a) $\displaystyle\int_3^7 6\,du$

b) $\displaystyle\int_{-2}^4 x\,dx$

Example. Evaluate the following integrals

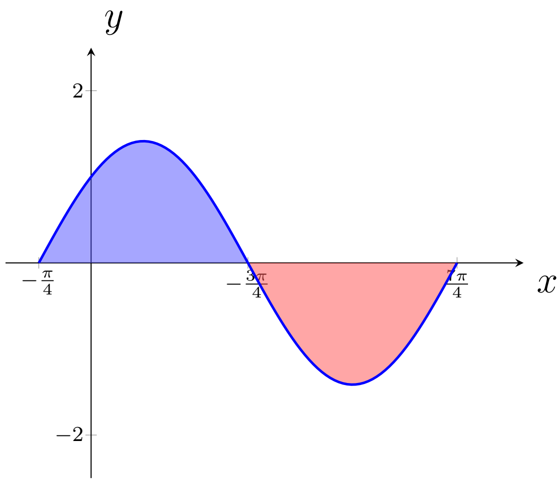
a) $\displaystyle\int_{-1}^3 x^2\,dx$

b) $\displaystyle\int_0^{\pi} (1 + \cos(x))\,dx$

c) $\displaystyle\int_{-5}^5 e\,dx$

d) $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$

Example. Evaluate the following integrals using the Fundamental Theorem of Calculus



$$\int_{-\pi/4}^{7\pi/4} (\sin(x) + \cos(x))\, dx$$

Example. Evaluate $\displaystyle\int_3^8 f'(t)\, dt$, where f' is continuous on $[3, 8]$, $f(3) = 4$, and $f(8) = 20$.

Example. Find $\displaystyle\frac{d}{dt}\int_0^{t^4} \sqrt{u}\, du$ by evaluating the integral directly and then differentiating the result.

Example. Find $\displaystyle\frac{d}{dt}\int_0^{t^4} \sqrt{u}\, du$ by differentiating the integral directly.

Example. Find $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan(\theta)} \sec^2(y)\,dy$ by evaluating the integral directly and then differentiating the result.

Example. Find $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan(\theta)} \sec^2(y)\,dy$ by differentiating the integral directly.

Example. Evaluate the following integrals:

a) $\displaystyle\int_1^8 \sqrt[3]{x}\,dx$

b) $\displaystyle\int_{-2}^{-1} \frac{2}{x^2}\,dx$

c) $\displaystyle\int_0^2 x(2+x^5)\,dx$

d) $\displaystyle\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}\,du$

e) $\displaystyle\int_0^2 (y-1)(2y+1)\,dy$

f) $\displaystyle\int_0^4 \left(1+3y-y^2-\frac{y^3}{4}\right)dy$

g) $\displaystyle\int_0^1 (x^e+e^x)\,dx$

h) $\displaystyle\int_0^3 (2\sin(x)-e^x)\,dx$

i) $\displaystyle\int_{\frac{\pi}{2}}^{0} \frac{1-\cos(2t)}{2}\,dt$

j) $\displaystyle\int_{\frac{1}{2}}^{2} \left(1-\frac{1}{x^2}\right) dx$

k) $\displaystyle\int_{1}^{2} \left(\frac{2}{s^2}-\frac{4}{s^3}\right) ds$

l) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

m) $\displaystyle\int_0^5 \left(x^2 - 9\right) dx$

n) $\displaystyle\int_{1/2}^2 \left(1 - \frac{1}{x^2}\right) dx$

o) $\displaystyle\int_{-1}^2 x^3\, dx$

p) $\displaystyle\int_{\pi/6}^{2\pi} \cos(x)\, dx$

Example. Find the area of the region bounded by $y = \sqrt{x}$ between $x = 1$ and $x = 4$.

Example. Find the area of the region below the x-axis bounded by $y = x^4 - 16$.

Example. Find the area of the region bounded by $y = 6\cos(x)$ between $x = -\pi/2$ and $x = \pi$.

Example. Find the area of the region bounded by $f(x) = x(x+1)(x-2)$ and the x-axis on the interval $[-1, 2]$.

Example. Find the total area between $y = 3x^2 - 3$ and the x-axis on $-2 \leq x \leq 2$.

Example. Find the total area between $y = x^3 - 3x^2 + 2x$ and the x axis on the interval $0 \leq x \leq 2$.

5.4: Working with Integrals

Theorem 5.4: Integrals of Even and Odd Functions
Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Example. Rewrite the following trig functions to determine if it is even or odd:

$\sin(-x) =$ $\qquad$ $\cos(-x) =$

$\tan(-x) =$ $\qquad$ $\cot(-x) =$

$\csc(-x) =$ $\qquad$ $\sec(-x) =$

Use this to rewrite and evaluate the following integrals:

$$\int_{-\pi}^{\pi} \sin(x)\,dx \qquad \int_{-\pi}^{\pi} \cos(x)\,dx$$

$$\int_{-\pi/4}^{\pi/4} \tan(x)\,dx \qquad \int_{-\pi/4}^{\pi/4} \sec(x)\,dx$$

Note: $\cot(x)$ and $\csc(x)$ are excluded here as they are not continuous on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Example. Use symmetry to evaluate the following integrals:

a) $\displaystyle\int_{-10}^{10} \frac{x}{\sqrt{200-x^2}}\,dx$

b) $\displaystyle\int_{-2}^{2} \left(x^9 - 3x^5 + 2x^2 - 10\right) dx$

c) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^5(x)\,dx$

d) $\displaystyle\int_{-1}^{1} (1 - |x|)\,dx$

e) $\displaystyle\int_{-2}^{2} \frac{x^3 - 4x}{x^2+1}\,dx$

Example. Given that $f(x)$ is even and $\int_{-8}^{8} f(x)\,dx = 18$, find

a) $\displaystyle\int_0^8 f(x)\,dx$

b) $\displaystyle\int_{-8}^{8} xf(x)\,dx$

Example. Given that $f(x)$ is odd and $\int_0^4 f(x)\,dx = 3$ and $\displaystyle\int_0^8 f(x)\,dx = 9$, find

a) $\displaystyle\int_{-4}^{8} f(x)\,dx$

b) $\displaystyle\int_{-8}^{4} f(x)\,dx$

Example. Use symmetry to explain why

$$\int_{-4}^{4} \left(5x^4 + 3x^3 + 2x^2 + x + 1\right) dx = \int_0^4 \left(5x^4 + 2x^2 + 1\right) dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos(2\theta) + \cos(\theta)\sin(\theta) - 3\sin(\theta^5)\right) d\theta$$

Example. While the following integrals are not on symmetric intervals, symmetry still applies here:

a) $\displaystyle\int_0^{\pi} \cos(x)\, dx$ b) $\displaystyle\int_0^{2\pi} \sin(x)\, dx$ c) $\displaystyle\int_0^{4\pi} \cos(x)\, dx$

Definition. (Average Value of a Function)
The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example. Find the average value of $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$.

Example. Find the average value of $f(x) = 3x^2 - 3$ on $[0, 1]$.

Example. Find the average value of $f(t) = t^2 - t$ on $[-2, 1]$.

Example. Find the average value of $f(x) = \dfrac{1}{x^2+1}$ on $[-1, 1]$.

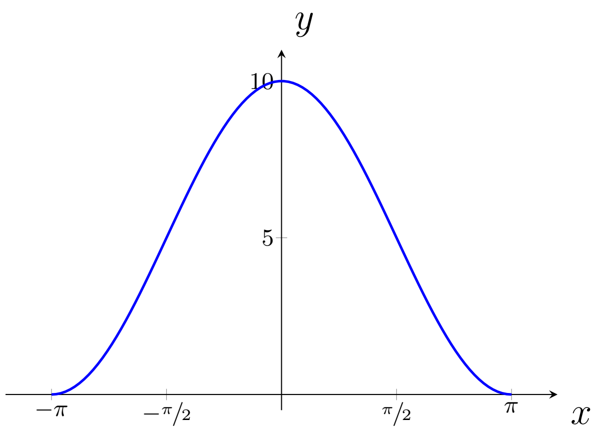
Example. Find the average value of $f(x) = \dfrac{1}{x}$ on $[1, e]$.

Example. Find the average value of $f(x) = x^{\frac{1}{n}}$ on $[0, 1]$.

Example. The velocity in m/s of an object moving along a line over the time interval $[0, 6]$ is $v(t) = t^2 + 3t$. Find the average velocity of the object over this time interval.

Example. A rock is launched vertically upward from the ground with a speed of $64 ft/s$. The height of the rock (in ft) above the ground after t seconds taskis given by the function $s(t) = -16t^2 + 64t$. Find its average velocity during its flight.

Example. The surface of a water wave is described by $y = 5(1+\cos(x)$, for $-\pi \leq x \leq \pi$, where $y = 0$ corresponds to a trough of the wave. Find the average height of the wave above the trough on $[-\pi, \pi]$.



Theorem 5.5: Mean Value Theorem for Integrals
Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

Example. For the following problems, find the point(s) that satisfy the Mean Value Theorem for Integrals.

a) $f(x) = \dfrac{1}{x^2}$ on $[1, 4]$.

b) $f(x) = e^x$ on $[0, 2]$.

c) $f(x) = \cos(x)$ on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

d) $f(x) = 1 - |x|$ on $[-1, 1]$.

5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

Example. We know

$$\frac{d}{dx}\left[\frac{(2x+1)^4}{4}\right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned}
\int 2(2x+1)^3\,dx &= \int f\big(g(x)\big)g'(x)\,dx \\
&= \int u^3\,du \\
&= \frac{u^4}{4} + C \\
&= \frac{(2x+1)^4}{4} + C
\end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f\big(g(x)\big)$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\displaystyle\int 2x(x^2+3)^4\,dx$

b) $\displaystyle\int (2x+1)^3\,dx$

c) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

d) $\displaystyle\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

e) $\displaystyle\int \sqrt{4-t}\,dt$

f) $\displaystyle\int (2-x)^6\,dx$

Example. Evaluate the following integrals:

a) $\displaystyle\int \sec(2\theta)\tan(2\theta)\,d\theta$

b) $\displaystyle\int \csc^2\left(\frac{t}{3}\right)dt$

c) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

d) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2\cos(\pi t)\ m/s^2$ for all t. If $s = 0$ and $v = 8\ m/s$ when $t = 0$, find the value of s when $t = 1$ sec.

Example. Evaluate the following integrals:

a) $\displaystyle\int (6x^2 + 2)\sin(x^3 + x + 1)\, dx$

b) $\displaystyle\int \frac{\sin(\theta)}{\cos^5(\theta)}\, d\theta$

c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

d) $\displaystyle\int \frac{2^t}{2^t + 3}\, dt$

e) $\displaystyle\int 6x^2 4^{x^3}\, dx$

f) $\displaystyle\int \frac{dx}{\sqrt{36 - 4x^2}}$

g) $\displaystyle\int \sin(t)\sec^2(\cos(t))\,dt$

h) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

i) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

j) $\displaystyle\int 5\cos(7x+5)\,dx$

k) $\displaystyle\int \frac{3}{\sqrt{1-25x^2}}\,dx$

l) $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

Example. Evaluate the following integrals using the recommended substitution:

a) $\int \sec^2(x)\tan(x)\,dx$
where $u = \tan(x)$.

b) $\int \sec^2(x)\tan(x)\,dx$
where $u = \sec(x)$.

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, y(0) = 0$.

Example. Evaluate the following integrals:

a) $\displaystyle\int xe^{-x^2}\,dx$

b) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

c) $\displaystyle\int \frac{dt}{8-3t}$

d) $\displaystyle\int 5^t\sin(5^t)\,dt$

e) $\displaystyle\int \frac{e^w}{36+e^{2w}}\,dw$

Theorem 5.7: Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example. Evaluate the integrals:

a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

b) $\displaystyle\int_1^3 \frac{dt}{(t-4)^2}$

c) $\displaystyle\int_0^3 \frac{v^2+1}{\sqrt{v^3+3v+4}}\,dv$

d) $\displaystyle\int_0^1 2x\big(4-x^2\big)\,dx$

e) $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$

g) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$

h) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y)\,dy$

i) $\displaystyle\int_0^1 (1 - 2x^9)\,dx$

j) $\displaystyle\int_0^1 (1 - 2x)^9\,dx$

k) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1 + 4x^2}\,dx$

l) $\displaystyle\int_0^4 \frac{x}{x^2 + 1}\,dx$

m) $\displaystyle\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

n) $\displaystyle\int_0^{\frac{\pi}{8}} \sec(2\theta)\tan(2\theta)\,d\theta$

o) $\displaystyle\int_0^{1} (3t-1)^{50}\,dx$

p) $\displaystyle\int_0^{3} \frac{1}{5x+1}\,dx$

q) $\displaystyle\int_0^1 xe^{-x^2}\,dx$

r) $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$

s) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$

t) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

u) $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

v) $\displaystyle\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln\left(\frac{\pi}{2}\right)} e^w \cos(e^w)\, dw$

w) $\displaystyle\int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}}\, dx$

x) $\displaystyle\int_1^{e^2} \frac{\ln(p)}{p}\, dp$

y) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x)\, dx$

z) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin(7x^3)\, dx$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8\sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

Example. Prove $\int \tan(x)\,dx = \ln|\sec(x)| + C$.

Example. Evaluate the integrals:

a) $\displaystyle\int \frac{x}{(x-2)^3}\,dx$

b) $\displaystyle\int x\sqrt{x-1}\,dx$

c) $\displaystyle\int x^3(1+x^2)^{\frac{3}{2}}\,dx$

d) $\displaystyle\int \frac{y^2}{(y+1)^4}\,dy$

e) $\displaystyle\int (z+1)\sqrt{3z+2}\,dz$

f) $\displaystyle\int_0^1 \frac{x}{(x+2)^3}\,dx$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\displaystyle\int \cos^2(x)\,dx$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx$

c) $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

d) $\displaystyle\int x\sin^2(x^2)\,dx$

e) $\displaystyle\int \sin^2\left(\theta+\frac{\pi}{6}\right)d\theta$

f) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(8\theta)\,d\theta$

Example. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

Example. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

Example. Suppose f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2)\,dx$.

b) $\int_{-2}^2 x^2 f(x^3)\,dx$.

Example. Evaluate the integrals:

a) $\displaystyle\int \sec^2(10x)\,dx$

b) $\displaystyle\int \tan^{10}(4x)\sec^2(4x)\,dx$

c) $\displaystyle\int \left(x^{\frac{3}{2}} + 8\right)^5 \sqrt{x}\,dx$

d) $\displaystyle\int \frac{2x}{\sqrt{3x+2}}\,dx$

e) $\displaystyle\int \frac{7x^2+2x}{x}\,dx$

f) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

g) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9+x^2}\,dx$

h) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y)+2}\,dy$

i) $\displaystyle\int \frac{\sec(z)\tan(z)}{\sqrt{\sec(z)}}\,dz$

j) $\displaystyle\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$

k) $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$

l) $\displaystyle\int \frac{x}{1+x^4}\,dx$

m) $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

n) $\displaystyle\int x^2\sqrt{2+x}\,dx$

o) $\displaystyle\int \big(\sin^5(x) + 3\sin^3(x) - \sin(x)\big)\cos(x)\,dx$

p) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 + x^4 \tan(x)\right) dx$

q) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x) \sin\left(\sin(x)\right) dx$

r) $\displaystyle\int \frac{1+x}{1+x^2}\, dx$

Note: Sometimes multiple substitutions may be needed:

Example. Evaluate the integrals:

a) $\displaystyle\int x\sin^4(x)\cos^2(x)\,dx$

b) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$