MATH 1080 **Exam 2 – Version A**

Calculus of One Variable II Second Summer 2019

Student's Printed Name: KEY **CUID:** ____________

Instructor: ____________ **Section:** ____________

Instructions: You are not permitted to use a calculator of any kind on any portion of this exam. You are not allowed to use a textbook, notes, cell phone, computer, or any other technology on any portion of this exam. All devices must be turned off and stored away while you are in the examination room.

During this exam, any form of communication with any person other than the instructor or designated proctor is understood to be a violation of academic integrity.

No part of this exam may be removed from the examination room.

Read each question carefully. To receive full credit in the free response, you must:

- ☐ show legible, logical, and relevant justification which supports your final answer,
- ☐ use complete and correct mathematical notation,
- ☐ include proper units, where necessary, and
- ☐ give answers as exact values whenever possible.

You have ninety (90) minutes to complete this entire examination. Good luck!

For instructor or teaching assistant use only.

Question	Points Possible	Points Earned	Question	Points Possible	Points Earned
1	10		6	10	
2	10		7	10	
3	10		Notation	5	
4a	5				
4b	5				
5	5				
			Multiple Choice	30	
			Free Response	70	
			Exam Total	100	

Multiple Choice. For each question, clearly circle your answer to each of the following questions. Each question has only one correct answer. If you indicate more than one answer, or leave a blank, the question is marked as incorrect. For this exam, there are ten (10) multiple choice questions worth three (3) points each for a total of thirty (30) points.

1. Which of the following is the correct partial fraction decomposition for the following rational function?

$$f(x) = \frac{2x+5}{x^4+x^2}$$

a. $\frac{A}{x^2} + \frac{B}{x^4}$

b. $\frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+1}$

c. $\frac{A}{x} + \frac{Bx+C}{x^2} + \frac{Cx+D}{x^2+1}$

d. $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

$$\frac{2x+5}{x^4+x^2} = \frac{2x+5}{x^2(x^2+1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

2. Which of the following substitutions is necessary for the following integral?

$$\int \sqrt{9+16x^2}\,dx$$

$9+16x^2 = 3^2 + (4x)^2$

a. $x = 3\tan\theta$ b. $x = \frac{3}{4}\tan\theta$ c. $x = 3\sin\theta$ d. $x = 4\sec\theta$

$4x = 3\tan\theta$

$x = \frac{3}{4}\tan\theta$

3. Determine if the integral $\int_1^\infty \frac{1}{x^2+1}$ converges or diverges.

a. Converges to $\frac{\pi}{4}$

b. Converges to $\frac{\pi}{2}$

c. Converges to 0

d. Diverges

$$= \lim_{N\to\infty} \left(\arctan x \Big|_1^N \right)$$

$$= \lim_{N\to\infty} \left(\arctan N - \arctan 1 \right)$$

$\arctan N \to \pi/2$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

4. A sequence is defined recursively by $a_1 = 0$ and $a_{n+1} = a_n^2 + 1$. Find the next three terms of the sequence, i.e. a_2, a_3, and a_4.

   a. $a_2 = 3,\ a_3 = 3,\ a_4 = 4$
   b. $a_2 = 1,\ a_3 = 2,\ a_4 = 3$
   c. $a_2 = 1,\ a_3 = 2,\ a_4 = 5$
   d. $a_2 = 1,\ a_3 = 0,\ a_4 = 1$

$a_2 = a_1^2 + 1 = 1$

$a_3 = a_2^2 + 1 = 2$

$a_4 = a_3^2 + 1 = 5$

5. Which of the following integrals are improper?

   a. $\int_2^3 \frac{1}{x}\,dx$ b. $\int_1^3 \frac{x}{x-2}\,dx$ c. $\int_0^1 \frac{1}{e^x}\,dx$ d. $\int_e^\pi \frac{1}{\ln x}\,dx$

6. Consider the series $\sum_{k=2}^{\infty}\left(\frac{1}{k} - \frac{1}{k-1}\right)$. Determine the formula for the partial sum, S_n, for this series.

   a. $S_n = \frac{1}{n} - \frac{1}{n-1}$
   b. $S_n = \frac{1}{n}$
   c. $S_n = \frac{1}{n} - 1$
   d. It is not possible to find an expression for S_n.

$S_n = \left(\cancel{\frac{1}{2}} - 1\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\frac{1}{n} - \cancel{\frac{1}{n-1}}\right)$

$S_n = \frac{1}{n} - 1$

7. Does the sequence whose nth term is $a_n = \dfrac{\arctan 2n}{3n}$ converge? If so, to what value?

   a. Yes, it converges to 0.
   b. Yes, it converges to $\frac{2}{3}$.
   c. Yes, it converges to $\frac{\pi}{2}$.
   d. No, it diverges.

$$\lim_{n\to\infty} \frac{\arctan 2n \to \pi/2}{3n \to \infty} = 0$$

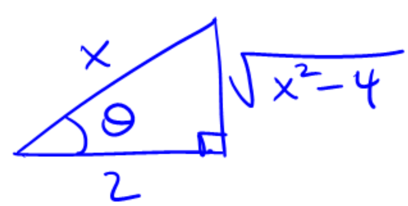
8. A trigonometric substitution is often referred to as what type of substitution?

   a. direct b. singular c. transcendental d. inverse

9. If $x = 2\sec\theta$, what is $\sin 2\theta$ in terms of x?

   a. $\dfrac{4}{x}$
   b. $\dfrac{4\sqrt{x^2-4}}{x^2}$
   c. $\sqrt{x^2-4}$
   d. $\dfrac{x^2\sqrt{x^2-4}}{4}$

$$\frac{x}{2} = \sec\theta$$



$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\,\frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x}$$
$$= \frac{4\sqrt{x^2-4}}{x^2}$$

10. The sequence $\left\{ \cos\left(\frac{(2n-1)\pi}{2} \right) \right\}_{n=1}^{\infty}$ is... $\longrightarrow = \{0, 0, 0, 0, \ldots\}$

a. ...neither bounded nor monotonic.

b. ...both bounded and monotonic.

c. ...unbounded.

d. ...divergent.

This concludes the multiple choice portion of this exam. Please be sure that you have circled a response to each of the questions.

Series Test Summary

Test	**Series**	**Series converges if:**	**Series diverges if:**
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	$\lvert r \rvert < 1$	$\lvert r \rvert \geqslant 1$

This table does not neccessarily include all details or conditions required for a given test.

Free Response. Answer all questions clearly and legibly. *If your work is unintelligible, the grader has no obligation to attempt to understand it and it may be marked incorrect.* Additionally, correct answers without appropriate work shown will receive little or no credit. For each question, show specific mathematical notation, work, and any appropriate units. Before you begin, read the directions at the beginning of the exam. This portion of the exam is worth seventy (70) points, five (5) of which are alloted for appropriate mathematical notation.

1. **(10 pts)** Determine if the integral $\int_1^\infty \frac{(\ln x)^2}{x}\,dx$ converges or diverges.

$$= \lim_{N\to\infty} \int_1^N \frac{(\ln x)^2}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$u(1) = 0$

$u(N) = \ln N$

$$= \lim_{N\to\infty} \int_0^{\ln N} u^2\,du$$

$$= \lim_{N\to\infty} \frac{1}{3}u^3 \Big|_0^{\ln N}$$

$$= \lim_{N\to\infty} \frac{1}{3}(\ln N)^3$$

$$= \infty$$

diverges

2. **(10 pts)** Evaluate the integral: $\displaystyle\int \frac{2x+5}{x(x^2+1)}\,dx.$ $= *$

$$\left(\frac{2x+5}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\right) x(x^2+1)$$

$$2x+5 = A(x^2+1) + (Bx+C)x$$

$$\underline{2x} + \underline{5} = Ax^2 + \underline{A} + Bx^2 + \underline{Cx}$$

$$\boxed{A=5}$$

$$A+B=0$$

$$B=-A \qquad \boxed{C=2}$$

$$\boxed{B=-5}$$

$$* = \int\left(\frac{5}{x} + \frac{-5x+2}{x^2+1}\right)dx$$

$$= \int\left(\frac{5}{x} - \frac{5x}{x^2+1} + \frac{2}{x^2+1}\right)dx$$

$$= 5\ln|x| - \frac{5}{2}\ln|x^2+1| + 2\arctan x + C$$

$x = a\sin\theta$

3. **(10 pts)** Evaluate the integral $\int_0^a \frac{4b}{a}\sqrt{a^2 - x^2}\, dx$ where a, b are positive, nonzero constants.

$dx = a\cos\theta\, d\theta$

$\theta = \arcsin\left(\frac{x}{a}\right)$

<u>trig sub</u>

$$= \int_{\theta(0)}^{\theta(a)} \frac{4b}{\cancel{a}}\sqrt{a^2 - a^2\sin^2\theta}\ \cancel{a}\cos\theta\, d\theta$$

$$= 4b\int_0^{\pi/2} \sqrt{a^2(\underbrace{1-\sin^2\theta}_{=\cos^2\theta})}\ \cos\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= 4ab\left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= 4ab\left(\frac{\pi}{4} + 0 - (0+0)\right)$$

$$= \pi ab$$

4. (a) **(5 pts)** Determine if the series $\displaystyle\sum_{k=2}^{\infty} \frac{2^k}{5^{k+1}}$ converges or diverges.

You must clearly justify your conclusion. If it converges, state the sum.

$$\sum_{k=2}^{\infty} \frac{2^k}{5^{k+1}} = \frac{1}{5}\sum_{k=2}^{\infty}\left(\frac{2}{5}\right)^k$$

$r = \frac{2}{5}$ geometric series

Since $r = \frac{2}{5}$ and $|r| < 1$, then the series <u>converges</u> by the GST.

$$\frac{1}{5}\sum_{k=2}^{\infty}\left(\frac{2}{5}\right)^k = \frac{1}{5}\cdot\frac{\frac{4}{25}}{1-\frac{2}{5}} = \frac{4}{75}$$

(b) **(5 pts)** Same as in part (a), but with $\displaystyle\sum_{k=1}^{\infty} \big(\arctan(k+1) - \arctan k\big)$.

$$S_n = \sum_{k=1}^{n} (\arctan(k+1) - \arctan k)$$

$$= (\cancel{\arctan 2} - \underbrace{\cancel{\arctan 1}}_{=\frac{\pi}{4}})$$

$$+ (\cancel{\arctan 3} - \cancel{\arctan 2}) + \cdots$$

$$+ (\arctan(n+1) - \cancel{\arctan n})$$

$$S_n = \arctan(n+1) - \frac{\pi}{4}$$

$$\lim_{n\to\infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$$

The series <u>converges</u> to $\frac{\pi}{4}$.

5. **(5 pts)** For each of the following statements, determine whether the statement is true or false.

You do not need to justify your conclusion.

☐ If $S_n = \sum_{k=1}^{n} a_k$ and $\lim_{n\to\infty} S_n = L$, then $\lim_{n\to\infty} a_n = L$.

False, $\sum_{k=1}^{\infty} a_k = L$

☐ If $\sum_{k=1}^{\infty} ar^k$ converges, then so does $\sum_{k=1}^{\infty} a(r-0.0001)^k$.

False, $r = -0.9999$

☐ If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} a_k = \sum_{k=1000}^{\infty} a_k$.

False, $\sum_{k=1}^{\infty} a_k \neq \sum_{k=1000}^{\infty} a_k$

☐ If $\{a_n\}_{n=1}^{\infty}$ converges, then $\left\{\frac{1}{a_n}\right\}_{n=1}^{\infty}$ must also converge.

False, $a_n \to 0$

$\frac{1}{a_n} \to \infty$

☐ Every divergent sequence is unbounded.

False

$\{1, 0, 1, 0, \ldots\}$

6. **(10 pts)** Use a geometric series to write $0.\overline{52}$ as an irreducible fraction.

$$0.\overline{52} = \frac{52}{100}\sum_{k=0}^{\infty}\left(\frac{1}{100}\right)^k$$

$r = \frac{1}{100}$ Converges

$$= \frac{52}{100}\cdot\frac{1}{1-\frac{1}{100}}$$

$$= \frac{52}{\cancel{100}}\cdot\frac{\cancel{100}}{99}$$

$$= \frac{52}{99}$$

7. **(10 pts)** If $k \geqslant 0$, show that $\displaystyle\int_{-\infty}^{0} \frac{\arctan^k x}{1+x^2}\,dx = -\frac{1}{k+1}\left(-\frac{\pi}{2}\right)^{k+1}$.

$$= \lim_{N\to-\infty} \int_N^0 \frac{\arctan^k x}{1+x^2}\,dx \qquad u = \arctan x \quad du = \frac{1}{1+x^2}\,dx$$

$$= \lim_{N\to-\infty} \int_{\arctan N}^{0} u^k\,du$$

$$= \lim_{N\to-\infty} \frac{1}{k+1} u^{k+1} \Big|_{\arctan N}^{0}$$

$$= \lim_{N\to-\infty} \left(0 - \frac{1}{k+1}(\arctan N)^{k+1}\right)$$

$$= -\frac{1}{k+1} \lim_{N\to-\infty} (\arctan N)^{k+1} \to -\frac{\pi}{2}$$

$$= -\frac{1}{k+1}\left(-\frac{\pi}{2}\right)^{k+1}$$

Alternatively, convert the integral to $\displaystyle\int_{-\frac{\pi}{2}}^{0} u^k\,du$ and evaluate.

This page intentionally left blank.