4.5: Optimization Problems

An **optimization problem** is a problem where we want to maximize or minimize and objective function subject to given constraints. So, we want to find the absolute maximum or absolute minimum value of a function.

Steps for Optimization Problems:

1. Read the problem carefully and define all variables
   - What quantities are given?
   - What constraints are given?
   - What objective function are we trying to maximize or minimize?
2. Draw a diagram if possible.
3. Express the function to be optimized in terms of the variables.
4. Express the constraints in terms of the variables.
5. Use the constraints to eliminate all but one variable of the objective function.
6. Find the domain of the objective function.
7. Use methods of calculus to find the absolute extreme of the objective function on the domain.
8. Verify that you have found the location of the absolute extreme. Use can use:
   - Closed Interval Method (Extreme Values Theorem)
   - First Derivative Test
   - Second Derivative Test
9. Write a summary sentence answering the question.

Three Methods to Verify the Absolute Extreme

- The Closed Interval Method (Extreme Value Theorem) for finding absolute extrema of a continuous function f on a closed interval $[a, b]$.

  Steps:

  1. Find the critical points of f
  2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
  3. The smallest function value is the absolute minimum and the largest function value is the absolute maximum.

  Example: Find two nonnegative numbers x and y whose sum is 10 which minimize the quantities $x^2 + y^2$. Use The Closed Interval Method to verify absolute minimum.
  objective function: $x^2 + y^2$
  constraints: $x + y = 10$ and $x, y \geq 0$

- The First Derivative Test for Absolute Extrema of a Continuous Function f.

  Steps:

  1. Find the critical points of f
  2. Determine where f' is negative and positive
  3. If c is a critical points,
     - If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f
     - If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

  Example: Find two nonnegative numbers x and y whose sum is 10 which minimizes the quantity $x^2 + y^2$. Use the first derivative to verify the absolute minimum.
  objective function: $x^2 + y^2$
  constraints: $x + y = 10$ and $x, y \geq 0$

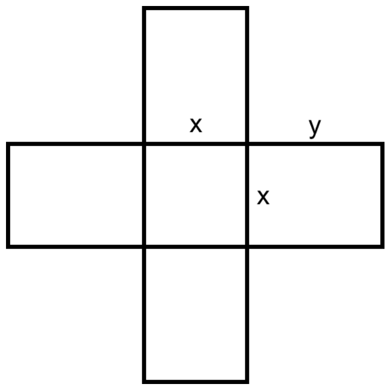
- Second Derivative Test for Absolute Extrema of a Continuous Function f.

  Steps:

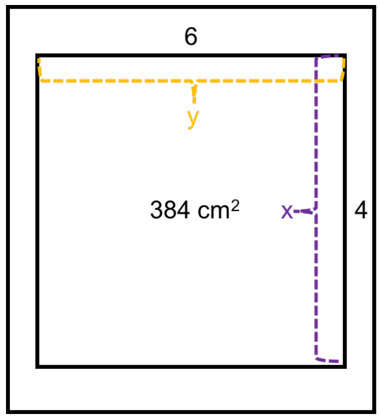
  1. Determin the local extreme values of f.
  2.
     - If $f(c)$ is a local minimum value AND $f''(x) > 0$ for all x, then $f(c)$ is the absolute minimum value of f.
     - If $f(c)$ is a local maximum value AND $f''(x) < 0$ for all x, then $f(c)$ is the absolute maximum value of f.

  Example: Find two nonnegative numbers x and y whose sum is 10 which minimizes the quantity $x^2 + y^2$. Use the second derivative to verify the absolute minimum.
  objective function: $x^2 + y^2$
  constraints: $x + y = 10$ and $x, y \geq 0$

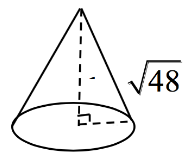
Example: A box with a square base and an open top must have a volume of $32,000\ \mathrm{cm}^3$. Find the dimensions of the box that minimizes the amount of material.



Example: The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.



Example: A right triangle whose hypotenuse is $\sqrt{48}$ meters long is revolved about one of its legs to generate a right circular one. Find the radius and height that maximizes the volume of the cone. Note that the volume of a right circular cone is $V = \frac{\pi}{3}r^2h$.



4.6: Linear Approximation and Differentials

Linear Approximation

The tangent line to a function $y = f(x)$ at a point $(a, f(a))$ can be used to approximate function values, $f(x)$, when x is near a.

Equation of the tangent line to f at $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$
$$y = f(a) + f'(a)(x - a)$$

To approximate $f(x)$ when x is near a, use the equation:

$$L(x) = f(a) + f'(a)(x - a)$$

This linear function is called the **linear approximation** or **linearization** of f at a. The value a is sometimes called the **center of the approximation**.

Example: Consider the function $f(x) = \frac{x}{x+1}$.

(a) Find the linearization of f at $x = 1$.

(b) Graph the function and its linearization at $a = 1$. Using the linearization to approximate f near $x = 1$ will be an overestimate.

c) Use the linear approximation to estimate $f(1.1)$

(d) Compute $f''(1)$ to confirm the conclusion from part *b*- that the linearization is an overestimate.

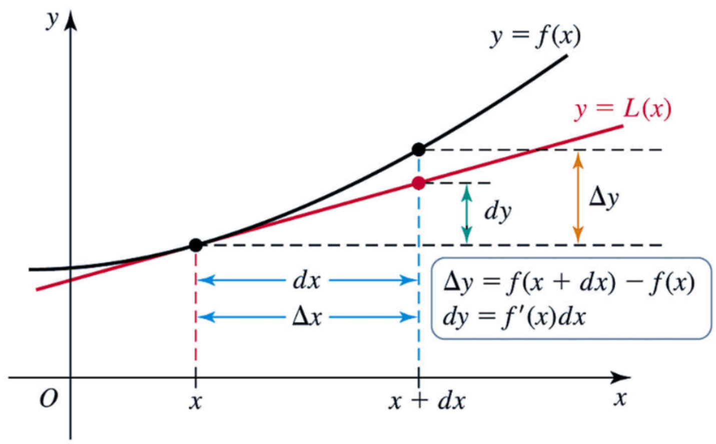
Example: Use a linear approximation to estimate the quantity $\sqrt{146}$. Choose a value of a to produce a small error.
We want to approximate the square root function, $\sqrt{x}$ at 146. So, our function is $(x) = \sqrt{x}$. To produce small error let's choose an a value that will be close to 146 that we know things about. We know $\sqrt{144} = 12$, so let's choose $a = 144$.

Differentials

Another way to perform the approximation of a function value (or a change in the function value) is to use differentials.

Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx$$



To approximate $f(a + dx) = f(a) + \Delta y$, use the equation

$f(a + dx) \approx f(a) + dy$ where $dy = f'(a)dx$

$$f(a + dx) = f(a) + f'(a)dx$$

Note that we are using dy as an approximation for Δy.

Remark:

- Using the linear approximation of a function and using differentials to approximate the function value are equivalent methods.
- The focus of differentials is teh small difference in the x-coordinate, $dx = \Delta x$.

Example: Find the differential dy.

Example: Find the differential dy.

Example: The radius of a circle is given to be 2.00 m with a possible error of 0.02 m. Use differentials to estimate the maximum possible error in computing the area of the circle.
Area of a circle: $A = \pi r^2$ m^2

4.7: L'Hopital's Rule

Indeterminate Forms: $\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty, 1^{\infty}, 0^0, \infty^0$

Indeterminate forms sometimes arise when we take limits.

Example: $\lim\limits_{x \to 0} \frac{\sin 5x}{x}$
Taking the limit of the numerator and denominator as $x \to 0$ gives the indeterminate form $\frac{0}{0}$. To combat this, we can use L'Hopital's Rule.

Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

For indeterminate forms, $\frac{0}{0}$ and $\frac{\infty}{\infty}$, we use L'Hopital's Rule.

L'Hopital's Rule: Suppose that you are taking the limit of a rational expression and that the limit of the numerator over the limit of the denominator results in indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit exists, where a can be finite or infinite.
Note:

- L'Hopital's Rule only applies for the given indeterminate forms. You can use it again and again and again as long as you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- DON'T USE QUOTIENT RULE!!!

Example: Use L'Hopital's Rule to evaluate the limit $\lim_{x \to 0} \frac{\sin 5x}{x}$

Example: Use L'Hopital's Rule to evaluate the limit $\lim_{x \to 0} \frac{8x^2}{\cos x - 1}$

Related Indeterminate Forms $\infty \cdot 0$ and $\infty - \infty$

For indeterminate forms $\infty \cdot 0$ and $\infty - \infty$, sometimes algebra can be used to find the limit.
Example: Evaluate the limit $\lim_{x \to 0^+} \sin x \ln x$.

Example: Evaluate the limit $\lim_{x \to 0^+} \left(\cot x - \frac{1}{x}\right)$.

Indeterminate Powers: $1^\infty, 0^0, \infty^0$

Use the natural logarithm to change the form of the function.

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = \lim_{x\to a} \ln\left(e^{f(x)}\right)$$

Example: Evaluate the limit $\lim_{x\to 0^+} x^x$.

Example: Evaluate the limit $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

4.9: Antiderivatives

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example: Find two different antiderivatives of $f(x) = 2x$.
$F(x) = x^2 + 4$ because $F'(x) = 2x + 0 = 2x$
$F(x) = x^2 + 1738$ because $F'(x) = 2x + 0 = 2x$

Theorem: If F is an antiderivative of f, then the most general antiderivative of f is

$$F(x) + C$$

where C is an arbitrary constant.

Note: The most general antiderivative of a function f is often referred to the set of all antiderivative of f.

Example: Find the most general antiderivative of $f(x) = 2x$.

Indefinite Integrals

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x denoted by

$$\int f(x)dx$$

- The symbol $\int$ is the **integral sign**
- The function $f(x)$ is the **integrand** of the integral.
- Differential dx indicates that x is the **variable of integration**

Basically, the indefinite integral is all of the antiderivatives.

$$\int f(x)dx = F(x) + C \text{ means } F'(x) = f(x)$$

The arbitrary constant C is called the **constant of integration**

Example: Being asked to "Find the most general antiderivative of $f(x) = 2x$" is equivalent to "Determine $\displaystyle\int 2xdx$"

Example: Verify the indefinite integral by differentiation.
$\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\sin\sqrt{x} + C$

Example: Determine $\int \sin 2x \; dx$. Check your work by differentiation.

Power Rule for Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

where $n \neq -1$ is a real number and C is an arbitrary constant.

To find antiderivative, apply derivative rules opposite and in reverse.

Derivative	Antiderivative
1. Multiply by power 2. Subtract 1 from exponent	1. Add 1 to the exponent 2. Divide by new exponent

Example: Find the most general antiderivative of
$f(x) = x^7$.

Constant Multiple Rule

$$\int af(x)dx = a\int f(x)dx$$

Sum Rule

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

Indefinite Integral of $1/x$

$$\int \frac{1}{x}dx = \ln|x| + C, \text{ for all } x \neq 0$$

Tables of Examples on pg 325 and 327

Example: Determine the indefinite integral. Check your work.

$$\int \left(\frac{5}{x^5} + 4x^2 \right) dx$$

Example: Determine the indefinite integral. Check your work.
$\int \sqrt{x}\left(2x^6 - 4\sqrt[3]{x}\right)dx.$

Differential Equations

A **differential equation** is an equation that involves derivatives.

So far, we've found general solutions to differential equations.

A particular solution can be found when extra conditions are given that allow you to determine the constant C. This is called in **initial value problem (IVP)**

Example: For the function $f(t) = \sec^2 t$, find the particular antiderivative F that satisfies the given condition $F\left(\frac{\pi}{4}\right) = 1$.

Example: Solve the initial value problem.
$g'(x) = 7x^6 - 4x^3 + 12$; $g(1) = 24$

Rectilinear Motion

Recall: If an object has position $x = f(t)$, then the velocity is $v(t) = s'(t)$, and acceleration is $a(t) = v'(t) = s''(t)$.

So, the velocity is the antiderivative of the acceleration. And, the position is the antiderivative of the velocity.

If we have initial conditions given, $s(0)$ and $v(0)$, then we can find the particular position function.

Recall: Acceleration due to gravity in this class is $g = 9.8$ m/s^2 or $g = 32$ ft/s^2.

Example: A stone is thrown vertically upwards with a velocity of 30 m/s from the edge of a cliff 200 m above the river.

(a) Find the velocity of the object at all relevant times.

(b) Find the position of the object at all relevant times.

(c) Find the time when the object reaches its highest point.

(d) What is the highest point?

(e) Find the time when the object strikes the ground.

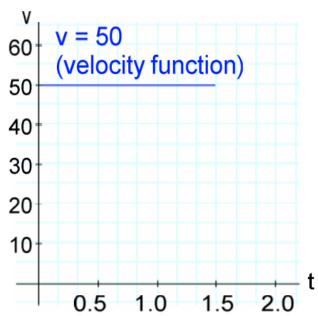
Example: A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?

5.1: Approximating Areas under Curves

Area under a Velocity Curve

Recall that displacement = velocity $\times$ time
So, if we want to know about displacement, we can use the velocity and the formula above to calculate estimates.
Remember that if an object changes direction, this formula will give us displacement. If the velocity of an object is always positive, its displacement equals the distance traveled.

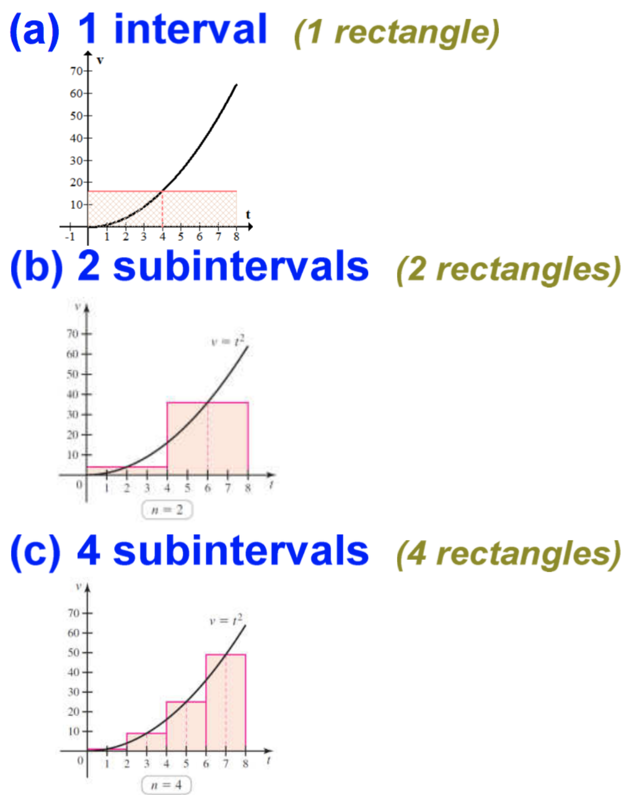
Example: Imagine a train traveling a constant velocity of 50 mi/hr along a straight track over a 1.5 hour period. Find the displacement.



So, given a constant velocity function, the displacement of the object is the area of the rectangle formed by the function, the x-axis, and the time traveled.

What if the velocity isn't constant?

Example: Suppose the velocity in m/s of an object moving along a line is given by the function $v(t) = t^2$, where $0 \leq t \leq 8$.
We can approximate the displacement of the object by dividing the time interval $[0, 8]$ into a number of subintervals of equal length. On each subinterval, we'll use the velocity at the midpoint as a constant velocity over the subinterval.



So, the more subintervals we use, the better our approximation is.

The Area Problem

Suppose we want to find the area of the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis from $x = a$ to $x = b$.
In other words, find the area of the region that lies under the curve $y = f(x)$ from a to b.

We'll learn how to find this area exactly in the next section. For now, our goal is to approximate the area under the curve. To do this, we'll use a speacial sum called a **Riemann Sum**.

Approximating Areas by Riemann Sums

1. Subdivide the interval $[a, b]$ into n subintervals of equal length with $x_0 = a$ and $x_n = b$
2. The length of each subinterval is $\Delta x = \frac{b-a}{n}$

DEFINITION **Regular Partition**
Suppose $[a, b]$ is a closed interval containing n subintervals

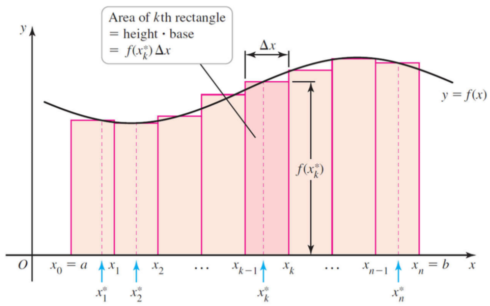
$$\boxed{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]}$$

of equal length $\Delta x = \dfrac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general, the kth grid point is

$$\boxed{x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \ldots, n.}$$

3. In each subinterval, choose *any point* you like and build a rectangle whose height is the function evaluated at the point you've chosen (we like to either choose right endpoint, midpoint, or left endpoint)

4. For the k^{th} subinterval $[x_{k-1}, x_k]$, choose a point x_k^* (this is your right, midpoint, or left)



5. Sum the areas of the n rectangles. This is an approximation of the area under the curve of the function f from a to b.

DEFINITION Riemann Sum

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \dots, n$, then

$$\boxed{f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x}$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \dots, n$.

The three most common Riemann Sums are

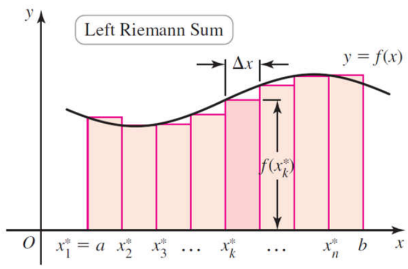


Figure 5.9

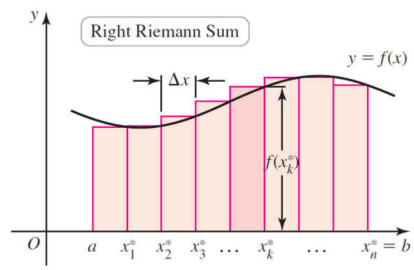


Figure 5.10

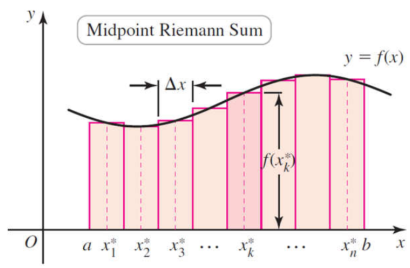
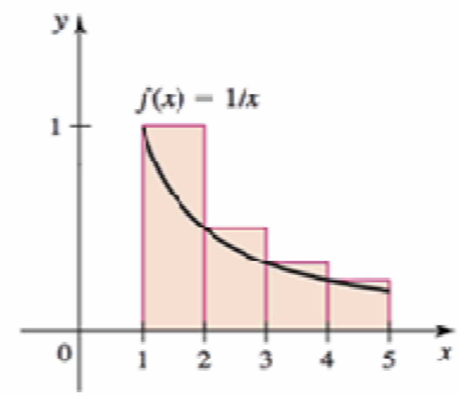


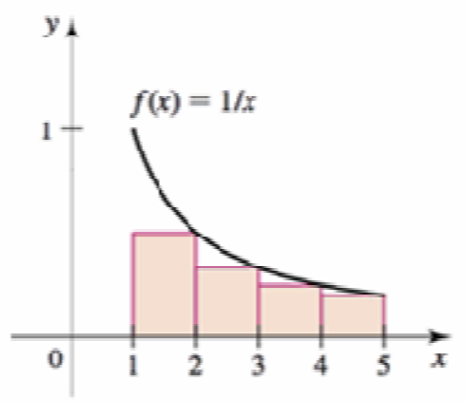
Figure 5.11

Example: Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$ using 4 approximating rectangles as indicated.

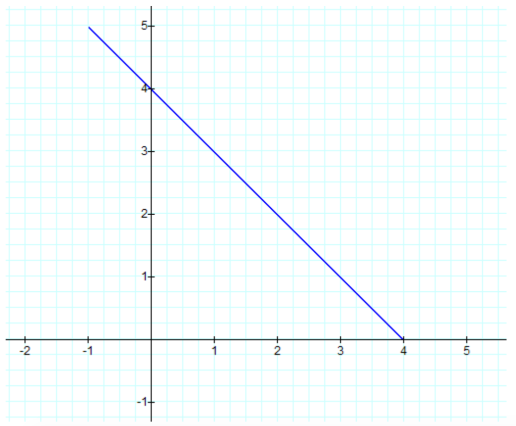
1. Use left endpoints



2. Use right endpoints



Example: Consider $f(x) = 4 - x$ on $[-1, 4]$ with $n = 5$.



(a) Calculate Δx and the points $x_0, x_1, \cdots, x_n$.

(b) Illustrate the midpoint Riemann sum by sketching rectangles on the graph above.

(c) Calculate the midpoint Riemann Sum.

Example: The accompanying table shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance traveled using 5 subintervals of equal length 2 and the indicated endpoint.

Time (sec)	Velocity (in/sec)
0	0
2	22
4	5
6	11
8	4
10	2

Our subintervals are as follows:

- $[0, 2], [2, 4], [4, 6], [6, 8], [8, 10]$

(a) left endpoints

(b) right endpoints

Sigma (Summation) Notation

We can write a sum with many terms in a compact form instead of writing out all the terms.

summation symbol

$$\sum_{k=1}^{n} a_k$$

formula for the k-th term

index of summation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Example: Express the sum in sigma notation: $1^2 + 2^2 + 3^2 + 4^2 + 5^2$

Example: Write the sum without sigma notation and evaluate it.

(a) $\sum\limits_{k=1}^{3} \frac{k-1}{k}$

(b) $\sum\limits_{k=1}^{4} (-1)^k \cos(k\pi)$

Algebra Rules and Formulas for Finite Sums

Constant Multiple Rule: $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

Addition Rule: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Sums of Powers of Integers: Let n be a positive integer and c a real number. Then, we have:

- $\sum_{k=1}^{n} c = cn$
- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Example: Evaluate the sums.

(a) $\sum_{k=1}^{5} \frac{\pi k}{15}$

(b) $\sum_{k=1}^{7} k(2k+1)$

Riemann Sums Using Sigma Notation

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

For $k = 1, 2, \cdots n$ and recalling that $\Delta x = \frac{b-a}{n}$, we have that

- **Left Riemann Sum:** (left endpoint of $[x_{k-1}, x_k]$)

$$x_k^* = a + (k-1)\Delta x$$

- **Right Riemann Sum** (right endpoint of $[x_{k-1}, x_k]$

$$x_k^* = a + k\Delta x$$

- **Midpoint Riemann Sum** (midpoint of $[x_{k-1}, x_k]$

$$s_k^* = a + (k - 1/2)\Delta x$$

Example: For $n = 10$, use sigma notation to write the left, right, and midpoint Riemann sums. Then, evaluate the right Riemann sum.

$$f(x) = x^2 + 1 \text{ on the interval } [-1, 1]$$

5.2: The Definite Integral

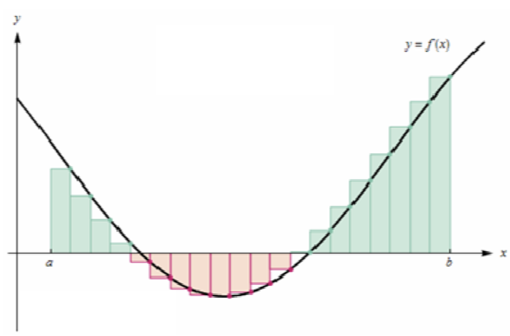
Area under a Velocity Curve

Recall from Section 5.1 we learned the following:

Let f be an arbitrary function on a closed interval $[a, b]$. Note that f can have positive and negative values.
Subdivide the interval $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ where $x_0 = a$ and $x_n = b$
So, the k^{th} subinterval is $[x_{k-1}, x_k]$ where $k = 1, \cdots, n$
Choose a value from each subinterval where the function f will be evaluated (height of rectangle). Denote this value x_k^* for the k^{th} subinterval.

$$\text{Riemann Sum: } f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

Geometrically, think of standing a vertical rectangle of length Δx that stretches from the x=axis up (or down) to the curve at $(x_k^*, f(x_k^*))$.
This is an example of a right Riemann Sum: $\sum\limits_{k=1}^{n} f(x_k)\Delta x$

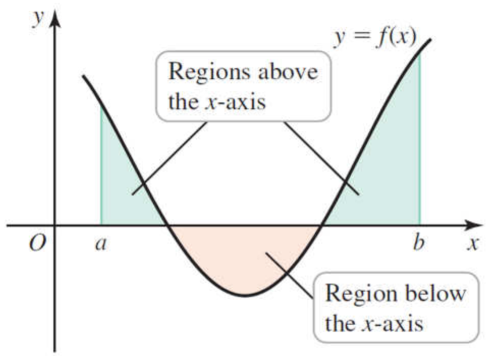


Note:

- If f is positive at x_k^*, then $f(x_k^*)\Delta x$ is the area of the k^{th} rectangle.
- If f is negative at x_k^*, then $f(x_k^*)\Delta x$ is the negative of the area of the k^{th} rectangle.

Net Area

The **net area** of the region bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$ is the area of the parts of the region above the x-axis *minus* the area of the parts that lie below the x-axis.



A Riemann sum for f on $[a, b]$ gives an approximation to the net area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.
We can increase n (the number of subintervals) to get a *better estimate* of the net area. Recall: more rectangles $\rightarrow$ better estimate.

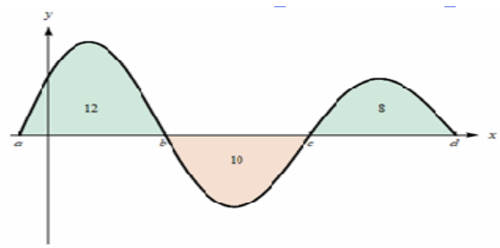
The Definite Integral

We can calculate the exact value of the net area by taking the *limit* of the Riemann sum as $n\infty$.

The **definite integral** of a function f from a to b is defined to be the limit of a Riemann sum as $n \to \infty$, provided the limit exists.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta a = \text{area above } x\text{-axis - area below } x\text{-axis}$$

Example: The figure below shows the graph of a function f with the areas of the regions bounded by its graph and the x-axis given. Find the values of the following definite integrals.



(a) $\int_a^b f(x)dx$

(b) $\int_a^c f(x)dx$

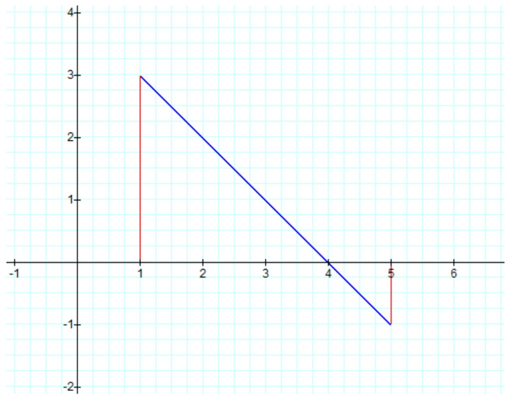
(c) $\int_a^d f(x)dx$

(d) $\int_b^c f(x)dx$

(e) $\int_b^d f(x)dx$

Example: Use familiar area formulas to evaluate the definite integral geometrically.

$$\int_1^5 (4 - x)dx$$



Example: Compute the definite integral as the limit of a right Riemann sum.

$$\int_1^5 (4-x)dx$$

(a) Let n be the number of subintervals into which the interval $[1,5]$ is to be divided. State Δx, the width of each subinterval.

(b) Find an expression for the right endpoint of the k^{th} subinterval.

(c) State the right Riemann sum for $f(x) = 4 - x$ on the interval $[1,5]$.

(d) Simplify the Riemann sum from part (c) using the sum formulas. State your final answer in terms of n only.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \tfrac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \tfrac{n(n+1)(2n+1)}{6}$$

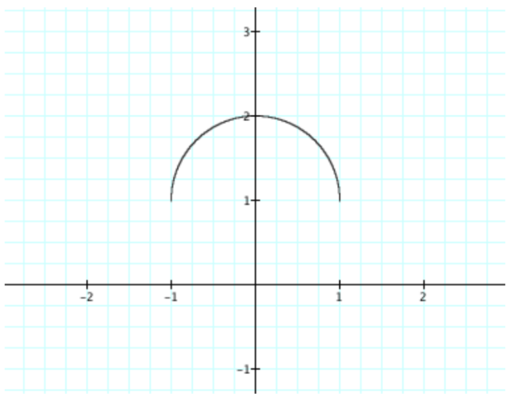
(e) Find the exact value of the definite integral by taking a limit.

Area vs Net Area

If the function f is positive (above the x-axis), then $\int_a^b |f(x)|dx$ is the **(total) area** under the curve of f from a to b

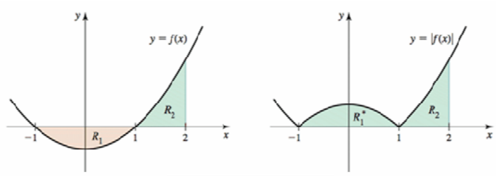
If the function f is not always positive, then $\int_a^b f(x)dx$ is the **net area** of the region bounded by the graph of f and the x-axis from a to b.

Example: Graph the integrand and use areas to evaluate the integral.
$\int_{-1}^{1} \left(1+\sqrt{1-x^2}\right) dx$



Properties of Definite Integrals

- **Identical Limits:** $\int_a^a f(x)dx = 0$
- **Reversing Limits:** $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- **Constant Multiple:** $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
- **Sum:** $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- **Additivity:** $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ and $\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$
- **Absolute Value:** ${}_a^b|f(x)|dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a,b]$



Example: Given $\int_1^9 f(x)dx = -1$, $\int_7^9 f(x)dx = 5$, and $\int_7^9 h(x)dx = 4$, find the following:

(a) $\int_1^9 -2f(x)dx$

(b) $\int_7^9 [2f(x) - 3h(x)]dx$

(c) $\int_9^1 f(x)dx$

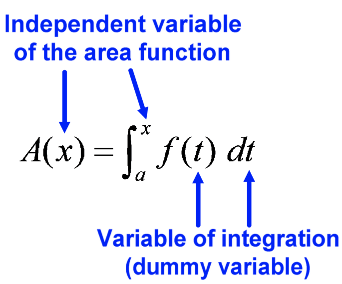
(d) $\int_1^7 f(x)dx$

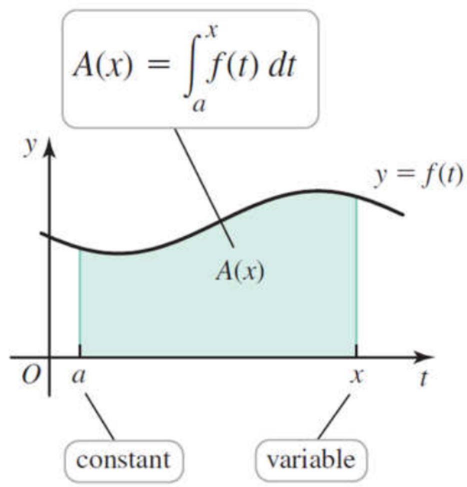
(e) $\int_9^7 [h(x) - f(x)]dx$

5.3: The Fundamental Theorem of Calculus

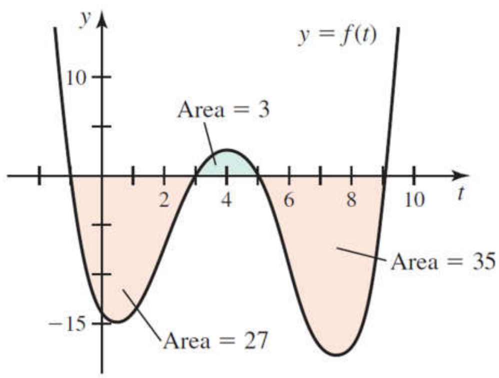
Area Functions

Suppose $y = f(t)$ is a continuous function defined for $t \geq a$ where a is a fixed number. The **area function** for f with left endpoint a is denoted $A(x)$ and gives the net area of the region bounded by the graph of f and the t-axis between $t = a$ and $t = x$. Note that $x \geq a$.





Example: The graph of f is shown in the figure below with area of various regions marked. Let $A(x) = \int_{-1}^{x} f(t)dt$ and $F(x) = \int_{3}^{x} f(t)dt$ be the two area functions for f.



Evaluate the following area functions:

- $A(3)$

- $F(3)$

- $A(5)$

- $F(5)$

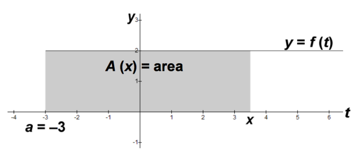
- $A(9)$

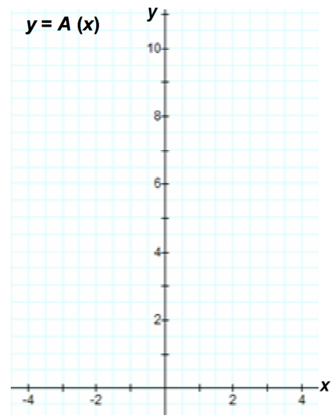
- $F(9)$

Example: Consider the constant function $f(t) = 2$ with $a = -3$.

(a) Find and graph the area function $A(x) = \displaystyle\int_{-3}^{x} 2dt$.

The function graphed is the constant function $f(t) = 2$. As x increases, our area under our function will increase.





(b) Verify that $A'(x) = f(x)$

The slope of the Area function is $A'(x)$. We can tell from the graph that we drew that the slope of $A(x)$ is 2, which is $f(t)$!

The Fundamental Theorem of Calculus (Part I)

FTOC (Part I): If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)dt, \text{ for } a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b).
The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \tfrac{d}{dx} \int_a^x f(t)dt = f(x)$$

This means that the area function of f is an antiderivative of f.

Example: Use Part I of the Fundamental Theorem of Calculus to find the derivative of the function.
$g(x) = \displaystyle\int_3^x (t^3 + \sqrt{t})dt$

We know from the FTOC (Part I) that $g'(x) = \tfrac{d}{dx} \displaystyle\int_3^x f(t)dt = f(x)$

So, we have that

Derivatives of Integrals: Note that our upper limit was just x. However, if your upper limit of integration is a function of x, you must use chain rule to take the derivative of the integral.

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$$

Note: Your function of x also must be the upper limit of integration.

Example: Find the Derivative of the following functions:

(a) $F(x) = \displaystyle\int_2^{x^3} \frac{1}{p^2} dp$

(b) $y = \displaystyle\int_{\tan x}^0 \frac{1}{1+t^2} dt$

(c) $h(x) = \displaystyle\int_{x}^{\ln x} e^t dt$

Rules for Using the FTOC Part I

Given Integral	How to Get Derivative
$f(x) = \displaystyle\int_a^x f(t)dt$	plug in x for t
$f(x) = \displaystyle\int_x^a f(t)dt$	reverse limits of integration, multiply by -1 plug in x for t
$f(x) = \displaystyle\int_a^{g(x)} f(t)dt$	plug in $g(x)$ for t, multiply by $g'(x)$
$f(x) = \displaystyle\int_{g(x)}^a f(t)dt$	reverse limits of integration, multiply by -1 plug in $g(x)$ for t, multiply by $g'(x)$
$f(x) = \displaystyle\int_{g(x)}^{h(x)} f(t)dt$	split the limits of integration as $\displaystyle\int_{g(x)}^0 f(t)dt + \int_0^{h(x)} f(t)dt$ reverse limits of integration of $\displaystyle\int_{g(x)}^0 f(t)dt$ and multiply by -1 plug $g(x)$ and $h(x)$ in for t and multiply by $g'(x)$ and $h'(x)$

The Fundamental Theorem of Calculus (Part II)

FTOC (Part II): If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example: Evaluate the integral $\displaystyle\int_{-2}^{2}(x^3 - 2x + 3)dx$

Example: Evaluate the integral $\displaystyle\int_1^2 \frac{2}{x^2}dx$

Example: Evaluate the integral $\int_{\pi/6}^{5\pi/6} \csc^2 x dx$

Example: Evaluate the integral $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}} du$

Total Area

To find the total area between the graph $y = f(x)$ and the x-axis over $[a, b]$

1. Find all x-values where $f(x) = 0$ (find x-intercepts)
2. Subdivide $[a, b]$ using these values
3. Integrate f over each subinterval and take the absolute value of each result

Example: Graph the function and find the total area of the region bounded by the function and the x-axis on the given interval.
$y = -2x + 6$, $1 \leq x \leq 4$