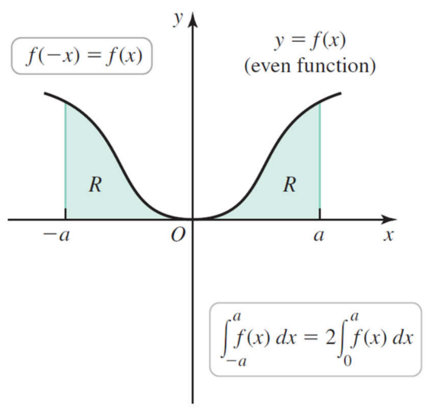
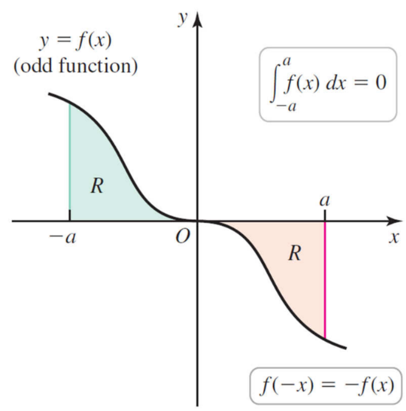
5.4: Working with Integrals

Theorem: Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a,a]$.

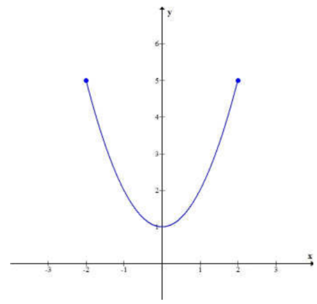
- If f is even, $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
- If f is odd, $\displaystyle\int_{-a}^{a} f(x)dx = 0$



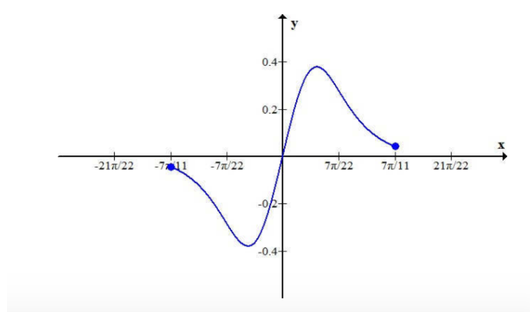


Example: Use symmetry to evaluate the following integrals

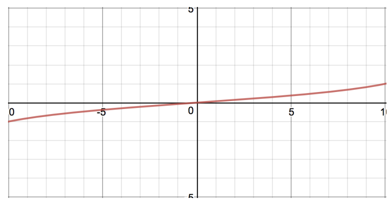
- $\displaystyle\int_{-2}^{2}(x^2+1)dx$



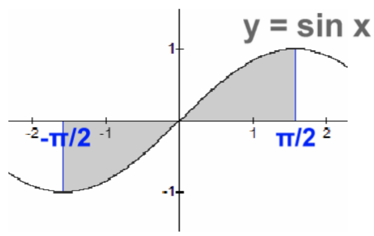
- $\displaystyle\int_{-7\pi/11}^{7\pi/11}\frac{\sin x}{1+x^2+x^4}dx$



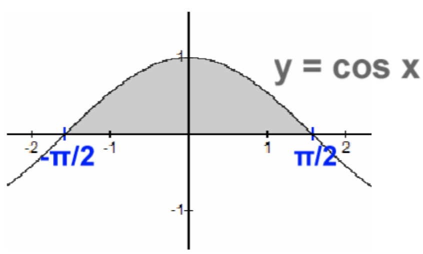
- $\displaystyle\int_{-10}^{10} \frac{x}{\sqrt{200 - x^2}} dx$



- $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x dx$



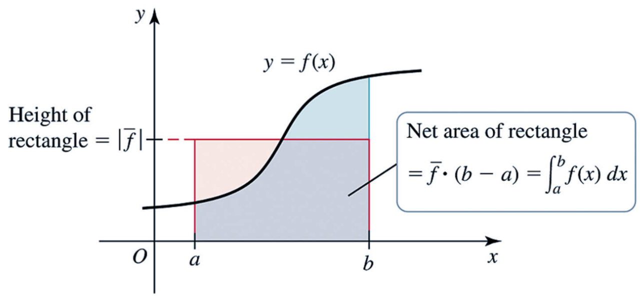
- $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x dx$



Average Value of a Function: The average value of a function f on the interval $[a, b]$ is

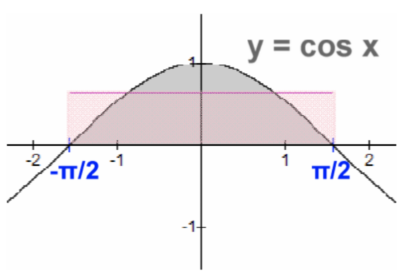
$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)dx$$

Geometric Interpretation of Average Value: The average value is the height of the rectangle with base $[a, b]$ that has the same net area as the region bounded by the graph of f and the x-axis on the same interval $[a, b]$

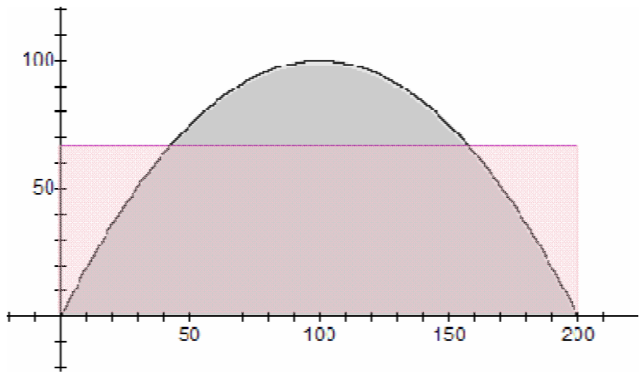


Example: Find the average value of the following function on the given interval. Draw a graph of the function and indicate the average value.

$f(x) = \cos x$; $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



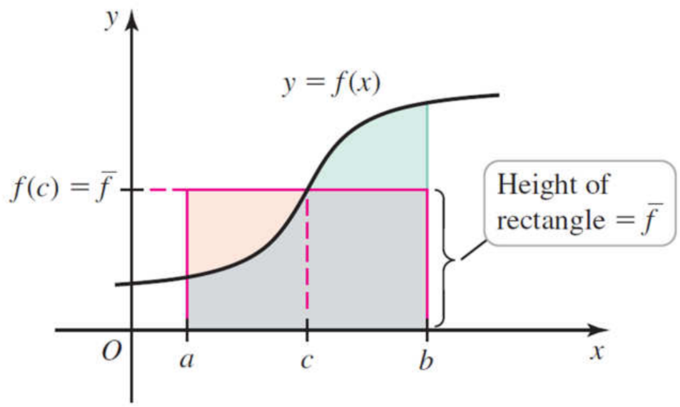
Example: A baseball is launched into the outfield on a parabolic trajectory given by $y = 0.01x(200 - x)$. Find the average height of the baseball over the horizontal extent of its flight. You may assume that the height is given in feet.



Mean Value Theorem for Integrals: Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

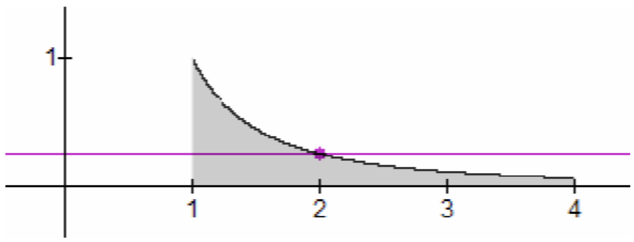
$$f(x) = \bar{f} = \tfrac{1}{b-a} \int_a^b f(t)dt$$

Geometric Interpretation of the MVT for Integrals: The horizontal line drawn at the average value must intersect the graph of the function at some point in the interval.



Example: Find the point(s) at which the given function equals its average values on the given interval.

$f(x) = \frac{1}{x^2}$; $[1, 4]$



First, we find the average value of the function over the interval:

Then, we set the function value to our average value and solve for x:

5.5: Substitution Rule

Substitution Rule for Indefinite Integrals

Substitution Rule for Indefinite Integrals: If $u = g(x)$ is a differentiable function whose range is an interval I and F is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Note: The substitution rule says that it is permissible to operate with dx and du after the integral signs as if they were differentials. The variable of integration is a "dummy variable".

Steps for Substitution Rule:

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $u'(x) = g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = u'(x)dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example: Use the given substitution to find the following indefinite integral.

$\int (6x + 1)\sqrt{3x^2 + x}dx$; $u = 3x^2 + x$

Example: Use a change of variables to find the following indefinite integral.

$$\int x \sin(2x^2) dx$$

Example: Evaluate the integral.

$$\int \frac{x}{\sqrt{4-9x^2}} dx$$

Substitution Rule for Definite Integrals

Substitution Rule for Definite Integrals: If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{u(a)}^{u(b)} f(u)du = \int_{g(a)}^{g(b)} f(u)du$$

Note: The net areas calculated by these integrals are exactly the same. Don't forget to change the limits of integration on a definite integral if you use substitution.

Example: Evaluate the integral.

$\displaystyle\int_0^1 2x(4 - x^2)dx.$

Don't believe me? Try doing the regular integration.

$$\begin{aligned}
\int_0^1 2x(4-x^2)dx &= \int_0^1 (8x-2x^3)dx \\
&= \left[\frac{8x^2}{2} - \frac{2x^4}{4}\right|_0^1 \\
&= \left[\frac{8}{2} - \frac{2}{4}\right] - [0] \\
&= \frac{8}{2} - \frac{1}{2} \\
&= \frac{7}{2}
\end{aligned}$$

Example: Evaluate the integral.
$\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} dx.$

Example: Evaluate the integral.

$$\int (x^6 - 3x^2)^4 (x^5 - x) dx.$$

Note: Sometimes the choice for a u-substitution is not so obvious or more than one u-substitution is required.

Example: Evaluate the integral.

$$\int \frac{x}{\sqrt[3]{x+4}}.$$

Example: Evaluate the integral.

$$\int_0^{\pi/2} \frac{\cos x \sin x}{\sqrt{\cos^2 x + 16}} dx.$$

Example: Evaluate the integral.

$$\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}.$$