$$J_1 = \begin{bmatrix} -L_1 s_1 \\ L_1 c_1 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -L_1 s_1 - L_2 s_2 & -L_2 s_{12} \\ L_1 c_1 + L_2 c_2 & L_2 c_{12} \end{bmatrix}$$

$$J_3 = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} \end{bmatrix}$$

$$\mathcal{L} = K - \cancel{P}^{\,0}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta_k} - \frac{\partial \mathcal{L}}{\partial \theta_k} = \tau_k$$

$$K = \frac{1}{2}\sum_{i=1}^{3} m_i \|\dot{x}_i\|^2$$

$$\dot{x}_i = J_i(\theta)\dot\theta$$

$$\mathcal{L}(\theta, \dot\theta) = \frac{1}{2}\dot\theta^T \dot\theta \left(\sum_{i=1}^{3} m_i J_i(\theta)^T J_i(\theta)\right)\dot\theta$$

For $i = 1$ $\ldots$ $k=1$

$$-\frac{\partial \mathcal{L}}{\partial \theta_k} \Rightarrow \frac{1}{2}\begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}\left(m_1\begin{bmatrix}-L_1S_1\\ L_1C_1\end{bmatrix}\begin{bmatrix}-L_1S_1 & L_1C_1\end{bmatrix}\right)$$

$$\Rightarrow \frac{1}{2}\dot\theta^2\left(m_1\left(L_1S_1^2+L_1C_1^2\right)\right)$$

$$\Rightarrow \frac{\dot\theta^2}{2}\left(m_1\left(L_1\right)\right) \Longrightarrow \boxed{0}$$

$$\frac{d}{dt}\frac{\partial L}{\cancel{\partial}\dot\theta_k} \Rightarrow \frac{1}{2}\dot\theta_1^2 m_1\left(L_1S_1^2+L_1C_1^2\right)+\frac{1}{2}\dot\theta_2^2 m_1\left(L_1S_1^2+L_1C_1^2\right)$$

$$+\frac{1}{2}\dot\theta_3^2 m_1\left(L_1S_1^2+L_1C_1^2\right)$$

$$\Rightarrow \frac{1}{2}\dot\theta_1^2 m_1L_1 + \cancel{\frac{1}{2}\dot\theta_2^2 m_1L_1} + \cancel{\frac{1}{2}\dot\theta_3^2 m_1L_1}$$

$$\Rightarrow \frac{d}{dt}\dot\theta_1 m_1L_1 \Rightarrow \boxed{\ddot\theta_1 m_1 L_1}$$

For $i=2$ $\ldots$ $k=2$

$$-\frac{\partial \mathcal{L}}{\partial \theta_2} \Rightarrow \frac{1}{2}\begin{bmatrix}\dot\theta_1 & \dot\theta_2 & \dot\theta_3\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \theta_3\end{bmatrix}\Bigg(m_2 \cdot$$

$$\begin{bmatrix}-L_1 s_1 - L_2 s_2 & -L_2 s_{12}\\ L_1 c_1 + L_2 c_{12} & L_2 c_{12}\end{bmatrix}\begin{bmatrix}-L_1 s_1 - L_2 s_2 & L_1 c_1 + L_2 c_{12}\\ -L_2 s_{12} & L_2 c_{12}\end{bmatrix}\Bigg)$$

$$\Rightarrow \frac{1}{2}\dot\theta^2 m_2 \begin{bmatrix} \cancel{(-L_1\sin(\theta_1)} \cancel{- L_2\sin(\theta_2))^2 + L_2^2\sin^2(\theta_1+\theta_2)} & -L_1\sin(\theta_1) - \cancel{L_2\sin(\theta_2)})(L_1\cos(\theta_1) + \cancel{L_2\cos(\theta_1+\theta_2)}) - \cancel{L_2^2 \sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2)} \\ (\cancel{L_1\cos(\theta_1)} + \cancel{L_2\cos(\theta_1+\theta_2)})(\cancel{-L_1\sin(\theta_1)} - \cancel{L_2\sin(\theta_2)}) - \cancel{L_2^2\cos(\theta_1+\theta_2)\sin(\theta_1+\theta_2)} & \cancel{L_1^2\cos^2(\theta_1)} + \cancel{2L_1L_2\cos(\theta_1)\cos(\theta_1+\theta_2)} + \cancel{2L_2^2\cos^2(\theta_1+\theta_2)} \end{bmatrix}$$

$$\boxed{\Rightarrow \frac{1}{2}\dot\theta^2 m_2 \begin{bmatrix} -L_2\cdot 2\sin(\theta_2)\cos(\theta_2) + L_2^2\sin(2(\theta_1+\theta_2)) \\ -L_2^3\cos(2\theta_2+\theta_1) - L_2^2\cos(2(\theta_1+\theta_2)) \end{bmatrix}}$$

$$-L_2^2\cos(\theta_1+2\theta_2) - L_2^2\cos(2(\theta_1+\theta_2))$$

$$-2L_1L_2\sin(\theta_1+\theta_2)\cos(\theta_1) - 4L_2^2\cos(\theta_1+\theta_2)\cdot\sin(\theta_1+\theta_2)\Big]$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_2} \Rightarrow \frac{d}{dt}\dot{\theta}_2 m_2\begin{bmatrix} -L_1 s_1 - L_2 s_2^2 + L_2^2 s_{12}^2 & (-L_1 s_1 - L_2 s_2)(L_1 c_1 + L_2 c_{12}) - L_2^2 s_{12}\cdot c_{12} \\ (L_1 c_1 + L_2 c_{12})(-L_1 s_1 - L_2 s_2) - L_2^2 c_{12} s_{12} & L_1^2 c_1^2 + 2L_1L_2 c_1 c_{12} + 2L_2^2 c_{12}^2 \end{bmatrix}$$

$$\Rightarrow \ddot{\theta}_2 m_2 \begin{bmatrix} \quad \end{bmatrix}$$

For $i=3 \ldots k=3$

$$-\frac{\partial \mathcal{L}}{\partial \theta_3} \Rightarrow \frac{1}{2}\dot{\theta}^2 m_3 \cdot$$

$$\begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & L_2 c_{12} + L_3 c_{123} \\ -L_2 s_{12} - L_2 s_{123} & L_2 c_{12} + L_2 c_{123} \end{bmatrix} \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_2 s_{123} \\ L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_2 c_{123} \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 m_3 \cdot \begin{bmatrix} (\cancel{-L_1 s_1} \cancel{-L_2 s_{12}} - L_3 s_{123}) \cdot (\cancel{-L_1 s_1} \cancel{-L_2 s_{12}} - L_3 s_{123}) + (\cancel{L_2 c_{12}} + L_3 c_{123}) \cdot (\cancel{-L_2 c_{12}} + L_3 c_{123}) \\ (\cancel{-L_2 s_{12}} - L_2 s_{123})(\cancel{-L_1 s_1} \cancel{-L_2 s_{12}} - L_3 s_{123}) + (\cancel{L_2 c_{12}} + L_2 c_{123})(\cancel{-L_2 c_{12}} + L_3 c_{123}) \end{bmatrix}$$

$$\begin{matrix} (\cancel{-L_1 s_1} \cancel{-L_2 s_{12}} - L_3 s_{123})(\cancel{-L_2 s_{12}} - L_2 s_{123}) + (\cancel{-L_2 c_{12}} + L_3 c_{123})(\cancel{L_2 c_{12}} + L_2 c_{123}) \\ (\cancel{-L_2 s_{12}} - L_2 s_{123})(\cancel{-L_2 s_{12}} - L_2 s_{123}) + (\cancel{L_2 c_{12}} + L_2 c_{123})(\cancel{L_2 c_{12}} + L_2 c_{123}) \end{matrix} \Bigg]$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 m_3 \cdot \quad L_3 \sin(2(\theta_1+\theta_2+\theta_3)) - L_3^2 \sin(2(\theta_1+\theta_2+\theta_3))$$

$$L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3)) - L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3))$$

$$L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3)) - L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3))$$

$$L_2^2 \sin(2(\theta_1+\theta_2+\theta_3)) - L_2^2 \sin(2(\theta_1+\theta_2+\theta_3))$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_3} \Rightarrow \frac{d}{dt}\dot{\theta}_3 m_3 \begin{bmatrix} (-L_1 s_1 - L_2 s_{12} - L_3 s_{123})\cdot(-L_1 s_1 - L_2 s_{12} - L_3 s_{123}) + (L_2 c_{12} + L_3 c_{123})\cdot(-L_2 c_{12} + L_3 c_{123}) & (-L_1 s_1 - L_2 s_{12} - L_3 s_{123})(-L_2 s_{12} - L_2 s_{123}) + (-L_2 c_{12} + L_3 c_{123})(L_2 c_{12} + L_2 c_{123}) \\ (-L_2 s_{12} - L_2 s_{123})\cdot(-L_1 s_1 - L_2 s_{12} - L_3 s_{123}) + (L_2 c_{12} + L_2 c_{123})(-L_2 c_{12} + L_3 c_{123}) & (-L_2 s_{12} - L_2 s_{123})(-L_2 s_{12} - L_2 s_{123}) + (L_2 c_{12} + L_2 c_{123})(L_2 c_{12} + L_2 c_{123}) \end{bmatrix}$$

$$\Rightarrow (\ddot{\theta}_3 m_3)\cdot \left(\begin{bmatrix} (-L_1 s_1 - L_2 s_{12} - L_3 s_{123})\cdot(-L_1 s_1 - L_2 s_{12} - L_3 s_{123}) + (L_2 c_{12} + L_3 c_{123})\cdot(-L_2 c_{12} + L_3 c_{123}) & (-L_1 s_1 - L_2 s_{12} - L_3 s_{123})(-L_2 s_{12} - L_2 s_{123}) + (-L_2 c_{12} + L_3 c_{123})(L_2 c_{12} + L_2 c_{123}) \\ (-L_2 s_{12} - L_2 s_{123})\cdot(-L_1 s_1 - L_2 s_{12} - L_3 s_{123}) + (L_2 c_{12} + L_2 c_{123})(-L_2 c_{12} + L_3 c_{123}) & (-L_2 s_{12} - L_2 s_{123})(-L_2 s_{12} - L_2 s_{123}) + (L_2 c_{12} + L_2 c_{123})(L_2 c_{12} + L_2 c_{123}) \end{bmatrix} \right)$$

Thus,

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T\left(\sum_{i=1}^{3} m_i J_i(\theta)^T J_i(\theta)\right)\dot{\theta} =$$

$$\ddot{\theta} m_1 L_1 + \ddot{\theta}_2 m_2 \begin{bmatrix} -L_1 s_1 - L_2 s_2^2 + L_2^2 s_{12}^2 & (-L_1 s_1 - L_2 s_2)(L_1 c_1 + L_2 c_{12}) - L_2^2 s_{12} \cdot c_{12} \\ (L_1 c_1 + L_2 c_{12})(-L_1 s_1 - L_2 s_2) - L_2^2 c_{12} s_{12} & L_1^2 c_1^2 + 2 L_1 L_2 c_1 c_{12} + 2 L_2^2 c_{12}^2 \end{bmatrix}$$

$$- \frac{1}{2}\dot{\theta}^2 m_2 \begin{bmatrix} -L_2 \cdot 2\sin(\theta_2)\cos(\theta_2) + L_2^2 \sin(2(\theta_1+\theta_2)) & -L_2^2\cos(\theta_1+2\theta_2) - L_2^2\cos(2(\theta_1+\theta_2)) \\ -L_2^3\cos(2\theta_2+\theta_1) - L_2^2\cos(2(\theta_1+\theta_2)) & -2L_1L_2\sin(\theta_1+\theta_2)\cos(\theta_1) - 4L_2^2\cos(\theta_1+\theta_2)\cdot\sin(\theta_1+\theta_2) \end{bmatrix} +$$

$$\ddot{\theta} m_3 \begin{bmatrix} (-L_1 s_1 - L_2 s_{12} - L_3 s_{123})\cdot(-L_1 s_1 - L_2 s_{12} - L_3 s_{123}) + (L_2 c_{12} + L_3 c_{123})\cdot(-L_2 c_{12} + L_3 c_{123}) & (-L_1 s_1 - L_2 s_{12} - L_3 s_{123})(-L_2 s_{12} - L_2 s_{123}) + (-L_2 c_{12} + L_3 c_{123})(L_2 c_{12} + L_2 c_{123}) \\ (-L_2 s_{12} - L_2 s_{123})\cdot(-L_1 s_1 - L_2 s_{12} - L_3 s_{123}) + (L_2 c_{12} + L_2 c_{123})(-L_2 c_{12} + L_3 c_{123}) & (-L_2 s_{12} - L_2 s_{123})(-L_2 s_{12} - L_2 s_{123}) + (L_2 c_{12} + L_2 c_{123})(L_2 c_{12} + L_2 c_{123}) \end{bmatrix}$$

$$- \frac{1}{2}\dot{\theta}^2 m_3 \begin{bmatrix} L_3 \sin(2(\theta_1+\theta_2+\theta_3)) - L_3^2 \sin(2(\theta_1+\theta_2+\theta_3)) & L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3)) - L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3)) \\ L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3)) - L_2 L_3 \sin(2(\theta_1+\theta_2+\theta_3)) & L_2^2 \sin(2(\theta_1+\theta_2+\theta_3)) - L_2^2 \sin(2(\theta_1+\theta_2+\theta_3)) \end{bmatrix}$$

$$= \tau_K$$