Graph Theory Exam

Shakeel Gavioli-Akilagun

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Question 1

Let G_0 be the four cycle C_4. For $k > 1$ G_k is defined recursively by 'adding' G_{k-1} with vertex set $\{u_1, ..., u_k\}$ to vertices $v_1, ..., v_k, w$; an edge is added between vertices u and v having the same index, and between vertex w and every v. G_0 and G_1 are drawn below:

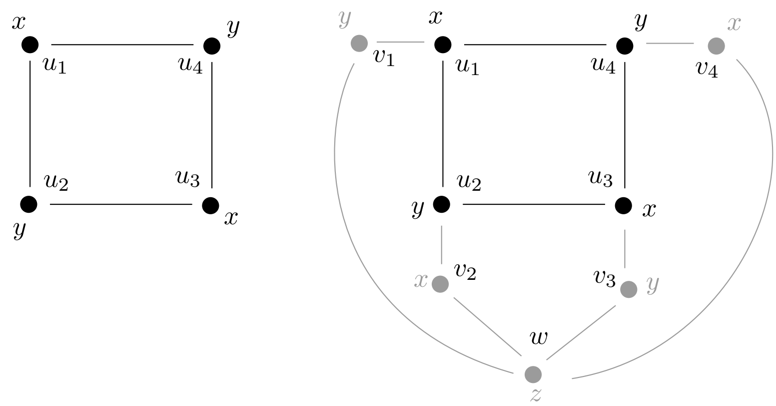

Figure 1: Graphs G_0 (left) and G_1 (right) coloured with $\{x, y, z\}$.

$\chi(G_0) = \chi(C_4) = 2$ and by inspection $\chi(G_1) = 3$. Assume $\chi(G_k) = 3$ for all $k \geqslant 1$, and for simplicity that the vertices of G_k are coloured $z, y, z, x, y, z, ...$ Ignoring vertex w, in G_{k+1} v_1 accepts colours (y, z), v_2 accepts (x, z), v_3 accepts (x, y), and so on. This is shown below:

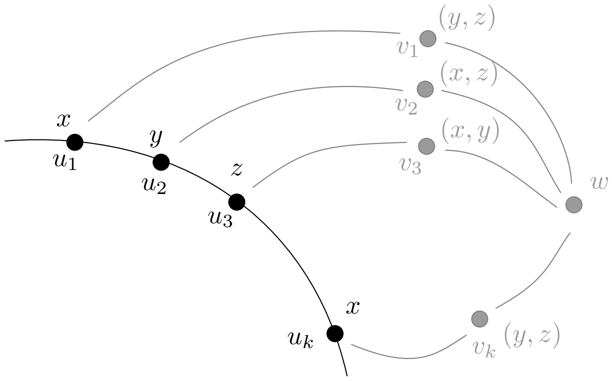

Figure 2: Colouring vertices $v_1, v_2, ..., v_k$ in G_{k+1}.

Picking any two colours it possible to assign a valid colouring to $v_1, v_2, ..., v_k$ and w can be safely coloured with the remaining colour. By induction for all $k \geqslant 1$ we have that $\chi(G_k) = 3$.

Question 2

Part 1

Let $G = (V, E)$ be a graph with n vertices and adjacency matrix A such that $A_{i,j} = 1$ if $ij \in E$ and 0 else. By construction G is undirected and A is therefore symmetric, so $A = A^T$. All the entries in A are real so $A = \overline{A}$. Let (λ, v) be any eigenvalue-eigenvector pair for A. By definition:

$$\overline{v}^T Av = \overline{v}^T (Av) = \overline{v}^T \lambda v = \lambda \langle \overline{v}, v \rangle$$
$$\overline{v}^T Av = \left(\overline{Av}\right)^T v = \overline{\lambda}\overline{v}^T v = \overline{\lambda} \langle \overline{v}, v \rangle$$

For any eigenvalue of A it therefore holds that $\lambda = \overline{\lambda}$, so A must have all real eigenvalues.

Part 2

An (i,j)-walk of length $k \geqslant 0$ on G is a sequence of vertices $u_1, ..., u_{k+1}$ with u_1 incident to i and u_{k+1} incident to j. $(A^0)_{i,j}$ is 1 if $i = j$ and 0 else and so gives the number of (i,j)-walks of length 0. By definition $A_{i,j}$ gives the number of (i,j)-walks of length 1. Note that the set of (i,j)-walks of length k consists of the concatenation of all (i,r)-walks of length $k-1$ with all (r,j)-walks of length 1 over every $r \in V$. Assume $(A^{k-1})_{i,j}$ gives the number of length $k-1$ (i,j)-walks, then the number of length k walks will be given by:

$$\sum_{r=1}^{n} \left(A^{k-1}\right)_{i,r} A_{r,j} = \left(A^{k-1}A\right)_{i,j} = \left(A^{k}\right)_{i,j}$$

Since an (i,r)-walk can only contribute to the set of (i,j)-walks if G has an edge incident to both r and j. By induction it therefore follows that for any $k \geqslant 0$ the number of length k walks is given by $(A^k)_{i,j}$.

Part 3

The diameter of a graph G is the maximum over all pairs of vertices (i,j) of the length of the shortest (i,j)-path. If vertices i and j have loops and the shortest (i,j)-path has length t then $(A^k)_{i,j} > 0$ for all $k \geqslant t$. This motivates the following algorithm:

Algorithm 1 Graph diameter

```
M_0 ← A + I
diam ← ∞

for k in range(1, n) do
    M_k ← M_{k−1} * M_0
    if M_k is full then
        diam ← k
        break

return diam
```

The algorithm has worst case time complexity $\mathcal{O}(n^{1+\omega})$, which occurs when G has two vertices having shortest walk of length $n-1$. The worst case complexity for matrix multiplication is $\mathcal{O}(n^{\omega})$, and **?** proves the upper bound $\omega < 2.3728639$.

Question 3

Part 1

Let $G(n,p)$ be a an Erdős graph and let $\eta > 0$ be a constant. To show that there exists a constant $C > 0$ such that if $p \geqslant Cn^{-1}$ as $n \to \infty$ w.p.a. 1 there do not exists disjoint vertex sets X, Y with $|X|, |Y| \geqslant n\eta$ and $e(X,Y) \geqslant 2p\,|X|, |Y|$ consider the following. Let $\mathcal{S}$ be the set of all pairs of disjoint vertex sets having size at least $n\eta$. For any single element of $\mathcal{S}$ the probability of the event occurring can be upper bounded as follows:

$$
\begin{aligned}
P\left(e(X,Y) \geqslant 2p\,|X|\,|Y|\right) &= P\left(|e(X,Y) - p\,|X|\,|Y|| \geqslant p\,|X|\,|Y|\right) \\
&\leqslant 2\exp\left(-\frac{|X|\,|Y|\,p}{3}\right) \\
&\leqslant 2\exp\left(-\left[\frac{C\eta^2}{3}\right]n\right)
\end{aligned}
$$

Where the first inequality follows from a Chernoff bound for the sum of independent Bernoulli trials, which may be applied since $e(X,Y) \sim \mathrm{Bin}(|X|\,|Y|\,, p)$. The probability of the event occurring on at least one element of $\mathcal{S}$ can therefore be upper bounded as follows:

$$
\begin{aligned}
P\left(\{\text{event occurs on } \mathcal{S}\}\right) &\leqslant \binom{n}{\eta n}^2 P\left(e(X,Y) \geqslant 2p\,|X|\,|Y|\right) \\
&\leqslant 2\left(\frac{e}{\eta}\right)^{2\eta n} \exp\left(-\left[\frac{C\eta^2}{3}\right]n\right) \\
&\leqslant 2\exp\left(\left[2 - \frac{C\eta^2}{3}\right]n\right)
\end{aligned}
$$

A sufficient condition for G not containing any such vertex sets w.p.a. 1 is therefore that the right hand side of the final expression goes to zero as $n \to \infty$. This can be guaranteed by choosing $C > 6\eta^{-2}$.

Parts 2,3,4

Unfortunately I have run out of time. For question (b) I would have tried to apply a probabilistic argument to Szemeredi's regularity lemma. For questions (c) and (d) I would have used the triangle counting lemma together with the fat that w.p.a. 1 as $n \to \infty$ any n vertex sub-graph of $G(n,p)$ can be split into pairwise regular disjoint vertex sets.