LTCC Exam: Harmonic Analysis

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Question 1

Part a

Write $f(x) = x^2 - x$ for $0 \leq x < 1$. For $p \neq 0$ the Fourier coefficients are given by

$$\begin{aligned}
\hat{f}(p) &= \int_{\mathbb{T}} \left(x^2 - x\right) \overline{\exp\left(2\pi i p x\right)} \mathrm{d}x && (1) \\
&= \int_{\mathbb{T}} x^2 e^{-2\pi i p x} \mathrm{d}x - \int_{\mathbb{T}} x e^{-2\pi i p x} \mathrm{d}x && (2) \\
&= -\left(\frac{x^2 e^{-2\pi i p x}}{2\pi i p}\right)\Bigg|_0^1 + \left(\frac{1}{\pi i p}\right) \int_{\mathbb{T}} x e^{-2\pi i p x} \mathrm{d}x - \int_{\mathbb{T}} x e^{-2\pi i p x} \mathrm{d}x && (3) \\
&= -\left(\frac{x^2 e^{-2\pi i p x}}{2\pi i p}\right)\Bigg|_0^1 + \left(\frac{1}{\pi i p} - 1\right)\left(-\left(\frac{x e^{-2\pi i p x}}{2\pi i p}\right)\Bigg|_0^1 + \left(\frac{1}{2\pi i p}\right) \int_{\mathbb{T}} e^{-2\pi i p x} \mathrm{d}x\right) && (4) \\
&= -\left(\frac{1}{2\pi i p}\right)\left(x^2 e^{-2\pi i p x} + \left(\frac{1}{\pi i p} - 1\right)\left(x e^{-2\pi i p x} + \frac{e^{-2\pi i p x}}{2\pi i p}\right)\right)\Bigg|_0^1 && (5) \\
&= \frac{1}{2\pi^2 p^2} && (6)
\end{aligned}$$

Finally using $\hat{f}(0) = \int_{\mathbb{T}} f(x) \mathrm{d}x$ I have that

$$\hat{f}(p) = \begin{cases} -\frac{1}{6} & p = 0 \\ \frac{1}{2\pi^2 p^2} & \text{else} \end{cases} \tag{7}$$

Part b

By Plancherel's theorem $\sum_{p\in\mathbb{Z}} \left|\hat{f}(p)\right|^2 = \int_{\mathbb{T}} |f(x)|^2 \, \mathrm{d}x$. Therefore I have that

$$\int_{\mathbb{T}} \left(x^2 - x\right) \mathrm{d}x = \left(-\frac{1}{6}\right)^2 + 2\sum_{p\geq 1} \left(\frac{1}{2\pi^2 p^2}\right)^2 \tag{8}$$

$$\Rightarrow \sum_{p\geq 1} \frac{1}{p^4} = \frac{\pi^4}{90} \tag{9}$$

Question 2

Part a

A function $f : \mathbb{T} \to \mathbb{C}$ is uniformly continuous if for any ϵ there is a $\delta > 0$ such that $\sup_{|x-y|\leq\delta} |f(x) - f(y)| < \epsilon$. For the χ_X I have that

$$|\chi_X(x) - \chi_X(y)| = \left|\sum_{p\in\mathbb{Z}} \mathbb{P}(X = p)\, e^{2\pi i p x} - \sum_{p'\in\mathbb{Z}} \mathbb{P}(X = p')\, e^{2\pi i p' x}\right| \tag{10}$$

$$\leq \left| i \sum_{p\in\mathbb{Z}} \mathbb{P}(X = p) \left(\sin(2\pi p x) - \sin(2\pi p y)\right)\right| + \left|\sum_{p\in\mathbb{Z}} \mathbb{P}(X = p) \left(\cos(2\pi p x) - \cos(2\pi p y)\right)\right| \tag{11}$$

$$= \left| 2i \sum_{p\in\mathbb{Z}} \mathbb{P}(X = p) \sin(\pi p(x-y)) \cos(\pi p(x+y))\right| + \left| 2 \sum_{p\in\mathbb{Z}} \mathbb{P}(X = p) \sin(\pi p(x-y)) \sin(\pi p(x+y) \tag{12}$$

$$\leq 2\left| i \sum_{p\in\mathbb{Z}} \mathbb{P}(X = p) \sin(\pi p(x-y))\right| + 2\left|\sum_{p\in\mathbb{Z}} \mathbb{P}(X = p) \sin(\pi p(x-y))\right| \tag{13}$$

$$\leq 2\pi |x-y| \left(\left| i \sum_{p\in\mathbb{Z}} \mathbb{P}(X = p)\, p\right| + \left|\sum_{p\in\mathbb{Z}} \mathbb{P}(X = p)\, p\right|\right) \tag{14}$$

$$= 2\pi |\mathbb{E}X| |x-y| (1+i) \tag{15}$$

By the above discussion χ_X is uniformly continuous as long as $|\mathbb{E}X| < \infty$ which is assumed later in the question.

Part b

In class it was proved that for $f : \mathbb{T} \to \mathbb{C}$ if $f \in C^{\infty}(\mathbb{T})$ and $\left\| f^{(k)} \right\|_2 \leq C\tilde{a}^{-k}k!$ for some $\tilde{a} > 0$ and $C = C(\tilde{a})$ for all $k > 1$ then f admits an analytic extension to the strip $|\mathrm{Im}\,(z)| < a$ where $0 < \tilde{a} < a$. For χ_X and any $k > 0$ I have that

$$
\begin{aligned}
\sum_{p \in \mathbb{Z}} \left| \widehat{\chi_X^{(k)}}(p) \right|^2 &= \sum_{p \in \mathbb{Z}} \left| (2\pi i p)^k \, \widehat{\chi_X}(p) \right|^2 && (16) \\
&\leq (2\pi)^{2k} \sum_{p \in \mathbb{Z}} p^{2k} \mathbb{P}(X = p) && (17) \\
&= (2\pi)^{2k} \, \mathbb{E} X^{2k} && (18) \\
&\leq (2\pi)^{2k} \, C A^{2k} (2k)! && (19) \\
&= C (2\pi A)^{2k} (k!) (k+1) \times \cdots \times (2k) && (20) \\
&\leq C (2\pi A)^{2k} (k!) (2k)^k && (21) \\
&\leq C (4\pi A)^{2k} (k!)^2 && (22)
\end{aligned}
$$

Therefore for each $k > 0$ using Plancherel's theorem I have that $\left\| \chi_X^{(k)} \right\|_2 \leq C (4\pi A)^k k!$ and the above discussion gives that χ_X admits an analytic extension to the strip $|\mathrm{Im}\,(z)| < (4\pi A)^{-1}$.

Part c

By part b for each $i \in \{1, 2\}$ each χ_{X_i} admits an analytic extension and so can be expressed as a convergent Taylor series. That is for each $x \in \mathbb{T}$ we must have

$$
\begin{aligned}
\chi_{X_i}(x) &= \sum_{k\geq 0} \frac{x^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \chi_{X_i}(t)\Big|_{t=0} \\
&= \sum_{k\geq 0} \frac{x^k}{k!} \left(\sum_{p\in\mathbb{Z}} \mathbb{P}(X_i = p) \frac{\mathrm{d}^k}{\mathrm{d}t^k} e^{2\pi i t p}\Big|_{t=0} \right) \\
&= \sum_{k\geq 0} \frac{x^k}{k!} \left(\sum_{p\in\mathbb{Z}} \mathbb{P}(X_i = p) (2\pi i p)^k \right) \\
&= \sum_{k\geq 0} \frac{x^k}{k!} (2\pi i)^k \sum_{p\in\mathbb{Z}} p^k \mathbb{P}(X_i = p) \\
&= \sum_{k\geq 0} \frac{x^k}{k!} (2\pi i)^k \, \mathbb{E}X_i^k
\end{aligned}
$$

Then the fact that $\mathbb{E}X_1^k = \mathbb{E}X_2^k$ for all $k > 0$ gives $\chi_{X_1} \equiv \chi_{X_2}$.

Part d

By definition for each $i \in \{1, 2\}$ I have that

$$
\chi_{X_i} = \sum_{p\in\mathbb{Z}} \mathbb{P}(X_i = p) \, e^{2\pi i x p} \equiv \sum_{p\in\mathbb{Z}} \widehat{\chi_{X_i}}(p) \, e^{2\pi i x p} \tag{23}
$$

Therefore $\mathbb{P}(X_i = p) = \widehat{\chi_{X_i}}(p)$. Form part c I have that $\chi_{X_1} \equiv \chi_{X_2}$ therefore the uniqueness of the Fourier transform gives $\widehat{\chi_{X_1}}(p) = \widehat{\chi_{X_2}}(p)$ which implies that $\mathbb{P}(X_1 = p) = \mathbb{P}(X_2 = p)$ for all p.

Question 3

The spectral representation theorem for unitary operators states that if $\mathcal{H}$ is a Hilbert space and U is a unitary operator acting on $\mathcal{H}$ then there exists a family of measures $(\mu_{a,b})_{a,b\in\mathcal{H}}$ on $\mathbb{T}$ such that for every bounded function $g : \mathbb{T} \to \mathbb{C}$ it holds that

$$
\langle g(U)\, a, b \rangle = \int_{\mathbb{T}} g(x) \, \mathrm{d}\mu_{a,b} \tag{24}
$$

Moreover each measure $\mu_{a,b}$ is characterized by its Fourier coefficients which are given by

$$\widehat{\mu_{a,b}}\left(p\right) = \langle U^{p}a, b\rangle \tag{25}$$

Let $U \in \mathbb{C}^{n\times n}$ be a unitary tridiagonal matrix and let $f \in C^{1}\left(\mathbb{T}\right)$ be such that $g\left(e^{2\pi i x}\right) = f(x)$. Write $\mathbf{v}_k$ for the unit vector in $\mathbb{R}^n$ with k-th entry equal to 1. I have that

$$\left|g\left(U\right)_{jk}\right| = \left|\langle g\left(U\right)\mathbf{v}_j, \mathbf{v}_k\rangle\right| = \left|\int_{\mathbb{T}} g\left(x\right)\mathrm{d}\mu_{\mathbf{v}_1,\mathbf{v}_2}\right| \tag{26}$$

Next, since the discrete part of a measure can be recovered from its Fourier-Stieltjes series

$$\begin{aligned}
\left|\int_{\mathbb{T}} g\left(x\right)\mathrm{d}\mu_{\mathbf{v}_1,\mathbf{v}_2}\right| &= \lim_{N\to\infty}\left|\int_{\mathbb{T}} g(x)\mathrm{d}\left(\frac{1}{2N+1}\sum_{p=-N}^{N}\widehat{\mu}_{\mathbf{v}_1,\mathbf{v}_2}\left(p\right)e^{2\pi i p x}\right)\right| && (27)\\
&= \lim_{N\to\infty}\left|\int_{\mathbb{T}} g(x)\left(\frac{1}{2N+1}\sum_{p=-N}^{N}\langle U^{p}\mathbf{v}_j, \mathbf{v}_k\rangle e^{2\pi i p x}2\pi i p\right)\mathrm{d}x\right| && (28)\\
&= \lim_{N\to\infty}\left|\left(\frac{2\pi i}{2N+1}\right)\sum_{p=-N}^{N}\langle U^{p}\mathbf{v}_j, \mathbf{v}_k\rangle p\int_{\mathbb{T}} g(x)e^{2\pi i p x}\mathrm{d}x\right| && (29)\\
&\le \|f'\|_{\infty}\lim_{N\to\infty}\left|2\pi i\left\langle\left(\frac{1}{2N+1}\sum_{p=-N}^{N} p\times U^{p}\right)\mathbf{v}_j, \mathbf{v}_k\right\rangle\right| && (30)
\end{aligned}$$

The last line follows from this discussion on mathoverflow[1] and unfortunately from here I am not sure how to proceed. I would like to invoke the Ergodic theorem, namely that if U is as above then $N^{-1}\sum_{p=0}^{N-1}U^{p}$ converges to its orthogonal projection onto the subspace of U-invariant vectors. However, the presence of "p" in the sum means I cannot do this.

Question 4

Part a

For $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)\setminus\mathbb{Q}$ write $\mu_{\alpha}\left(A\right) = \#\left(A\cap\{-\alpha, \alpha\}\right)/2|\alpha|$. For $p \neq 0$ I have that

[1] https://mathoverflow.net/questions/108418/functional-calculus-of-unitary-matrices-and-commutator-norms-reference-request

$$
\begin{aligned}
\left|\widehat{\mu_\alpha}(p)\right| &= \left|\int_{\mathbb{T}} e^{-2\pi i p x} \mathrm{d}\mu_\alpha(x)\right| && (31)\\
&= \left|\left(\frac{1}{2\alpha}\right)\int_{-\alpha}^{\alpha} e^{-2\pi i p x} \mathrm{d}x\right| && (32)\\
&= \left|\left(\frac{1}{4\pi i p\alpha}\right) e^{-2\pi i p x}\Big|_{-\alpha}^{\alpha}\right| && (33)\\
&= \left|\frac{\sin(2\pi\alpha p)}{2\pi p\alpha}\right| && (34)\\
&\leq \frac{1}{2\pi\left|p\alpha\right|} < 1 && (35)
\end{aligned}
$$

Since μ_α is a probability measure for $p = 0$ I have that

$$
\left|\widehat{\mu_\alpha}(0)\right| = \left|\int_{\mathbb{T}} \mathrm{d}\mu_\alpha(x)\right| = 1 \tag{36}
$$

Part b

Writing $\mu_\alpha^{*k} = \mu_\alpha * \cdots * \mu_\alpha$ and using recursively the property $\widehat{\mu * \nu}(p) = \widehat{\mu}(p)\,\widehat{\nu}(p)$ for two measures μ, ν on $\mathbb{T}$ from part a above $\widehat{\mu_\alpha^{*k}}(0) = 1$ for all k. For $p \neq 0$ I have that

$$
\left|\mu_\alpha^{*k}(p)\right| \leq \left(\frac{1}{2\pi\left|p\alpha\right|}\right)^k \to 0 \qquad (\text{as } k \to \infty) \tag{37}
$$

Let mes be the Lebesgue measure on $\mathbb{T}$. I have that $\widehat{\mathrm{mes}}(p) = 0$ for all $p \neq 0$ and $\widehat{\mathrm{mes}}(0) = 1$. It was proved in class that a sequence of measures $(\mu_n)_{n\geq 1}$ converges weakly to a measure μ (as $n \to \infty$) if for all p we have that $\widehat{\mu_n}(p) \to \widehat{\mu}(p)$. Therefore μ_α^{*k} converges weakly to mes.