ST2133 ZA

BSc DEGREES AND GRADUATE DIPLOMAS IN ECONOMICS, MANAGEMENT, FINANCE AND THE SOCIAL SCIENCES, THE DIPLOMA IN ECONOMICS AND SOCIAL SCIENCES AND THE CERTIFICATE IN EDUCATION IN SOCIAL SCIENCES

Summer 2022 Online Assessment Instructions

ST2133 Advanced Statistics: Distribution Theory

The assessment will be an **open-book take-home online assessment within a 4-hour window**. The requirements for this assessment remain the same as the closed-book exam, with an expected time/effort of **2 hours**.

Candidates should answer all **FOUR** questions: **QUESTION 1** of Section A (40 marks) **and all THREE** questions from Section B (60 marks in total). **Candidates are strongly advised to divide their time accordingly**.

You should complete this examination paper using **pen and paper**. Please use **BLACK INK only**.

Handwritten work then needs to be scanned, converted to PDF and then uploaded to the VLE as **ONE individual file** including the coversheet. Each scanned sheet should have your **candidate number** written clearly in the header. Please **do not write your name anywhere** on any sheet.

The paper will be available at 00:00 (BST) on XXXday XX XXXX 2022.

You have **until** 00:00 (BST) on XXXday XX XXXX 2022 to upload your file into the VLE submission portal. However, you are advised not to leave your submission to the last minute. *We will deduct 5 marks if your submission is up to one hour late, 10 marks if your submission is more than one hour late but less than two hours late (etc.)*.

Workings should be submitted for all questions requiring calculations. Any necessary assumptions introduced in answering a question are to be stated.

You may use any calculator for any appropriate calculations, but you may not use any computer software to obtain solutions. Credit will only be given if all workings are shown.

If you think there is any information missing or any error in any question, then you should indicate this but proceed to answer the question stating any assumptions you have made.

The assessment has been designed with a duration of 4 hours to provide a more flexible window in which to complete the assessment and to appropriately test the course learning outcomes. As an open-book exam, the expected amount of effort required to complete all questions and upload your answers during this window is no more than 2 hours. Organise your time well.

You are assured that there will be no benefit in you going beyond the expected 2 hours of effort. Your assessment has been carefully designed to help you show what you have learned in the hours allocated.

This is an open book assessment and as such you may have access to additional materials including but not limited to subject guides and any recommended reading. But the work you submit is expected to be 100% your own. Therefore, unless instructed otherwise, you must not collaborate or confer with anyone during the assessment. The University of London will carry out checks to ensure the academic integrity of your work. Many students that break the University of Londons assessment regulations did not intend to cheat but did not properly understand the University of Londons regulations on referencing and plagiarism. The University of London considers all forms of plagiarism, whether deliberate or otherwise, a very serious matter and can apply severe penalties that might impact on your award. The University of London 2021-22 Procedure for the Consideration of Allegations of Assessment Offences is available online at:

[Include web link]

The University of Londons Rules for Taking Online Timed Assessments have been included in an update to the University of London General Regulations and are available at:

[Include web link]

Section A

Answer all $\mathbf{three}$ parts of question 1 (40 marks in total)

1. (a) The probability density function of a random variable X is given by

$$f_X(x) = \begin{cases} x, & 0 \leq x < 1; \\ 2 - x, & 1 \leq x < 2; \\ 0, & \text{otherwise.} \end{cases}$$

i. Find $E(X)$. You can state the answer without derivations, but you will receive no marks if the stated answer is wrong. [4 marks]

ii. Work out the cumulative distribution function (CDF) of X. [6 marks]

iii. Find $P(|2X - 1| < 2)$. [5 marks]

(b) Let $W = \sum_{i=1}^{N} X_i$, where the X_i's are independent and identically distributed Bernoulli random variables with probability of success p. The random variable N has a Poisson distribution with mean μ, and is independent of the X_i's.

i. Work out the moment-generating function of W. You can use standard formulae for random sums, as long as you can state them clearly.

[9 marks]

ii. State the probability mass function of W. [3 marks]

(c) There are two tills in a convenience store. Let T_1 and T_2 denote the waiting times until a customer is served in till 1 and till 2 respectively. Assume that T_1 and T_2 are independent of each other, and are both identically distributed to T, with density function

$$f_T(t) = \lambda \exp(-\lambda t), \quad t > 0.$$

Tills 1 and 2 simultaneously start serving customers (exactly one per till). There are no more customers in the queue, and no other customers enter the store.

Let W be the waiting time until both tills are free of customers.

i. Explain briefly why $W = \max(T_1, T_2)$. [2 marks]

ii. Show that $E(W) \geq 1/\lambda$. (Hint: You need to work out $E(T)$ first.) [5 marks]

iii. Work out the probability density function of W. (Hint: Find $P(W < w)$ first for $w > 0$.) [6 marks]

Section B

Answer all three questions in this section (60 marks in total)

2. Let V and W be two random variables with joint probability density function given by

$$f_{V,W}(v,w) = aw\exp(-\delta v + (\delta - \lambda)w^2), \quad w > 0, \ \ v > w^2,$$

where λ, δ are positive constants.

(a) Find the value of a. [6 marks]

(b) Consider the transformation

$$X = W^2, \quad Y = V - W^2.$$

Find the joint density of X and Y. Hence show that X and Y are independent exponential random variables. [10 marks]

(c) State the distribution of $\lambda X + \delta Y$. [4 marks]

3. Consider throws of a biased coin with probability p of showing a head and $q = 1-p$ of showing a tail. Assume that throws are independent of each other, and let X denote the number of throws until you obtain a head.

(a) Write down the probability mass function of X. [3 marks]

(b) Find the probability of observing an odd number of tails before the first head (0 is counted as even). [4 marks]

(c) Now consider the following game. In each round you throw the biased coin until you get a head, which is the end of the round. If the number of tails before the head is even, the game ends. If it is odd, you start another round. Let N denote the total number of rounds you play.

i. Show that the probability mass function of N is given by

$$p_N(n) = Q^{n-1}P, \quad n = 1, 2, \ldots,$$

where Q and P have to be specified in terms of p and q. [5 marks]

ii. Let Y_i denote the total number of throws required to obtain the ith head (after obtaining the $(i-1)$th head), $i = 1, \ldots, N$.

Let W be the total number of throws when the game ends. Write W in terms of the Y_i's. [2 marks]

iii. If $E(Y_i) = \mu$ for $i = 1, \ldots, N-1$ and $E(Y_N) = \gamma$, write $E(W)$ in terms of μ, γ, p and q. [6 marks]

4. Suppose there are N multiple-choice questions in an examination. Each question has 3 choices. You have a probability of 0.7 of knowing the correct answer to a particular question. If you do not know the answer, you pick one at random. Your answer to different questions are independent of each other.

   (a) For a particular question, find the probability that you answer it correctly.

   [3 marks]

   (b) Suppose $N = 3$. Given that you have answered all questions correctly, what is the probability that you only know the answer to exactly one question?

   [5 marks]

   (c) The time T to answer a particular question has the following density function:

$$f_T(t) = \begin{cases} \lambda \exp(-\lambda t), & \text{if you know the answer;} \\ 5\lambda \exp(-5\lambda t), & \text{otherwise.} \end{cases}$$

   i. Find $E(T)$. You can use the mean of an exponential distribution without proof, but you need to state it clearly.

   [5 marks]

   ii. Let T_i be the time to answer the ith question. The T_i's are independent of each other and identically distributed to T.
   Let W be the total time to answer all questions. Find $E(W)$ and Var(W) if you are given that Var$(T) = 529/(625\lambda^2)$ and $E(N) =$ Var$(N) = 10$.

   [7 marks]

END OF PAPER