ST2133 Distribution Theory Exam 2022 - Solutions for Zone A Paper

Section A

1. (a) i. The probability density is symmetric around 1, hence $E(X) = 1$. **(4 marks)**. Or, we have

$$E(X) = \int_{\mathbb{R}} x f_X(x) dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx \quad \textbf{(2 marks)}$$
$$= [x^3/3]_0^1 + [x^2 - x^3/3]_1^2 \quad \textbf{(1 mark)}$$
$$= 1/3 + 2/3 = 1. \quad \textbf{(1 mark)}$$

ii. The distribution function of X is

$$P(X < x) = \int_{-\infty}^{x} f_X(s) ds$$
$$= \begin{cases} 0, & x < 0; \\ \int_0^x s ds, & 0 \le x < 1; \\ \int_0^1 s ds + \int_1^x (2-s) ds, & 1 \le x < 2; \\ 1, & x \ge 2. \end{cases} \quad \textbf{(3 marks)}$$
$$= \begin{cases} 0, & x < 0; \\ x^2/2, & 0 \le x < 1; \\ 2x - x^2/2 - 1, & 1 \le x < 2; \\ 1, & x \ge 2. \end{cases} \quad \textbf{(3 marks)}$$

iii.

$$
\begin{aligned}
P(|2X-1|<2) &= P(-1 < X - 1/2 < 1) \quad \textbf{(1 mark)} \\
&= P(-1/2 < X < 3/2) \quad \textbf{(1 mark)} \\
&= 1 - P(3/2 < X < 2) \quad \textbf{(1 mark)} \\
&= 1 - P(0 < X < 1/2) \text{ (symmetry of the density around 1)} \quad \textbf{(1 mark)} \\
&= 1 - 1/8 = 7/8. \quad \textbf{(1 mark)}
\end{aligned}
$$

1. (b) i. We have

$$
\begin{aligned}
M_W(t) = E(\exp(tW)) &= E\left(\exp\left(t\sum_{i=1}^{N} X_i\right)\right) \\
&= E\left(\prod_{i=1}^{N} e^{tX_i}\right) \quad \textbf{(1 mark)} \\
&= E\left(E\left(\prod_{i=1}^{N} e^{tX_i}\middle| N\right)\right) \quad \textbf{(1 mark)} \\
&= E(M_{X_1}(t)^N) \quad \textbf{(1 mark)} \\
&= E\left((pe^t+q)^N\right), \quad q := 1-p \quad \textbf{(2 marks)} \\
&= M_N(t'), \quad t' := \log(pe^t+q) \quad \textbf{(1 mark)} \\
&= \sum_{j\geq 0} (\mu e^{t'})^j e^{-\mu}/j! \quad \textbf{(1 mark)} \\
&= \exp(\mu(e^{t'}-1)), \quad \textbf{(1 mark)} \\
&= \exp(\mu p(e^t-1)), \quad t \in \mathbb{R}. \quad \textbf{(1 mark)}
\end{aligned}
$$

ii. From part (b)i, W has a Poisson distribution with mean μp. Hence the probability mass function of W is

$$
p_W(w) = \frac{(\mu p)^w e^{-\mu p}}{w!}, \quad w = 0, 1, \ldots. \quad \textbf{(3 marks)}
$$

1. (c) i. If $T_1 \geq T_2$, then the last customer served will be from till 1, and hence the waiting time until both tills are free is the waiting time until till 1 is free, so that $W = T_1$ **(1 mark)**. Same argument for $T_2 \geq T_1$, then $W = T_2$. **(1 mark)**. Hence $W = \max(T_1, T_2)$.

ii. For $i = 1, 2$,

$$\begin{aligned} W &\geq T_i \quad \textbf{(1 mark)} \\ \Rightarrow E(W) &\geq E(T_i) = E(T) \quad \textbf{(1 mark)} \\ &= \int_0^\infty \lambda t e^{-\lambda t} dt \\ &= [-te^{-\lambda t}]_0^\infty + \int_0^\infty e^{-\lambda t} dt \quad \textbf{(2 marks)} \\ &= 1/\lambda. \quad \textbf{(1 mark)} \end{aligned}$$

iii.

$$\begin{aligned} P(W < w) &= P(\max(T_1, T_2) < w) = P(\{T_1 < w\} \text{ and } \{T_2 < w\}) \quad \textbf{(1 mark)} \\ &= P(T_1 < w)P(T_2 < w) \quad \textbf{(1 mark)} \\ &= (P(T < w))^2 \quad \textbf{(1 mark)} \\ &= \left(\int_0^w \lambda e^{-\lambda t} dt \right)^2 \\ &= (1 - e^{-\lambda w})^2, \quad w > 0. \quad \textbf{(1 mark)} \end{aligned}$$

Hence

$$f_W(w) = \frac{dP(W < w)}{dw} = 2\lambda e^{-\lambda w}(1 - e^{-\lambda w}), \quad w > 0. \quad \textbf{(2 marks)}$$

2. (a) We have

$$
\begin{aligned}
1 &= \int_0^\infty \int_{w^2}^\infty aw \exp(-\delta v + (\delta - \lambda) w^2) dv dw \\
&= a \int_0^\infty w e^{(\delta-\lambda)w^2} \cdot [-e^{-\delta v}/\delta]_{w^2}^\infty dw \quad \textbf{(1 mark)} \\
&= (a/\delta) \int_0^\infty w e^{(\delta-\lambda)w^2} \cdot e^{-\delta w^2} dw \quad \textbf{(1 mark)} \\
&= (a/\delta) \int_0^\infty w e^{-\lambda w^2} dw \\
&= (a/\delta)[-e^{-\lambda w^2}/(2\lambda)]_0^\infty \quad \textbf{(2 marks)} \\
&= a/(2\lambda\delta). \quad \textbf{(1 mark)}
\end{aligned}
$$

Hence $a = 2\lambda\delta$. **(1 mark)**

2. (b) We can invert the transformation to obtain

$$W = \sqrt{X}, \quad \textbf{(1 mark)} \quad V = X + Y. \quad \textbf{(1 mark)}$$

The solution $W = -\sqrt{X}$ is rejected since $W > 0$. With this,

$$
\begin{aligned}
f_{X,Y}(x, y) &= f_{V,W}(v, w) \left\| \begin{matrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{matrix} \right\| \quad \textbf{(1 mark)} \\
&= f_{V,W}(x + y, \sqrt{x}) \left\| \begin{matrix} 1 & 1 \\ 1/(2\sqrt{x}) & 0 \end{matrix} \right\| \quad \textbf{(2 marks)} \\
&= a\sqrt{x} \exp(-\delta(x + y) + (\delta - \lambda)x)/(2\sqrt{x}) \quad \textbf{(1 mark)} \\
&= \lambda\delta \exp(-\lambda x - \delta y), \quad x, y > 0. \quad \textbf{(1 mark)}
\end{aligned}
$$

It is easy to see that the density for X is $f_X(x) = \lambda \exp(-\lambda x)$ for $x > 0$ (i.e., $X \sim \text{Exp}(\lambda)$) **(1 mark)** and the density for Y is $f_Y(y) = \delta \exp(-\delta y)$ for $y > 0$ (i.e., $Y \sim \text{Exp}(\delta)$). **(1 mark)** Hence $f_{X,Y}(x, y) = f_X(x) f_Y(y)$, and so X is independent of Y. **(1 mark)**

2. (c) $\lambda X + \delta Y$ has a Gamma distribution with parameter $n = 2$ and rate 1. **(4 marks)**

3.(a)

$$p_X(x) = q^{x-1}p, \quad x = 1, 2, \dots. \quad \textbf{(3 marks)}$$

3.(b)

$$\begin{aligned} P(\text{odd tails until head}) &= P(1T1H) + P(3T1H) + \cdots \quad \textbf{(1 mark)} \\ &= qp + q^3p + \cdots \quad \textbf{(1 mark)} \\ &= qp \sum_{j \geq 0} q^{2j} \quad \textbf{(1 mark)} \\ &= \frac{qp}{1 - q^2} = \frac{q}{1+q}. \quad \textbf{(1 mark)} \end{aligned}$$

3.(c)i. To obtain n heads, we need to obtain exactly $n-1$ heads first, which means each time throwing odd number of tails and then a head, and repeat the process exactly $n-1$ times. **(2 marks)** The probability of throwing odd number of tails until obtaining a head is obtained in (b), which is $Q = q/(1+q)$. **(1 mark)** After the $n-1$ heads, we need to throw even number of tails until getting a head, which has probability $P = 1 - Q = 1/(1+q)$. **(1 mark)**

Hence the probability mass function of N is

$$p_N(n) = Q^{n-1}P = q^{n-1}/(1+q)^n, \quad n = 1, 2, \dots. \quad \textbf{(1 mark)}$$

3.(c)ii. $W = \sum_{i=1}^{N} Y_i$. **(2 marks)**

3.(c)iii. We have

$$E(W) = E\left(\sum_{i=1}^{N} E(Y_i)\right) \quad \textbf{(1 mark)}$$
$$= E\left((N-1)\mu + \gamma\right) \quad \textbf{(2 marks)}$$
$$= (E(N)-1)\mu + \gamma. \quad \textbf{(1 mark)}$$

But $E(N) = 1/P = 1 + q$. **(1 mark)** Hence

$$E(W) = q\mu + \gamma. \quad \textbf{(1 mark)}$$

4.(a) Let C be the event of answering a question correctly, and K be the event of knowing the answer to a particular question. Then by the law of total probability,

$$P(C) = P(C|K)P(K) + P(C|K^c)P(K^c) \quad \textbf{(1 mark)}$$
$$= 1 \cdot 0.7 + (1/3) \cdot (1 - 0.7) \quad \textbf{(1 mark)}$$
$$= 0.8. \quad \textbf{(1 mark)}$$

4.(b) First, $P(K^c \cap C) = 0.3 \cdot 1/3 = 0.1$. **(1 mark)** Then for a general N,

$$P(\text{know 1}|\text{All correct}) = \frac{P(\text{know 1, all correct})}{P(\text{All correct})} \quad \textbf{(1 mark)}$$
$$= \frac{\binom{N}{1} \cdot (0.7)(0.1)^{N-1}}{(0.8)^N}. \quad \text{(For your information only)}$$

Hence for $N = 3$,

$$P(\text{know 1}|\text{All correct}) = \frac{3 \cdot 0.7 \cdot (0.1)^2}{0.8^3} \quad \textbf{(2 marks)}$$
$$= \frac{21}{512} = 0.0410. \quad \textbf{(1 mark)}$$

4.(c)i. For $X \sim$Exp(λ), $E(X) = 1/\lambda$ and var$(X) = 1/\lambda^2$. Let $Q_1 \sim$ Exp(λ) and $Q_2 \sim$ Exp(5λ). Then

$$\begin{aligned} E(T) &= E(T|K)P(K) + E(T|K^c)P(K^c) \quad \textbf{(1 mark)} \\ &= E(Q_1)\cdot 0.7 + E(Q_2)\cdot 0.3 \quad \textbf{(2 marks)} \\ &= 0.7/\lambda + 0.3/(5\lambda) \quad \textbf{(1 mark)} \\ &= 0.76/\lambda. \quad \textbf{(1 mark)} \end{aligned}$$

4.(c)ii. We can write $W = \sum_{i=1}^{N} T_i$. **(1 mark)** Hence

$$E(W) = E(N)E(T_1) = 10 \cdot 0.76/\lambda = 7.6/\lambda. \quad \textbf{(2 marks)}$$

Also,

$$\begin{aligned} \text{var}(W) &= E(\text{var}(W|N)) + \text{var}(E(W|N)) \quad \textbf{(1 mark)} \\ &= E(N\text{var}(T_1)) + \text{var}(NE(T_1)) \quad \textbf{(1 mark)} \\ &= 10 \cdot \text{var}(T) + 10 \cdot (0.76/\lambda)^2 \quad \textbf{(1 mark)} \\ &= 8900/(625\lambda^2) \quad \textbf{(1 mark)} \\ &= 356/(25\lambda^2) = 14.24/\lambda^2. \end{aligned}$$