$n\,I(\theta)$

$\tilde{\theta}_1: \frac{\theta^2}{3} \geq \frac{\theta^2}{3} \quad \forall \theta \in \Theta$

по дост. усл. $\tilde{\theta}_1$ эффект., т.к. $D[\tilde{g}] = \frac{g'^2(\theta)}{n\,I(\theta)}$

по т. о ∃! → $\tilde{\theta}$ эфф. $\Rightarrow \tilde{\theta}_3'$ не эфф.

(Т5) $\xi \sim R[\theta, 2\theta]$

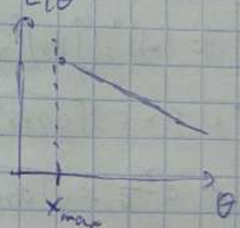
$p(x) = \frac{1}{\theta}\{\theta \leq x \leq 2\theta\} + 0\{x \in \mathbb{R} \setminus [\theta, 2\theta]\}$

ОММ:

$\alpha_1 = \int_\theta^{2\theta} \frac{x}{\theta}dx = \frac{x^2}{2\theta}\Big|_\theta^{2\theta} = 2\theta - \frac{1}{2}\theta = \frac{3}{2}\theta$

$\alpha_2 = \int_\theta^{2\theta} \frac{x^2}{\theta}dx = \frac{x^3}{3\theta}\Big|_\theta^{2\theta} = \frac{8}{3}\theta^2 - \frac{1}{3}\theta^2 = \frac{7}{3}\theta^2$

$D[\xi] = \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{1}{12}\theta^2$




$\tilde{\alpha}_1 = \overline{x}$ $\qquad \tilde{\theta}_1 = \frac{2}{3}\overline{x}$

$\frac{3}{2}\tilde{\theta}_1 = \overline{x}$

$\tilde{\theta}_1: M[\tilde{\theta}_1] = \frac{2}{3}M[\overline{x}] = \frac{2}{3n}\sum_{i=1}^{n} M[x_i] = \theta$

$D[\tilde{\theta}_1] = D[\frac{2}{3}\overline{x}] = \frac{4 \cdot n}{9n^2} \cdot \frac{1}{12}\theta^2 = \frac{\theta^2}{27n} \to 0,\ n \to \infty \Rightarrow \tilde{\theta}_1$ сост. по дост. усл.

ОМП: $\rho(x, \theta) = \begin{cases} \frac{1}{\theta}; & x \in [\theta, 2\theta] \\ 0; & \text{иначе} \end{cases}$

$X_{max}$: $p(t) = \frac{n}{\theta}\left(\frac{t}{\theta} - \frac{\theta}{\theta}\right)^{n-1}\{t\in[\theta;2\theta]\} + 0\{t\in R\setminus[0;2\theta]\}$

$\tilde{\theta}_2$: $M[\tilde{\theta}_2] = \frac{1}{2}M[x_{max}] + \frac{n}{2\theta^n}\int_\theta^{2\theta}(t-\theta)^{n-1}t\,dt = \begin{Bmatrix} k=t-\theta \\ t=k+\theta \end{Bmatrix} =$

$= \frac{n}{2\theta^n}\int_0^\theta k^{n-1}(k+\theta)\,dk = \frac{n}{2\theta^n(n+1)}k^{n+1}\Big|_0^\theta + \frac{n}{2(n+1)\theta^{n-1}}k^n\Big|_0^\theta = \frac{\theta n}{2(n+1)} + \frac{\theta}{2} = \frac{\theta(2n+1)}{2(n+1)}$

$\tilde{\theta}_2' = \frac{2(n+1)}{2n+1}\tilde{\theta}_2$ – несмещ

$D[\tilde{\theta}_2'] = \frac{4(n+1)^2}{(2n+1)^2}D[\tilde{\theta}_2] = \frac{(n+1)^2}{(2n+1)^2}D[x_{max}]$

$M[x_{max}] = \frac{\theta(2n+1)}{n+1}$

$M[x^2_{max}] = \frac{n}{\theta^n}\int_0^\theta k^{n-1}(k^2+2k\theta+\theta^2)\,dk = \frac{n}{\theta^n}\left[\frac{\theta^{n+2}}{n+2} + \frac{2\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n}\right] =$

$= n\left[\frac{\theta^2(n^2+1) + 2\theta^2(n^2+2n) + \theta^2(n^2+3n+2)}{n^3+3n^2+2n}\right] = \theta^2\left(\frac{4n^2+8n+2}{n^2+3n+2}\right)$

$D[x_{max}] = \theta^2\left(\frac{4n^2+8n+2}{n^2+3n+2} - \frac{4n^2+4n+1}{n^2+2n+1}\right) = \theta^2\left(\frac{n}{(n+1)^2(n+2)}\right)$

$D[\tilde{\theta}_2'] = \frac{(n+1)^2}{(2n+1)^2}\cdot\frac{n}{(n+1)^2(n+2)}\theta^2 \to 0$, $n\to\infty$ сост по дост. усл.

лучше $\tilde{\theta}_2$ эффективнее

эффективность:

$\forall\theta\in\Theta \hookrightarrow D[\tilde{\theta}_1] > D[\tilde{\theta}_2]$ $\qquad \frac{\theta^2}{27n} > \frac{n}{(2n+1)^2(n+2)}$ при $n\geq 4$

$x_i \in [\theta, 2\theta]$ $\qquad \frac{x_i}{\theta}\in[1,2]$

$\varphi(x_{max}) = (F(x))^n = \left(\int_1^x dx\right)^n = (x-1)^n$

$\sqrt[n]{0,025}+1 < x < \sqrt[n]{0,975}+1$

$\sqrt[n]{0,025}+1 < \frac{x_{max}}{\theta} < \sqrt[n]{0,975}+1$

$\frac{x_{max}}{\sqrt[n]{0,975}+1} < \theta < \frac{x_{max}}{\sqrt[n]{0,025}+1}$

по ОММ: $\sqrt{n}\,\frac{g(\tilde{\alpha})-g(\alpha)}{\sigma(\alpha)} \sim N(0,1)$

$\sigma(\alpha) = \sqrt{\nabla^T g K \nabla g} = \frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}$

$x=\theta$ $\qquad \sqrt{n}\,\frac{\frac{2}{3}\tilde{\alpha} - \frac{2}{3}\alpha_1}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}} =$

$= \sqrt{n}\,\frac{\tilde{\theta}-\theta}{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}\cdot\underbrace{\frac{\frac{2}{3}\sqrt{\alpha_2-\alpha_1^2}}{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}}}_{\to 1} \sim N(0,1)$

по т. Слуцкого $\sim N(0,1)$

$-1,96 < \sqrt{n}\,\frac{\tilde{\theta}-\theta}{\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}} < 1,96$

$-\frac{1,96}{\sqrt{n}}\cdot\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}+\tilde{\theta} < \theta < \frac{1,96}{\sqrt{n}}\cdot\frac{2}{3}\sqrt{\tilde{\alpha}_2-\tilde{\alpha}_1^2}+\tilde{\theta}$

№4 $\theta \in (0,1)$

$\xi \sim \frac{\theta}{2}\{(-1,1)\setminus\{0\}\} + \frac{1-\theta}{2}\{0\} + \frac{1-\theta}{2}\{2\}$

$\alpha_1 = M[\xi] = \int_{-\infty}^{\infty} x\,p(x,\theta)dx = \underbrace{\int_{-1}^{1} \frac{\theta}{2} x\,dx}_{=0} + \frac{1-\theta}{2}\cdot 0 + \frac{1-\theta}{2}\cdot 2 = 1-\theta$

$\alpha_2 = M[\xi^2] = \frac{\theta}{2}\int_{-1}^{1} x^2 dx + \frac{1-\theta}{2}\cdot 2^2 = \frac{\theta}{3} + 2(1-\theta) = 2 - \frac{5}{3}\theta$

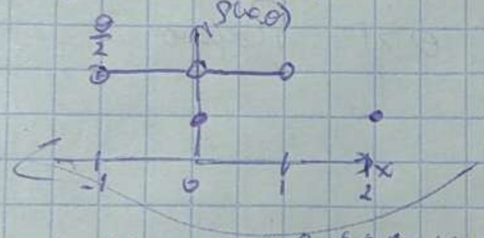
$\mu_2 = \alpha_2 - \alpha_1^2 = D[\xi] = 2 - \frac{5}{3}\theta - \theta^2 + 2\theta - 1 = 1 + \frac{\theta}{3} - \theta^2$

ОММ: $\alpha_1(\theta) = \bar{\alpha}_1 = \bar{x} \Rightarrow 1-\theta = \bar{x} \Rightarrow \tilde{\theta} = 1 - \bar{x}$

$M[\tilde{\theta}_1] = M[1-\bar{x}] = 1 - M[\xi] = 1 - 1 + \theta = \theta \Rightarrow$ несмещ

$D[\tilde{\theta}_1] = D[1-\bar{x}] = \frac{1}{n} D[\xi] \xrightarrow[n\to\infty]{} 0$ сост по дост. усл.

$\alpha_2$ ОМП




$L = \left(\frac{\theta}{2}\right)^{n-m_1-m_2} \left(\frac{1-\theta}{2}\right)^{m_1+m_2}$

$\ln L = (n-m_1-m_2)\ln\frac{\theta}{2} + (m_1+m_2)\ln\left(\frac{1-\theta}{2}\right)$

$(\ln L)' = \frac{n-m_1-m_2}{\theta} + \frac{m_1+m_2}{\theta-1} = \frac{n\theta - n + m_1 + m_2}{\theta(\theta-1)} = 0 \Rightarrow \theta = 1 - \nu_1 - \nu_2$

$(\ln L)'' = \frac{m_1+m_2-n}{\theta^2} - \frac{m_1+m_2}{(\theta-1)^2} = \frac{(m_1+m_2-n)(\theta-1)^2 - (m_1+m_2)\theta^2}{\theta^2(\theta-1)^2} = \ldots = \frac{(\nu_1+\nu_2)(\nu_1+\nu_2-1)}{n\theta^2(\theta-1)^2} < 0$

$\theta = 1 - \nu_1 - \nu_2$ — т max $\Rightarrow \tilde{\theta}_2 = 1 - \nu_1 - \nu_2$

$M[\tilde{\theta}_2] = 1 - M[\nu_1] - M[\nu_2] = 1 - \frac{1-\theta}{2} - \frac{1-\theta}{2} = \theta$ несмещ

$D[\tilde{\theta}_2] = D[\nu_1 + \nu_2] = \frac{(1-\theta)(\theta)}{n} \to 0$, $n\to\infty$ сост.

в) Условия Р-Рао

1) $p(x,\theta) \in C^\infty_\theta((0,1))$ опр

2) $\frac{\partial}{\partial\theta}\int_{A=\mathbb{R}} p(x,\theta)dx = \int_{\mathbb{R}} \frac{\partial}{\partial\theta} p(x,\theta)dx$ на $(0,1)$

$\int_{-\infty}^{\infty} \frac{\partial p(x,\theta)}{\partial\theta} = \frac{1}{2}\int_{-1}^{1} dx + \left(\frac{1-\theta}{2}\right)'_\theta + \left(\frac{1-\theta}{2}\right)'_\theta = 1 - \frac{1}{2} - \frac{1}{2} = 0$

3) $I(\theta) \in C(0,1)$, $I(\theta) > 0$ на $(0\ 1]$

$$\frac{\partial \ln p(x,\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\ln\frac{\theta}{2}\right) = \frac{1}{\theta} \quad \frac{\partial \ln p_{-1}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\ln\left(\frac{1-\theta}{2}\right)\right) = \frac{1}{\theta-1}$$

$$I(\theta) = \int_{-1}^{1} \frac{1}{2\theta}dx = \frac{1}{(\theta-1)^2}\left(\frac{1-\theta}{2}\right)2 = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)} \overset{\in C(0,1)}{>} 0 \text{ на } (0,1)$$

$\tilde{\theta}_1 = 1 - \bar{x}$ - несмещ $D[\tilde{\theta}_1]$ - огр на $\forall$ компакте $\in (0\ 1)$ $\tilde{\theta}_1$ - рег (регулярна)

$\tilde{\theta}_2 = 1 - \nu_1 - \nu_2$ - несмещ, $D[\tilde{\theta}_2]$ - огр на $\forall$ компакте $\in (0 1)$ $\Rightarrow \theta_2$ - рег

$\Rightarrow$ Неравенство Крамера Рао

$\theta_2$ : $D[\hat{\theta}_2] \geqslant \frac{1}{n I(\theta)} = \frac{\theta(1-\theta)}{n}$ подойт. для эффективна

$\Rightarrow \tilde{\theta}_1$ неэф. т.к. $\tilde{\theta}_1 \neq \tilde{\theta}_2$

(Т6) $p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}$, $\theta > 1$ Парето

а) $L(x,\theta) = \frac{(\theta-1)^n}{\prod_{i=1}^{n} x_i^\theta}$

$\ln L = n\ln(\theta-1) - \theta\sum_{i=1}^{n}\ln x_i$

$(\ln L)'_\theta = \frac{n}{\theta-1} - \sum_{i=1}^{n}\ln x_i = 0$

$\tilde{\theta} = 1 + \frac{1}{\overline{\ln x_i}}$ $\quad 1 + \frac{1}{\overline{\ln x_i}}$

$(\ln L)''_{\theta\theta} = \frac{-n}{(\theta-1)^2} < 0 \Rightarrow \theta = 1 + \frac{1}{\overline{\ln x_i}}$ - т. max.

б) $\left(\int \frac{\theta-1}{x^\theta}dx\right)'_\theta = x^{1-\theta}\ln x$ $\Rightarrow$ модель сильно рег

$\int \frac{\partial}{\partial\theta}\left(\frac{\theta-1}{x^\theta}\right)dx = x^{1-\theta}\ln x$

$\int_1^{\tilde{x}} \frac{\theta-1}{x^\theta}dx = -\frac{1}{\tilde{x}^{\theta-1}} + 1 = \frac{1}{2}$ $\quad \tilde{x} = 2^{\frac{1}{\theta-1}}$

$g(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}}$

$\sqrt{n}\cdot\frac{g(\tilde{\theta}) - g(\theta)}{\sigma'(\theta)} \sim N(0,1)$ $\quad \sigma \xrightarrow{P} \sigma'(\tilde{\theta})$

$$\sigma(\tilde{\theta}) = \sqrt{\nabla^T g(\tilde{\theta})\ I^{-1}(\tilde{\theta})\ \nabla g(\theta)}$$

$$I(\theta) = M\left[\left((\ln p)'_\theta\right)^2\right] = M\left[\left(\frac{1}{\theta-1} - \ln x\right)^2\right] = \int_1^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 p(x,\theta)\,dx =$$

$$= \int_1^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 \frac{\theta-1}{x^2}\,dx = \frac{1}{(\theta-1)^2}$$ непр на $\theta > 1$

$$\nabla g(\tilde{\theta}) = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\tilde{\theta}-1} \qquad \sigma(\tilde{\theta}) = -\frac{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\tilde{\theta}-1}$$

$$\sqrt{n}\,\frac{g(\tilde{\theta}) - g(\theta)}{\sigma(\tilde{\theta})} \sim N(0,1)$$

$$\frac{1.96\,\sigma(\tilde{\theta})}{\sqrt{n}} + g(\tilde{\theta}) < g(\theta) < \frac{-1.96\,\sigma(\tilde{\theta})}{\sqrt{n}} + g(\theta)$$

в) $\sqrt{n}\,\dfrac{\tilde{\theta} - \theta}{\sigma(\theta)} \sim N(0,1)$

$$\sigma(\theta) \xrightarrow{P} \sigma(\theta)$$

$$\sigma(\theta) = \theta - 1 \quad \Rightarrow \quad \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\theta - 1} \sim N(0,1)$$

$$\frac{-1.96(\theta-1)}{\sqrt{n}} + 1 + \frac{1}{\overline{\ln x_i}} < \theta < \frac{1.96(\theta-1)}{\sqrt{n}} + 1 + \frac{1}{\overline{\ln x_i}}$$