A confluent λ-calculus with a catch/throw mechanism

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Abstract

We derive a confluent λ-calculus with a catch/throw mechanism (called $\lambda_{\mathbf{ct}}$-calculus) from M. Parigot's $\lambda\mu$-calculus. We also present several translations from one calculus into the other which are morphisms for the reduction. We use them to show that the $\lambda_{\mathbf{ct}}$-calculus is a retract of $\lambda\mu$-calculus (these calculi are isomorphic if we consider only convertibility). As a by-product, we obtain the subject reduction property for the $\lambda_{\mathbf{ct}}$-calculus, as well as the strong normalization for $\lambda_{\mathbf{ct}}$-terms typable in the second order classical natural deduction.

1 Introduction

In the last four years, several extensions of the λ-calculus with some catch/throw mechanism have been proposed by H. Nakano (1994a; 1994b; 1995) and by M. Sato (1997) and Y. Kameyama (1997; 1998). In these papers, the authors consider the catch/throw mechanism as "intrinsically non-deterministic" and thus investigate non-confluent calculi or confine themselves to some specific evaluation strategy. For instance in (Nakano, 1994b), the non-deterministic feature of the catch/throw mechanism is introduced by the following rule:

$$C[\mathbf{throw}\ \alpha\ t] \mapsto \mathbf{throw}\ \alpha\ t$$

where the context $C[\bullet]$ does not capture α or any individual/tag variable occurring freely in t. Let us now look at the following example from H. Nakano (1994b):

$$M \equiv \mathbf{catch}\ \alpha\ ((\lambda x.\lambda y.1\ (\mathbf{throw}\ \alpha\ 2)\ (\mathbf{throw}\ \alpha\ 3))$$

If we assume that we also have the two following rules $\mathbf{catch}\ \alpha\ \mathbf{throw}\ \alpha\ t \rightarrow \mathbf{catch}\ \alpha\ t$ and $\mathbf{catch}\ \alpha\ t \rightarrow t$ when α does not occur free in t, we have three possible normal forms depending on the evaluation strategy:

$$M \rightarrow_\beta \mathbf{catch}\ \alpha\ ((\lambda y.1\ (\mathbf{throw}\ \alpha\ 2)) \rightarrow_\beta \mathbf{catch}\ \alpha\ 1 \rightarrow 1$$
$$M \mapsto \mathbf{catch}\ \alpha\ \mathbf{throw}\ \alpha\ 2 \rightarrow \mathbf{catch}\ \alpha\ 2 \rightarrow 2$$
$$M \mapsto \mathbf{catch}\ \alpha\ \mathbf{throw}\ \alpha\ 3 \rightarrow \mathbf{catch}\ \alpha\ 3 \rightarrow 3$$

In this paper, we will see however that it if we weaken the rule $\mapsto$, it is possible to define a confluent λ-calculus with a catch/throw mechanism. This calculus, called $\lambda_{\mathbf{ct}}$-calculus, is in fact derived from M. Parigot's $\lambda\mu$-calculus. We will present several "canonical" morphisms for the reduction from one calculus to the other (this notion of canonical translation will be formalized).

Then we show that the $\lambda_{\mathbf{ct}}$-calculus is a canonical retract of the $\lambda\mu$-calculus. This will enable us to derive the confluence of the $\lambda_{\mathbf{ct}}$-calculus from the confluence of the $\lambda\mu$-calculus. We also prove that the converse is not true (the $\lambda\mu$-calculus is not a canonical retract of the $\lambda_{\mathbf{ct}}$-calculus) since there is no surjective canonical morphism from the $\lambda_{\mathbf{ct}}$-calculus to the $\lambda\mu$-calculus. Both calculi are however isomorphic if we consider terms up to renaming/simplification.

As a by-product of these translations, we will also obtain the subject reduction property as well as the strong normalization for terms typable in the second order classical natural deduction.

As usual with control operators, the catch/throw mechanism is easier to introduce in the framework of abstract stack machines. Indeed, control operators are aimed to handle the continuation (*i.e.* the rest of the computation to be performed, see Felleisen *at al.* (1986; 1987) or Reynolds (1993) for a survey), and precisely, in abstract stack machines, the continuation is represented by the stack. In the remainder of this introduction, we will thus consider two simple extensions of Krivine's abstract machine: the $\lambda\mu$-machine and the $\lambda_{\mathbf{ct}}$-machine. The former is designed for evaluating $\lambda\mu$-terms, as suggested in (Parigot, 1993) and developped in (Beus & Streicher, 1998; De Groote, 1999), while the latter is provided with a catch/throw mechanism.

In section 2, we derive the reduction rules of the $\lambda_{\mathbf{ct}}$-calculus from the rules of the $\lambda\mu$-calculus. In section 3, we prove that the $\lambda_{\mathbf{ct}}$-calculus is a canonical retract of the $\lambda\mu$-calculus and as a corollary we obtain the confluence of the $\lambda_{\mathbf{ct}}$-calculus. In section 4, we take advantage of the fact that the classical natural deduction may be seen as a type system for the $\lambda_{\mathbf{ct}}$-calculus, as well as for the $\lambda\mu$-calculus.

1.1 An abstract machine for the $\lambda\mu$-calculus

We first recall the syntax of $\lambda\mu$-terms (as usual, we use $x, y, z \ldots$ as λ-variables and $\alpha, \beta, \gamma \ldots$ as μ-variables). The set of $\lambda\mu$-terms is inductively defined as follows.

Definition 1.1.1
If t, u are $\lambda\mu$-terms then the following terms are also $\lambda\mu$-terms (where α and β range over μ-variables and x ranges over λ-variables):

$$x, \quad (t\ u), \quad \lambda x.t, \quad \mu\alpha[\beta]t$$

Remark. The μ-operator is a binder (the μ-variable α is bound in $\mu\alpha[\beta]t$).

We will now define the $\lambda\mu$-machine as a rewrite system. For that purpose we recall some common definitions (see (Beus & Streicher, 1998) for instance): a closure is inductively defined as a triple $\langle \lambda\mu\text{-term, closure-environment, stack-environment} \rangle$ where a closure-environment is a list of pairs (λ-variable, closure),

a stack-environment is a list of pairs (μ-variable, stack) and a stack is a list of closures.

The rules of the $\lambda\mu$-machine are given below. The variables CE and SE range over closure-environments and stack-environments, respectively. As usual, the notation $E(v)$, where E is a closure-environment (resp. stack-environment) and v a λ-variable (resp. μ-variable) stands for the closure (resp. stack) assigned to v in E.

- $(\langle x, CE, SE\rangle, S) \to (CE(x), S)$
- $(\langle(u\ v), CE, SE\rangle, S) \to (\langle u, CE, SE\rangle, \langle v, CE, SE\rangle :: S)$
- $(\langle\lambda x.t, CE, SE\rangle, c :: S) \to (\langle t, (x, c) :: CE, SE\rangle, S)$
- $(\langle\mu\alpha[\beta]t, CE, SE\rangle, S) \to (\langle t, CE, (\alpha, S) :: SE\rangle, ((\alpha, S) :: SE)(\beta))$

Remark. We wrote $((\alpha, S) :: SE)(\beta)$ in the last rule (and not just $SE(\beta)$) in order to deal with the case $\alpha = \beta$.

An instruction of the form $\mu\alpha[\beta]t$ is carried out as expected: the machine first binds the current continuation to name α, then restores the continuation whose name is β in the current environment (discarding the current continuation) and eventually evaluates t.

Remark. Notice that this abstract machine actually evaluates only the weak head normal form of a $\lambda\mu$-term. For further details, see (Beus & Streicher, 1998; De Groote, 1999).

1.2 An abstract machine with a **catch**/**throw** *mechanism*

We still consider two separate name-spaces for λ-variables and μ-variables (or "tag-variables"). The set of $\lambda_{\mathbf{ct}}$-terms is inductively defined as follows.

Definition 1.2.1
If t, u are $\lambda_{\mathbf{ct}}$-terms then the following terms are also $\lambda_{\mathbf{ct}}$-terms (where α ranges over μ-variables and x ranges over λ-variables):

$$x, \quad (t\ u), \quad \lambda x.t, \quad \mathbf{catch}\ \alpha\ t, \quad \mathbf{throw}\ \alpha\ t$$

Remark. The **catch** operator is a binder (the μ-variable α is bound in **catch** α t).

The intended behaviour of the **catch** and **throw** operators is intuitively clear. To evaluate **catch** α t, the machine should bind the current continuation to name α and then evaluate t. To evaluate **throw** α t, the machine should discard the current continuation, restore the continuation whose name is α in the current environment, and eventually evaluate t. Formally, to define the $\lambda_{\mathbf{ct}}$-machine, just replace the last rule of the $\lambda\mu$-machine by the following rules:

- $(\langle\mathbf{catch}\ \alpha\ t, CE, SE\rangle, S) \to (\langle t, CE, (\alpha, S) :: SE\rangle, S)$
- $(\langle\mathbf{throw}\ \alpha\ t, CE, SE\rangle, S) \to (\langle t, CE, SE\rangle, SE(\alpha))$

1.3 Simulating one machine by the other

It is easy to simulate the behaviour of $\mu\alpha[\beta]t$ in the $\lambda_{\mathbf{ct}}$-machine. Indeed, let us consider a term of the form **catch** α **throw** β t. To evaluate such a term, the $\lambda_{\mathbf{ct}}$-machine binds the current stack to name α, restores the stack whose name is β in the current environment and then evaluates t: this is exactly what does the $\lambda\mu$-machine when it evaluates $\mu\alpha[\beta]t$.

Conversely, the behaviour of the **catch** and **throw** operators can also be simulated in the $\lambda\mu$-machine.

- Let us first consider a term of the form $\mu\alpha[\alpha]t$. When the $\lambda\mu$-machine evaluates such a term, it first binds the current stack to name α, then restores *this very stack*, before it evaluates t. This is exactly what does the $\lambda_{\mathbf{ct}}$-machine when it evaluates **catch** α t.
- Let us now consider the instruction $\mu\alpha[\beta]t$ where α does not occur in t. To carry out this term, the $\lambda\mu$-machine binds the current stack to name α, then restores the stack whose name is β in the current environment before it evaluates t. Since α does not occur in t, the current stack should have been discarded, and this is exactly what does the $\lambda_{\mathbf{ct}}$-machine whenever it evaluates **throw** β t.

2 The catch and throw operators and the $\lambda\mu$-calculus

In the previous section, we have discussed how abstract machines can simulate one another. However, restricting ourselves to some abstract machine amounts exactly to considering a specified evaluation strategy (weak head reduction for Krivine's abstract machines). In this section, we show that the simulation of the **catch** and **throw** operators defined in the framework of abstract machines work as well when we consider the $\lambda\mu$-calculus as a confluent rewriting system: we will take advantage of this in the next section to derive a confluent λ-calculus with some catch/throw mechanism. Let us first recall the reduction rules of the $\lambda\mu$-calculus (note that the original proof of confluence of the $\lambda\mu$-calculus given in (Parigot, 1992) is broken by the renaming rule, however the fix is easy and presented in (Py, 1998)).

Reduction rules of the $\lambda\mu$-calculus

- The β-reduction:

$$(\lambda x.t\ u) \to t\{u/x\}$$

- The *structural* rule:

$$(\mu\alpha.t\ u) \to \mu\alpha.t\big\{[\alpha](w\ u)/[\alpha]w\big\}$$

- The *renaming* rule:

$$[\beta]\mu\alpha.t \to t\{\beta/\alpha\}$$

- The *simplification* rule:

$$\mu\alpha[\alpha]t \to t \text{ if } \alpha \text{ does not occur free in } t$$

The notation $t\{u/x\}$ stands for the usual capture-avoiding substitution of the λ-variable x by u in t. The *structural substitution* $t\{[\alpha](w\ u)/[\alpha]w\}$ is defined inductively by:

- $x\{[\alpha](w\ v)/[\alpha]w\} = x$
- $(\lambda x.t)\{[\alpha](w\ v)/[\alpha]w\} = \lambda x.t\{[\alpha](w\ v)/[\alpha]w\}$
- $(t\ u)\{[\alpha](w\ v)/[\alpha]w\} = (t\{[\alpha](w\ v)/[\alpha]w\}\ u\{[\alpha](w\ v)/[\alpha]w\})$
- $(\mu\beta.t)\{[\alpha](w\ v)/[\alpha]w\} = \mu\beta.t\{[\alpha](w\ v)/[\alpha]w\}$
- $([\alpha]t)\{[\alpha](w\ v)/[\alpha]w\} = [\alpha](t\{[\alpha](w\ v)/[\alpha]w\}\ v)$
- $([\beta]t)\{[\alpha](w\ v)/[\alpha]w\} = [\beta]t\{[\alpha](w\ v)/[\alpha]w\}$ if $\alpha \neq \beta$.

Remark. We will sometimes use the more explicit notation $t\{\mu\beta[\alpha](w\ u)/\mu\beta[\alpha]w\}$ for structural substitution above since any occurrence of $[\alpha]w$ in t has actually the shape $\mu\beta[\alpha]w$.

Definition 2.0.1
We call *simple λμ-term* a term which contains no renaming/simplification redex.

Remark. The simple form of a $\lambda\mu$-term t (which is obtained from t by applying only renaming/simplification rules) is unique (modulo α-conversion) and is reached in a linear number of reduction steps. Indeed, it is easy to check that renaming and simplification rules commute with any other rule. Moreover, the application of renaming/simplification rules strictly decreases the size of the term.

Notation. We denote by $\overline{t}$ the simple form of a $\lambda\mu$-term t.

2.1 Deriving the rules

We saw in the previous section that the **catch** and **throw** operators can be simulated respectively by $catch\ \alpha\ t \equiv\ \mu\alpha[\alpha]t$ and $throw\ \alpha\ t \equiv \mu\beta[\alpha]t$ where β is a μ-variable different from α and which does not occur free in t.

Notation. We will use the abbreviation $\mu\_[\alpha]t$ in the latter case, where _ stands for any μ-variable different from α which does not occur free in t. Moreover, to avoid any confusion, we will use italic font for macros (while we use boldface font for built-in operators).

We will call $\lambda\mu ct$-calculus the sublanguage of the $\lambda\mu$-calculus containing only the "macros" *catch* and *throw* (in other words, where any occurrence of a subterm $\mu\alpha[\beta]t$ is either of the form $\mu\alpha[\alpha]t$ or of the form $\mu\_[\beta]t$). Unfortunately the subset consisting of all $\lambda\mu ct$-terms is not closed under reduction because of the following rule:

$$catch\ \alpha\ throw\ \beta\ t = \mu\alpha[\alpha]\mu\_[\beta]t \rightarrow \mu\alpha[\beta]t\{\alpha/\_\} = \mu\alpha[\beta]t$$

We will therefore restrict ourselves to instances of rules for which the contractum is still a $\lambda\mu ct$-term. We obtain these rules by enumerating all the redexes that may occur in a $\lambda\mu ct$-term.

- The subset of $\lambda\mu ct$-terms is clearly closed under substitution. We thus obtain the β-reduction as a derived rule (since the contractum is always a $\lambda\mu ct$-term):

$$(\lambda x.t\ u) \to t\{u/x\}$$

- Any redex of the form $\mu\alpha[\alpha]t$ that occurs in a $\lambda\mu ct$-term has the form $catch\ \alpha\ t$. The rule $\mu\alpha[\alpha]t \to t$, if α does not occur free in t, thus yields a unique derived rule (since the contractum is still a $\lambda\mu ct$-term):

$$catch\ \alpha\ t = \mu\alpha[\alpha]t \to t$$

- Any redex of the form $(\mu\alpha.w\ v)$ (to be more specific $(\mu\alpha[\beta]u\ v)$) that occurs in a $\lambda\mu ct$-term is either of the form $(catch\ \alpha\ u)\ v$ or of the form $(throw\ \alpha\ u)\ v$. The rule $(\mu\alpha.w\ v) \to \mu\alpha.w\big\{[\alpha](t\ v)/[\alpha]t\big\}$ yields two derived rules (since in both cases the contractum is still a $\lambda\mu ct$-term):

$$\begin{aligned}((catch\ \alpha\ u)\ v) = (\mu\alpha[\alpha]u\ v) &\to \mu\alpha([\alpha]u)\big\{[\alpha](t\ v)/[\alpha]t\big\}\\ &= \mu\alpha[\alpha](u(\big\{\mu\_[\alpha](t\ v)/\mu\_[\alpha]t\big\}\ v)\\ &= catch\ \alpha\ (u\big\{throw\ \alpha\ (t\ v)/throw\ \alpha\ t\big\}\ v)\\ ((throw\ \alpha\ u)\ v) = (\mu\delta[\alpha]u\ v) &\to \mu\delta([\alpha]u)\big\{[\delta](t\ v)/[\delta]t\big\}\\ &= \mu\delta[\alpha]u = throw\ \alpha\ u\end{aligned}$$

  since, by definition of *throw*, the variable δ is different from α and does not occur free in u.
- Any redex of the form $[\alpha]\mu\beta.w$ (to be more specific $\mu\gamma[\alpha]\mu\beta[\delta]t$) that occurs in a $\lambda\mu ct$-term has one of the four following forms: $catch\ \alpha\ catch\ \beta\ t$, $throw\ \alpha\ throw\ \beta\ t$, $throw\ \alpha\ catch\ \beta\ t$, $catch\ \alpha\ throw\ \beta\ t$. The rule $[\alpha]\mu\beta.w \to w\big\{\alpha/\beta\big\}$ yields four cases:

$$\begin{aligned}&catch\ \alpha\ catch\ \beta\ t = \mu\alpha[\alpha]\mu\beta[\beta]t \to \mu\alpha[\alpha]t\{\alpha/\beta\} = catch\ \alpha\ t\{\alpha/\beta\}\\ &throw\ \alpha\ throw\ \beta\ t = \mu\_[\alpha]\mu\_[\beta]t \to \mu\_[\beta]t\{\alpha/\_\} = throw\ \beta\ t\\ &throw\ \alpha\ catch\ \beta\ t = \mu\_[\alpha]\mu\beta[\beta]t \to \mu\_[\alpha]t\{\alpha/\beta\} = throw\ \alpha\ t\{\alpha/\beta\}\\ &catch\ \alpha\ throw\ \beta\ t = \mu\alpha[\alpha]\mu\_[\beta]t \to \mu\alpha[\beta]t\{\alpha/\_\} = \mu\alpha[\beta]t\end{aligned}$$

The first three cases yield three derived rules, but in the last case (as we already saw) the contractum is no more a $\lambda\mu ct$-term. Nevertheless, in the special case $\alpha = \beta$ we obtain the following derived rule:

$$catch\ \alpha\ throw\ \alpha\ t = \mu\alpha[\alpha]\mu\_[\alpha]t \to \mu\alpha[\alpha]t = catch\ \alpha\ t$$

Remark. The rule $catch\ \alpha\ throw\ \beta\ t \to \mu\alpha[\beta]t$ shows that the two "macros" *catch* and *throw* are enough to express all the $\lambda\mu$-terms up to renaming.

Let us summarize the derived rules we obtained above in the following definition (where **catch** and **throw** are now native operators):

Definition 2.1.1
We call $\lambda_{\mathbf{ct}}$-calculus the λ-calculus together with the operators **catch** and **throw** defined by the 8 following rules:

1. $(\lambda x.t\ u) \to t\{u/x\}$

2. $((\mathbf{catch}\ \alpha\ t)\ u) \to \mathbf{catch}\ \alpha\ (t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}\ u)$
3. $((\mathbf{throw}\ \alpha\ t)\ u) \to \mathbf{throw}\ \alpha\ t$
4. $\mathbf{catch}\ \alpha\ \mathbf{catch}\ \beta\ t \to \mathbf{catch}\ \alpha\ t\{\alpha/\beta\}$
5. $\mathbf{throw}\ \alpha\ \mathbf{throw}\ \beta\ t \to \mathbf{throw}\ \beta\ t$
6. $\mathbf{throw}\ \alpha\ \mathbf{catch}\ \beta\ t \to \mathbf{throw}\ \alpha\ t\{\alpha/\beta\}$
7. $\mathbf{catch}\ \alpha\ \mathbf{throw}\ \alpha\ t \to \mathbf{catch}\ \alpha\ t$
8. $\mathbf{catch}\ \alpha\ t \to t$ if α does not occur free in t.

Notation. The *structural substitution* $t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}$ is defined inductively by:

- $x\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\} = x$
- $(\lambda x.t)\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\} =$
  $\lambda x.t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}$
- $(s\ t)\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\} =$
  $(s\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}\ t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\})$
- $(\mathbf{catch}\ \beta\ t)\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\} =$
  $\mathbf{catch}\ \beta\ t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}$
- $(\mathbf{throw}\ \alpha\ t)\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\} =$
  $\mathbf{throw}\ \alpha\ (t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}\ u)$
- $(\mathbf{throw}\ \beta\ t)\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\} =$
  $\mathbf{throw}\ \beta\ t\{\mathbf{throw}\ \alpha\ (w\ u)/\mathbf{throw}\ \alpha\ w\}$ if $\alpha \neq \beta$.

Definition 2.1.2
Rules 4, 5, 6, 7 are called *renaming rules*. Rule 8 is called *simplification rule*. A *simple* $\lambda_{\mathbf{ct}}$*-term* is a term which contains no renaming/simplification redex.

Remark. As for $\lambda\mu$-terms, the simple form of a $\lambda_{\mathbf{ct}}$-term t (which is obtained from t by applying only renaming and simplification rules) is unique (modulo α-conversion) and is reached in a linear number of reduction steps. Two $\lambda_{\mathbf{ct}}$-terms are said to be *equal up to renaming/simplification* if they have the same simple form.

Notation. We denote by $\overline{t}$ the simple form of a $\lambda_{\mathbf{ct}}$-term t.

3 Canonical morphisms

The construction of the $\lambda_{\mathbf{ct}}$-calculus lets us expect the existence of some canonical translations that embedd each calculus into the other and which are morphisms for the reduction. We first formalize this notion of canonical translation. Then we show that the $\lambda_{\mathbf{ct}}$-calculus is a canonical retract of the $\lambda\mu$-calculus. This will enable us to derive the confluence of the $\lambda_{\mathbf{ct}}$-calculus from the confluence of the $\lambda\mu$-calculus. We also show that the converse is not true (the $\lambda\mu$-calculus is not a canonical retract of the $\lambda_{\mathbf{ct}}$-calculus) since there is no surjective canonical morphism from the $\lambda_{\mathbf{ct}}$-calculus to the $\lambda\mu$-calculus. Both calculi are however isomorphic if we consider terms up to renaming/simplification (and consequently up to convertibility).

To be more specific, we will define two canonical translations $\Lambda^{\iota}_{\mathbf{ct}}$ and $\Lambda^{s}_{\mathbf{ct}}$ of $\lambda\mu$-terms into $\lambda_{\mathbf{ct}}$-terms and two canonical translations Λ^{ι}_{μ} and Λ^{s}_{μ} of $\lambda_{\mathbf{ct}}$-terms into $\lambda\mu$-terms such that:

- $\Lambda^{s}_{\mathbf{ct}} \circ \Lambda^{\iota}_{\mu} = \mathrm{Id}_{\mathbf{ct}}$ and thus $\Lambda^{s}_{\mathbf{ct}}$ is surjective and Λ^{ι}_{μ} is injective.
- $\Lambda^{s}_{\mu} \circ \Lambda^{\iota}_{\mathbf{ct}} = \mathrm{Id}_{\mu}$ and thus Λ^{s}_{μ} is surjective and $\Lambda^{\iota}_{\mathbf{ct}}$ is injective.
- Λ^{ι}_{μ} and $\Lambda^{s}_{\mathbf{ct}}$ are morphisms and thus $\langle \Lambda^{\iota}_{\mu}, \Lambda^{s}_{\mathbf{ct}} \rangle$ is a retraction pair.
- $\Lambda^{\iota}_{\mathbf{ct}}$ is a morphism, but Λ^{s}_{μ} is not a morphism (since there is no surjective canonical morphism from the $\lambda_{\mathbf{ct}}$-calculus to the $\lambda\mu$-calculus).

3.1 Morphisms

We give here the formal definition of a morphism. As usual, for any relation $\rightarrow$ we denote by $\rightarrow^{\star}$ the reflexive, transitive closure of $\rightarrow$.

Definition 3.1.1
Given two calculus $\mathbf{c}_1$ and $\mathbf{c}_2$ and a mapping Φ from $\mathbf{c}_1$ to $\mathbf{c}_2$, we say that Φ is a morphism for $\rightarrow_{\mathbf{c_1}}$ iff for any terms t, u of $\mathbf{c}_1$:

$$t \rightarrow_{\mathbf{c_1}} u \quad \text{implies} \quad \Phi(t) \rightarrow^{\star}_{\mathbf{c_2}} \Phi(u)$$

Remarks

- A morphism for the reduction also preserves convertibility. In other words, if Φ is a morphism for $\rightarrow_{\mathbf{c_1}}$ and if $=_{\mathbf{c_i}}$ denotes the reflexive, symmetric, transitive closure of $\rightarrow_{\mathbf{c_i}}$:

$$t =_{\mathbf{c_1}} u \quad \text{implies} \quad \Phi(t) =_{\mathbf{c_2}} \Phi(u)$$

- A mapping Φ which preserves one-step reduction:

$$t \rightarrow_{\mathbf{c_1}} u \quad \text{implies} \quad \Phi(t) \rightarrow_{\mathbf{c_2}} \Phi(u)$$

  is of course a morphism according to the previous definition.
- If Φ is an injective morphism and the relation $\rightarrow_{\mathbf{c_1}}$ is irreflexive (i.e. $t \not\rightarrow_{\mathbf{c_1}} t$ for any t, which is usually the case for one-step reduction) then if $\rightarrow^{+}_{\mathbf{c_i}}$ denotes the transitive closure of $\rightarrow_{\mathbf{c_i}}$:

$$t \rightarrow_{\mathbf{c_1}} u \quad \text{implies} \quad \Phi(t) \rightarrow^{+}_{\mathbf{c_2}} \Phi(u)$$

3.2 Canonical translations

Let us notice that there is a very natural bijection between simple terms of both calculi (see proposition 3.2.3). A translation from one calculus into the other is thus said to be canonical if it extends this natural bijection. Conversely, we recover this natural bijection from any canonical translation when we consider terms equal up to renaming/simplification (in both calculi).

Definition 3.2.1
We define the mapping Λ^{s}_{μ} from $\lambda_{\mathbf{ct}}$-terms to λ_{μ}-terms by induction:

- $\Lambda^s_\mu(x) = x$, if x is a λ-variable,
- $\Lambda^s_\mu((u\ v)) = (\Lambda^s_\mu(u)\ \Lambda^s_\mu(v))$
- $\Lambda^s_\mu(\lambda x.t) = \lambda x.\Lambda^s_\mu(t)$
- $\Lambda^s_\mu(\mathbf{catch}\ \alpha\ t) = \begin{cases} \mu\alpha[\beta]\Lambda^s_\mu(u) & \text{if } t \text{ has the form } \mathbf{throw}\ \beta\ u \\ \mu\alpha[\alpha]\Lambda^s_\mu(t) & \text{otherwise} \end{cases}$
- $\Lambda^s_\mu(\mathbf{throw}\ \alpha\ t) = \mu\delta[\alpha]\Lambda^s_\mu(t)$ where δ is a fresh μ-variable.

Definition 3.2.2
We define the mapping $\Lambda^s_{\mathbf{ct}}$ from $\lambda\mu$-terms to $\lambda_{\mathbf{ct}}$-terms by induction:

- $\Lambda^s_{\mathbf{ct}}(x) = x$, if x is a λ-variable,
- $\Lambda^s_{\mathbf{ct}}((u\ v)) = (\Lambda^s_{\mathbf{ct}}(u)\ \Lambda^s_{\mathbf{ct}}(v))$
- $\Lambda^s_{\mathbf{ct}}(\lambda x.t) = \lambda x.\Lambda^s_{\mathbf{ct}}(t)$
- $\Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]t) = \begin{cases} \mathbf{catch}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t) & \text{if } \alpha = \beta \\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t) & \text{if } \alpha \neq \beta \text{ and } \alpha \text{ is not free in } t \\ \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t) & \text{otherwise} \end{cases}$

Remark. For any $\lambda_{\mathbf{ct}}$-term (resp λ_μ-term) t, the free λ-variables and μ-variables are the same in t and $\Lambda^s_{\mathbf{ct}}(t)$ (resp. $\Lambda^s_\mu(t)$).

Proposition 3.2.3
The mapping $\Lambda^s_{\mathbf{ct}}$ (resp. Λ^s_μ) is a bijection between simple forms of both calculi.

Proof
Check that if t is a simple $\lambda_{\mathbf{ct}}$-term then $\Lambda^s_{\mathbf{ct}}(\Lambda^s_\mu(t)) = t$ and conversely if t is a simple λ_μ-term then $\Lambda^s_\mu(\Lambda^s_{\mathbf{ct}}(t)) = t$. □

Remark. Notice that neither $\Lambda^s_{\mathbf{ct}}$ nor Λ^s_μ is a bijection (if we do not restrict the domain to simple terms). Indeed, $\Lambda^s_{\mathbf{ct}}$ is not injective since if t is a $\lambda\mu$-term such that $\alpha \neq \beta$ and α occurs free in t:

$$\Lambda^s_{\mathbf{ct}}(\mu\alpha[\alpha]\mu\_[\beta]t) = \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t) = \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]t)$$

Besides, Λ^s_μ is not injective since if t is a $\lambda_{\mathbf{ct}}$-term which has not the form $\mathbf{throw}\ \beta\ u$:

$$\Lambda^s_\mu(\mathbf{catch}\ \alpha\ \mathbf{throw}\ \alpha\ t) = (\mu\alpha[\alpha]\Lambda^s_\mu(t)) = \Lambda^s_\mu(\mathbf{catch}\ \alpha\ t)$$

Definition 3.2.4
A translation Ψ from the $\lambda_{\mathbf{ct}}$-calculus into the λ_μ-calculus is said to be *canonical* if for any $\lambda_{\mathbf{ct}}$-term t, $\overline{\Psi(t)} = \Lambda^s_\mu(\overline{t})$. Conversely, A translation Φ from the λ_μ-calculus into the $\lambda_{\mathbf{ct}}$-calculus is said to be *canonical* if for any $\lambda\mu$-term t, $\overline{\Phi(t)} = \Lambda^s_{\mathbf{ct}}(\overline{t})$.

Remark. A canonical translation is thus a translation which maps sequences of control operators of one calculus onto sequences of control operators of the other calculus and leaves the rest unchanged. In particular, a term without control operator is translated into itself.

Proposition 3.2.5
The translations Λ^s_μ and $\Lambda^s_{\mathbf{ct}}$ are canonical.

Proof
Check by induction on the $\lambda_{\mathbf{ct}}$-term (resp. $\lambda\mu$-term) t, $\overline{\Lambda^s_\mu(t)} = \Lambda^s_\mu(\overline{t})$ (resp. $\overline{\Lambda^s_{\mathbf{ct}}(t)} = \Lambda^s_{\mathbf{ct}}(\overline{t})$). □

3.3 Injective canonical morphisms

In this section we define two injective canonical morphisms $\Lambda^\iota_{\mathbf{ct}}$ and Λ^ι_μ from one calculus into the other. Since these translations are injective, they will allow us to derive the strong normalization of one calculus from the strong normalization of the other calculus (see section 4).

From the $\lambda_{\mathbf{ct}}$-calculus towards the $\lambda\mu$-calculus

Definition 3.3.1
We define the mapping Λ^ι_μ from $\lambda_{\mathbf{ct}}$-terms to $\lambda\mu$-terms by induction:

- $\Lambda^\iota_\mu(x) = x$, if x is a λ-variable,
- $\Lambda^\iota_\mu((u\ v)) = (\Lambda^\iota_\mu(u)\ \Lambda^\iota_\mu(v))$
- $\Lambda^\iota_\mu(\lambda x.t) = \lambda x.\Lambda^\iota_\mu(t)$
- $\Lambda^\iota_\mu(\mathbf{catch}\ \alpha\ t) = \mu\alpha[\alpha]\Lambda^\iota_\mu(t)$
- $\Lambda^\iota_\mu(\mathbf{throw}\ \alpha\ t) = \mu\delta[\alpha]\Lambda^\iota_\mu(t)$ where δ is a fresh μ-variable.

Remark. For any $\lambda_{\mathbf{ct}}$-terms t, $\Lambda^\iota_\mu(t)$ is a $\lambda\mu ct$-term.

Proposition 3.3.2
The mapping Λ^ι_μ is canonical.

Proof
Check by induction on the $\lambda_{\mathbf{ct}}$-term t that $\overline{\Lambda^\iota_\mu(t)} = \Lambda^s_\mu(\overline{t})$. □

We easily show that Λ^ι_μ is a morphism for the reduction.

Proposition 3.3.3
For any $\lambda_{\mathbf{ct}}$-terms t, t', if $t \to_{\mathbf{ct}} t'$ then $\Lambda^\iota_\mu(t) \to_{\lambda\mu} \Lambda^\iota_\mu(t')$.

Proof
By construction, since for any $\lambda_{\mathbf{ct}}$-term t, $\Lambda^\iota_\mu(t)$ is a $\lambda\mu ct$-term and any reduction rule of the $\lambda_{\mathbf{ct}}$-calculus comes from some reduction rule of the $\lambda\mu$-calculus. □

From the $\lambda\mu$-calculus towards the $\lambda_{\mathbf{ct}}$-calculus

Definition 3.3.4
We define the mapping $\Lambda^\iota_{\mathbf{ct}}$ from $\lambda\mu$-terms to $\lambda_{\mathbf{ct}}$-terms by induction:

- $\Lambda^\iota_{\mathbf{ct}}(x) = x$, if x is a λ-variable,
- $\Lambda^\iota_{\mathbf{ct}}((u\ v)) = (\Lambda^\iota_{\mathbf{ct}}(u)\ \Lambda^\iota_{\mathbf{ct}}(v))$
- $\Lambda^\iota_{\mathbf{ct}}(\lambda x.t) = \lambda x.\Lambda^\iota_{\mathbf{ct}}(t)$
- $\Lambda^\iota_{\mathbf{ct}}(\mu\alpha[\beta]t) = \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^\iota_{\mathbf{ct}}(t)$

Proposition 3.3.5
The mapping $\Lambda^{\iota}_{\mathbf{ct}}$ is canonical.

Proof
Check by induction on the $\lambda\mu$-term t that $\overline{\Lambda^{\iota}_{\mathbf{ct}}(t)} = \Lambda^{s}_{\mathbf{ct}}(\overline{t})$. □

We will now show that $\Lambda^{\iota}_{\mathbf{ct}}$ is a morphism for the reduction.

Lemma 3.3.6
For any $\lambda\mu$-terms u, v and any variable x free in u:

$$\Lambda^{\iota}_{\mathbf{ct}}(u\{v/x\}) = \Lambda^{\iota}_{\mathbf{ct}}(u)\{\Lambda^{\iota}_{\mathbf{ct}}(v)/x\}$$

Proof
By induction on the term u. □

Lemma 3.3.7
For any instance $u \to v$ of a rule of the $\lambda\mu$-calculus we have $\Lambda^{\iota}_{\mathbf{ct}}(u) \to^{\star}_{\mathbf{ct}} \Lambda^{\iota}_{\mathbf{ct}}(v)$.

Proof
We consider each rule of the $\lambda\mu$-calculus:

- Case of the β-reduction $(\lambda x.u\ v) \to_{\beta} u\{v/x\}$:

  $$\Lambda^{\iota}_{\mathbf{ct}}((\lambda x.u\ v)) = (\lambda x.\Lambda^{\iota}_{\mathbf{ct}}(u)\ \Lambda^{\iota}_{\mathbf{ct}}(v)) \to_{\beta} \Lambda^{\iota}_{\mathbf{ct}}(u)\{\Lambda^{\iota}_{\mathbf{ct}}(v)/x\} = \Lambda^{\iota}_{\mathbf{ct}}(u\{v/x\})$$

  by the substitution lemma.
- Case of the rule $(\mu\alpha.u\ v) \to \mu\alpha.u\big\{[\alpha](t\ v)/[\alpha]t\big\}$

  $$\begin{array}{rcl} \Lambda^{\iota}_{\mathbf{ct}}(\mu\alpha.u\ v) & & \\ & = & (\mathbf{catch}\ \alpha\ \Lambda^{\iota}_{\mathbf{ct}}(u)\ \Lambda^{\iota}_{\mathbf{ct}}(v)) \\ & \to_2 & \mathbf{catch}\ \alpha\ (\Lambda^{\iota}_{\mathbf{ct}}(u)\big\{\mathbf{throw}\ \alpha\ (t\ \Lambda^{\iota}_{\mathbf{ct}}(v))/\mathbf{throw}\ \alpha\ t\big\}\ \Lambda^{\iota}_{\mathbf{ct}}(v)) \\ & = & \Lambda^{\iota}_{\mathbf{ct}}(\mu\alpha.u\big\{[\alpha](t\ v)/[\alpha]t\big\}) \end{array}$$

- Case of the rule $[\beta]\mu\alpha t \to t\{\beta/\alpha\}$, i.e. $\mu\gamma[\beta]\mu\alpha[\delta]t \to \mu\gamma([\delta]t)\{\beta/\alpha\}$

  $$\begin{array}{rcl} \Lambda^{\iota}_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\delta]t) & = & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \mathbf{catch}\ \alpha\ \mathbf{throw}\ \delta\ \Lambda^{\iota}_{\mathbf{ct}}(t) \\ & \to_6 & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ (\mathbf{throw}\ \delta\ \Lambda^{\iota}_{\mathbf{ct}}(t))\{\beta/\alpha\} \\ & \to_5 & \mathbf{catch}\ \gamma\ (\mathbf{throw}\ \delta\ \Lambda^{\iota}_{\mathbf{ct}}(t))\{\beta/\alpha\} \\ & = & \Lambda^{\iota}_{\mathbf{ct}}(\mu\gamma([\delta]t)\{\beta/\alpha\}) \end{array}$$

- Case of the rule $\mu\alpha[\alpha]t \to t$ if α does not occur free in t.

  $$\begin{array}{rcl} \Lambda^{\iota}_{\mathbf{ct}}(\mu\alpha[\alpha]t) & = & \mathbf{catch}\ \alpha\ \mathbf{throw}\ \alpha\ \Lambda^{\iota}_{\mathbf{ct}}(t) \\ & \to_7 & \mathbf{catch}\ \alpha\ \Lambda^{\iota}_{\mathbf{ct}}(t) \\ & \to_8 & \Lambda^{\iota}_{\mathbf{ct}}(t) \end{array}$$

□

In the sequel, we will need the usual concept of context (*i.e.* a term with a *hole*). Given a context, denoted by $C[\bullet]$, the notation $C[t]$ stands for the term obtained by replacing in $C[\bullet]$ the symbol $\bullet$ (the hole) by the term t. Let us now prove the following lemma:

Lemma 3.3.8

For any context $C[\bullet]$ and any $\lambda\mu$-terms u, v, if $\Lambda^{\iota}_{\mathbf{ct}}(u) \rightarrow_{\mathbf{ct}} \Lambda^{\iota}_{\mathbf{ct}}(v)$ then:

$$\Lambda^{\iota}_{\mathbf{ct}}(C[u]) \rightarrow_{\mathbf{ct}} \Lambda^{\iota}_{\mathbf{ct}}(C[v])$$

Proof
If we extend $\Lambda^{\iota}_{\mathbf{ct}}$ to $\lambda\mu$-contexts by taking $\Lambda^{\iota}_{\mathbf{ct}}(\bullet) = \bullet$, we easily prove that for any $\lambda\mu$-context C and any $\lambda\mu$-term t, we have $\Lambda^{\iota}_{\mathbf{ct}}(C[t]) = \Lambda^{\iota}_{\mathbf{ct}}(C)\{\Lambda^{\iota}_{\mathbf{ct}}(t)\}$. □

Proposition 3.3.9
For any $\lambda\mu$-terms u, v, if $u \rightarrow_{\lambda\mu} v$ then $\Lambda^{\iota}_{\mathbf{ct}}(u) \rightarrow^{\star}_{\mathbf{ct}} \Lambda^{\iota}_{\mathbf{ct}}(v)$.

Proof
By lemma 3.3.7 and lemma 3.3.8. □

To round off this section, let us prove two useful lemmas:

Lemma 3.3.10
$\Lambda^{s}_{\mathbf{ct}} \circ \Lambda^{\iota}_{\mu} = \mathrm{Id}_{\mathbf{ct}}$.

Proof
Let us prove that $\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(t)) = t$ by induction on the $\lambda_{\mathbf{ct}}$-term t:

- $\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(x)) = \Lambda^{s}_{\mathbf{ct}}(x) = x$, if x is a λ-variable,
- $\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}((u\ v))) = \Lambda^{s}_{\mathbf{ct}}((\Lambda^{\iota}_{\mu}(u)\ \Lambda^{\iota}_{\mu}(v)) = (\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(u))\ \Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(v))) = (u\ v)$
- $\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(\lambda x.t)) = \Lambda^{s}_{\mathbf{ct}}(\lambda x.\Lambda^{\iota}_{\mu}(t)) = \lambda x.\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(t)) = \lambda x.t$
- $\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(\mathbf{catch}\ \alpha\ t)) = \Lambda^{s}_{\mathbf{ct}}(\mu\alpha[\alpha]\Lambda^{\iota}_{\mu}(t)) = \mathbf{catch}\ \alpha\ \Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(t)) = \mathbf{catch}\ \alpha\ t$
- $\Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(\mathbf{throw}\ \alpha\ t)) = \Lambda^{s}_{\mathbf{ct}}(\mu\delta[\alpha]\Lambda^{\iota}_{\mu}(t))$ where δ is a fresh μ-variable
  $= \mathbf{throw}\ \alpha\ \Lambda^{s}_{\mathbf{ct}}(\Lambda^{\iota}_{\mu}(t))$ since δ does not occur in $\Lambda^{\iota}_{\mu}(t)$
  $= \mathbf{throw}\ \alpha\ t$

□

Remark. The translation $\Lambda^{s}_{\mathbf{ct}}$ is thus surjective, and the translation Λ^{ι}_{μ} is injective. However, Λ^{ι}_{μ} is clearly not surjective since Λ^{ι}_{μ} maps any $\lambda_{\mathbf{ct}}$-term to some $\lambda\mu ct$-term. We already saw that $\Lambda^{s}_{\mathbf{ct}}$ is not injective.

Lemma 3.3.11
$\Lambda^{s}_{\mu} \circ \Lambda^{\iota}_{\mathbf{ct}} = \mathrm{Id}_{\mu}$.

Proof
Let us prove that $\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(t)) = t$ by induction on the $\lambda\mu$-term t:

- $\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(x)) = \Lambda^{s}_{\mu}(x) = x$, if x is a λ-variable,
- $\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}((u\ v))) = \Lambda^{s}_{\mu}((\Lambda^{\iota}_{\mathbf{ct}}(u)\ \Lambda^{\iota}_{\mathbf{ct}}(v)) = (\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(u))\ \Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(v))) = (u\ v)$
- $\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(\lambda x.t)) = \Lambda^{s}_{\mu}(\lambda x.\Lambda^{\iota}_{\mathbf{ct}}(t)) = \lambda x.\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(t)) = \lambda x.t$
- $\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(\mu\alpha[\beta]t)) = \Lambda^{s}_{\mu}(\mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^{\iota}_{\mathbf{ct}}(t)) = \mu\alpha[\beta]\Lambda^{s}_{\mu}(\Lambda^{\iota}_{\mathbf{ct}}(t)) = \mu\alpha[\beta]t$

□

Remark. The translation Λ^{s}_{μ} is thus surjective, and the translation $\Lambda^{\iota}_{\mathbf{ct}}$ is injective. However, $\Lambda^{\iota}_{\mathbf{ct}}$ is clearly not surjective since $\Lambda^{\iota}_{\mathbf{ct}}$ maps any $\lambda\mu$-term to some $\lambda_{\mathbf{ct}}$-term in which any **catch** is followed by a **throw**. We already saw that Λ^{s}_{μ} is not injective.

3.4 The $\lambda_{\mathbf{ct}}$-calculus is a canonical retract of the $\lambda\mu$-calculus

In this section we show that $\Lambda^s_{\mathbf{ct}}$ is a morphism and thus $\langle \Lambda^\iota_\mu, \Lambda^s_{\mathbf{ct}} \rangle$ is a retraction pair (the $\lambda_{\mathbf{ct}}$-calculus is a canonical retract of the $\lambda\mu$-calculus).

Lemma 3.4.1
For any $\lambda\mu$-terms u, v and any variable x free in u:

$$\Lambda^s_{\mathbf{ct}}(u\{v/x\}) = \Lambda^s_{\mathbf{ct}}(u)\{\Lambda^s_{\mathbf{ct}}(v)/x\}$$

Proof
By induction on the $\lambda\mu$-term u. □

Lemma 3.4.2
For any instance $u \to v$ of any rule of the $\lambda\mu$-calculus we have $\Lambda^s_{\mathbf{ct}}(u) \to^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(v)$.

Proof
We consider each rule of the $\lambda\mu$-calculus:

- Case of the β-reduction $(\lambda x.u\ v) \to_\beta u\{v/x\}$:

  $$\Lambda^s_{\mathbf{ct}}((\lambda x.u\ v)) = (\lambda x.\Lambda^s_{\mathbf{ct}}(u)\ \Lambda^s_{\mathbf{ct}}(v)) \to_\beta \Lambda^s_{\mathbf{ct}}(u)\{\Lambda^s_{\mathbf{ct}}(v)/x\} = \Lambda^s_{\mathbf{ct}}(u\{v/x\})$$

  by the substitution lemma.
- Case of the rule $(\mu\alpha[\beta]u\ v) \to \mu\alpha([\beta]u)\big\{[\alpha](t\ v)/[\alpha]t\big\}$
  1. If $\alpha = \beta$ then $\mu\alpha([\beta]u)\big\{[\alpha](t\ v)/[\alpha]t\big\} = \mu\alpha[\alpha](u\big\{[\alpha](t\ v)/[\alpha]t\big\}\ v)$,

     $$\begin{array}{lll} & \Lambda^s_{\mathbf{ct}}(\mu\alpha[\alpha]u\ v) & \\ = & (\mathbf{catch}\ \alpha\ \Lambda^s_{\mathbf{ct}}(u)\ \Lambda^s_{\mathbf{ct}}(v)) & \\ \to_2 & \mathbf{catch}\ \alpha\ (\Lambda^s_{\mathbf{ct}}(u)\big\{\mathbf{throw}\ \alpha\ (t\ \Lambda^s_{\mathbf{ct}}(v))/\mathbf{throw}\ \alpha\ t\big\}\ \Lambda^s_{\mathbf{ct}}(v)) & \\ = & \Lambda^s_{\mathbf{ct}}(\mu\alpha[\alpha](u\{[\alpha](t\ v)/[\alpha]t\}\ v)) & \end{array}$$

     since any occurrence of a subterm $\mu\beta[\alpha]t$ (for some $\beta \neq \alpha$) is translated, by definition of $\Lambda^s_{\mathbf{ct}}$, either into $\mathbf{catch}\ \beta\ \mathbf{throw}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t)$ or into $\mathbf{throw}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t)$.
  2. If $\alpha \neq \beta$ and α does not occur in u, $\mu\alpha([\beta]u)\{[\alpha](t\ v)/[\alpha]t\}) = \mu\alpha[\beta]u$,

     $$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]u\ v) & = & ((\mathbf{throw}\ \alpha\ \Lambda^s_{\mathbf{ct}}(u))\ \Lambda^s_{\mathbf{ct}}(v)) \\ & \to_3 & \mathbf{throw}\ \alpha\ \Lambda^s_{\mathbf{ct}}(u) \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]u) \end{array}$$

  3. If $\alpha \neq \beta$ and α occurs in u, $\mu\alpha([\beta]u)\big\{[\alpha](t\ v)/[\alpha]t\big\} = \mu\alpha[\beta]u\big\{[\alpha](t\ v)/[\alpha]t\big\}$,

     $$\begin{array}{lll} & \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]u\ v) & \\ = & ((\mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(u))\ \Lambda^s_{\mathbf{ct}}(v)) & \\ \to_2 & \mathbf{catch}\ \alpha\ ((\mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(u))\big\{\mathbf{throw}\ \alpha\ (t\ \Lambda^s_{\mathbf{ct}}(v))/\mathbf{throw}\ \alpha\ t\big\}\ \Lambda^s_{\mathbf{ct}}(v)) & \\ = & \mathbf{catch}\ \alpha\ (\mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(u)\big\{\mathbf{throw}\ \alpha\ (t\ \Lambda^s_{\mathbf{ct}}(v))/\mathbf{throw}\ \alpha\ t\big\}\ \Lambda^s_{\mathbf{ct}}(v)) & \\ \to_3 & \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(u)\big\{\mathbf{throw}\ \alpha\ (t\ \Lambda^s_{\mathbf{ct}}(v))/\mathbf{throw}\ \alpha\big\} & \\ = & \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]u\{[\alpha](t\ v)/[\alpha]t\}) & \end{array}$$

     Again, since any occurrence of a subterm $\mu\beta[\alpha]t$ (for some $\beta \neq \alpha$) is translated, by definition of $\Lambda^s_{\mathbf{ct}}$, either into $\mathbf{catch}\ \beta\ \mathbf{throw}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t)$ or into $\mathbf{throw}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t)$.

- Case of the rule $[\beta]\mu\alpha t \to t\{\beta/\alpha\}$, i.e. $\mu\gamma[\beta]\mu\alpha[\delta]t \to \mu\gamma([\delta]t)\{\beta/\alpha\}$
  1. If $\gamma = \beta$ and $\alpha = \delta$ then $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\beta[\beta]t\{\beta/\alpha\}$

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]\mu\alpha[\alpha]t) & = & \mathbf{catch}\ \beta\ \mathbf{catch}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t)\\ & \to_4 & \mathbf{catch}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\}\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]t\{\beta/\alpha\})\end{array}$$

  2. If $\gamma = \beta$ and α thus does not occur in $[\delta]t$, $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\beta[\delta]t$
     (a) If $\beta = \delta$ then $\mu\beta[\delta]t = \mu\beta[\beta]t$

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]\mu\alpha[\beta]t) & = & \mathbf{catch}\ \beta\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\\ & \to_7 & \mathbf{catch}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]t)\end{array}$$

     (b) If $\beta \neq \delta$ and β does not occur in t

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]\mu\alpha[\delta]t) & = & \mathbf{catch}\ \beta\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\\ & \to_8 & \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\beta[\delta]t)\end{array}$$

     (c) If $\beta \neq \delta$ and β occurs in t

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]\mu\alpha[\delta]t) & = & \mathbf{catch}\ \beta\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\beta[\delta]t)\end{array}$$

  3. If $\gamma = \beta$ and α occurs then in $[\delta]t\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\beta([\delta]t)\{\beta/\alpha\}$
     (a) If $\alpha = \delta$, this case has already been delt with.
     (b) If $\alpha \neq \delta$ then $\mu\beta([\delta]t)\big\{\beta/\alpha\big\} = \mu\beta[\delta]t\big\{\beta/\alpha\big\}$
     — If $\beta = \delta$ then $\mu\beta[\delta]t\big\{\beta/\alpha\big\} = \mu\beta[\beta]t\big\{\beta/\alpha\big\}$

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]\mu\alpha[\beta]t) & = & \mathbf{catch}\ \beta\ \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\\ & \to_4 & \mathbf{catch}\ \beta\ (\mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t))\{\beta/\alpha\}\\ & = & \mathbf{catch}\ \beta\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\}\\ & \to_7 & \mathbf{catch}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\}\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]t\{\beta/\alpha\})\end{array}$$

     — If $\beta \neq \delta$ then

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\beta[\beta]\mu\alpha[\delta]t) & = & \mathbf{catch}\ \beta\ \mathbf{catch}\ \alpha\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\\ & \to_4 & \mathbf{catch}\ \beta\ (\mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t))\{\beta/\alpha\}\\ & = & \mathbf{catch}\ \beta\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\}\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\beta[\delta]t\{\beta/\alpha\})\end{array}$$

  4. If $\gamma \neq \beta$ and γ does not occur in $\mu\alpha[\delta]t$ (then in particular $\gamma \neq \delta$)
     (a) If $\alpha = \delta$ then $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\gamma[\beta]t\{\beta/\alpha\}$

$$\begin{array}{rcl}\Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\alpha]t) & = & \mathbf{throw}\ \beta\ \mathbf{catch}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t)\\ & \to_6 & \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\}\\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]t\{\beta/\alpha\})\end{array}$$

     since $\gamma \neq \beta$ in the last equality.

(b) If $\alpha \neq \delta$ and α does not occur in t (and thus $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\gamma[\delta]t$)

$$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\delta]t) & = & \mathbf{throw}\ \beta\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_5 & \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t) \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\delta]t) \end{array}$$

since $\gamma \neq \delta$ in the last equality.

(c) If $\alpha \neq \delta$ and α occurs in t (and thus $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\gamma[\delta]t\{\beta/\alpha\}$)

$$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\delta]t) & = & \mathbf{throw}\ \beta\ \mathbf{catch}\ \alpha\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_6 & \mathbf{throw}\ \beta\ (\mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t))\{\beta/\alpha\} \\ & = & \mathbf{throw}\ \beta\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ & \to_5 & \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\delta]t\{\beta/\alpha\}) \end{array}$$

5. If $\gamma \neq \beta$ and γ occurs in $\mu\alpha[\delta]t$

(a) If $\alpha = \delta$ then $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\gamma[\beta]t\{\beta/\alpha\}$

$$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\alpha]t) & = & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \mathbf{catch}\ \alpha\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_6 & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]t\{\beta/\alpha\}) \end{array}$$

(b) If $\alpha \neq \delta$ and α does not occur in t (and thus $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\gamma[\delta]t$)

— If $\gamma = \delta$ then $\mu\gamma[\delta]t = \mu\gamma[\gamma]t$

$$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\gamma]t) & = & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \mathbf{throw}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_5 & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_7 & \mathbf{catch}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t) \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\gamma]t) \end{array}$$

— If $\gamma \neq \delta$

$$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\delta]t) & = & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_5 & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \delta\ \Lambda^s_{\mathbf{ct}}(t) \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\delta]t) \end{array}$$

(c) If $\alpha \neq \delta$ and α occurs in t (and thus $\mu\gamma([\delta]t)\{\beta/\alpha\} = \mu\gamma[\delta]t\{\beta/\alpha\}$)

— If $\gamma = \delta$ then $\mu\gamma[\delta]t\{\beta/\alpha\} = \mu\gamma[\gamma]t\{\beta/\alpha\}$

$$\begin{array}{l} \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\gamma]t) \\ \quad\begin{array}{rcl} & = & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \mathbf{catch}\ \alpha\ \mathbf{throw}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t) \\ & \to_6 & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ (\mathbf{throw}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t))\{\beta/\alpha\} \\ & = & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \beta\ \mathbf{throw}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ & \to_5 & \mathbf{catch}\ \gamma\ \mathbf{throw}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ & \to_7 & \mathbf{catch}\ \gamma\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\gamma]t\{\beta/\alpha\}) \end{array} \end{array}$$

— If $\gamma \neq \delta$ then $\mu\gamma[\gamma]t\{\beta/\alpha\}$

$$\begin{array}{rl} & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\beta]\mu\alpha[\delta]t) \\ = & \textbf{catch } \gamma \textbf{ throw } \beta \textbf{ catch } \alpha \textbf{ throw } \delta\ \Lambda^s_{\mathbf{ct}}(t) \\ \rightarrow_6 & \textbf{catch } \gamma \textbf{ throw } \beta\ (\textbf{throw } \delta\ \Lambda^s_{\mathbf{ct}}(t))\{\beta/\alpha\} \\ = & \textbf{catch } \gamma \textbf{ throw } \beta \textbf{ throw } \delta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ \rightarrow_5 & \textbf{catch } \gamma \textbf{ throw } \delta\ \Lambda^s_{\mathbf{ct}}(t)\{\beta/\alpha\} \\ = & \Lambda^s_{\mathbf{ct}}(\mu\gamma[\delta]t\{\beta/\alpha\}) \end{array}$$

- Case of the rule $\mu\alpha[\alpha]t \rightarrow t$ if α does not occur free in t.

$$\Lambda^s_{\mathbf{ct}}(\mu\alpha[\alpha]t) = \textbf{catch } \alpha\ \Lambda^s_{\mathbf{ct}}(t) \rightarrow_8 \Lambda^s_{\mathbf{ct}}(t)$$

□

Lemma 3.4.3
For any $\lambda\mu$-context $C[\bullet]$ and any $\lambda\mu$-terms u, v, if $\Lambda^s_{\mathbf{ct}}(u) \rightarrow^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(v)$ then:

$$\Lambda^s_{\mathbf{ct}}(C[u]) \rightarrow^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(C[v])$$

Proof
By induction on the context:

- If the context is $\bullet$, $\Lambda^s_{\mathbf{ct}}(C[u]) = \Lambda^s_{\mathbf{ct}}(u) \rightarrow^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(v) = \Lambda^s_{\mathbf{ct}}(C[v])$.
- If the context is an application or an abstraction, just apply the induction hypothesis.
- If the context has the form $\mu\alpha[\beta]C[\bullet]$ then, by definition of the translation $\Lambda^s_{\mathbf{ct}}$, on the one hand,

$$\Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[u]) = \begin{cases} \textbf{catch } \alpha\ \Lambda^s_{\mathbf{ct}}(C[u]) & \textit{if } \alpha = \beta \\ \textbf{throw } \beta\ \Lambda^s_{\mathbf{ct}}(C[u]) & \textit{if } \alpha \neq \beta \textit{ and } \alpha \textit{ is not free in } C[u] \\ \textbf{catch } \alpha \textbf{ throw } \beta\ \Lambda^s_{\mathbf{ct}}(C[u]) & \textit{otherwise} \end{cases}$$

  and on the other hand,

$$\Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[v]) = \begin{cases} \textbf{catch } \alpha\ \Lambda^s_{\mathbf{ct}}(C[v]) & \textit{if } \alpha = \beta \\ \textbf{throw } \beta\ \Lambda^s_{\mathbf{ct}}(C[v]) & \textit{if } \alpha \neq \beta \textit{ and } \alpha \textit{ is not free in } C[v] \\ \textbf{catch } \alpha \textbf{ throw } \beta\ \Lambda^s_{\mathbf{ct}}(C[v]) & \textit{otherwise} \end{cases}$$

  By induction hypothesis $\Lambda^s_{\mathbf{ct}}(C[u]) \rightarrow^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(C[v])$, and then we consider each case:

  1. If $\alpha = \beta$, by applying the induction assumption:

$$\begin{array}{l} \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[u]) = \textbf{catch } \alpha\ \Lambda^s_{\mathbf{ct}}(C[u]) \\ \qquad\qquad \rightarrow^\star_{\mathbf{ct}} \textbf{catch } \alpha\ \Lambda^s_{\mathbf{ct}}(C[v]) = \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[v]) \end{array}$$

  2. If $\alpha \neq \beta$ and α does not occur free in $C[u]$, then α cannot occur free in $C[v]$ because no reduction of the $\lambda\mu$-calculus can introduce a free μ-variable, hence by applying the induction hypothesis:

$$\begin{array}{l} \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[u]) = \textbf{throw } \beta\ \Lambda^s_{\mathbf{ct}}(C[u]) \\ \qquad\qquad \rightarrow^\star_{\mathbf{ct}} \textbf{throw } \beta\ \Lambda^s_{\mathbf{ct}}(C[v]) = \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[v]) \end{array}$$

3. If $\alpha \neq \beta$ and α occurs free in $C[u]$, and α do not occur free then in $C[v]$ by applying the induction hypothesis and then rule (8):

$$\begin{array}{rcl} \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[u]) & = & \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(C[u]) \\ & \rightarrow^\star_{\mathbf{ct}} & \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(C[v]) \\ & \rightarrow_1 & \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(C[v]) \\ & = & \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[v]) \end{array}$$

4. If $\alpha \neq \beta$ and α occurs free in $C[u]$ and in $C[v]$ by applying the induction hypothesis:

$$\begin{array}{l} \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[u]) = \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(C[u]) \\ \qquad \rightarrow^\star_{\mathbf{ct}} \mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(C[v]) = \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]C[v]) \end{array}$$

□

Proposition 3.4.4
For any $\lambda\mu$-terms u, v, if $u \rightarrow_{\lambda\mu} v$ then $\Lambda^s_{\mathbf{ct}}(u) \rightarrow^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(v)$.

Proof
By lemma 3.4.1 and lemma 3.4.3. □

Remark. The morphism $\Lambda^s_{\mathbf{ct}}$ does not map any reduction step onto at least one reduction step since $\mu\alpha[\alpha]\mu\_[\beta]t \rightarrow \mu\alpha[\beta]t$ while $\Lambda^s_{\mathbf{ct}}(\mu\alpha[\alpha]\mu\_[\beta]t) = \Lambda^s_{\mathbf{ct}}(\mu\alpha[\beta]t)$.

Corollary 3.4.5
The $\lambda_{\mathbf{ct}}$-calculus and the $\lambda\mu$-calculus are isomorphic if we consider terms up renaming/simplification.

Proof
Indeed, for any $\lambda_{\mathbf{ct}}$-term t, we have:

$$\overline{\lambda_{\mathbf{ct}}(\Lambda^\iota_\mu(t))} = \lambda_{\mathbf{ct}}(\overline{\Lambda^\iota_\mu(t)}) = \lambda_{\mathbf{ct}}(\Lambda^s_\mu(\overline{t})) = \overline{t}$$

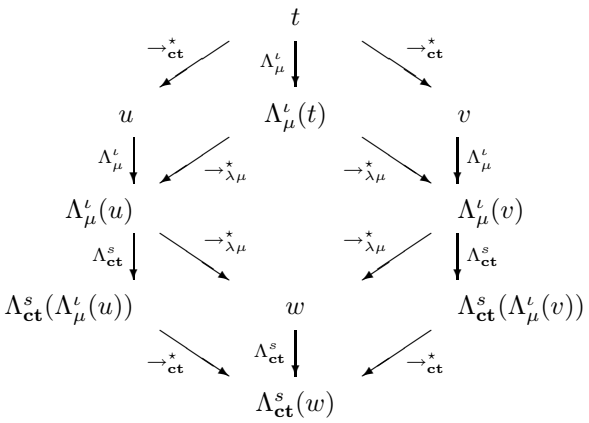
since $\lambda_{\mathbf{ct}}$ and Λ^ι_μ are canonical. □

Corollary 3.4.6
The $\lambda_{\mathbf{ct}}$-calculus and the $\lambda\mu$-calculus are isomorphic if we consider terms up convertibility.

Proof
Since $\overline{\lambda_{\mathbf{ct}}(\Lambda^\iota_\mu(t))} = \overline{t}$ implies $\lambda_{\mathbf{ct}}(\Lambda^\iota_\mu(t)) =_{\mathbf{ct}} t$. □

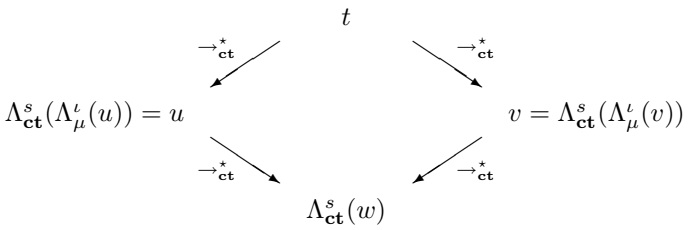
3.5 Confluence of the $\lambda_{\mathbf{ct}}$-calculus

We are now able to show the confluence of the $\lambda_{\mathbf{ct}}$-calculus. Indeed, let us consider following diagram (where w exists since the $\lambda\mu$-calculus is confluent):



Theorem 3.5.1
The $\lambda_{\mathbf{ct}}$-calculus is confluent.

Proof
We have shown that for any $\lambda_{\mathbf{ct}}$-term t, $t = \Lambda^s_{\mathbf{ct}}(\Lambda^\iota_\mu(t))$, (only the weaker property $t \to^\star_{\mathbf{ct}} \Lambda^s_{\mathbf{ct}}(\Lambda^\iota_\mu(t))$ was actually needed) and the confluence of the $\lambda_{\mathbf{ct}}$-calculus results from following diagram:



□

3.6 The $\lambda\mu$-calculus is not a canonical retract of the $\lambda_{\mathbf{ct}}$-calculus

Proposition 3.6.1
There is no surjective canonical morphism from the $\lambda_{\mathbf{ct}}$-calculus to the $\lambda\mu$-calculus.

Proof
Let us assume that Ψ is a surjective canonical morphism from the $\lambda_{\mathbf{ct}}$-calculus to the $\lambda\mu$-calculus. Let r be a simple $\lambda\mu$-term of the form $(\mu\alpha[\beta]u\ v)$ where $\alpha \neq \beta$ and β occurs free in u. Since Ψ is surjective, there is a $\lambda_{\mathbf{ct}}$-term s such that $\Psi(s) = r$. Since Ψ is canonical, $\overline{\Psi(s)} = \Lambda^s_\mu(\overline{s})$. But $\overline{\Psi(s)} = \overline{r} = r$ since r is simple. Thus the simple $\lambda_{\mathbf{ct}}$-term $\overline{s} = \Lambda^s_{\mathbf{ct}}(r)$ is $((\mathbf{catch}\ \alpha\ \mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(u)\ \Lambda^s_{\mathbf{ct}}(v))$. Now let c be the contractum of $\overline{s}$:

$$c = \mathbf{catch}\ \alpha\ (\mathbf{throw}\ \beta\ \Lambda^s_{\mathbf{ct}}(u)\{\mathbf{throw}\ \alpha\ (t\ \Lambda^s_{\mathbf{ct}}(v))/\mathbf{throw}\ \alpha\ t\}\ \Lambda^s_{\mathbf{ct}}(v))$$

Since Ψ is a morphism, we should have $\Psi(\overline{s}) \to^\star \Psi(c)$ and thus $\overline{\Psi(\overline{s})} \to^\star \overline{\Psi(c)}$ (since the renaming/simplification rules commute with any other rule). Finally, notice that

$r = \overline{\Psi(s)} = \overline{\Psi(\overline{s})}$ but we do not have in general $r \to^\star \mu\alpha[\alpha](\mu\delta[\beta]u\{[\alpha](t\ v)/t\}\ v)$ (for instance if the contractum of r which is $\mu\alpha[\beta]u\{[\alpha](t\ v)/t\}$ is a normal form). Whence the contradiction. □

4 The typed $\lambda_{\mathbf{ct}}$-calculus

Typing control operators is strongly related to classical logic. This striking fact has been first noticed by T. G. Griffin (1990) and has been widely investigated since, for instance, by C. R. Murthy (1990; 1991), F. Barbanera and S. Berardi (1994a; 1994b), N. J. Rehof and M. H. Sørensen (1994), P. De Groote (1995; 1994a), J.-L. Krivine (1994), H. Nakano, (1994b; 1994a; 1995), M. Sato and Y. Kamayema (1997; 1998; 1997) and M. Parigot (1992; 1993).

As far as the author knows, M. Parigot's $\lambda\mu$-calculus is the only λ-calculus with control operators for which strong normalization has been proved in the second order framework.

We first recall the typing rules of this calculus (for more details see (Parigot, 1992)). Then we derive the typing rules of the **catch** and **throw** operators: they correspond respectively to right contraction and weakening rules of classical natural deduction. The subject reduction property and strong normalization are straightforward consequences.

4.1 The typed $\lambda\mu$-calculus

Axiom

$$x : A^x \vdash A$$

Rules for $\to$

$$\frac{t : \Gamma, A^x \vdash \Delta; B}{\lambda x.t : \Gamma \vdash \Delta; A \to B} \qquad \frac{u : \Gamma \vdash \Delta; A \to B \qquad v : \Gamma \vdash \Delta; A}{(u\ v) : \Gamma \vdash \Delta; B}$$

Rules for $\forall$ (where x does not occur free in Γ, Δ in the introduction rule)

$$\frac{u : \Gamma \vdash \Delta; A}{u : \Gamma \vdash \Delta; \forall x A} \qquad \frac{u : \Gamma \vdash \Delta; \forall x A}{u : \Gamma \vdash \Delta; A\{t/x\}}$$

Rules for $\forall^2$ (where X does not occur free in Γ, Δ in the introduction rule)

$$\frac{u : \Gamma \vdash \Delta; A}{u : \Gamma \vdash \Delta; \forall X A} \qquad \frac{u : \Gamma \vdash \Delta; \forall X A}{u : \Gamma \vdash \Delta; A\{T/X\}}$$

Contraction and weakening rules (where Π is either empty or a single formula B)

$$\frac{t : \Gamma, A^x, A^y \vdash \Delta; \Pi}{t\{x/y\} : \Gamma, A^x \vdash \Delta; \Pi} \qquad \frac{t : \Gamma \vdash \Delta; \Pi}{t : \Gamma, A^x \vdash \Delta; \Pi}$$

$$\frac{t : \Gamma \vdash \Delta, A^\alpha, A^\beta; \Pi}{t\{\beta/\alpha\} : \Gamma \vdash \Delta, A^\beta; \Pi} \qquad \frac{t : \Gamma \vdash \Delta; \Pi}{t : \Gamma \vdash \Delta, A^\alpha; \Pi}$$

Remark. As usual in natural deduction, these explicit contraction and weakening rules are not actually needed if we allow for "generalized axioms":

$$x : \Gamma, A^x \vdash \Delta; A$$

Naming rules

These are the rules of the $\lambda\mu$-calculus that allow for multiple conclusions:

$$\frac{t : \Gamma \vdash \Delta; A}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha;} \qquad \frac{t : \Gamma \vdash \Delta, A^\alpha;}{\mu\alpha.t : \Gamma \vdash \Delta; A}$$

4.2 Typing the **catch** *and* **throw** *operators*

Let us now use the naming rules to derive type judgments for the $\lambda\mu ct$-terms *throw* α t and *catch* α t.

- We recall that *catch* α $t = \mu\alpha[\alpha]t$:

$$\frac{\dfrac{t : \Gamma \vdash \Delta, A^\alpha; A}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha;}}{\mu\alpha[\alpha]t : \Gamma \vdash \Delta; A}$$

- We recall that *throw* α $t = \mu\beta[\alpha]t$ where β does not occur free in t:

$$\frac{\dfrac{\dfrac{t : \Gamma \vdash \Delta; A}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha;}}{[\alpha]t : \Gamma \vdash \Delta, A^\alpha, B^\beta;}}{\mu\beta[\alpha]t : \Gamma \vdash \Delta, A^\alpha; B}$$

Hence, we are now able to type the native **throw** and **catch** operators.

The catch rule

$$\frac{t : \Gamma \vdash \Delta, A^\alpha; A}{\mathbf{catch}\ \alpha\ t : \Gamma \vdash \Delta; A}$$

The throw rule

$$\frac{t : \Gamma \vdash \Delta; A}{\mathbf{throw}\ \alpha\ t : \Gamma \vdash \Delta, A^\alpha; B}$$

Remark. As for the $\lambda\mu$-calculus, there is no need for explicit rules for contracting and weakening "named" conclusions. Consequently, one can see these catch and throw rules respectively as explicit right-hand contraction and weakening rules for classical natural deduction.

Proposition 4.2.1

A $\lambda_{\mathbf{ct}}$-term t is typable of type $\Gamma \vdash \Delta; A$ if and only if $\lambda\mu ct$-term $\Lambda^{\iota}_{\mu}(t)$ is typable of type $\Gamma \vdash \Delta; A$.

Example. The $\lambda_{\mathbf{ct}}$-term $\lambda y.\mathbf{catch}\ \alpha\ (y\ \lambda x.\mathbf{throw}\ \alpha\ x)$, which represents the famous **call/cc** of the Scheme language just as the corresponding $\lambda\mu ct$-term (Parigot, 1992), can be typed by Peirce's axiom:

$$\dfrac{\dfrac{\dfrac{y : ((A \to B) \to A)^{y} \vdash (A \to B) \to A \qquad \dfrac{\dfrac{x : A^{x} \vdash A}{\mathbf{throw}\ \alpha\ x : A^{x} \vdash A^{\alpha}; B}}{\lambda x.\mathbf{throw}\ \alpha\ x \vdash A^{\alpha}; A \to B}}{(y\ \lambda x.\mathbf{throw}\ \alpha\ x) : ((A \to B) \to A)^{y} \vdash A^{\alpha}; A}}{\mathbf{catch}\ \alpha\ (y\ \lambda x.\mathbf{throw}\ \alpha\ x) : ((A \to B) \to A)^{y} \vdash A}}{\lambda y.\mathbf{catch}\ \alpha\ (y\ \lambda x.\mathbf{throw}\ \alpha\ x) \vdash ((A \to B) \to A) \to A}$$

4.3 Subject reduction property

The subject reduction property holds for the second order $\lambda\mu$-calculus: if a $\lambda\mu$-term t is typable of the type $\Gamma \vdash \Delta; A$ and $t \to_{\lambda\mu} t'$ then t' is also typable of the type $\Gamma \vdash \Delta; A$. This property extends directly to the $\lambda_{\mathbf{ct}}$-calculus:

Proposition 4.3.1
Given a $\lambda_{\mathbf{ct}}$-term t, if t is typable of type $\Gamma \vdash \Delta; A$ and $t \to_{\mathbf{ct}} t'$ then t' is also typable of the type $\Gamma \vdash \Delta; A$.

Proof
By proposition 4.2.1, if t is typable of type $\Gamma \vdash \Delta; A$, then $\Lambda^{\iota}_{\mu}(t)$ is also typable of type $\Gamma \vdash \Delta; A$. We know that $\Lambda^{\iota}_{\mu}(t) \to_{\lambda\mu} \Lambda^{\iota}_{\mu}(t')$, and since the subject reduction property holds for the $\lambda\mu$-calculus, $\Lambda^{\iota}_{\mu}(t')$ is typable of type $\Gamma \vdash \Delta; A$, and again by proposition 4.2.1, $t^{'}$ is also typable of type $\Gamma \vdash \Delta; A$. □

4.4 Strong normalization of the second order $\lambda_{\mathbf{ct}}$-calculus

The $\lambda\mu$-calculus is strongly normalizing in the second order framework, *i.e.* if a $\lambda\mu$-term t is typable of type $\Gamma \vdash \Delta; A$ then there is no infinite sequence of reductions starting from t. This property extends directly to the $\lambda_{\mathbf{ct}}$-calculus.

Proposition 4.4.1
Given a $\lambda_{\mathbf{ct}}$-term t, if t is typable of type $\Gamma \vdash \Delta; A$ then there is no infinite sequence of reductions starting from t.

Proof
If t is typable of type $\Gamma \vdash \Delta; A$, then the $\lambda\mu\mathbf{ct}$-term $\Lambda^{\iota}_{\mu}(t)$ is also typable of type $\Gamma \vdash \Delta; A$. If there was an infinite sequence of reductions $t_1 \to_{\mathbf{ct}} t_2 \ldots \to_{\mathbf{ct}} t_n \ldots$ then, since Λ^{ι}_{μ} preserves one-step reduction (proposition 3.3.3), there would be an infinite sequence of reductions $\Lambda^{\iota}_{\mu}(t_1) \to_{\lambda\mu} \Lambda^{\iota}_{\mu}(t_2) \ldots \to_{\lambda\mu} \Lambda^{\iota}_{\mu}(t_n) \ldots$, which contradicts the strong normalization of the $\lambda\mu$-calculus. □

Remark. The converse is also true: the strong normalization of the $\lambda_{\mathbf{ct}}$-calculus implies the strong normalization of the $\lambda\mu$-calculus. Indeed, the translation $\Lambda^{\iota}_{\mathbf{ct}}$ is also an injective morphism. Moreover, a $\lambda\mu$-term t is typable of type $\Gamma \vdash \Delta; A$ if and only if $\Lambda^{\iota}_{\mathbf{ct}}(t)$ is typable of type $\Gamma \vdash \Delta; A$.

5 Conclusion

We have defined a confluent λ-calculus with a catch/throw mechanism. Any $\lambda_{\mathbf{ct}}$-term typable in the second order classical natural deduction is strongly normalizing. We have also seen that the **call/cc** of Scheme can be defined as:

$$\mathbf{call/cc}\ t \equiv \mathbf{catch}\ \alpha\ (t\ \lambda x.\mathbf{throw}\ \alpha\ x)$$

P. De Groote has shown in (1994b) that the $\lambda\mu$-calculus is isomorphic (modulo convertibility) to the $\lambda\mathcal{C}$-calculus. Similarly, it would be interesting to study how the λ-calculus with a "native" **call/cc** is related to the $\lambda_{\mathbf{ct}}$-calculus. Besides, we have only investigated here M. Parigot "call-by-name" $\lambda\mu$-calculus. C.-H. Ong and C. A. Stewart (1996; 1997) have proposed a "call-by-value" $\lambda\mu$-calculus. It is likely that a "call-by-value" $\lambda_{\mathbf{ct}}$-calculus can be derived from their work. Notice that P. De Groote (1994b) and C.-H. Ong (1996; 1997) separate the μ and the [ ] in their $\lambda\mu$-calculus. Nevertheless, this separation does not define a catch/throw mechanism (since in $\mu\alpha.t$ the type of t is $\bot$).

We did not consider tag-abstraction as in the work of H. Nakano, Y. Kamayema and M. Sato, since there is no need for tag-abstraction in the classical framework where a tag (μ-variable) α can be reified as the term $\lambda x.\mathbf{throw}\ \alpha\ x$ whose type is $\vdash A^{\alpha}; \neg A$ (first-class continuations are typed by the negation $\neg A \equiv A \to \bot$). Of course, this is not sound anymore in intuitionistic logic since this type is the excluded-middle. We will consider tag-abstraction in a constructive framework in a forthcoming paper, but where subtraction (the connector dual to implication, see Crolard (1996; 1999)) will be used instead of disjunction.

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