

## Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ ?

2. What algebraic structures with one operation (groupoid, semigroup, monoid or group) are the numerical sets  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  together with addition or multiplication?

3. Give examples of:

(i) a groupoid which is not a semigroup.

(ii) a semigroup which is not a monoid.

(iii) a monoid which is not a group.

4. Give example of a groupoid with identity element in which there exists an element having two different symmetric elements.

5. Let  $A = \{a_1, a_2, a_3\}$  be a set. Determine the number of:

(i) operations on  $A$ ;

(ii) commutative operations on  $A$ ;

(iii) operations on  $A$  with identity element.

Generalization for a set  $A$  with  $n$  elements ( $n \in \mathbb{N}^*$ ).

6. Let “ $*$ ” be the operation on  $\mathbb{R}$  defined by:

$$x * y = x + y + xy.$$

Show that:

(i)  $(\mathbb{R}, *)$  is a commutative monoid.

(ii) The interval  $[-1, \infty)$  is a stable subset of  $(\mathbb{R}, *)$ .

7. Let “ $*$ ” be the operation on  $\mathbb{N}$  defined by  $x * y = \text{g.c.d.}(x, y)$ .

(i) Prove that  $(\mathbb{N}, *)$  is a commutative monoid.

(ii) Show that  $D_n = \{x \in \mathbb{N} \mid x/n\}$  ( $n \in \mathbb{N}^*$ ) is a stable subset of  $(\mathbb{N}, *)$  and  $(D_n, *)$  is a commutative monoid.

(iii) Fill in the table of the operation “ $*$ ” on  $D_6$ .

8. Determine the finite stable subsets of  $(\mathbb{Z}, \cdot)$ .

9. Let  $A$  be a set and let  $\mathcal{P}(A)$  be the power set of  $A$  (that is, the set of all subsets of  $A$ ). What algebraic structure with one operation (groupoid, semigroup, monoid or group) is  $\mathcal{P}(A)$  together with the operation “ $\cup$ ” or “ $\cap$ ”?

10. Let  $(A, \cdot)$  be a groupoid and  $X, Y \subseteq A$ . Let “ $\cdot$ ” be the operation on the power set  $\mathcal{P}(A)$  defined by:

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Show that:

(i) If  $(A, \cdot)$  is commutative, then  $(\mathcal{P}(A), \cdot)$  is commutative.

(ii) If  $(A, \cdot)$  is a semigroup, then  $(\mathcal{P}(A), \cdot)$  is a semigroup.

(iii) If  $(A, \cdot)$  is a monoid, then  $(\mathcal{P}(A), \cdot)$  is a monoid.

(iv) If  $(A, \cdot)$  is a group, then in general  $(\mathcal{P}(A), \cdot)$  is not a group (for  $A \neq \emptyset$ ).

## Seminar 2

1. Let “ $*$ ” be the operation on  $\mathbb{R}$  defined by:

$$x * y = xy - 5x - 5y + 30.$$

Is  $(\mathbb{R}, *)$  a group? What about  $(\mathbb{R} \setminus \{5\}, *)$ ?

2. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Show that the set

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

is a stable subset of the monoid  $(M_n(\mathbb{R}), \cdot)$  and  $(GL_n(\mathbb{R}), \cdot)$  is a group.

3. Let  $n \in \mathbb{N}^*$ . Show that the set

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}$$

is a stable subset of the group  $(\mathbb{C}^*, \cdot)$ ,  $(U_n, \cdot)$  is an abelian group, and determine the elements of  $U_n$ .

4. Let  $n \in \mathbb{N}$  and  $\mathbb{Z}_n = \{\hat{x} \mid x \in \mathbb{Z}\}$ , where  $\hat{x} = x + n\mathbb{Z} = \{x + nk \mid k \in \mathbb{Z}\}$ . Let “ $+$ ” be the operation on  $\mathbb{Z}_n$  defined by:

$$\hat{x} + \hat{y} = \widehat{x + y}, \quad \forall \hat{x}, \hat{y} \in \mathbb{Z}_n.$$

Show that  $(\mathbb{Z}_n, +)$  is an abelian group and determine its cardinal (discussion on  $n$ ).

5. Let  $M \neq \emptyset$  be a set and

$$S_M = \{f : M \rightarrow M \mid f \text{ bijective}\}.$$

(i) Show that  $(S_M, \circ)$  is a group.

(ii) If  $|M| = n \in \mathbb{N}^*$ , then we denote  $S_M$  by  $S_n$ . Determine the operation table for the group  $(S_3, \circ)$ .

6. Determine the operation table for the dihedral group  $(D_3, \cdot)$  of rotations and symmetries of an equilateral triangle.

7. Determine the operation table for the dihedral group  $(D_4, \cdot)$  of rotations and symmetries of a square.

8. Let  $(G, \cdot)$  and  $(G', \cdot)$  be groups with identity elements  $e$  and  $e'$  respectively. Let “ $\cdot$ ” be the operation on  $G \times G'$  defined by:

$$(g_1, g'_1) \cdot (g_2, g'_2) = (g_1 \cdot g_2, g'_1 \cdot g'_2), \quad \forall (g_1, g'_1), (g_2, g'_2) \in G \times G'.$$

Show that  $(G \times G', \cdot)$  is a group, called the *direct product* of the groups  $G$  and  $G'$ .

9. Determine the group of invertible elements of the monoids  $(\mathbb{N}, +)$ ,  $(\mathbb{N}, \cdot)$ ,  $(\mathbb{Z}, \cdot)$ ,  $(\mathbb{Q}, \cdot)$ ,  $(\mathbb{R}, \cdot)$ ,  $(\mathbb{C}, \cdot)$ ,  $(M_n(\mathbb{R}), \cdot)$  ( $n \in \mathbb{N}$ ,  $n \geq 2$ ) and  $(M^M, \circ)$ , where  $M \neq \emptyset$  is a set and  $M^M$  denotes the set of all functions  $f : M \rightarrow M$ .

10. Let  $(G, \cdot)$  be a group. Show that:

(i)  $G$  is abelian  $\iff \forall x, y \in G, (xy)^2 = x^2y^2$ .

(ii)  $\forall x \in G, x^2 = 1 \implies G$  is abelian.

## Seminar 3

1. Which ones of the numerical sets  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  are subgroups of the groups  $(\mathbb{C}, +)$  and  $(\mathbb{C}^*, \cdot)$ ?

2. Show that  $H = \{z \in \mathbb{C} \mid |z| = 1\}$  is a subgroup of  $(\mathbb{C}^*, \cdot)$ , but not of  $(\mathbb{C}, +)$ .

3. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Show that:

(i)  $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\}$  is a stable subset of the monoid  $(M_n(\mathbb{C}), \cdot)$ .

(ii)  $(GL_n(\mathbb{C}), \cdot)$  is a group.

(iii)  $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) = 1\}$  is a subgroup of the group  $(GL_n(\mathbb{C}), \cdot)$ .

4. Let  $n \in \mathbb{N}^*$ . Show that  $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$  is a subgroup of the group  $(\mathbb{C}^*, \cdot)$ .

5. Consider the set  $S(\mathbb{Z}, +) = \{n\mathbb{Z} \mid n \in \mathbb{N}\}$  of subgroups of the group  $(\mathbb{Z}, +)$  and  $m, n \in \mathbb{N}$ . Show that:

(i)  $n\mathbb{Z} \subseteq m\mathbb{Z} \iff m \mid n$ .

(ii)  $m\mathbb{Z} \cap n\mathbb{Z} = [m, n]\mathbb{Z}$ , where  $[m, n]$  denotes the least common multiple of  $m$  and  $n$ .

(iii)  $m\mathbb{Z} + n\mathbb{Z} = (m, n)\mathbb{Z}$ , where  $(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ .

6. Let  $(G, \cdot)$  be a group and  $H, K \leq G$ . Show that:

$$H \cup K \leq G \iff H \subseteq K \text{ or } K \subseteq H.$$

7. Let  $(G, \cdot)$  be a group and let  $\emptyset \neq H \subseteq G$  be a finite set. Show that:

$$H \leq G \iff H \text{ is a stable subset of } (G, \cdot).$$

8. Let  $(G, \cdot)$  be a group. Prove that:

$$Z(G) = \{x \in G \mid x \cdot g = g \cdot x, \forall g \in G\}$$

is a subgroup of  $G$ , called *the center of  $G$* . When does the equality  $Z(G) = G$  hold?

9. Prove that:

$$Z(GL_2(\mathbb{R}), \cdot) = \{a \cdot I_2 \mid a \in \mathbb{R}^*\},$$

where  $I_2$  is the identity matrix. Generalization for  $GL_n(\mathbb{R})$  with  $n \in \mathbb{N}$ ,  $n \geq 2$ .

10. Prove that  $Z(S_3, \circ) = \{e\}$ , where  $e$  is the identity permutation. Generalization for  $S_n$  with  $n \in \mathbb{N}$ ,  $n \geq 3$ .

## Seminar 4

**1.** (i) Let  $f : \mathbb{C}^* \rightarrow \mathbb{R}^*$  be defined by  $f(z) = |z|$ . Show that  $f$  is a group homomorphism between  $(\mathbb{C}^*, \cdot)$  and  $(\mathbb{R}^*, \cdot)$ .

(ii) Let  $n \in \mathbb{N}$  and  $g : \mathbb{Z} \rightarrow \mathbb{Z}_n$  be defined by  $g(x) = \widehat{x}$ . Prove that  $g$  is a group homomorphism between  $(\mathbb{Z}, +)$  and  $(\mathbb{Z}_n, +)$ .

**2.** (i) Let  $n \in \mathbb{N}$ ,  $n \geq 2$  and let  $\alpha : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^*$  be defined by  $\alpha(A) = \det(A)$ . Show that  $\alpha$  is a group homomorphism between  $(GL_n(\mathbb{R}), \cdot)$  and  $(\mathbb{R}^*, \cdot)$ .

(ii) Let  $n \in \mathbb{N}$ ,  $n \geq 2$  and  $\beta : M_n(\mathbb{R}) \rightarrow \mathbb{R}$  be defined by  $\beta(A) = \det(A)$ . Show that  $\beta$  is not a group homomorphism between  $(M_n(\mathbb{R}), +)$  and  $(\mathbb{R}, +)$ .

**3.** For a group homomorphism  $f : G \rightarrow G'$  between groups  $(G, \cdot)$  and  $(G', \cdot)$  the *kernel* of  $f$  is  $\text{Ker } f = \{x \in G \mid f(x) = 1'\}$  and the *image* of  $f$  is  $\text{Im } f = \{f(x) \mid x \in G\}$ . Determine the kernel and the image of the group homomorphisms from Ex. **1.** and **2.**

**4.** Let  $f : \mathbb{C}^* \rightarrow GL_2(\mathbb{R})$  be defined by  $f(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ . Show that  $f$  is a group homomorphism between  $(\mathbb{C}^*, \cdot)$  and  $(GL_2(\mathbb{R}), \cdot)$ .

**5.** Let  $a, b \in \mathbb{N}$  and  $f : \mathbb{C}^* \rightarrow \mathbb{R}^*$  be defined by  $f(z) = a \cdot |z| + b$ . Determine  $a, b$  such that  $f$  is a group homomorphism between  $(\mathbb{C}^*, \cdot)$  and  $(\mathbb{R}^*, \cdot)$ .

**6.** Let  $(G, \cdot)$  be a group and let  $f : G \rightarrow G$  be defined by  $f(x) = x^{-1}$ . Show that  $f \in \text{End}(G) \iff G$  is abelian.

**7.** Show that the following groups are isomorphic:  $(\mathbb{Z}_n, +)$  and  $(U_n, \cdot)$  ( $n \in \mathbb{N}^*$ ).

**8.** Show that the following groups are isomorphic: Klein's group  $(K, \cdot)$  and  $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ .

**9.** Show that the following groups are isomorphic:  $(\mathbb{R}, +)$  and  $(\mathbb{R}_+^*, \cdot)$ .

**10.** Let  $(G, \cdot)$  be a group with 3 elements. Determine  $\text{End}(G)$  and  $\text{Aut}(G)$ .

**11.** Determine  $\text{Aut}(U_4, \cdot)$ .

**12.** (i) Let  $f \in \text{End}(\mathbb{Z}, +)$ . Show that  $f(n) = f(1) \cdot n$ ,  $\forall n \in \mathbb{Z}$ .

(ii)  $\forall a \in \mathbb{Z}$ , let  $t_a : \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $t_a(n) = a \cdot n$ . Prove that:

$$\text{End}(\mathbb{Z}, +) = \{t_a \mid a \in \mathbb{Z}\}$$

and determine  $\text{Aut}(\mathbb{Z}, +)$ .

## Seminar 5

**1.** Determine the order of each element and all generators of the cyclic groups  $(\mathbb{Z}_8, +)$  and  $(U_6, \cdot)$ .

**2.** Let  $(G, \cdot)$  be a cyclic group, where  $G = \langle x \rangle$  and  $|G| = n \in \mathbb{N}^*$ , and let  $k \in \mathbb{N}^*$ . Show that:

$$G = \langle x^k \rangle \iff (n, k) = 1.$$

**3.** Determine the order of each element of Klein's group  $(K, \cdot)$ , permutation group  $(S_3, \circ)$  and quaternion group  $(Q, \cdot)$ . Are they cyclic groups?

**4.** (i) Consider the matrices  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$  in the group  $(GL_2(\mathbb{R}), \cdot)$ . Determine  $\text{ord } A$ ,  $\text{ord } B$ ,  $\text{ord } (A \cdot B)$  and  $\text{ord } (B \cdot A)$ .

(ii) Give an example of group in which there exist two elements of infinite order, whose product has finite order.

**5.** Let  $(G, \cdot)$  be a group and  $x, y \in G$ . Show that:

$$\text{ord}(xy) = \text{ord}(yx).$$

**6.** Let  $(G, \cdot)$  be an abelian group. Show that

$$t(G) = \{x \in G \mid \text{ord } x \text{ is finite}\}$$

is a subgroup of  $G$ . Is the property still true if  $G$  is not abelian?

**7.** Let  $(G, \cdot)$  and  $(G', \cdot)$  be abelian groups. Show that if  $G \simeq G'$ , then  $t(G) \simeq t(G')$ .

**8.** Using **7.** show that the following groups are not isomorphic:

(i)  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}^*, \cdot)$ .

(ii)  $(\mathbb{R}, +)$  and  $(\mathbb{R}^*, \cdot)$ .

**9.** Let  $f : G \rightarrow G'$  be a group homomorphism and let  $x \in G$  be an element of finite order. Prove that:

(i)  $\text{ord } f(x)$  is finite and  $\text{ord } f(x) \mid \text{ord } x$ .

(ii) If  $f$  is injective, then  $\text{ord } f(x) = \text{ord } x$ .

**10.** Using **9.** show that the groups  $(\mathbb{Z}_4, +)$  and  $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$  are not isomorphic.

## Seminar 6

1. Let  $n \in \mathbb{N}$ ,  $n \geq 2$  and

$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\},$$

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}.$$

Show that  $SL_n(\mathbb{R})$  is a normal subgroup of the group  $(GL_n(\mathbb{R}), \cdot)$ .

2. For  $a \in \mathbb{R}$ , let  $t_a : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $t_a(x) = a \cdot x$ . Is the set  $H = \{t_a \mid a \in \mathbb{R}^*\}$  a normal subgroup of the symmetric group  $(S_{\mathbb{R}}, \circ)$ ?

3. Show that the center

$$Z(G) = \{x \in G \mid x \cdot g = g \cdot x, \forall g \in G\}$$

of a group  $(G, \cdot)$  is a normal subgroup.

4. Determine the (normal) subgroups and the factor groups of the group  $(\mathbb{Z}, +)$ .

5. Determine the (normal) subgroups and the factor groups of the group  $(\mathbb{Z}_6, +)$ . Fill in the operation table for one of the factor groups.

6. Determine the (normal) subgroups and the factor groups of Klein's group  $(K, \cdot)$ . Fill in the operation table for one of the factor groups.

7. Determine the (normal) subgroups of the group  $(S_3, \circ)$  (compute  $S_3/r_H$  and  $S_3/r'_H$  for  $H \leq S_3$ ). Determine the factor groups  $S_3/N$ , where  $N$  is a normal subgroup of  $S_3$ , and fill in the operation table for one of them.

8. Determine the (normal) subgroups of the quaternion group  $(Q, \cdot)$ . Determine the factor groups  $Q/N$ , where  $N$  is a normal subgroup of  $Q$ , and fill in the operation table for the group  $(Q/N, \cdot)$ , where  $N = \{-1, 1\}$ .

## Seminar 7

1. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Prove the group isomorphism

$$(GL_n(\mathbb{R})/SL_n(\mathbb{R}), \cdot) \simeq (\mathbb{R}^*, \cdot)$$

by using the first isomorphism theorem.

2. Prove the group isomorphism

$$(\mathbb{C}/\mathbb{R}, +) \simeq (\mathbb{R}, +)$$

by using the first isomorphism theorem.

3. Let  $m, n \in \mathbb{N}$  be such that  $(m, n) = 1$ . Prove the group isomorphism

$$(\mathbb{Z}_{mn}, +) \simeq (\mathbb{Z}_m \times \mathbb{Z}_n, +).$$

4. Consider the group  $(\mathbb{Z}_{24}, +)$  and its cyclic subgroups  $H = \langle \hat{4} \rangle$  and  $N = \langle \hat{6} \rangle$ . Determine  $H \cap N$ ,  $H + N$  and apply the second isomorphism theorem.

5. Determine the (normal) subgroups and the factor groups of the group  $(\mathbb{Z}_{12}, +)$  by using the third isomorphism theorem.

6. Determine the subgroups of the groups  $(\mathbb{Z}_n, +)$  for  $n = 1, \dots, 12$ , and then draw the Hasse diagram of the subgroup lattice of each of them.

7. Determine the subgroups of the group  $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ , and then draw the Hasse diagram of its subgroup lattice.

8. Determine the subgroups of the quaternion group  $(Q, \cdot)$ , and then draw the Hasse diagram of its subgroup lattice.

## Seminar 8

1. Compute the composition (product) of the following permutations of 4 elements, and then determine the signature and the inverse of the result:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

2. Determine the orbits of each element of the set  $\{1, 2, 3, 4, 5\}$  relative to the permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}.$$

3. Decompose into products of disjoint cycles and into products of transpositions the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 5 & 7 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 7 & 2 & 5 & 1 & 8 & 9 & 3 \end{pmatrix}.$$

4. Write down all elements of the alternating groups  $A_3$  and  $A_4$ , and then decompose these elements into products of disjoint cycles.

5. Let  $H = \{\sigma \in S_5 \mid \sigma(1) = 1 \text{ or } \sigma(5) = 5\}$ . Is  $H$  a subgroup of the group  $(S_5, \circ)$ ?

6. Show the isomorphism  $(D_3, \cdot) \simeq (S_3, \circ)$ , where  $D_3$  is the 3-rd dihedral group.

7. Determine the order of each element and the cyclic subgroups of the group  $(S_3, \circ)$ .

8. Determine the subgroups of the group  $(S_3, \circ)$ , and then draw the Hasse diagram of its subgroup lattice.



## Seminar 9

**1.** Show that the sets  $\mathbb{Z}_n$  (residue classes modulo  $n$ ),  $M_n(\mathbb{R})$  (matrices  $n \times n$ ) and  $\mathbb{R}[X]$  (polynomials) form rings together with the corresponding addition and multiplication. Are they commutative rings, integral domains, division rings or fields? Generalization.

**2.** Show that the set  $\mathbb{R}^{\mathbb{R}}$  of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  forms a ring together with the addition and the multiplication defined by:  $\forall f, g \in \mathbb{R}^{\mathbb{R}}, (f + g)(x) = f(x) + g(x), (f \cdot g)(x) = f(x) \cdot g(x), \forall x \in \mathbb{R}$ . Is it a commutative ring, integral domain, division ring or field? Generalization.

**3.** Let  $(G, +)$  be an abelian group. Show that  $(\text{End}(G), +, \circ)$  is a ring with identity.

**4.** Let  $(R, +, \cdot)$  be a ring. Consider on the set  $\mathbb{Z} \times R$  the addition and the multiplication defined by:

$$\begin{aligned}(m, a) + (n, b) &= (m + n, a + b), \\ (m, a) \cdot (n, b) &= (mn, ab + na + mb),\end{aligned}$$

$\forall (m, a), (n, b) \in \mathbb{Z} \times R$ . Show that  $(\mathbb{Z} \times R, +, \cdot)$  is a ring with identity.

**5.** Let  $n \in \mathbb{N}, n \geq 2$  and  $\hat{0} \neq \hat{a} \in \mathbb{Z}_n$ . Prove that:

$$\hat{a} \text{ is invertible in the ring } (\mathbb{Z}_n, +, \cdot) \iff (a, n) = 1.$$

When is  $(\mathbb{Z}_n, +, \cdot)$  a field?

**6.** Solve the following equations in the ring  $(\mathbb{Z}_{12}, +, \cdot)$ :  $\hat{4}x + \hat{5} = \hat{9}$  and  $\hat{5}x + \hat{5} = \hat{9}$ .

**7.** Solve the following system of equations in the ring  $(\mathbb{Z}_{12}, +, \cdot)$ :

$$\begin{cases} \hat{3}x + \hat{4}y = \hat{11} \\ \hat{4}x + \hat{9}y = \hat{10} \end{cases}.$$

**8.** Solve the equation  $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$  in the ring  $(M_2(\mathbb{C}), +, \cdot)$ .

**9.** Let  $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{Q} \right\}$ . Show that  $\mathcal{M}$  is a stable subset of the ring  $(M_2(\mathbb{Q}), +, \cdot)$  and  $(\mathcal{M}, +, \cdot)$  is a field.

**10.** Let  $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$ . Show that  $\mathcal{M}$  is a stable subset of the ring  $(M_2(\mathbb{R}), +, \cdot)$  and  $(\mathcal{M}, +, \cdot)$  is a field.

## Seminar 10

**1.** Are the following sets subrings of the field  $\mathbb{C}$ :

- (i)  $A = \{bi \mid b \in \mathbb{R}\}$ ;
- (ii)  $B = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$ ;
- (iii)  $C = \{z \in \mathbb{C} \mid |z| \leq 1\}$ ?

**2.** Show that the set  $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$  is a subring of the field  $\mathbb{C}$ , called *the ring of Gauss integers*. Determine its invertible elements.

**3.** Are the following sets subrings of the ring  $M_2(\mathbb{R})$ :

- (i)  $\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ ;
- (ii)  $\mathcal{B} = \left\{ \begin{pmatrix} a & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ ;
- (iii)  $\mathcal{C} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ ?

**4.** Are the following sets subrings of the ring  $\mathbb{R}[X]$ :

- (i)  $A = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 0\}$ ;
- (ii)  $B = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 1\}$ ;
- (iii)  $C = \{f \in \mathbb{R}[X] \mid \text{the coefficient of the term of degree 1 of } f \text{ is } 0\}$ ?

**5.** Give examples of:

- (i) subring without identity of a ring with identity.
- (ii) subring with identity of a ring with identity, which have different identities.
- (iii) non-commutative finite ring.

**6.** Show that the set  $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$  is a subfield of the field  $\mathbb{R}$ . Generalization.

**7.** Is the set  $A = \{a + b\sqrt[3]{2} \mid a, b \in \mathbb{Q}\}$  a subring of the field  $\mathbb{R}$ ?

**8.** Let  $m, n \in \mathbb{N}$ . Show that  $n\mathbb{Z}$  is a subring of the ring  $m\mathbb{Z} \Leftrightarrow m \mid n$ .

**9.** Let  $(R, +, \cdot)$  be a ring. Show that:

$$Z(R) = \{a \in R \mid a \cdot r = r \cdot a, \forall r \in R\}$$

is a subring of  $R$ , called the *center of  $R$* . When does the equality  $Z(R) = R$  hold?

**10.** Show that:

$$Z(M_2(\mathbb{R}), +, \cdot) = \{a \cdot I_2 \mid a \in \mathbb{R}\},$$

where  $I_2$  is the identity matrix. Generalization for  $M_n(\mathbb{R})$  with  $n \in \mathbb{N}$ ,  $n \geq 2$ .

## Seminar 11

**1.** Let  $R$  be a ring. An element  $a \in R$  is called idempotent if  $a^2 = a$ .

Determine the idempotents of the ring  $\mathbb{Z}_{12}$ , and write down 4 idempotents of the ring  $M_2(\mathbb{Z})$ .

**2.** Let  $R$  be a ring. An element  $a \in R$  is called nilpotent if there exists  $n \in \mathbb{N}$  such that  $a^n = 0$ .

Determine the nilpotent elements of the ring  $\mathbb{Z}_{12}$ , and write down 2 nilpotent elements of the ring  $M_2(\mathbb{Z})$ .

**3.** Are the following functions ring homomorphism between the corresponding rings:

(i)  $f : \mathbb{Z} \rightarrow \mathbb{Z}_n$  defined by  $f(x) = \widehat{x}$ ;

(ii)  $f : \mathbb{R} \rightarrow M_2(\mathbb{R})$  defined by  $f(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ ;

(iii)  $g : M_2(\mathbb{R}) \rightarrow \mathbb{R}$  defined by  $g(A) = \det(A)$ ?

**4.** Let  $f : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_4$  be defined by  $f(\widehat{x}) = \overline{x}$ . Prove that  $f$  is well defined (that is,  $f$  is a function) and  $f$  is a ring homomorphism.

**5.** Consider the fields  $(\mathcal{M}, +, \cdot)$  and  $(\mathbb{Q}(\sqrt{2}), +, \cdot)$ , where  $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{Q} \right\}$  and  $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ . Show that the fields  $(\mathcal{M}, +, \cdot)$  and  $(\mathbb{Q}(\sqrt{2}), +, \cdot)$  are isomorphic.

**6.** Consider the field  $(\mathcal{M}, +, \cdot)$ , where  $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ . Show that the fields  $(\mathcal{M}, +, \cdot)$  and  $(\mathbb{C}, +, \cdot)$  are isomorphic.

**7.** For  $a \in \mathbb{Z}$ , let  $t_a : \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $t_a(x) = ax$ . Using the result  $\text{End}(\mathbb{Z}, +) = \{t_a \mid a \in \mathbb{Z}\}$ , show that  $\text{End}(\mathbb{Z}, +, \cdot) = \{t_0, t_1\}$  and  $\text{Aut}(\mathbb{Z}, +, \cdot) = \{t_1\}$ .

**8.** Consider the field  $(\mathbb{Q}(\sqrt{2}), +, \cdot)$ , where  $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ . Determine  $\text{Aut}(\mathbb{Q}(\sqrt{2}), +, \cdot)$ .

## Seminar 12

1. Are the following sets (left, right, two-sided) ideals of the ring  $M_2(\mathbb{R})$ :

(i)  $\mathcal{A} = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\};$

(ii)  $\mathcal{B} = \left\{ \begin{pmatrix} a & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\};$

(iii)  $\mathcal{C} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}?$

2. In an arbitrary ring, is the intersection of a left ideal and a right ideal a two-sided ideal?

3. Are the following sets ideals of the ring  $\mathbb{R}[X]$ :

(i)  $A = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 0\};$

(ii)  $B = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 1\};$

(iii)  $C = \{f \in \mathbb{R}[X] \mid \text{the coefficient of the term of degree 1 of } f \text{ is } 0\}?$

4. Let  $R = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is odd}\}$  and  $U = \{\frac{m}{n} \in R \mid m \text{ is even}\}$ . Show that  $R$  is a subring of the field  $\mathbb{Q}$  and  $U$  is an ideal of  $R$ .

5. Let  $R$  be a ring and  $a \in R$ . Show that  $Ra = \{ra \mid r \in R\}$  is a left ideal of  $R$  and  $aR = \{ar \mid r \in R\}$  is a right ideal of  $R$ .

6. Let  $R$  be a ring and

$$\text{Ann}(R) = \{a \in R \mid \forall x \in R, ax = 0 = xa\}.$$

Show that  $\text{Ann}(R)$  is an ideal of  $R$ , called the *annihilator* of  $R$ .

7. Determine the ideals of the ring  $\mathbb{Z}_8$ , and draw the Hasse diagram of its ideal lattice.

8. Determine the ideals of the ring  $\mathbb{Z}_{12}$  and draw the Hasse diagram of its ideal lattice.

## Seminar 13

1. Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Prove the ring isomorphism

$$\mathbb{Z}[X]/(n) \cong \mathbb{Z}_n[X]$$

by using the first isomorphism theorem.

2. Prove the ring isomorphism

$$\mathbb{Q}[X]/(X+1) \cong \mathbb{Q}$$

by using the first isomorphism theorem.

3. Prove the ring isomorphism

$$\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}$$

by using the first isomorphism theorem.

4. Let

$$R = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Q} \right\}, \quad I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Q} \right\}.$$

Show that  $R$  is a subring of the ring  $M_2(\mathbb{Q})$ ,  $I$  is an ideal of  $R$  and  $R/I \cong \mathbb{Q}$ .

5. Determine the factor rings of the ring  $\mathbb{Z}_{12}$  by using the third isomorphism theorem.

6. Determine the characteristic of the ring  $\mathbb{Z}_4 \times \mathbb{Z}_6$ . Generalization for the ring  $\mathbb{Z}_m \times \mathbb{Z}_n$  ( $m, n \in \mathbb{N}$ ,  $m, n \geq 2$ ).

7. Give examples of:

- (i) Infinite ring having finite characteristic.
- (ii) Commutative ring with identity which is not a field but has a prime characteristic.

8. Let  $R$  be a unitary commutative ring with  $1 \neq 0$  and  $\text{char}(R) = p$  for some prime  $p$ . Prove that:

$$(a+b)^p = a^p + b^p, \quad \forall a, b \in R.$$