Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

2. What algebraic structures with one operation (groupoid, semigroup, monoid or group) are the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ together with addition or multiplication?

3. Give examples of:
(i) a groupoid which is not a semigroup.
(ii) a semigroup which is not a monoid.
(iii) a monoid which is not a group.

4. Give example of a groupoid with identity element in which there exists an element having two different symmetric elements.

5. Let $A = \{a_1, a_2, a_3\}$ be a set. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

6. Let "$*$" be the operation on $\mathbb{R}$ defined by:

$$x * y = x + y + xy.$$

Show that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

7. Let "$*$" be the operation on $\mathbb{N}$ defined by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

8. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

9. Let A be a set and let $\mathcal{P}(A)$ be the power set of A (that is, the set of all subsets of A). What algebraic structure with one operation (groupoid, semigroup, monoid or group) is $\mathcal{P}(A)$ together with the operation "$\cup$" or "$\cap$"?

10. Let $(A, \cdot)$ be a groupoid and $X, Y \subseteq A$. Let "$\cdot$" be the operation on the power set $\mathcal{P}(A)$ defined by:

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Show that:
(i) If $(A, \cdot)$ is commutative, then $(\mathcal{P}(A), \cdot)$ is commutative.
(ii) If $(A, \cdot)$ is a semigroup, then $(\mathcal{P}(A), \cdot)$ is a semigroup.
(iii) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), \cdot)$ is a monoid.
(iv) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), \cdot)$ is not a group (for $A \neq \emptyset$).

Seminar 2

1. Let “ $*$ ” be the operation on $\mathbb{R}$ defined by:

$$x * y = xy - 5x - 5y + 30.$$

Is $(\mathbb{R}, *)$ a group? What about $(\mathbb{R} \setminus \{5\}, *)$?

2. Let $n \in \mathbb{N}$, $n \geq 2$. Show that the set

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

is a stable subset of the monoid $(M_n(\mathbb{R}), \cdot)$ and $(GL_n(\mathbb{R}), \cdot)$ is a group.

3. Let $n \in \mathbb{N}^*$. Show that the set

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}$$

is a stable subset of the group $(\mathbb{C}^*, \cdot)$, $(U_n, \cdot)$ is an abelian group, and determine the elements of U_n.

4. Let $n \in \mathbb{N}$ and $\mathbb{Z}_n = \{\widehat{x} \mid x \in \mathbb{Z}\}$, where $\widehat{x} = x + n\mathbb{Z} = \{x + nk \mid k \in \mathbb{Z}\}$. Let “$+$” be the operation on $\mathbb{Z}_n$ defined by:

$$\widehat{x} + \widehat{y} = \widehat{x + y}, \quad \forall\, \widehat{x}, \widehat{y} \in \mathbb{Z}_n.$$

Show that $(\mathbb{Z}_n, +)$ is an abelian group and determine its cardinal (discussion on n).

5. Let $M \neq \emptyset$ be a set and

$$S_M = \{f : M \to M \mid f \text{ bijective}\}.$$

(i) Show that $(S_M, \circ)$ is a group.

(ii) If $|M| = n \in \mathbb{N}^*$, then we denote S_M by S_n. Determine the operation table for the group $(S_3, \circ)$.

6. Determine the operation table for the dihedral group $(D_3, \cdot)$ of rotations and symmetries of an equilateral triangle.

7. Determine the operation table for the dihedral group $(D_4, \cdot)$ of rotations and symmetries of a square.

8. Let $(G, \cdot)$ and $(G', \cdot)$ be groups with identity elements e and e' respectively. Let “ $\cdot$ ” be the operation on $G \times G'$ defined by:

$$(g_1, g_1') \cdot (g_2, g_2') = (g_1 \cdot g_2, g_1' \cdot g_2'), \quad \forall (g_1, g_1'), (g_2, g_2') \in G \times G'.$$

Show that $(G \times G', \cdot)$ is a group, called the *direct product* of the groups G and G'.

9. Determine the group of invertible elements of the monoids $(\mathbb{N}, +)$, $(\mathbb{N}, \cdot)$, $(\mathbb{Z}, \cdot)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{C}, \cdot)$, $(M_n(\mathbb{R}), \cdot)$ $(n \in \mathbb{N},\ n \geq 2)$ and $(M^M, \circ)$, where $M \neq \emptyset$ is a set and M^M denotes the set of all functions $f : M \to M$.

10. Let $(G, \cdot)$ be a group. Show that:

(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2 y^2$.

(ii) $\forall x \in G$, $x^2 = 1 \Longrightarrow G$ is abelian.

Seminar 3

1. Which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are subgroups of the groups $(\mathbb{C},+)$ and $(\mathbb{C}^*,\cdot)$?

2. Show that $H=\{z\in\mathbb{C}\mid |z|=1\}$ is a subgroup of $(\mathbb{C}^*,\cdot)$, but not of $(\mathbb{C},+)$.

3. Let $n\in\mathbb{N}$, $n\geq 2$. Show that:
(i) $GL_n(\mathbb{C})=\{A\in M_n(\mathbb{C})\mid \det(A)\neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}),\cdot)$.
(ii) $(GL_n(\mathbb{C}),\cdot)$ is a group.
(iii) $SL_n(\mathbb{C})=\{A\in M_n(\mathbb{C})\mid \det(A)=1\}$ is a subgroup of the group $(GL_n(\mathbb{C}),\cdot)$.

4. Let $n\in\mathbb{N}^*$. Show that $U_n=\{z\in\mathbb{C}\mid z^n=1\}$ is a subgroup of the group $(\mathbb{C}^*,\cdot)$.

5. Consider the set $S(\mathbb{Z},+)=\{n\mathbb{Z}\mid n\in\mathbb{N}\}$ of subgroups of the the group $(\mathbb{Z},+)$ and $m,n\in\mathbb{N}$. Show that:
(i) $n\mathbb{Z}\subseteq m\mathbb{Z}\Longleftrightarrow m|n$.
(ii) $m\mathbb{Z}\cap n\mathbb{Z}=[m,n]\mathbb{Z}$, where $[m,n]$ denotes the least common multiple of m and n.
(iii) $m\mathbb{Z}+n\mathbb{Z}=(m,n)\mathbb{Z}$, where (m,n) denotes the greatest common divisor of m and n.

6. Let $(G,\cdot)$ be a group and $H,K\leq G$. Show that:

$$H\cup K\leq G\Longleftrightarrow H\subseteq K \text{ or } K\subseteq H\,.$$

7. Let $(G,\cdot)$ be a group and let $\emptyset\neq H\subseteq G$ be a finite set. Show that:

$$H\leq G\Longleftrightarrow H \text{ is a stable subset of } (G,\cdot).$$

8. Let $(G,\cdot)$ be a group. Prove that:

$$Z(G)=\{x\in G\mid x\cdot g=g\cdot x\,,\forall g\in G\}$$

is a subgroup of G, called *the center of* G. When does the equality $Z(G)=G$ hold?

9. Prove that:

$$Z(GL_2(\mathbb{R}),\cdot)=\{a\cdot I_2\mid a\in\mathbb{R}^*\},$$

where I_2 is the identity matrix. Generalization for $GL_n(\mathbb{R})$ with $n\in\mathbb{N}$, $n\geq 2$.

10. Prove that $Z(S_3,\circ)=\{e\}$, where e is the identity permutation. Generalization for S_n with $n\in\mathbb{N}$, $n\geq 3$.

Seminar 4

1. (i) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $n \in \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{Z}_n$ be defined by $g(x) = \widehat{x}$. Prove that g is a group homomorphism between $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$.

2. (i) Let $n \in \mathbb{N}$, $n \geq 2$ and let $\alpha : GL_n(\mathbb{R}) \to \mathbb{R}^*$ be defined by $\alpha(A) = \det(A)$. Show that α is a group homomorphism between $(GL_n(\mathbb{R}), \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $n \in \mathbb{N}$, $n \geq 2$ and $\beta : M_n(\mathbb{R}) \to \mathbb{R}$ be defined by $\beta(A) = \det(A)$. Show that β is not a group homomorphism between $(M_n(\mathbb{R}), +)$ and $(\mathbb{R}, +)$.

3. For a group homomorphism $f : G \to G'$ between groups $(G, \cdot)$ and $(G', \cdot)$ the *kernel* of f is $\operatorname{Ker} f = \{x \in G \mid f(x) = 1'\}$ and the *image* of f is $\operatorname{Im} f = \{f(x) \mid x \in G\}$. Determine the kernel and the image of the group homomorphisms from Ex. **1.** and **2.**

4. Let $f : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $f(a + b\mathrm{i}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

5. Let $a, b \in \mathbb{N}$ and $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = a \cdot |z| + b$. Determine a, b such that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

6. Let $(G, \cdot)$ be a group and let $f : G \to G$ be defined by $f(x) = x^{-1}$. Show that $f \in \operatorname{End}(G) \iff G$ is abelian.

7. Show that the following groups are isomorphic: $(\mathbb{Z}_n, +)$ and $(U_n, \cdot)$ $(n \in \mathbb{N}^*)$.

8. Show that the following groups are isomorphic: Klein's group $(K, \cdot)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

9. Show that the following groups are isomorphic: $(\mathbb{R}, +)$ and $(\mathbb{R}_+^*, \cdot)$.

10. Let $(G, \cdot)$ be a group with 3 elements. Determine $\operatorname{End}(G)$ and $\operatorname{Aut}(G)$.

11. Determine $\operatorname{Aut}(U_4, \cdot)$.

12. (i) Let $f \in \operatorname{End}(\mathbb{Z}, +)$. Show that $f(n) = f(1) \cdot n$, $\forall n \in \mathbb{Z}$.

(ii) $\forall a \in \mathbb{Z}$, let $t_a : \mathbb{Z} \to \mathbb{Z}$ be defined by $t_a(n) = a \cdot n$. Prove that:

$$\operatorname{End}(\mathbb{Z}, +) = \{t_a \mid a \in \mathbb{Z}\}$$

and determine $\operatorname{Aut}(\mathbb{Z}, +)$.