Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.1

# Analysis for CS

GROUPWORK:

**(G 4)**

a) Let $S \subseteq \mathbb{R}$. Using the definition of the *lower* (respectively, *upper*) *bound* of $S$ write down what it does mean that an element $x \in \mathbb{R}$ is not a lower (respectively, upper) bound of $S$.

b) Fill in the following table:

| S | LB($S$) | UB($S$) | $\min S$ | $\max S$ | $\inf S$ | $\sup S$ |
|---|---|---|---|---|---|---|
| $\emptyset$ | | | | | | |
| $(-5, 3) \cup [4, +\infty)$ | | | | | | |
| $(-2, 4) \cup \{5\}$ | | | | | | |
| $(-\infty, 0] \cup \{1, 2\}$ | | | | | | |
| $(-2, 3) \cap \mathbb{Z}$ | | | | | | |
| $\mathbb{N}$ | | | | | | |
| $(-2, \sqrt{3}) \cap \mathbb{Q}$ | | | | | | |
| $\{x \in \mathbb{R} \mid x^3 - x^2 - 6x \geq 0\}$ | | | | | | |

c) Give an example of a subset $S$ of $\mathbb{R}$ that satisfies simultaneously the following conditions: it is not an interval, it is unbounded below, it doesn't have a greatest element, and $\sup S = -1$.

**(G 5) (Train your brain)**

Let $S \subseteq \mathbb{R}$.

a) Prove that if UB($S$) $\neq \emptyset$, then UB($S$) contains infinitely many elements.

b) Prove that if $S$ has a greatest element, then $\max S = \sup S$.

c) Prove that $S$ has at most one greatest element. (In other words, $S$ cannot have two distinct greatest elements.)

d) Prove that $S$ has at most one supremum. (In other words, $S$ cannot have two distinct suprema.)

HOMEWORK:

**(H 5) (To be delivered in the next exercise-class)**

a) Fill in the following table:

| A | LB($A$) | UB($A$) | $\min A$ | $\max A$ | $\inf A$ | $\sup A$ |
|---|---|---|---|---|---|---|
| $\mathbb{R}_+$ | | | | | | |
| $\mathbb{Q}^*$ | | | | | | |
| $[-2,1)\cup(2,\infty)$ | | | | | | |
| $(-\infty,-1)\cup(2,3)$ | | | | | | |
| $(-2,5)\cap\mathbb{N}$ | | | | | | |
| $\mathbb{Z}$ | | | | | | |
| $(-\infty,5]\cap\mathbb{Q}$ | | | | | | |
| $\left\{x\in\mathbb{R}\mid \frac{x+1}{x^2+1}<1\right\}$ | | | | | | |

b) Give an example of a subset $S$ of $\mathbb{R}$ that satisfies simultaneously the following conditions: it is not an interval, it is unbounded above, it doesn't have a least element, and $\inf S = 3$.

**(H 6) (Train your brain)**

Let $S \subseteq \mathbb{R}$.

a) Prove that if $\mathrm{LB}(S) \neq \emptyset$, then $\mathrm{LB}(S)$ contains infinitely many elements.

b) Prove that if $S$ has a least element, then $\min S = \inf S$.

c) Prove that $S$ has at most one least element. (In other words, $S$ cannot have two distinct least elements.)

d) Prove that $S$ has at most one infimum. (In other words, $S$ cannot have two distinct infima.)

**(H 7) (Train your brain)**

Having the proof of **C2** in the first course as a model, prove **C4**.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.1

# Analysis for CS

**(G 4)**

a) An element $x \in \mathbb{R}$ is not a lower bound of $S$ if $\exists s \in S$ such that $s < x$. An element $x \in \mathbb{R}$ is not an upper bound of $S$ if $\exists s \in S$ such that $s > x$.

b)

| S | LB($S$) | UB($S$) | $\min S$ | $\max S$ | $\inf S$ | $\sup S$ |
|---|---|---|---|---|---|---|
| $\emptyset$ | $\mathbb{R}$ | $\mathbb{R}$ | $\nexists$ | $\nexists$ | $\infty$ | $-\infty$ |
| $(-5,3) \cup [4,+\infty)$ | $(-\infty,-5]$ | $\emptyset$ | $\nexists$ | $\nexists$ | -5 | $\infty$ |
| $(-2,4) \cup \{5\}$ | $(-\infty,-2]$ | $[5,\infty)$ | $\nexists$ | 5 | -2 | 5 |
| $(-\infty,0] \cup \{1,2\}$ | $\emptyset$ | $[2,\infty)$ | $\nexists$ | 2 | $-\infty$ | 2 |
| $(-2,3) \cap \mathbb{Z}$ | $(-\infty,-1)$ | $[2,\infty)$ | -1 | 2 | -1 | 2 |
| $\mathbb{N}$ | $(-\infty,0]$ | $\emptyset$ | 0 | $\nexists$ | 0 | $\infty$ |
| $(-2,\sqrt{3}) \cap \mathbb{Q}$ | $(-\infty,-2]$ | $[\sqrt{3},\infty)$ | $\nexists$ | $\nexists$ | -2 | $\sqrt{3}$ |
| $\{x \in \mathbb{R} \mid x^3 - x^2 - 6x \geq 0\}$ | $(-\infty,-2]$ | $\emptyset$ | -2 | $\nexists$ | -2 | $\infty$ |

For the last set note that $x^3 - x^2 - 6x \geq 0 \Leftrightarrow x(x-3)(x+2) \geq 0 \Leftrightarrow x \in [-2,0] \cup [3,\infty)$.

c) Take for instance $S = (-\infty,-5] \cup (-2,-1)$.

**(G 5)**

Let $S \subseteq \mathbb{R}$.

a) If $x \in$UB($S$) then $[x,\infty) \subseteq$UB($S$), thus UB($S$) contains infinitely many elements.

b) Since $S$ has a greatest element, $S$ is nonempty and bounded above, thus $\sup S \in \mathbb{R}$. As $\max S \in S$ and $\sup S \in UB(S)$ we have from the definition of the upper bound that

$$\max S \leq \sup S. \tag{1}$$

From the definition of the greatest element we have that $\max S \in S \cap UB(S)$. Since $\sup S$ is the least upper bound, and $\max S \in UB(S)$, it holds

$$\sup S \leq \max S. \tag{2}$$

Hence, from (1) and (2), we get the desired conclusion, i.e., $\max S = \sup S$.

c) When there is no greatest element, the statement is obvious. Let us now consider the case when $m_1 \in S$ and $m_2 \in S$ are such that they are both greatest elements of $S$.

From $m_1$ being the greatest element of $S$ and $m_2 \in S$ we have

$$m_2 \leq m_1. \tag{3}$$

From $m_2$ being the greatest element of $S$ and $m_1 \in S$ we have

$$m_1 \leq m_2. \tag{4}$$

Hence, from (3) and (4), we get the desired conclusion, i.e., $m_1 = m_2$.

d) If $S = \emptyset$, then, by definition, $-\infty$ is the supremum of $S$. If $S$ is unbounded above, then $\infty$ is the supremum of $S$. In this case the supremum cannot be a real number, since that would imply the boundedness from above of $S$.

Suppose now that $S$ is nonempty and bounded above. In this case the supremum of $S$ cannot be $\infty$. Assume that $a$ and $b$ are reals and both suprema of $S$. Note that in particular both $a$ and $b$ are then upper bounds of $S$. Since $a$ is a least upper bound of $S$ and $b$ is an upper bound of $S$, $a \leq b$. Similarly, since $b$ is a least upper bound and $a$ is an upper bound of $S$, $b \leq a$. Thus $a = b$.

In conclusion, the supremum of a set is unique.

HOMEWORK:

### (H 5)

a)

| A | LB($A$) | UB($A$) | $\min A$ | $\max A$ | $\inf A$ | $\sup A$ |
|---|---|---|---|---|---|---|
| $\mathbb{R}_+$ | $(-\infty, 0]$ | $\emptyset$ | 0 | $\nexists$ | 0 | $\infty$ |
| $\mathbb{Q}^*$ | $\emptyset$ | $\emptyset$ | $\nexists$ | $\nexists$ | $-\infty$ | $\infty$ |
| $[-2, 1) \cup (2, \infty)$ | $(-\infty, -2]$ | $\emptyset$ | -2 | $\nexists$ | -2 | $\infty$ |
| $(-\infty, -1) \cup (2, 3)$ | $\emptyset$ | $[3, +\infty]$ | $\nexists$ | $\nexists$ | $-\infty$ | 3 |
| $(-2, 5) \cap \mathbb{N}$ | $(-\infty, 0]$ | $[4, +\infty)$ | 0 | 4 | 0 | 4 |
| $\mathbb{Z}$ | $\emptyset$ | $\emptyset$ | $\nexists$ | $\nexists$ | $-\infty$ | $\infty$ |
| $(-\infty, 5] \cap \mathbb{Q}$ | $\emptyset$ | $[5, +\infty)$ | $\nexists$ | 5 | $-\infty$ | 5 |
| $\left\{x \in \mathbb{R} \mid \frac{x+1}{x^2+1} < 1\right\}$ | $\emptyset$ | $\emptyset$ | $\nexists$ | $\nexists$ | $-\infty$ | $\infty$ |

For the last set note that $\frac{x+1}{x^2+1} < 1 \Leftrightarrow (x+1) - (x^2+1) < 0 \Leftrightarrow x \in (-\infty, 0) \cup (1, +\infty)$.

b) Take for instance $S = (3, 4] \cup (5, +\infty)$ or $S = (3, +\infty) \cap \mathbb{Q}$.

### (H 6)

The proofs are almost similar to those done in (G 5).

### (H 7)

**C4**: Let $T$ be a subset of $\mathbb{R}$ which is bounded below and let $S$ be a nonempty subset of $T$. Then $S$ is also bounded below and the inequality $\inf T \leq \inf S$ does hold.

**Proof:** From $\emptyset \neq S \subseteq T \Rightarrow T \neq \emptyset$. As $T$ is nonempty, from **Th3** in the first lecture we know that $\exists \inf T \in \mathbb{R}$. Since $\inf T \in \mathrm{LB}(T)$, we have that $\inf T \leq t, \forall t \in T$, and thus, due to the fact that $S \subseteq T$, we have $\inf T \leq s$ for all $s \in S$. This means that $\inf T \in \mathrm{LB}(S)$, implying that $S$ is bounded below. As $S \neq \emptyset$, **Th3** in the first lecture yields that $\exists \inf S \in \mathbb{R}$. Since $\inf T \in \mathrm{LB}(S)$, we finally get $\inf T \leq \inf S$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.2

# Analysis for CS

GROUPWORK:

**(G 6)**

Using the rules of calculation for limits, compute the limit of the sequences having the general term defined as follows:

a) $\frac{5^n+1}{7^n+1}$, b) $\frac{4^n+(-2)^n}{4^{n-1}+2}$, c) $\left(\sin\frac{\pi}{10}\right)^n$, d) $\sqrt{9n^2+2n+1}-3n$, e) $\left(5+\frac{1-2n^3}{3n^4+2}\right)^2$,

f) $\sqrt{n^2+3}-\sqrt{n^3+1}$, g) $\left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}}$, h) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right)$.

**(G 7)**

Study the monotonicity, the boundedness properties and the convergence of the sequence $(x_n)_{n\in\mathbb{N}^*}$ in each of the following cases, also motivating your statements:

a) $x_n=\frac{2^n+3^n}{5^n}$, b) $x_n=\frac{(-1)^n}{n}$, c) $x_n=\frac{2^n}{n!}$, d) $x_n=\frac{n}{n^2+1}$.

**(G 8) (Train your brain)**

Treat cases 3 and 4 in the proof of **Th2** in the second course.

HOMEWORK:

**(H 8) (To be delivered in the next exercise-class)**

1) Using the rules of calculation for limits, compute the limit of the sequences having the general term defined as follows:

a) $\frac{3^n}{4^n}$, b) $\frac{2^n+(-2)^n}{3^n}$, c) $\frac{5-n^3}{n^2+1}$, d) $\left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2}$, e) $\frac{1+2+\dots+n}{n^2}$,

f) $\left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}$, g) $(\cos(-2013))^n$, h) $\left(\frac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}}$.

2) Study the monotonicity, the boundedness properties and the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n=\sqrt{n+1}-\sqrt{n}$, and motivate your statements.

**(H 9)**

Study the convergence of the sequence with general term defined as follows

$$a_n=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{n^2}\right),\ n\geq 2.$$

In case the sequence is convergent, determine its limit.

**(H 10) (Train your brain)**

Using the density property of the set $\mathbb{Q}$ (see **Th5** in the first course), prove that every real number is the limit of a sequence of rational numbers. Moreover, prove that this sequence can be chosen to be increasing (or decreasing).

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.2

# Analysis for CS

**(G 6)**

a) $\lim\limits_{n\to\infty} \dfrac{5^n+1}{7^n+1} = \lim\limits_{n\to\infty} \left( \left(\dfrac{5}{7}\right)^n \cdot \dfrac{1+\frac{1}{5^n}}{1+\frac{1}{7^n}} \right) = 0.$

b) $\lim\limits_{n\to\infty} \dfrac{4^n+(-2)^n}{4^{n-1}+2} = \lim\limits_{n\to\infty} \dfrac{4^n(1+\left(-\frac{1}{2}\right)^n)}{4^{n-1}\left(1+\frac{2}{4^{n-1}}\right)} = 4.$

c) $\lim\limits_{n\to\infty} \left(\sin\dfrac{\pi}{10}\right)^n = 0$, since $-1 < \sin\frac{\pi}{10} < 1$.

d) We have that

$$\lim_{n\to\infty} \sqrt{9n^2+2n+1} - 3n = \lim_{n\to\infty} \frac{2n+1}{\sqrt{9n^2+2n+1}+3n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{9+\frac{2}{n}+\frac{1}{n^2}}+3} = \frac{1}{3}.$$

e) $\lim\limits_{n\to\infty} \left(5+\dfrac{1-2n^3}{3n^4+2}\right)^2 = 25.$

f) $\lim\limits_{n\to\infty} \sqrt{n^2+3} - \sqrt{n^3+1} = \lim\limits_{n\to\infty} \sqrt{n^3+1}\left(\sqrt{\dfrac{n^2+3}{n^3+1}} - 1\right) = -\infty.$

g) $\lim\limits_{n\to\infty} \left(\dfrac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}} = \infty^{-\frac{5}{6}} = 0.$

h) Since

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right) = \frac{1}{n},$$

we get $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{2}\right)\left(1-\dfrac{1}{3}\right)\dots\left(1-\dfrac{1}{n}\right) = 0.$

**(G 7)**

a) We first note that, if $q \in (0,1)$, then $q^n > q^{n+1}$, for every $n \in \mathbb{N}$. Thus the sequence $(q^n)_{n\in\mathbb{N}}$ is strictly decreasing. In our case $x_n = \left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n$, for $n \in \mathbb{N}^*$. By the previous observation, we conclude that the sequence $(x_n)_{n\in\mathbb{N}^*}$ is strictly decreasing, hence it is bounded above. Since $x_n > 0$, for every $n \in \mathbb{N}^*$, the sequence is also bounded below. The sequence converges to 0.

b) We notice that for $n = 2k$, $x_{2k} = \frac{1}{k}$, while for $n = 2k-1$, $x_{2k-1} = -\frac{1}{2k-1}$, for $k \in \mathbb{N}^*$. Since $x_1 < x_2$ and $x_2 > x_3$, the sequence $(x_n)_{n\in\mathbb{N}^*}$ is not monotonic. From $|x_n| = \frac{1}{n} \in [-1,1]$, for every $n \in \mathbb{N}^*$, we conclude that $(x_n)_{n\in\mathbb{N}^*}$ is bounded. Since $\lim\limits_{n\to\infty} |x_n| = 0$, the sequence $(x_n)_{n\in\mathbb{N}^*}$ converges to 0.

c) We notice that

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \leq 1, \forall n \in \mathbb{N}^*.$$

Thus $(x_n)_{n\in\mathbb{N}^*}$ is a decreasing sequence. Moreover $x_n \leq x_1 = 2$ for all $n \in \mathbb{N}^*$. It is obvious that $x_n > 0$ for all $n \in \mathbb{N}^*$, implying thus that $(x_n)_{n\in\mathbb{N}^*}$ is bounded. But then $(x_n)_{n\in\mathbb{N}^*}$ converges to a real number $\ell$. Assuming, by contradiction, that $\ell \neq 0$, we get that

$$1 = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2}{n+1} = 0,$$

a contradiction. Thus $\ell = 0$.

d) We notice that

$$x_{n+1} - x_n = \frac{n+1}{n^2+2n+2} - \frac{n}{n^2+1} = \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \text{ for all } n \in \mathbb{N}^*.$$

Thus $(x_n)_{n\in\mathbb{N}^*}$ is a strictly decreasing sequence. Moreover $x_n \leq x_1 = \frac{1}{2}$ for all $n \in \mathbb{N}^*$. It is obvious that $x_n > 0$ for all $n \in \mathbb{N}^*$, implying thus that $(x_n)_{n\in\mathbb{N}^*}$ is bounded. The sequence $(x_n)_{n\in\mathbb{N}^*}$ converges to 0.

**(G 8)**

**Case 3** $x \in \mathbb{R}$ and $y = -\infty$.

From $\lim\limits_{n\to\infty} x_n = x$ we have that $\exists n(1) \in \mathbb{N}$ such that $|x_n - x| < 1$, $\forall n \geq n(1)$.

From $\lim\limits_{n\to\infty} x_n = -\infty$ we have that $\exists n(x-1) \in \mathbb{N}$ such that $x_n < x-1$, $\forall n \geq n(x-1)$.

Fix a natural number $n \geq \max\{n(1), n(x-1)\}$, then we have

$$|x_n - x| < 1 \text{ and } x_n < x - 1.$$

As $x_n < x-1$ we have that $|x_n - x| = x - x_n$. Hence we get

$$x - x_n < 1 \text{ and } 1 < x - x_n$$

which is a contradiction.

**Case 3** $x = -\infty$ and $y = \infty$.

From $\lim\limits_{n\to\infty} x_n = -\infty$ we have that $\exists n(1) \in \mathbb{N}$ such that $x_n < 1$, $\forall n \geq n(1)$.

From $\lim\limits_{n\to\infty} x_n = \infty$ we have that $\exists m(1) \in \mathbb{N}$ such that $x_n > 1$, $\forall n \geq m(1)$.

Fix a natural number $n \geq \max\{n(1), m(1)\}$, then we have

$$x_n < 1 \text{ and } x_n > 1,$$

which is a contradiction.

Homework:

**(H 8)**

1) a) $\lim\limits_{n\to\infty} \dfrac{3^n}{4^n} = \lim\limits_{n\to\infty} \left(\dfrac{3}{4}\right)^n = 0$, since $-1 < \frac{3}{4} < 1$.

b) $\lim\limits_{n\to\infty} \dfrac{2^n + (-2)^n}{3^n} = \lim\limits_{n\to\infty} \left(\left(\dfrac{2}{3}\right)^n + \left(-\dfrac{2}{3}\right)^n\right) = 0.$

c) $\lim\limits_{n\to\infty} \dfrac{5-n^3}{n^2+1} = -\infty$.

d) $\lim\limits_{n\to\infty} \left(2+\dfrac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2} = 2^\infty = \infty$.

e) Since $1+2+\cdots+n = \frac{n(n+1)}{2}$, we get

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \frac{1}{2}.$$

f) $\lim\limits_{n\to\infty} \left(\dfrac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}} = \left(\dfrac{1}{2}\right)^{-2} = 4$.

g) $\lim\limits_{n\to\infty} (\cos(-2013))^n = 0$, since $-1 < \cos(-2013) < 1$.

h) $\lim\limits_{n\to\infty} \left(\dfrac{n^5+3n+1}{2n^5-n^4+3}\right)^{\frac{3n-n^4}{n^3+1}} = \left(\dfrac{1}{2}\right)^{-\infty} = \infty$.

2) Note that $x_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, for every $n \in \mathbb{N}$. The sequence $(\sqrt{n})_{n\in\mathbb{N}}$ being strictly increasing, we conclude that $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing, hence bounded above. Since $x_n > 0$, for every $n \in \mathbb{N}$, the sequence is also bounded below. We have that $\lim\limits_{n\to\infty} x_n = 0$, thus the sequence converges to 0.

**(H 9)**

For every $n \geq 2$ we have that

$$(2^2-1)\cdot\dots\cdot(n^2-1) = \prod_{k=2}^{n}(k^2-1) = \prod_{k=2}^{n}(k-1)(k+1) = (n-1)!\frac{(n+1)!}{2}.$$

We obtain that

$$a_n = \frac{(n-1)!(n+1)!}{2n!\,n!} = \frac{n+1}{2n},$$

thus $\lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$.

**(H 10)**

Let $x \in \mathbb{R}$ be arbitrary. We prove that $x$ is the limit of a decreasing sequence of real numbers. By the density property of the set $\mathbb{Q}$, there exists a rational number $x_1 \in (x, x+1)$. Using once again this property, there exists a rational number $x_2 \in (x, \min\{x_1, x+\frac{1}{2}\})$. Thus $x < x_2 < x_1$ and $x_2 < x + \frac{1}{2}$. We continue inductively this procedure: Assuming that $x_n \in \mathbb{Q}$, $n \geq 2$, has been chosen such that $x < x_n < x_{n-1}$ and $x_n < x + \frac{1}{n}$, there exists (according to the density property of the set $\mathbb{Q}$) a rational number $x_{n+1}$ such that $x_{n+1} \in (x, \min\{x_n, x+\frac{1}{n+1}\})$. Hence $x < x_{n+1} < x_n$ and $x_{n+1} < x + \frac{1}{n+1}$. This way we obtain the strictly decreasing sequence $(x_n)_{n\in\mathbb{N}^*}$ of rational numbers with the property that

$$x < x_n < x + \frac{1}{n}, \forall\, n \in \mathbb{N}^*.$$

Applying the Sandwich-Theorem, we conclude that $\lim\limits_{n\to\infty} x_n = x$. Hence every real number is the limit of a decreasing sequence of rational numbers.

In order to prove that every real number is the limit of an increasing sequence of rationals, consider an arbitrary $x \in \mathbb{R}$. We already know that $-x$ is the limit of a decreasing sequence $(x_n)_{n\in\mathbb{N}^*}$ of rationals. Then $(-x_n)_{n\in\mathbb{N}^*}$ is an increasing sequence of rationals converging to $x$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.3

# Analysis for CS

GROUPWORK:

**(G 9)**

Compute the limit of the sequences having the general term defined as follows:

a) $\left(1+\frac{1}{-n^3+3n}\right)^{n^2-n^3}$, b) $(3n^2+5)\ln\left(1+\frac{1}{n^2}\right)$, c) $\frac{n^n}{1+2^2+3^3+\cdots+n^n}$,

d) $\frac{x_1+2x_2+\cdots+nx_n}{n^2}$, where $(x_n)_{n\geq 1}$ is a sequence converging to $x \in \mathbb{R}$.

**(G 10) (A sequence approximating $\frac{1}{a}$)**

Let $a > 0$, and fix $x_0 \in \mathbb{R}$ such that $0 < x_0 < \frac{1}{a}$. Define $(x_n)_{n\in\mathbb{N}}$ recursively as

$$x_{n+1} = 2x_n - ax_n^2,\ \forall n \in \mathbb{N}.$$

Prove that $(x_n)_{n\in\mathbb{N}}$ converges to $\frac{1}{a}$, keeping in mind the following steps:

(i) Prove (using mathematical induction) that $x_n < \frac{1}{a}$, $\forall n \in \mathbb{N}$.

(ii) Prove (using mathematical induction) that $0 < x_n$, $\forall n \in \mathbb{N}$.

(iii) Using (i) and (ii), prove that $(x_n)_{n\in\mathbb{N}}$ is strictly increasing.

(iv) Finally conclude that $(x_n)_{n\in\mathbb{N}}$ is convergent and that $\lim\limits_{n\to\infty} x_n = \dfrac{1}{a}$.

**(G 11) (Train your brain)**

Prove Th 6 (concerning limits and boundedness properties) in the third course.

Homework:

**(H 11) (To be delivered in the next exercise-class)**

1) Compute the limit of the sequences having the general term defined as follows:

a) $\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n}$, b) $\frac{\sqrt{1+2^2}+\sqrt{1+3^2}+\cdots+\sqrt{1+n^2}}{1+n^2}$,

c) $\frac{x_0+2^1x_1+2^2x_2+\cdots+2^nx_n}{2^{n+1}}$, where $(x_n)_{n\geq 0}$ is a sequence converging to $x\in\mathbb{R}$.

2) Prove statement 2° of Th 7 in the third course: If $(x_n)_{n\in\mathbb{N}^*}$ is a decreasing sequence and if $X$ is the set consisting of all its terms, then $\lim\limits_{n\to\infty} x_n = \inf X$.

**(H 12)**

a) Prove (either by a direct computation or by mathematical induction) that the following equalities hold for every natural number $n\geq 2$

$$\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2\left(1+\frac{1}{3}\right)^3\cdots\left(1+\frac{1}{n-1}\right)^{n-1}=\frac{n^n}{n!},$$

$$\left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4\cdots\left(1+\frac{1}{n-1}\right)^{n}=\frac{n^n}{(n-1)!}.$$

b) Using a) and the following inequalities (proved in the third course)

$$\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1},\ \forall n\in\mathbb{N}^*,$$

show that

$$e\left(\frac{n}{e}\right)^n\leq n!\leq en\left(\frac{n}{e}\right)^n,\ \forall n\in\mathbb{N}^*.$$

c) Using b) and the Sandwich-Theorem, prove that $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.3

# Analysis for CS

**(G 9)**

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{-n^3+3n}\right)^{n^2-n^3} = \lim\limits_{n\to\infty}\left(\left(1+\dfrac{1}{-n^3+3n}\right)^{-n^3+3n}\right)^{\frac{n^2-n^3}{-n^3+3n}} = e.$

b) $\lim\limits_{n\to\infty}(3n^2+5)\ln\left(1+\dfrac{1}{n^2}\right) = \lim\limits_{n\to\infty}\ln\left(1+\dfrac{1}{n^2}\right)^{3n^2+5} = \ln\left(\left(1+\dfrac{1}{n^2}\right)^{n^2}\right)^{\frac{3n^2+5}{n^2}} = \ln e^3 = 3.$

c) For $n \geq 1$ let $x_n = n^n$ and $y_n = 1+2^2+3^3+\dots+n^n$. The sequence $(y_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit $\infty$. Since

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} = 1-\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{n+1} = 1,$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = 1$.

d) For $n \geq 1$ let $a_n = x_1+2x_2+\dots+nx_n$ and $b_n = n^2$. The sequence $(b_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit $\infty$. Since

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{(n+1)x_{n+1}}{2n+1} = \frac{x}{2},$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{x}{2}$.

**(G 10)**

(i) The proposition we prove, for $n \in \mathbb{N}$, through mathematical induction is

$$P(n): x_n < \frac{1}{a}.$$

I. $P(0)$ is true from the hypothesis.

II. We assume that $P(k)$ is true and prove that $P(k+1)$ is also true, for some $k \in \mathbb{N}$. Thus we know that $x_k < \frac{1}{a}$. Since $a > 0$, let us further notice the following chain of equivalences

$$x_{k+1} < \frac{1}{a} \iff 2x_k - ax_k^2 < \frac{1}{a} \iff 2ax_k - a^2x_k^2 < 1 \iff (ax_k-1)^2 > 0.$$

Thus $P(k+1)$ is true, since $ax_k - 1 \neq 0$, fact known from $P(k)$.

(ii) The proposition we prove, for $n \in \mathbb{N}$, through mathematical induction is

$$Q(n): x_n > 0.$$

I. $Q(0)$ is true from the hypothesis.

II. We assume that $Q(k)$ is true and prove that $Q(k+1)$ is also true, for some $k \in \mathbb{N}$. Thus we know that $x_k > 0$. Let us further notice the following chain of equivalences

$$x_{k+1} > 0 \Longleftrightarrow 2x_k - ax_k^2 > 0 \Longleftrightarrow 2 - ax_k > 0 \Longleftrightarrow \frac{2}{a} > x_k.$$

Thus $P(k+1)$ is true, since $\frac{2}{a} > \frac{1}{a} > x_k$, fact known from $P(k)$.

(iii) Let $n \in \mathbb{N}$ be arbitrarily chosen. Then, using (i) and (ii), we get

$$x_{n+1} - x_n = x_n - ax_n^2 = x_n(1 - ax_n) > 0.$$

Thus $(x_n)_n \in \mathbb{N}$ is a strictly increasing sequence.

(iv) From (i), (ii) and (iii) we have that the sequence $(x_n)_{n\in\mathbb{N}}$ is both strictly increasing and bounded. Thus, it is convergent, so there exists $l = \lim\limits_{n\to\infty} x_n \in \mathbb{R}$. Hence, we may pass to limit in the recurrence relation

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (2x_n - ax_n^2) \Longleftrightarrow l = 2l - al^2 \Longleftrightarrow l(1 - al) = 0 \Longleftrightarrow l = 0 \text{ or } l = \frac{1}{a}.$$

As $(x_n)_{n\in\mathbb{N}}$ is strictly increasing with positive terms, we get the conclusion that $l = \frac{1}{a}$.

**(G 11)**

**Th6** (Limits and boundedness properties)

For a sequence $(x_n)_{n\in\mathbb{N}}$, the following assertions hold:

1) If $(x_n)_{n\in\mathbb{N}}$ is convergent, then $(x_n)_{n\in\mathbb{N}}$ is bounded.
2) If $\lim\limits_{n\to\infty} x_n = \infty$ then $(x_n)_{n\in\mathbb{N}}$ is unbounded above.
3) If $\lim\limits_{n\to\infty} x_n = -\infty$ then $(x_n)_{n\in\mathbb{N}}$ is unbounded below.

**Proof:** 1) We have $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$. This means, from the definition, that

$$\forall \varepsilon > 0, \exists n(\varepsilon) \in \mathbb{N} \text{ such that } |x_n - x| < \varepsilon, \forall n \geq n(\varepsilon).$$

For $\varepsilon = 1, \exists\ n(1) \in \mathbb{N}^*$ such that $|x_n - x| < 1, \forall n \geq n(1)$, which can be rewritten as

$$-1 < x_n - x < 1 \Longleftrightarrow x - 1 < x_n < x + 1 \Longleftrightarrow |x_n| < \max\{|x+1|, |x-1|\}, \forall n \geq n(1)$$
$$\Longleftrightarrow |x_n| < \max\{|x+1|, |x-1|\}, \forall n \geq n(1).$$

Thus we know that all the terms of the sequence $(x_n)_{n\in\mathbb{N}}$, which have an index $\geq n(1)$, are bounded. As well, the other elements, i.e., $x_0, x_1, ..., x_{n(1)-1}$ are all bounded by their maximum, maximum that exists as it is taken from a set of finite elements. Hence we have that

$$|x_n| < \max\{|x+1|, |x-1|, |x_0|, |x_1|, ..., |x_{n(1)-1}|\}, \forall\ n \in \mathbb{N},$$

showing that $(x_n)_{n\in\mathbb{N}}$ is bounded.

2) Since $\lim\limits_{n\to\infty} x_n = \infty$, we have that, for every $t \in \mathbb{R}$, there exists an index $n(t)$ such that $x_n > t$. This fact shows that $(x_n)_{n\in\mathbb{N}}$ is unbounded above.

3) Since $\lim\limits_{n\to\infty} x_n = -\infty$, we have that, for every $t \in \mathbb{R}$, there exists an index $n(t)$ such that $x_n < t$. This fact shows that $(x_n)_{n\in\mathbb{N}}$ is unbounded below.

Homework:

### (H 11)

1) a) For $n \geq 1$ let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $y_n = n$. The sequence $(y_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit $\infty$. Since

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 0$.

b) For $n \geq 1$ let $x_n = \sqrt{1+2^2} + \sqrt{1+3^2} + \cdots + \sqrt{1+n^2}$ and $y_n = 1 + n^2$. The sequence $(y_n)_{n\in\mathbb{N}^*}$ is strictly increasing and has limit $\infty$. Since

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{\sqrt{1+(n+1)^2}}{2n+1} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}{2+\frac{1}{n}} = \frac{1}{2},$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{1}{2}$.

c) For $n \in \mathbb{N}$ let $a_n = x_0 + 2^1 x_1 + 2^2 x_2 + \cdots + 2^n x_n$ and $b_n = 2^{n+1}$. The sequence $(b_n)_{n\in\mathbb{N}}$ is strictly increasing and has limit $\infty$. Since

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{2^{n+1} x_{n+1}}{2^{n+2} - 2^{n+1}} = \lim_{n\to\infty} x_{n+1} = x$$

the Theorem of Stolz-Cesàro yields that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = x$.

2) Case 1: The set $X$ is unbounded below. In this case $\inf X = -\infty$. Let $t \in \mathbb{R}$ be arbitrary. Since $X$ is unbounded below, there exists an index $n(t) \in \mathbb{N}$ such that $x_{n(t)} < t$. The sequence being decreasing, we get that $x_n \leq x_{n(t)} < t$, for every $n \geq n(t)$. Thus $\lim\limits_{n\to\infty} x_n = -\infty$.

Case 2: The set $X$ is bounded below. In this case $x := \inf X \in \mathbb{R}$. Let $\varepsilon > 0$. Then $x + \varepsilon > x$, hence $x + \varepsilon$ cannot be a lower bound of $X$. Thus there exists $n(\varepsilon) \in \mathbb{N}$ such that $x_{n(\varepsilon)} < x + \varepsilon$. The sequence being decreasing, we get that $x_n \leq x_{n(\varepsilon)} < x + \varepsilon$, for every $n \geq n(\varepsilon)$. Hence $x \leq x_n < x + \varepsilon$, for every $n \geq n(\varepsilon)$. We conclude that $|x_n - x| = x_n - x < \varepsilon$, for every $n \geq n(\varepsilon)$, showing that $\lim\limits_{n\to\infty} x_n = x$.

### (H 12)

a) The equalities follow by a direct computation.

b) We have

$$\left(1+\frac{1}{1}\right)^1 < e < \left(1+\frac{1}{1}\right)^2,$$
$$\left(1+\frac{1}{2}\right)^2 < e < \left(1+\frac{1}{2}\right)^3,$$
$$\cdots$$
$$\left(1+\frac{1}{n-1}\right)^{n-1} < e < \left(1+\frac{1}{n-1}\right)^n.$$

All the terms involved in the equalities written above are positive, therefore we may multiply them and keep the inequalities. Hence we get

$$\left(1+\frac{1}{1}\right)^1\cdot\left(1+\frac{1}{2}\right)^2\cdot\ldots\cdot\left(1+\frac{1}{n-1}\right)^{n-1}<e^{n-1}<\left(1+\frac{1}{1}\right)^2\cdot\left(1+\frac{1}{2}\right)^3\cdot\ldots\cdot\left(1+\frac{1}{n-1}\right)^n.$$

Applying now a) we obtain the following

$$\frac{n^n}{n!}<e^{n-1}<\frac{n^n}{(n-1)!}\Longleftrightarrow e\frac{n^n}{n!}<e^n<e\frac{n^n}{(n-1)!}\Longleftrightarrow e\left(\frac{n}{e}\right)^n\leq n!\leq en\left(\frac{n}{e}\right)^n,\ \forall n\in\mathbb{N}^*.$$

c) From the first inequality in the above chain of inequalities we get that

$$\frac{n^n}{n!}<e^{n-1}\text{ and }e^{n-1}<n\frac{n^n}{n!},\ \forall n\in\mathbb{N}^*,$$

thus

$$\frac{e^{n-1}}{n}<\frac{n^n}{n!}<e^{n-1},\forall n\in\mathbb{N}^*.$$

Since all the terms in the above inequality are positive, by taking the $n$-th root we do not affect them, hence

$$\frac{e^{\frac{n-1}{n}}}{\sqrt[n]{n}}<\frac{n}{\sqrt[n]{n!}}<e^{\frac{n-1}{n}},\forall n\in\mathbb{N}^*.$$

By passing to limit, knowing from a previous seminar that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$, and computing $\lim\limits_{n\to\infty}e^{\frac{n-1}{n}}=e$, we obtain, using the Sandwich-Theorem, the desired conclusion, i.e.,

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e.$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.4

# Analysis for CS

GROUPWORK:

**(G 12)**

Compute the sum of the following series, indicating in each case the results you are using.

a) $\displaystyle\sum_{n\geq 1} \frac{1}{\sqrt[3]{n}}$ b) $\displaystyle\sum_{n\geq 1} \frac{3}{4^n}$, c) $\displaystyle\sum_{n\geq 2} \frac{1}{3^{n-1}}$, d) $\displaystyle\sum_{n\geq 1} \frac{1}{4n^2-1}$, e) $\displaystyle\sum_{n\geq 1} \frac{2n+1}{n!}$,

f) $\displaystyle\sum_{n\geq 1} \frac{1}{\sqrt{n}+\sqrt{n+1}}$, g) $\displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$, h) $\displaystyle\sum_{n\geq 0} \frac{1}{n!+(n+1)!}$.

HOMEWORK:

**(H 13) (To be delivered in the next exercise-class)**

Compute the sum of the following series, indicating in each case the results you are using.

a) $\displaystyle\sum_{n\geq 0} \frac{(-3)^n}{4^n}$, b) $\displaystyle\sum_{n\geq 1} \frac{1}{\sqrt[5]{n}}$, c) $\displaystyle\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right)$, d) $\displaystyle\sum_{n\geq 0} \left(-\frac{2}{(n+1)!}+\frac{(-1)^{n+1}}{3^{n+2}}\right)$.

**(H 14)**

Compute the sum of the following telescopic series

a) $\displaystyle\sum_{n\geq 1} \frac{n}{(n+1)(n+2)(n+3)}$, b) $\displaystyle\sum_{n\geq 2} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)}$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.4

# Analysis for CS

**(G 12)**

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \infty$, being a generalized harmonic series with $\alpha := \frac{1}{3} \leq 1$.

b) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get $\sum_{n=1}^{\infty} \frac{3}{4^n} = \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{3}{4} \cdot \frac{1}{1-\frac{1}{4}} = 1$.

c) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get, $\sum_{n=2}^{\infty} \frac{1}{3^{n-1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$.

d) The following equalities hold for every $n \geq 1$

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

Put $a_n := \frac{1}{2} \cdot \frac{1}{2n-1}$ for $n \geq 1$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n = \frac{1}{2}.$$

e) Since for every $n \geq 1$

$$\frac{2n+1}{n!} = \frac{2}{(n-1)!} + \frac{1}{n!},$$

we get, using the rules of calculation for convergent series,

$$\sum_{n=1}^{\infty} \frac{2n+1}{n!} = 2\sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} = 2e + e - 1 = 3e - 1.$$

f) The following equalities hold for every $n \geq 1$

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})} = \sqrt{n+1}-\sqrt{n}.$$

Set $a_n := \sqrt{n}$ for $n \geq 1$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sum_{n=1}^{\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} a_n - a_1 = \infty.$$

g) The following equalities hold for every $n \geq 1$

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)\frac{1}{n+1} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right).$$

Let $a_n := \frac{1}{2} \cdot \frac{1}{n(n+1)}$ for $n \geq 1$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n = \frac{1}{4}.$$

h) The following equalities hold for every $n \geq 1$

$$\frac{1}{n! + (n+1)!} = \frac{1}{n!(n+2)} = \frac{n+1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}.$$

Put $a_n := \frac{1}{(n+1)!}$ for $n \geq 0$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=0}^{\infty} \frac{1}{n! + (n+1)!} = \sum_{n=0}^{\infty} (a_n - a_{n+1}) = a_0 - \lim_{n\to\infty} a_n = 1.$$

Homework:

**(H 13)**

a) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n = \frac{1}{1+\frac{3}{4}} = \frac{4}{7}$.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \infty$, being a generalized harmonic series with $\alpha := \frac{1}{5} \leq 1$.

c) The following equalities hold for every $n \geq 2$

$$\begin{aligned}\ln\left(1 - \frac{1}{n^2}\right) &= \ln(n^2-1) - \ln n^2 = \ln(n+1)(n-1) - 2\ln n = \ln(n+1) + \ln(n-1) - 2\ln n \\ &= (\ln(n+1) - \ln n) - (\ln n - \ln(n-1)).\end{aligned}$$

Let $a_n := \ln n - \ln(n-1) = \ln \frac{n}{n-1}$ for $n \geq 2$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} a_n - a_2 = -\ln 2.$$

d) Using the rules of calculation for convergent series and the formula for the sum of the geometric series, we get

$$\sum_{n=0}^{\infty} \left(-\frac{2}{(n+1)!} + \frac{(-1)^{n+1}}{3^{n+2}}\right) = (-2)\sum_{n=1}^{\infty} \frac{1}{n!} - \frac{1}{9}\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = -2(e-1) - \frac{1}{12} = \frac{23}{12} - 2e.$$

**(H 14)**

a) The following equalities hold for every $n \geq 1$

$$\begin{aligned}\frac{n}{(n+1)(n+2)(n+3)} &= \frac{n+1-1}{(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)} \\ &= \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \frac{1}{(n+1)(n+2)(n+3)}.\end{aligned}$$

Let $a_n := \frac{1}{n+2}$ for $n \geq 1$. Using the rules of calculation for convergent series, (G 12) g) and the formula for the sum of a telescopic series, we get

$$\sum_{n\geq 1} \frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=1}^{\infty}(a_n - a_{n+1}) - \sum_{n=2}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{3} - \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{4}.$$

b) The following equalities hold for every $n \geq 2$

$$\frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)} = \frac{\ln(n+1) - \ln n}{\ln n \cdot \ln(n+1)} = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}.$$

Put $a_n := \frac{1}{\ln n}$ for $n \geq 2$. Using the formula for the sum of a telescopic series, we get

$$\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)} = \sum_{n=2}^{\infty}(a_n - a_{n+1}) = a_2 - \lim_{n\to\infty} a_n = \frac{1}{\ln 2}.$$

Dr. Brigitte Breckner
Dr. Anca Grad

“Babeş–Bolyai” University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.5

# Analysis for CS

GROUPWORK:

**(G 13)**

Decide whether the following series are convergent or not, indicating in each case the criterion you are using.

a) $\displaystyle\sum_{n\geq 0}\frac{3^n}{4^n+5^n}$, b) $\displaystyle\sum_{n\geq 1}\frac{1}{(2n)^\alpha}$, where $\alpha\in\mathbb{R}$, c) $\displaystyle\sum_{n\geq 1}\frac{\sqrt{n+1}-\sqrt{n}}{n^{\frac{3}{4}}}$,

d) $\displaystyle\sum_{n\geq 1}\left(\frac{n}{n+1}\right)^{n^2}$, e) $\displaystyle\sum_{n\geq 1}\frac{x^n}{n^p}$, where $x>0$ and $p\in\mathbb{R}$, f) $\displaystyle\sum_{n\geq 1}\sin\frac{1}{n}$.

HINT for f): If $(x_n)_{n\in\mathbb{N}}$ is a sequence of nonzero reals converging to 0, then $\displaystyle\lim_{n\to\infty}\frac{\sin x_n}{x_n}=1$.

**(G 14)**

Study the convergence and the absolute convergence of the following series.

a) $\displaystyle\sum_{n\geq 1}(-1)^n\left(e-\left(1+\frac{1}{n}\right)^n\right)$, b) $\displaystyle\sum_{n\geq 1}\sin\frac{x}{n}$, where $x\in\mathbb{R}$.

HINT for a): For the absolute convergence of the series use the fact that $(*)\ \displaystyle\lim_{n\to\infty} n\left(e-\left(1+\frac{1}{n}\right)^n\right)=\frac{e}{2}$.

HOMEWORK:

**(H 15) (To be delivered in the next exercise-class)**

1) Decide whether the following series are convergent or not, indicating in each case the criterion you are using.

a) $\displaystyle\sum_{n\geq 0}\frac{1}{(2n+1)^\alpha}$, where $\alpha\in\mathbb{R}$, b) $\displaystyle\sum_{n\geq 1}\frac{1}{\sqrt{n(n+1)}}$, c) $\displaystyle\sum_{n\geq 1}\sin^3\frac{1}{n}$,

d) $\displaystyle\sum_{n\geq 2}\frac{1}{(\ln n)^n}$, e) $\displaystyle\sum_{n\geq 1}n^4e^{-n^2}$.

2) Study the convergence and the absolute convergence of the series $\displaystyle\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^\alpha}$, where $\alpha\in\mathbb{R}$.

**(H 16)**

Consider the series $\displaystyle\sum_{n\geq 1}n!\left(\frac{x}{n}\right)^n$, where $x>0$. Using the equality $(*)$ given in the hint for exercise (G 14), determine the set of those values of $x$ for which the series is convergent.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.5

# Analysis for CS

**(G 13)**

a) The inequality $\frac{3^n}{4^n+5^n} < \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$, for all $n \geq 0$, and the convergence of the geometric series $\sum_{n\geq 0} \left(\frac{3}{4}\right)^n$ imply, according to the first comparison criterion, the convergence of the given series.

b) Since $\sum_{n\geq 1} \frac{1}{(2n)^\alpha} = \frac{1}{2^\alpha} \sum_{n\geq 1} \frac{1}{n^\alpha}$, we conclude that the series is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

c) Since

$$\lim_{n\to\infty} \frac{n^{\frac{5}{4}}(\sqrt{n+1}-\sqrt{n})}{n^{\frac{3}{4}}} = \lim_{n\to\infty} \frac{n^{\frac{5}{4}}}{(\sqrt{n+1}+\sqrt{n})n^{\frac{3}{4}}} = \lim_{n\to\infty} \frac{n^{\frac{5}{4}}}{\left(\sqrt{1+\frac{1}{n}}+1\right) n^{\frac{5}{4}}} = \frac{1}{2},$$

the second comparison criterion yields that the given series is equivalent to the series $\sum_{n\geq 1} \frac{1}{n^{\frac{5}{4}}}$, hence it is convergent.

d) The relations

$$\lim \sqrt[n]{x_n} = \lim \left(\frac{n}{n+1}\right)^n = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

imply, according to the root criterion, the convergence of the series.

e) We put $x_n := \frac{x^n}{n^p}$, for $n \geq 1$ and apply the 2-steps-algorithm. We compute

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} x\left(\frac{n}{n+1}\right)^p = x.$$

Thus, if $x < 1$, the series is convergent, and, if $x > 1$, it is divergent. If $x = 1$ the series becomes the generalized harmonic series $\sum_{n\geq 1} \frac{1}{n^p}$, which is convergent if $p > 1$, and divergent if $p \leq 1$.

f) According to the hint,

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1.$$

Hence, by the second comparison criterion, the given series is equivalent to the harmonic series $\sum_{n\geq 1} \frac{1}{n}$, so it is divergent.

**(G 14)**

a) Let $x_n := e - \left(1+\frac{1}{n}\right)^n$, $n \geq 1$. The sequence $(x_n)_{n\in\mathbb{N}^*}$ is a strictly decreasing sequence converging to 0 (since, according to the third lecture, the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\geq 1}$ is a strictly increasing sequence converging to $e$). The criterion of Leibniz assures now the convergence of the given series. The equality $(*)$ given in the hint implies, according to the second comparison criterion, that the series $\sum\limits_{n\geq 1}\left(e-\left(1+\dfrac{1}{n}\right)^n\right)$ is equivalent to the harmonic series. This shows that the given series is not absolutely convergent.

b) It is obvious that the series is absolutely convergent (hence also convergent) if $x=0$. Assume now that $x>0$. Since $\lim\limits_{n\to\infty}\dfrac{x}{n}=0$, there exists an index $n_0\in\mathbb{N}^*$ such that $\frac{x}{n}\in(0,\pi)$, for all $n\geq n_0$. This means that the terms of the given series are positive up to the index $n_0$. Since, when studying the convergence type of a series it is not important where the summation starts, we may assume without any loss of generality that the summation starts at $n_0$ (in order to assure that all terms of the series are positive). Using the hint, we have

$$\lim_{n\to\infty}\frac{\sin\frac{x}{n}}{\frac{1}{n}}=x>0.$$

By the second comparison criterion the given series is equivalent to the harmonic series, so it is divergent. It follows that the series is not absolutely convergent.

If $x<0$ then, taking into account that $\sin\frac{x}{n}=-\sin\frac{-x}{n}$, for all $n\geq 1$, we can apply the previously obtained result. Thus the series is divergent in this case, too. It follows that it is not absolutely convergent.

HOMEWORK:

**(H 15)**

1) a) In view of $\lim\limits_{n\to\infty}\dfrac{n^\alpha}{(2n+1)^\alpha}=\left(\dfrac{1}{2}\right)^\alpha$, using the second comparison criterion, the given series is equivalent to the harmonic series $\sum\limits_{n\geq 1}\dfrac{1}{n^\alpha}$, hence it is convergent if $\alpha>1$, and divergent if $\alpha\leq 1$.

b) Since $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n(n+1)}}=1$, the second comparison criterion yields that the given series is equivalent to the series $\sum\limits_{n\geq 1}\dfrac{1}{n}$, hence it is divergent.

c) Since

$$\lim_{n\to\infty}\frac{\sin^3\frac{1}{n}}{\frac{1}{n^3}}=1,$$

the second comparison criterion yields that the given series is equivalent to the series $\sum\limits_{n\geq 1}\dfrac{1}{n^3}$, hence it is convergent.

d) The relations $\lim\limits_{n\to\infty}\sqrt[n]{x_n}=\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n}}{\ln n}=0<1$ imply, by the root criterion, the convergence of the given series.

e) The relations

$$\lim_{n\to\infty}D_n=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+1)^4}{n^4e^{2n+1}}=0<1$$

imply, in view of the quotient criterion, the convergence of the given series.

2) Denote by $x_n$ the general term of the given series. If $\alpha < 0$, then $\lim\limits_{n\to\infty} |x_n| = \infty \neq 0$, hence the series $\sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^\alpha}$ is divergent in this case. This series is divergent in the case $\alpha = 0$, too, since then $\lim\limits_{n\to\infty} |x_n| = 1 \neq 0$. If $\alpha > 0$, then $\left(\frac{1}{n^\alpha}\right)$ is a decreasing sequence which converges to 0. Using the criterion of Leibniz, we conclude that $\sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^\alpha}$ is convergent in this case.

Knowing the type of the generalized harmonic series, we conclude that $\sum\limits_{n\geq 1} \dfrac{(-1)^{n+1}}{n^\alpha}$ is absolutely convergent if $\alpha > 1$, and not absolutely convergent if $\alpha \leq 1$.

**(H 16)**

We have that

$$D_n = \frac{x_{n+1}}{x_n} = x\frac{n^n}{(n+1)^n} = x\frac{1}{\left(1+\frac{1}{n}\right)^n},$$

hence $\lim\limits_{n\to\infty} D_n = \dfrac{x}{e}$. According to the first step of the 2-steps-algorithm, we get that, if $x < e$, the given series is convergent, and, if $x > e$, it is divergent. If $x = e$ we continue the algorithm. We have that

$$R_n = n\left(\frac{1}{D_n} - 1\right) = \frac{n}{e}\left(\left(1+\frac{1}{n}\right)^n - e\right).$$

From $(*)$ given in the hint to (G 14) we obtain that

$$\lim_{n\to\infty} n\left(e - \left(1+\frac{1}{n}\right)^n\right) = \frac{e}{2},$$

thus $\lim\limits_{n\to\infty} R_n = -\dfrac{1}{2} < 1$, which yields that the series is divergent.

In conclusion, the given series is convergent if and only if $x \in (0, 1)$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.6

# Analysis for CS

GROUPWORK:

**(G 15)**

1) Decide whether the following subsets of $\mathbb{R}$ are neighborhoods of 0 or not, and motivate your answer: a) $[-1,1)$, b) $\mathbb{Q}$, c) $\bigcap_{n\in\mathbb{N}^*}[-\frac{1}{n},\frac{1}{n}]$.

2) Let $A \subseteq \mathbb{R}$. Determine the set $M$ of all reals $x$ with the property that $A \in \mathcal{V}(x)$, where $A$ is:
a) $A=[0,1]$, b) $A=(-\infty,-1)$, c) $A=(0,1]\cup[2,3]$, d) $A=\mathbb{R}$, e) $A=\mathbb{N}$.

3) Determine $A'$, where $A$ is: a) $A=\mathbb{Q}$, b) $A=(-\infty,1)\cup(2,\infty)$, c) $A=\mathbb{Z}$.

4) Finish the proof of **L1** in the 6th lecture.

**(G 16)**

Determine the one-sided limits of the function $f\colon D\to\mathbb{R}$ (with $D\subseteq\mathbb{R}$ the maximal domain of $f$) at $\alpha=1$, where

(1) $f(x)=e^{\frac{1}{x^2-1}}$, (2) $f(x)=e^{\frac{x^2-2}{x-1}}$, (3) $f(x)=e^{1+\frac{2}{|x-1|}}$, (4) $f(x)=\frac{|x|-1}{x-1}$.

**(G 17)**

Study the continuity of the following functions ($n\in\mathbb{N}$) and determine the type of their discontinuities:

$$f\colon\mathbb{R}\to\mathbb{R},\ f(x)=\lim_{n\to\infty}\frac{e^{nx}}{1+e^{nx}},\quad\text{and}\quad g\colon\mathbb{R}\setminus\{-1\}\to\mathbb{R},\ g(x)=\lim_{n\to\infty}\frac{x^n+x}{x^{2n}+1}.$$

HOMEWORK:

**(H 17)**

Compute the following limits: (1) $\lim\limits_{x\to4}(-x^3+5x)$, (2) $\lim\limits_{x\to-\infty}(-x^3+2x)$, (3) $\lim\limits_{x\to-3}\dfrac{x^2-9}{(x+3)^2}$,

(4) $\lim\limits_{x\to\infty}\dfrac{3x^k+5}{8x^3-2}$, with $k\in\mathbb{N}$, (5) $\lim\limits_{x\to0}\dfrac{e^{2x}-1}{x}$, (6) $\lim\limits_{x\to0}\left(\dfrac{1+4x+x^2}{1+x}\right)^{\frac{1}{x}}$,

(7) $\lim\limits_{x\to1}\dfrac{x^2-1}{x^2+x-2}$, (8) $\lim\limits_{\substack{x\to1\\x>1}}\left(\dfrac{1}{1-x}-\dfrac{1}{x^3-1}\right)$, (9) $\lim\limits_{x\to\infty}(x-\sqrt{x^2-1})$, (10) $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+1}}$,

(11) $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{x^2+1}}$, (12) $\lim\limits_{x\to1}\dfrac{x^3+x^2-x-1}{x^2-1}$, (13) $\lim\limits_{x\to0}\dfrac{1-\sqrt{1-x^2}}{x^2}$, (14) $\lim\limits_{x\to0}\dfrac{x^2}{|x|}$,

(15) $\lim\limits_{x\to\infty}\sqrt{x}(\sqrt{x+1}-\sqrt{x})$, (16) $\lim\limits_{x\to\infty}\dfrac{(-1)^{[x]}}{x}$, where $[x]$ denotes the largest integer not greater than $x$,

(17) $\lim\limits_{x\to-\infty}e^{\frac{|x|+1}{x-1}}$, (18) $\lim\limits_{x\to-\infty}\left(\dfrac{x^2+x+1}{x^2-x+1}\right)^{\sqrt{-x}}$, (19) $\lim\limits_{x\to0}\dfrac{\sqrt[3]{1+x}-1}{x}$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.6

# Analysis for CS

**(G 15)**

1) Bare in mind that $V \in \vartheta(\alpha)$ (where $\alpha \in \mathbb{R}$) $\iff$ $\exists r > 0$ such that $(\alpha - r, \alpha + r) \subseteq V$.

a) $[-1, 1) \in \vartheta(0)$ (take for example $r = 1$).

b) $\mathbb{Q} \notin \vartheta(0)$. Assume by contradiction that $\mathbb{Q} \in \vartheta(0)$. Thus, $\exists r > 0$ such that $(-r, r) \subseteq \mathbb{Q}$. From the density property of $\mathbb{R}\backslash\mathbb{Q}$, we know that there exists $p \in \mathbb{R}\backslash\mathbb{Q}$ such that $-r < p < r$, which contradicts $(-r, r) \subseteq \mathbb{Q}$. Therefore, $\mathbb{Q} \notin \vartheta(0)$.

c) We have that $\bigcap\limits_{n\in\mathbb{N}^*} \left[-\dfrac{1}{n}, \dfrac{1}{n}\right] = \{0\}$. For this observe that, on the one hand, $0 \in \bigcap\limits_{n\in\mathbb{N}^*} \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]$, and, on the other hand, if $x \in \bigcap\limits_{n\in\mathbb{N}^*} \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]$, then $-\frac{1}{n} \leq x \leq \frac{1}{n}$, $\forall\, n \in \mathbb{N}^*$, thus $x = 0$ (by the Sandwich Theorem). Thus $\bigcap\limits_{n\in\mathbb{N}^*} \left[-\dfrac{1}{n}, \dfrac{1}{n}\right] \notin \vartheta(0)$, since $\{0\}$ doesn't contain an open interval centered at 0.

2) a) $A = [0, 1] \Rightarrow M = (0, 1)$.

b) $A = (-\infty, -1) \Rightarrow M = A$.

c) $A = (0, 1] \cup [2, 3] \Rightarrow M = (0, 1) \cup (2, 3)$.

d) $A = \mathbb{R} \Rightarrow M = A$.

e) $A = \mathbb{N} \Rightarrow M = \emptyset$.

3) Bare in mind that $A' = \left\{x \in \overline{\mathbb{R}} \mid\ \forall\, V \in \vartheta(x),\ V \cap (A \setminus \{x\}) \neq \emptyset\right\}$.

a) $A = \mathbb{Q} \Rightarrow A' = \overline{\mathbb{R}}$.

b) $A = (-\infty, 1) \cup (2, +\infty) \Rightarrow A' = [-\infty, 1] \cup [2, +\infty]$.

c) $A = \mathbb{Z} \Rightarrow A' = \{-\infty, +\infty\}$.

Let us take a closer look to a), where $A = \mathbb{Q}$. Let $\alpha \in \mathbb{R}$ and let $V \in \vartheta(\alpha)$ be an arbitrary neighborhood of $\alpha$. Then, there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subseteq V$. It is easy to remark that $(\alpha - r, \alpha + r) \cap (\mathbb{Q} \setminus \{\alpha\}) \neq \emptyset$ (by the density property of $\mathbb{Q}$) and hence $V \cap (\mathbb{Q} \setminus \{\alpha\}) \neq \emptyset$. Therefore $\alpha \in A'$. Hence $\mathbb{R} \subseteq A'$.

For $\alpha = -\infty$ let $V \in \vartheta(\alpha)$ be an arbitrary neighborhood of $-\infty$. Then, there exists $t \in \mathbb{R}$ such that $[-\infty, t) \subseteq V$. It is easy to remark that $[-\infty, t) \cap (\mathbb{Q} \setminus \{-\infty\}) \neq \emptyset$ and hence $V \cap (\mathbb{Q} \setminus \{-\infty\}) \neq \emptyset$. Therefore $-\infty \in A'$. For $\alpha = \infty$ the proof is similar to the one for $-\infty$. Thus we come to the conclusion that $\mathbb{Q}' = \overline{\mathbb{R}}$.

4) We now finish the proof of **L1** in the 6th lecture.

**Case 2:** $x = -\infty$, $y \in \mathbb{R}$. Since $y \in \mathbb{R}$, $y - 1 < y$. Take $U := [-\infty, y-1) \in \vartheta(-\infty)$ and $V := (y-1, y+1) \in \vartheta(y)$. Then $U \cap V = \emptyset$.

**Case 3:** $x \in \mathbb{R}$, $y = \infty$. Since $x \in \mathbb{R}$, $x < x+1$. Take $U := (x-1, x+1) \in \vartheta(x)$ and $V := (x+1, \infty] \in \vartheta(\infty)$. Then $U \cap V = \emptyset$.

**Case 4:** $x = -\infty$, $y = \infty$. Take $U := [-\infty, -1) \in \vartheta(-\infty)$ and $V := (1, \infty] \in \vartheta(\infty)$. Then $U \cap V = \emptyset$.

**(G 16)**

(1) The one-sided limits of $f$ at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{\frac{1}{x^2-1}} = 0 \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} e^{\frac{1}{x^2-1}} = \infty.$$

(2) The one-sided limits of $f$ at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{\frac{x^2-2}{x-1}} = \infty \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} e^{\frac{x^2-2}{x-1}} = 0.$$

(3) The one-sided limits of $f$ at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} e^{1+\frac{2}{|x-1|}} = \infty \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} e^{1+\frac{2}{|x-1|}} = \infty$$

(4) The one-sided limits of $f$ at 1 are

$$\lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} \frac{|x|-1}{x-1} = \lim_{\substack{x\to 1\\ x<1}} \frac{x-1}{x-1} = 1 \text{ and } \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} \frac{|x|-1}{x-1} = \lim_{\substack{x\to 1\\ x>1}} \frac{x-1}{x-1} = 1.$$

**(G 17)**

We get

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ 0, & \text{if } x < 0. \end{cases}$$

The function $f$ is continuous at all points $x \in \mathbb{R} \setminus \{0\}$, and 0 is a jump discontinuity.

We have

$$g(x) = \begin{cases} 0, & \text{if } x > 1 \\ x, & \text{if } -1 < x \leq 1 \\ 0, & \text{if } x < -1. \end{cases}$$

The function $g$ is continuous at all points $x \in \mathbb{R} \setminus \{-1, 1\}$, the points $-1$ and 1 are both jump discontinuities.

HOMEWORK:

**(H 17)**

(1) $\lim_{x\to 4}(-x^3 + 5x) = -44$.

(2) $\lim_{x\to -\infty}(-x^3 + 2x) = \infty$.

(3) We have $\frac{x^2-9}{(x+3)^2} = \frac{x-3}{x+3}$ for all $x \in \mathbb{R} \setminus \{-3\}$. Since

$$\lim_{\substack{x\to -3\\ x>-3}} \frac{x-3}{x+3} = -\infty \text{ and } \lim_{\substack{x\to -3\\ x<-3}} \frac{x-3}{x+3} = \infty,$$

we conclude that the limit $\lim\limits_{x\to -3} \dfrac{x^2-9}{(x+3)^2}$ doesn't exist.

(4) $L := \lim\limits_{x\to\infty} \dfrac{3x^k+5}{8x^3-2} = \lim\limits_{x\to\infty} \dfrac{3x^k}{8x^3}$, hence

$$L = \begin{cases} 0, & \text{if } k<3 \\ \infty, & \text{if } k>3 \\ \frac{3}{8}, & \text{if } k=3. \end{cases}$$

(5) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x} = 2.$

(6) $\lim\limits_{x\to 0} \left(\dfrac{1+4x+x^2}{1+x}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0} \left(1+\dfrac{3x+x^2}{1+x}\right)^{\frac{1}{x}} = e^3.$

(7) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2+x-2} = \lim\limits_{x\to 1} \dfrac{(x+1)(x-1)}{(x-1)(x+2)} = \dfrac{2}{3}.$

(8) $\lim\limits_{\substack{x\to 1\\ x>1}} \left(\dfrac{1}{1-x} - \dfrac{1}{x^3-1}\right) = \lim\limits_{\substack{x\to 1\\ x>1}} \left(\dfrac{1}{1-x} + \dfrac{1}{1-x^3}\right) = -\infty.$

(9) $\lim\limits_{x\to\infty} (x-\sqrt{x^2-1}) = \lim\limits_{x\to\infty} \dfrac{1}{x+\sqrt{x^2-1}} = 0.$

(10) $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{x}{x\sqrt{1+\frac{1}{x^2}}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$

(11) $\lim\limits_{x\to -\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to -\infty} \dfrac{x}{-x\sqrt{1+\frac{1}{x^2}}} = -\lim\limits_{x\to -\infty} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} = -1.$

(12) $\lim\limits_{x\to 1} \dfrac{x^3+x^2-x-1}{x^2-1} = \lim\limits_{x\to 1} \dfrac{x(x^2-1)+x^2-1}{x^2-1} = \lim\limits_{x\to 1}(x+1) = 2.$

(13) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2} = \lim\limits_{x\to 0} \dfrac{(1+\sqrt{1-x^2})(1-\sqrt{1-x^2})}{(1+\sqrt{1-x^2})x^2} = \lim\limits_{x\to 0} \dfrac{x^2}{(1+\sqrt{1-x^2})x^2} = \dfrac{1}{2}.$

(14) Since $\left|\frac{x^2}{|x|}\right| = |x|$ for all $x \in \mathbb{R}^*$, we get that $\lim\limits_{x\to 0} \dfrac{x^2}{|x|} = 0$. The same result follows from the equalities

$$\lim_{\substack{x\to 0\\ x>0}} \frac{x^2}{|x|} = \lim_{\substack{x\to 0\\ x>0}} \frac{x^2}{x} = \lim_{\substack{x\to 0\\ x>0}} x = 0 \text{ and } \lim_{\substack{x\to 0\\ x<0}} \frac{x^2}{|x|} = \lim_{\substack{x\to 0\\ x<0}} \frac{x^2}{-x} = \lim_{\substack{x\to 0\\ x<0}} -x = 0.$$

(15) $\lim\limits_{x\to\infty} \sqrt{x}(\sqrt{x+1}-\sqrt{x}) = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})} =$
$\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x}}+1} = \dfrac{1}{2}.$

(16) Since

$$\left|\frac{(-1)^{[x]}}{x}\right| = \frac{1}{x}, \text{ for all } x>0,$$

we get $\lim\limits_{x\to\infty}\dfrac{(-1)^{[x]}}{x}=0$.

(17) $\lim\limits_{x\to-\infty} e^{\frac{|x|+1}{x-1}}=\lim\limits_{x\to-\infty} e^{\frac{-x+1}{x-1}}=\dfrac{1}{e}$.

(18) We get

$$\lim_{x\to-\infty}\left(\frac{x^2+x+1}{x^2-x+1}\right)^{\sqrt{-x}}=\lim_{x\to-\infty}\left(1+\frac{2x}{x^2-x+1}\right)^{\sqrt{-x}}=\lim_{x\to-\infty}\left(\left(1+\frac{2x}{x^2-x+1}\right)^{\frac{x^2-x+1}{2x}}\right)^{\frac{2x\sqrt{-x}}{x^2-x+1}}$$

$$=e^0=1.$$

(19) We get

$$\begin{aligned}\lim_{x\to0}\frac{\sqrt[3]{1+x}-1}{x}&=\lim_{x\to0}\frac{(\sqrt[3]{1+x}-1)(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1)}{x(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1)}=\\&=\lim_{x\to0}\frac{1}{\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1}=\frac{1}{3}.\end{aligned}$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.7

# Analysis for CS

GROUPWORK:

**(G 18)**

Let $f\colon [\frac{1}{2}, 3] \to \mathbb{R}$, $f(x) = \sin(\sqrt{x})$. Write down:

a) Taylor's polynomial $T_2(x, 1)$,

b) the remainder term $R_2(x, 1)$, for $x \in [\frac{1}{2}, 3] \setminus \{1\}$, according to Taylor's formula.

**(G 19)**

Consider the trigonometric functions sin, cos: $\mathbb{R} \to \mathbb{R}$.

a) Determine $\sin^{(n)}$ and $\cos^{(n)}$, for every $n \in \mathbb{N}$.

b) Write down the Taylor polynomials $T_n(x, 0)$, for every $n \in \mathbb{N}$, of these two functions.

c) Show that both sin and cos may be expanded as Taylor series around 0 on $\mathbb{R}$, and find the corresponding Taylor series expansions.

**(G 20)**

Determine the following higher order derivatives:

a) $(e^{3x})^{(n)}$, $n \in \mathbb{N}$, b) $(x^2 \sin 2x)^{(100)}$ , c) $((x^3 + 2x - 1)e^{2x})^{(n)}$, $n \in \mathbb{N}$.

HOMEWORK:

**(H 18)**

a) Show that the following equalities hold true for every $n \in \mathbb{N}$ and every $x \in \mathbb{R}$

$$\sin^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right), \quad \cos^{(n)}(x) = \cos\left(x + n\frac{\pi}{2}\right).$$

b) Determine the higher order derivatives $(e^x \sin x)^{(n)}$ and $(e^{-2x} \cos x)^{(n)}$, $n \in \mathbb{N}$.

**(H 19)**

Let $\alpha$, $\beta > 0$. Compute the following limits:

a) $\lim\limits_{x\to\infty} \dfrac{e^{\alpha x}}{x}$, b) $\lim\limits_{x\to\infty} \dfrac{e^{\alpha x}}{x^\beta}$, c) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^\alpha}$, d) $\lim\limits_{x\to\infty} \dfrac{(\ln x)^\beta}{x^\alpha}$, e) $\lim\limits_{\substack{x\to 0\\ x>0}} x^\alpha \ln x$, f) $\lim\limits_{\substack{x\to 0\\ x>0}} x^x$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.7

# Analysis for CS

**(G 18)**

a) We have for every $x \in \left[\frac{1}{2}, 3\right]$ that

$$f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}} \text{ and } f''(x) = -\frac{\cos\sqrt{x}}{2x\sqrt{x}} - \frac{\sin\sqrt{x}}{4x}.$$

Thus $f(1) = \sin 1$, $f'(1) = \frac{\cos 1}{2}$, $f''(1) = -\frac{\sin 1 + \cos 1}{4}$, hence

$$T_2(x, 1) = \sin 1 + \frac{\cos 1}{2}(x-1) - \frac{\sin 1 + \cos 1}{8}(x-1)^2.$$

b) We have for every $x \in \left[\frac{1}{2}, 3\right]$ that

$$f^{(3)}(x) = \frac{3\cos\sqrt{x}}{8x^2\sqrt{x}} + \frac{\sin\sqrt{x}}{8x^2} + \frac{\sin\sqrt{x}}{4x^2} - \frac{\cos\sqrt{x}}{8x\sqrt{x}}.$$

If $x \in \left[\frac{1}{2}, 3\right] \setminus \{1\}$, then, according to Taylor's formula, there exists $c$ strictly between $x$ and 1 such that

$$R_2(x, 1) = \frac{f^{(3)}(c)}{3!}(x-1)^3.$$

**(G 19)**

a) We have $\sin^{(2n)} = (-1)^n \sin$ and $\sin^{(2n+1)} = (-1)^n \cos$, for every $n \in \mathbb{N}$.

b) We get from a) that $\sin^{(2n)}(0) = 0$ and $\sin^{(2n+1)}(0) = (-1)^n$, for every $n \in \mathbb{N}$. Thus

$$T_{2n+1}(x, 0) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \forall\, n \in \mathbb{N}, \text{ and } T_{2n}(x, 0) = T_{2n-1}(x, 0), \forall\, n \in \mathbb{N}^*.$$

c) Let $x \in \mathbb{R}$. We know from Taylor's formula that there exists $c$ between 0 and $x$ such that

$$R_n(x, 0) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

It follows that

$$|R_n(x, 0)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Taking into account that $\lim\limits_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$, we conclude that $\lim\limits_{n\to\infty} R_n(x, 0) = 0$. Taylor's theorem finally yields the following Taylor series expansion for the sine function

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \forall\, x \in \mathbb{R}.$$

We analyze the same requirements for cos.

a) We have $\cos^{(2n)} = (-1)^n \cos$ and $\cos^{(2n+1)} = (-1)^{n+1} \sin$, for every $n \in \mathbb{N}$.

b) We get from a) that $\cos^{(2n)}(0) = (-1)^n$ and $\cos^{(2n+1)}(0) = 0$, for every $n \in \mathbb{N}$. Thus

$$T_{2n}(x,0) = 1 - \frac{1}{2!}x^2 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}, \forall\, n \in \mathbb{N}, \text{ and } T_{2n+1}(x,0) = T_{2n}(x,0), \forall\, n \in \mathbb{N}.$$

c) Let $x \in \mathbb{R}$. We know from Taylor's formula that there exists $c$ between 0 and $x$ such that

$$R_n(x,0) = \frac{\cos^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

It follows that

$$|R_n(x,0)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Taking into account that $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, we conclude that $\lim\limits_{n\to\infty} R_n(x,0) = 0$. Taylor's theorem finally yields the following Taylor series expansion for the cosine function

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}, \forall\, x \in \mathbb{R}.$$

**(G 20)**

a) $(e^{3x})^{(n)} = 3^n e^{3x}$, $\forall\, n \in \mathbb{N},\ x \in \mathbb{R}$.

b) We apply the formula of Leibniz for $f(x) = x^2$, $g(x) = \sin 2x$, and take also into account that, for all $n \in \mathbb{N}$, the following equalities hold true $(\sin 2x)^{(4n)} = 2^{4n} \sin 2x$, $(\sin 2x)^{(4n+1)} = 2^{4n+1} \cos 2x$, $(\sin 2x)^{(4n+2)} = -2^{4n+2} \sin 2x$, $(\sin 2x)^{(4n+3)} = -2^{4n+3} \cos 2x$, hence

$$\begin{aligned}(x^2 \sin 2x)^{(100)} &= -2^{99} C_{100}^{98} \sin 2x - 2^{100} C_{100}^{99} x \cos 2x + 2^{100} x^2 \sin 2x \\ &= 2^{100}(-2475 \sin 2x - 100x \cos 2x + x^2 \sin 2x).\end{aligned}$$

c) We apply the formula of Leibniz for $f(x) = x^3 + 2x - 1$, $g(x) = e^{2x}$. We notice that $f'(x) = 3x^2 + 2$, $f''(x) = 6x$, $f^{(3)}(x) = 6$ and $f^{(4)}(x) = 0$. Moreover, $f^{(n)}(x) = 0$ for all $n \geq 4$. Like in the case a) of this exercise, $(e^{2x})^{(n)} = 2^n e^{2x}$ for all $n \in \mathbb{N}$. If $n \geq 3$ we thus get

$$\begin{aligned}\left((x^3+2x-1)e^{2x}\right)^{(n)} &= C_n^{n-3} 6 \cdot 2^{n-3} \cdot e^{2x} + C_n^{n-2} 6x \cdot 2^{n-2} \cdot e^{2x} + C_n^{n-1}(3x^2+2) \cdot 2^{n-1} \cdot e^{2x} \\ &\quad + C_n^n (x^3+2x-1) \cdot 2^n \cdot e^{2x} = \\ &= 2^{n-3} e^{2x} \left(n(n-1)(n-2) + n(n-1)6x + n(3x^2+2)4 + (x^3+2x-1)8\right) = \\ &= 2^{n-3} e^{2x} \left(8x^3 + 12nx^2 + (6n(n-1)+16)\,x + n(n-1)(n-2) + 8n - 8\right).\end{aligned}$$

We further have

$$\left((x^3+2x-1)e^{2x}\right)' = (2x^3+3x^2+4x)e^{2x} \text{ and } \left((x^3+2x-1)e^{2x}\right)'' = (4x^3+12x^2+14x+4)e^{2x}.$$

Homework:

**(H 18)**

a) First of all bare in mind the following equalities:

$$\sin\left(x+\frac{\pi}{2}\right)=\sin x\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\cos x=\cos x$$

and

$$\cos\left(x+\frac{\pi}{2}\right)=\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}=-\sin x.$$

We use mathematical induction. First of all we are going to prove the proposition:

$$P(n):\sin^{(n)}(x)=\sin\left(x+n\frac{\pi}{2}\right),\ \text{ for } n\in\mathbb{N}.$$

I. P(1): $\sin' x=\cos x=\sin(x+1\cdot\frac{\pi}{2})$, thus $P(1)$ is verified.

II. $P(k)\Longrightarrow P(k+1)$. We know now that $\sin^{(k)}(x)=\sin\left(x+k\frac{\pi}{2}\right)$. Then

$$\sin^{(k+1)}(x)=\left(\sin^{(k)}(x)\right)'=\sin'\left(x+k\frac{\pi}{2}\right)=\cos\left(x+k\frac{\pi}{2}\right)=\sin\left(x+(k+1)\frac{\pi}{2}\right).$$

Thus $P(k+1)$ holds true.

We move on to the proposition:

$$Q(n):\cos^{(n)}(x)=\cos\left(x+n\frac{\pi}{2}\right),\ \text{ for } n\in\mathbb{N}.$$

I. Q(1): $\cos' x=-\sin x=\cos(x+1\cdot\frac{\pi}{2})$, thus $Q(1)$ is verified.

II. $P(k)\Longrightarrow P(k+1)$. We know now that $\cos^{(k)}(x)=\cos\left(x+k\frac{\pi}{2}\right)$. Then

$$\cos^{(k+1)}(x)=\left(\cos^{(k)}(x)\right)'=\cos'\left(x+k\frac{\pi}{2}\right)=-\sin\left(x+k\frac{\pi}{2}\right)=\cos\left(x+(k+1)\frac{\pi}{2}\right).$$

Thus $Q(k+1)$ holds true.

b) We use the Leibniz formula. In order to do that, first bare in mind that $(e^x)^{(n)}=e^x$ and $e^{(-2x)}=(-2)^n e^x$ for all $n\in\mathbb{N}$. Then

$$(e^x\sin x)^{(n)}=\sum_{k=0}^{n}C_n^k(e^x)^{(n-k)}(\sin x)^{(k)}=\sum_{k=0}^{n}C_n^k e^x\sin\left(x+k\frac{\pi}{2}\right)=e^x\sum_{k=0}^{n}C_n^k\sin\left(x+k\frac{\pi}{2}\right)$$

and

$$\begin{aligned}(e^{-2x}\cos x)^{(n)}&=\sum_{k=0}^{n}C_n^k(e^{-2x})^{(n-k)}(\cos x)^{(k)}=\sum_{k=0}^{n}C_n^k(-2)^{n-k}e^x\cos\left(x+k\frac{\pi}{2}\right)\\&=e^x\sum_{k=0}^{n}C_n^k(-2)^{n-k}\cos\left(x+k\frac{\pi}{2}\right).\end{aligned}$$

**(H 19)**

a) Since $\lim\limits_{x\to\infty}e^{\alpha x}=\lim\limits_{x\to\infty}x=\infty$ and since $\lim\limits_{x\to\infty}\dfrac{(e^{\alpha x})'}{x'}=\lim\limits_{x\to\infty}\alpha e^{\alpha x}=\infty$, L'Hospital's rules yield

$$\lim_{x\to\infty}\frac{e^{\alpha x}}{x}=\infty.$$

b) Since the limit obtained at a) holds true for every positive $\alpha$, we get that

$$\lim_{x\to\infty} \frac{e^{\alpha x}}{x^{\beta}} = \lim_{x\to\infty} \left( \frac{e^{\frac{\alpha}{\beta}x}}{x} \right)^{\beta} = \infty.$$

c) Since $\lim\limits_{x\to\infty} \ln x = \lim\limits_{x\to\infty} x^{\alpha} = \infty$ and since $\lim\limits_{x\to\infty} \dfrac{(\ln x)'}{(x^{\alpha})'} = \lim\limits_{x\to\infty} \dfrac{1}{\alpha x^{\alpha}} = 0$, L'Hospital's rules yield

$$\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = 0.$$

d) Since the limit obtained at c) holds true for every positive $\alpha$, we get that

$$\lim_{x\to\infty} \frac{(\ln x)^{\beta}}{x^{\alpha}} \lim_{x\to\infty} \left( \frac{\ln x}{x^{\frac{\alpha}{\beta}}} \right)^{\beta} = 0.$$

e) Let $x = \frac{1}{y}$. Using the result obtained at c), we then get

$$\lim_{\substack{x\to 0\\ x>0}} x^{\alpha} \ln x = \lim_{y\to\infty} \frac{-\ln y}{y^{\alpha}} = 0.$$

f) Using the result obtained at e), we get

$$\lim_{\substack{x\to 0\\ x>0}} x^{x} = \lim_{\substack{x\to 0\\ x>0}} e^{\ln x^{x}} = \lim_{\substack{x\to 0\\ x>0}} e^{x \ln x} = e^{0} = 1.$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.8

# Analysis for CS

GROUPWORK:

**(G 21)**

Let $u,\, v \in \mathbb{R}^n$. Denote by $\alpha = \langle u, v\rangle\,,\ \beta = ||u||$ and $\gamma = ||v||$.

a) Using the properties of the scalar product and the definition of the Euclidean norm, determine in terms of $\alpha$, $\beta$ and $\gamma$ the numbers $\langle u+v, v\rangle$, $\langle u, 2u-3v\rangle$ and $||u-v||$.

b) If $n = 3$, $u = (-1, 2, 3)$ and $v = (-2, 1, -3)$,

b1) compute $\alpha$, $\beta$ and $\gamma$,

b2) determine all reals $r > 0$ with the property that the open ball $B(u, r)$ doesn't contain the point $v$,

b3) determine all reals $t$ with the property that the closed ball $\overline{B}(u, 5)$ contains the vector $(1, -1, t)$.

**(G 22)**

Decide whether the following sequences $(x^k)_{k\in\mathbb{N}^*}$ in $\mathbb{R}^n$ are convergent or not, and, in case they are convergent, determine their limit.

a) $n = 2$ and $x^k = \left(\left(-\frac{1}{2}\right)^k, (-1)^k\right)$, b) $n = 3$ and $x^k = \left(\frac{2^k}{k!}, \frac{1-4k^7}{k^7+12k}, \frac{\sqrt{k}}{e^{3k}}\right)$,

c) $n = 2$ and $x^k = \left(\frac{\sin k}{k}, -k^3 + k\right)$, d) $n = 4$ and $x^k = \left(\frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}}, \frac{1}{\sqrt[k]{k!}}, (e^k + k)^{\frac{1}{k}}, \frac{\alpha^k}{k}\right)$, where $\alpha \in \mathbb{R}_+$ is fixed.

**(G 23) (Train your brain)**

Prove **TH2** in lecture no. 8 concerning the uniqueness of the limit of a convergent sequence in $\mathbb{R}^n$.

HOMEWORK:

**(H 20) (To be delivered in the next exercise-class)**

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{2x} \sin x$. Write down:

a) Taylor's polynomial $T_2(x, 0)$,

b) the remainder term $R_2(x, 0)$, for $x \in \mathbb{R} \setminus \{0\}$, according to Taylor's formula,

c) $(e^{2x})^{(n)}$, for $n \in \mathbb{N}$, $x \in \mathbb{R}$,

d) $f^{(n)}(x)$, for $n \in \mathbb{N}$, $x \in \mathbb{R}$, using also the formula for $\sin^{(n)}(x)$ given in exercise (H 18).

**(H 21)**

Let $f\colon (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$, and let $n \in \mathbb{N}$. Write down:

a) $f^{(n)}(x)$, for $x > 0$,

b) Taylor's polynomial $T_n(x, 1)$,

c) the remainder term $R_n(x, 1)$, for $x \in (0, \infty) \setminus \{1\}$, according to Taylor's formula.

**(H 22)**

Let $x$, $y \in \mathbb{R}^n$. Using the definition of the Euclidean norm and the properties of the scalar product, prove the following equality, known as the *parallelogram identity*

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

REMARK. A particular case (if $n = 2$) of the above identity is a result that belongs to elementary geometry. It states that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.8

# Analysis for CS

**(G 21)**

a) We have that

$\langle u+v, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \alpha + \gamma^2$,

$\langle u, 2u-3v\rangle = \langle u, 2u\rangle - \langle u, 3v\rangle = 2\langle u, u\rangle - 3\langle u, v\rangle = 2\beta^2 - 3\alpha$,

$$\begin{aligned} ||u-v|| &= \sqrt{\langle u-v, u-v\rangle} = \sqrt{\langle u-v, u\rangle - \langle u-v, v\rangle} = \sqrt{\langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle + \langle v, v\rangle} \\ &= \sqrt{\langle u, u\rangle - 2\langle v, u\rangle + \langle v, v\rangle} = \sqrt{\beta^2 - 2\alpha + \gamma^2}. \end{aligned}$$

b1) We have that

$\alpha = \langle(-1,2,3), (-2,1,-3)\rangle = (-1)(-2) + 2\cdot 1 + 3\cdot(-3) = 2+2-9 = -5$,

$\beta = \sqrt{\langle(-1,2,3),(-1,2,3)\rangle} = \sqrt{(-1)^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$,

$\gamma = \sqrt{\langle(-2,1,-3),(-2,1,-3)\rangle} = \sqrt{(-2)^2+1^2+(-3)^2} = \sqrt{4+1+9} = \sqrt{14}$.

b2) We have that

$$v \notin B(u,r) \Longleftrightarrow ||u-v|| \geq r.$$

We use a) and b1) to get that $||u-v|| = \sqrt{\beta^2-2\alpha+\gamma^2} = \sqrt{14+10+14} = \sqrt{38}$. In conclusion, $r \in (0, \sqrt{38}]$.

b3) We have that

$$(1,-1,t) \in \overline{B}(u,5) \Longleftrightarrow ||(1,-1,t)-u|| \leq 5$$

and

$$||(1,-1,t)-u|| = ||(1,-1,t)-(-1,2,3)|| = ||(2,-3,t-3)|| = \sqrt{4+9+(t-3)^2}.$$

Hence $||u-(1,-1,t)|| \leq 5 \Leftrightarrow \sqrt{13+(t-3)^2} \leq 5 \Leftrightarrow (t-3)^2 \leq 12 \Leftrightarrow t \in [3-2\sqrt{3}, 3+2\sqrt{3}]$.

**(G 22)**

a) Since the sequence $((-1)^k)_{k\in\mathbb{N}^*}$ is divergent, the sequence $(x^k)_{k\in\mathbb{N}^*}$ is divergent, too.

b) The equalities $\lim\limits_{k\to\infty}\frac{2^k}{k!} = 0$, $\lim\limits_{k\to\infty}\frac{1-4k^7}{k^7+12k} = -4$ and $\lim\limits_{k\to\infty}\frac{\sqrt{k}}{e^{3k}} = 0$ yield that the sequence $(x^k)_{k\in\mathbb{N}^*}$ converges to $(0,-4,0)$.

c) Since the sequence $(-k^3+k)_{k\in\mathbb{N}^*}$ is divergent, the sequence $(x^k)_{k\in\mathbb{N}^*}$ is divergent, too.

d) We have that

$$\lim_{k\to\infty}\frac{2^{2k}}{\left(2+\frac{1}{k}\right)^{2k}} = \lim_{k\to\infty}\frac{1}{\left(1+\frac{1}{2k}\right)^{2k}} = \frac{1}{e} \text{ and } \lim_{k\to\infty}\frac{1}{\sqrt[k]{k!}} = 0.$$

Denote by $a_k := (e^k + k)^{\frac{1}{k}}$, for $k \in \mathbb{N}^*$. Then $\ln a_k = \frac{\ln(e^k+k)}{k}$. Using L'Hospital's rules, we compute

$$\lim_{x\to\infty} \frac{\ln(e^x + x)}{x} = \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} = \lim_{x\to\infty} \frac{e^x}{e^x + 1} = 1.$$

Thus $\lim\limits_{k\to\infty} \ln a_k = 1$, so $\lim\limits_{k\to\infty} a_k = e$. Furthermore we have that

$$\lim_{k\to\infty} \frac{\alpha^k}{k} = \begin{cases} 0, & \text{if } \alpha \in [0,1) \\ \infty, & \text{if } \alpha > 1. \end{cases}$$

So, if $\alpha > 1$, the sequence $(x^k)_{k\in\mathbb{N}^*}$ is divergent, and, if $\alpha \in [0,1)$, the sequence $(x^k)_{k\in\mathbb{N}^*}$ converges to $\left(\frac{1}{e}, 0, e, 0\right)$.

## (G 23)

Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ having a limit. We assume by contradiction that this sequence has two limits $x, y \in \mathbb{R}^n$, $x \neq y$. By **L1** in the lecture no. 8, there exist $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ such that $U \cap V = \emptyset$. Using twice the definition of the limit of a sequence in terms of neighborhoods (given in the exercise-class), we have that there exist $k(U), k(V) \in \mathbb{N}$ such that $x^k \in U$, for every $k \geq k(U)$, and $x^k \in V$, for every $k \geq k(V)$. For $k := \max\{k(U), k(V)\}$ we then have that $x^k \in U \cap V$, a contradiction. Therefore, $(x^k)_{k\in\mathbb{N}}$ has exactly one limit.

HOMEWORK:

## (H 20)

a) Note that $f(0) = 0$. For every $x \in \mathbb{R}$ we have that

$$f'(x) = e^{2x}(2\sin x + \cos x) \text{ and } f''(x) = e^{2x}(3\sin x + 4\cos x),$$

hence $f'(0) = 1$ and $f''(0) = 4$. It follows that $T_2(x,0) = x + 2x^2$.

b) For every $x \in \mathbb{R}$ we have that $f^{(3)}(x) = e^{2x}(2\sin x + 11\cos x)$. By Taylor's formula there exists a point $c$ strictly between $x$ and $0$ such that $R_2(x,0) = \frac{f^{(3)}(c)}{3!}x^3$.

c) It follows easily (using, for instance, mathematical induction) that $(e^{2x})^{(n)} = 2^n e^{2x}$, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

d) Applying the formula of Leibniz, we get that

$$f^{(n)}(x) = e^{2x} \sum_{k=0}^{n} C_n^k 2^{n-k} \sin\left(x + k\frac{\pi}{2}\right),$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

## (H 21)

a) It is easy to prove by mathematical induction that

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}},$$

for all $n \in \mathbb{N}$ (even for $n = 0$) and for all $x > 0$.

b) Since $f^{(k)}(1) = (-1)^k k!$, we obtain that

$$T_n(x,1) = \sum_{k=0}^{n} \frac{f^k(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k = \sum_{k=0}^{n} (1-x)^k.$$

c) According to Taylor's formula, there exists $c$ strictly between $x$ and 1 such that

$$R_n(x,1) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = \frac{(-1)^{n+1}(x-1)^{n+1}}{c^{n+2}}.$$

**(H 22)**

We have that

$$||x+y||^2 = \langle x+y, x+y\rangle = ||x||^2 + \langle x,y\rangle + \langle y,x\rangle + ||y||^2$$

and

$$||x-y||^2 = \langle x-y, x-y\rangle = ||x||^2 - \langle x,y\rangle - \langle y,x\rangle + ||y||^2.$$

By adding up these two equalities we obtain the parallelogram identity.

Dr. Brigitte Breckner
Dr. Anca Grad

“Babeş–Bolyai” University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.9

# Analysis for CS

GROUPWORK:

**(G 24)**

Let $f\colon \mathbb{R}^* \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y,z) = \frac{z^2 e^y}{x}$. Determine all

a) first-order partial derivatives of $f$,

b) second-order partial derivatives of $f$.

**(G 25)**

For the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^4 - y^4}{2(x^4+y^4)}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases}$$

study

a) the partial differentiability with respect to both variables at $0_2$,

b) the continuity at $0_2$.

**(G 26) (Train your brain)**

Prove **P5** in the 9th lecture: If $M \subseteq \mathbb{R}^n$ then $\operatorname{int} M \subseteq M$ and $\operatorname{int} M \subseteq M'$.

HOMEWORK:

**(H 23) (To be delivered in the next exercise-class)**

a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2, \end{cases}$$

is partially differentiable with respect to both variables on $\mathbb{R}^2$, and determine both first-order partial derivatives of $f$.

b) Determine all first-order and all second-order partial derivatives of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z\sin(x-y)$.

**(H 24)**

Determine the gradient of the function $f$ at the point $a$ in the following cases:

a) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{-x}\sin(x+2y)$, $a = \left(0, \frac{\pi}{4}\right)$,

b) $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (x-y)\cos \pi z$, $a = \left(1, 0, \frac{1}{2}\right)$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.9

# Analysis for CS

**(G 24)**

a) Let $(x, y, z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y,z) = -\frac{z^2 e^y}{x^2},\ \frac{\partial f}{\partial y}(x,y,z) = \frac{z^2 e^y}{x} \text{ and } \frac{\partial f}{\partial z}(x,y,z) = \frac{2ze^y}{x}.$$

b) Let $(x, y, z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2\frac{z^2 e^y}{x^3},\ \frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{z^2 e^y}{x} \text{ and } \frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{2e^y}{x}.$$

The mixed second-order partial derivatives are

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = -\frac{z^2 e^y}{x^2} = \frac{\partial^2 f}{\partial x \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = \frac{2ze^y}{x} = \frac{\partial^2 f}{\partial z \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = -\frac{2ze^y}{x^2} = \frac{\partial^2 f}{\partial x \partial z}(x,y,z).$$

**(G 25)**

a) First we analyze the partial differentiability with respect to $x$ at $0_2$:

$$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{\frac{x^4-0}{2(x^4)+0} - 0}{x} = \lim_{x\to 0} \frac{1}{2x}.$$

Since $\lim\limits_{\substack{x\to 0\\ x<0}} \frac{1}{2x} = -\infty$ and $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{1}{2x} = +\infty$, we conclude that $\lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0}$ does not exist. Hence $f$ is not partially differentiable with respect to $x$ at $0_2$.

Then we analyze the partial differentiability with respect to $y$ at $0_2$:

$$\lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{\frac{0-y^4}{2(0+y^4)} - 0}{y} = \lim_{y\to 0} -\frac{1}{2y}.$$

Since $\lim\limits_{\substack{y\to 0\\ y<0}} -\frac{1}{2y} = +\infty$ and $\lim\limits_{\substack{y\to 0\\ y>0}} -\frac{1}{2y} = -\infty$ we conclude that $\lim\limits_{y\to 0} \frac{f(0,y)-f(0,0)}{x-0}$ does not exist. Hence $f$ is not partially differentiable with respect to $y$ at $0_2$.

b) We will prove that $f$ is not continuous at $0_2$. Assume by contradiction that $f$ is continuous at $0_2$. Then, according to **Th3** in Lecture 9, for every sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^2$, with $\lim\limits_{k\to\infty} x^k = 0_2$, one should have that $\lim\limits_{k\to\infty} f(x^k) = f(0_2)$.

If we consider the sequence with the general term $a^k = (\frac{1}{k}, 0)$, then $\lim\limits_{k\to\infty} a^k = 0_2$ and $\lim\limits_{k\to\infty} f(a^k) = \lim\limits_{k\to\infty} \dfrac{\left(\frac{1}{k}\right)^4 - 0}{2\left(\left(\frac{1}{k}\right)^4 + 0\right)} = \lim\limits_{k\to\infty} \dfrac{1}{2} = \dfrac{1}{2} \neq 0 = f(0_2)$. Hence we have obtained a contradiction. Thus $f$ is not continuous at $0_2$.

**(G 26)**

Let $x \in \operatorname{int} M$. Then there exists $r > 0$ such that $B(x, r) \subseteq M$. Since $x \in B(x, r)$, we conclude that $x \in M$.

If $V \in \mathcal{V}(x)$ then there exists $r' > 0$ such that $B(x, r') \subseteq V$. Let $r_0 := \min\{r, r'\}$. Then

$$B(x, r_0) \setminus \{x\} \subseteq V \cap (M \setminus \{x\}),$$

thus $V \cap (M \setminus \{x\}) \neq \emptyset$, showing that $x \in M'$.

Hence $\operatorname{int} M \subseteq M$ and $\operatorname{int} M \subseteq M'$

HOMEWORK:

**(H 23)**

a) First we analyze the partial differentiability with respect to $x$ at $0_2$:

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{0-0}{x} = \lim_{x\to 0} 0 = 0.$$

This means that $f$ is partially differentiable with respect to $x$ at $0_2$, and that $\frac{\partial f}{\partial x}(0,0) = 0$.

Then we analyze the partial differentiability with respect to $y$ at $0_2$:

$$\lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{0-0}{y} = \lim_{y\to 0} 0 = 0.$$

This means that $f$ is partially differentiable with respect to $y$ at $0_2$, and that $\frac{\partial f}{\partial y}(0,0) = 0$.

Second we analyze the case when $(x, y) \in \mathbb{R}^2 \setminus \{0_2\}$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(x^2+y^2) - xy^3(2x)}{(x^2+y^2)^2} = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{3xy^2(x^2+y^2) - xy^3(2y)}{(x^2+y^2)^2} = \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}.$$

Thus

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2. \end{cases}$$

b) Let $(x, y, z)$ be arbitrarily chosen in $\mathbb{R}^3$. Then

$$\frac{\partial f}{\partial x}(x,y,z) = z\cos(x-y), \ \frac{\partial f}{\partial y}(x,y,z) = -z\cos(x-y) \text{ and } \frac{\partial f}{\partial z}(x,y,z) = \sin(x-y).$$

Moreover

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = -z\sin(x-y),\ \frac{\partial^2 f}{\partial y^2}(x,y,z) = -z\sin(x-y) \text{ and } \frac{\partial^2 f}{\partial z^2}(x,y,z) = 0$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = z\sin(x-y) = \frac{\partial^2 f}{\partial x \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = \cos(x-y) = \frac{\partial^2 f}{\partial x \partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = -\cos(x-y) = \frac{\partial^2 f}{\partial z \partial y}(x,y,z).$$

**(H 24)**

a) Let $(x,y) \in \mathbb{R}^2$ be arbitrarily chosen. Then $\frac{\partial f}{\partial x}(x,y) = -e^{-x}\sin(x+2y) + e^{-x}\cos(x+2y)$ and $\frac{\partial f}{\partial y}(x,y) = 2e^{-x}\cos(x+2y)$. Thus

$$\begin{aligned}\nabla f\left(0,\frac{\pi}{4}\right) &= \left(\frac{\partial f}{\partial x}\left(0,\frac{\pi}{4}\right), \frac{\partial f}{\partial y}\left(0,\frac{\pi}{4}\right)\right) \\ &= \left(-e^{-0}\sin\left(0+\frac{\pi}{2}\right) + e^{-0}\cos\left(0+\frac{\pi}{2}\right), 2e^{-0}\cos\left(0+\frac{\pi}{2}\right)\right) = (-1,0).\end{aligned}$$

b) Let $(x,y,z) \in \mathbb{R}^3$ be arbitrarily chosen. Then $\frac{\partial f}{\partial x}(x,y,z) = \cos \pi z$, $\frac{\partial f}{\partial x}(x,y,z) = -\cos \pi z$ and $\frac{\partial f}{\partial x}(x,y,z) = -(x-y)\pi \sin \pi z$. Thus

$$\nabla f\left(1,0,\frac{1}{2}\right) = \left(\cos\frac{\pi}{2}, -\cos\frac{\pi}{2}, -(1-0)\pi\sin\frac{\pi}{2}\right) = (0,0,-\pi).$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.10

# Analysis for CS

GROUPWORK:

**(G 27)**

Determine all local extrema, their type (minima or maxima) and the corresponding extreme values of the following functions:

a) $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^3 - 3x + y^2 + z^2$; b) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4 + y^4 - 4(x-y)^2$;

c) $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2(1+xy) + xy$; d) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 3xy^2 - 15x - 12y$.

HOMEWORK:

**(H 25) (To be delivered in the next exercise-class)**

Determine all local extrema, their type (minima or maxima) and the corresponding extreme values of the following functions:

a) $f\colon \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$; b) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2(1-x)^3$.

**(H 26)**

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined in statement a) of exercise (H 23).

a) Show that $f$ is twice partially differentiable with respect to $(x,y)$ and $(y,x)$ on $\mathbb{R}^2$, and determine the second-order partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$. Observe that

$$\frac{\partial^2 f}{\partial y \partial x}(0_2) \neq \frac{\partial^2 f}{\partial x \partial y}(0_2).$$

b) Show that the functions $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are not continuous at $0_2$, thus $f \notin C^2(\mathbb{R}^2)$.

**(H 27)**

Let $M := \{(x,y) \in \mathbb{R}^2 \mid x + y > 0\}$ and let $f\colon M \to \mathbb{R}$ be defined by $f(x,y) = \frac{x^2+y^2}{x+y}$. Show that

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = f(x,y), \forall\ (x,y) \in M.$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.10

# Analysis for CS

**(G 27)**

a) For all $(x,y,z) \in \mathbb{R}^3$ it holds $\frac{\partial f}{\partial x}(x,y,z) = 3x^2 - 3$, $\frac{\partial f}{\partial y}(x,y,z) = 2y$ and $\frac{\partial f}{\partial z}(x,y,z) = 2z$. Thus the stationary points of the function $f$ are the solutions of the system

$$\begin{cases} 3x^2 - 3 = 0 \\ 2y = 0 \\ 2z = 0. \end{cases}$$

Therefore $(-1,0,0)$ and $(1,0,0)$ are the only stationary points of the function.

For all $(x,y,z) \in \mathbb{R}^3$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 6x, \quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 2, \quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = 2,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0,$$

thus

$$H_f(-1,0,0) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad H_f(1,0,0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The matrix $H_f(1,0,0)$ is positive definite, thus $(1,0,0)$ is a local minimum point of the function. The corresponding extreme value is $f(1,0,0) = -2$.

Since Sylvester's criterion cannot be applied to the matrix $H_f(-1,0,0)$, we will study its type by other means. The quadratic form associated to this matrix is $\Phi \colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\Phi(h_1,h_2,h_3) = -6(h_1)^2 + 2(h_2)^2 + 2(h_3)^2.$$

Since $\Phi(1,0,0) = -6 < 0$ and $\Phi(0,1,0) = 2 > 0$ it follows that this quadratic form (and therefore its associated matrix) is indefinite. Thus $(-1,0,0)$ is not a local extremum of $f$.

b) For all $(x,y) \in \mathbb{R}^2$ it holds $\frac{\partial f}{\partial x}(x,y) = 4x^3 - 8(x-y)$ and $\frac{\partial f}{\partial y}(x,y) = 4y^3 + 8(x-y)$. The stationary points of the function $f$ are the solutions of the system

$$\begin{cases} 4x^3 - 8(x-y) = 0 \\ 4y^3 + 8(x-y) = 0. \end{cases}$$

By adding the two equations we get $x^3 = -y^3$, thus $x = -y$. By replacing this in the first equation, we get $x^3 - 4x = 0$, thus $x \in \{-2,0,2\}$. Therefore the stationary points of the function are $(-2,2)$, $(0,0)$ and $(2,-2)$.

For all $(x,y) \in \mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 8, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 8, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 8.$$

For the three stationary points we get

$$H_f(-2,2) = H_f(2,-2) = \begin{pmatrix} 40 & 8 \\ 8 & 40 \end{pmatrix} \quad \text{and} \quad H_f(0,0) = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}.$$

Since the matrix $H_f(-2,2) = H_f(2,-2)$ is positive definite, the points $(-2,2)$ and $(2,-2)$ are local minimum points. The corresponding extreme values are, respectively, $f(-2,2) = -32$ and $f(2,-2) = -32$.

Due to the fact that the determinant of the matrix $H_f(0,0)$ is zero (more exactly, by computing the quadratic form associated with this matrix, we get that it is negative semidefinite), the algorithm for determining the stationary points provides us with no useful information connected to the stationary point $(0,0)$. Nevertheless we notice that $f(0,0) = 0$, $f(x,x) = 2x^4$, $f(x,0) = x^2(x^2-4)$. It follows that in each neighborhood of $(0,0)$ the function $f$ takes both positive and negative values (for example, for all $n \in \mathbb{N}^*$ it holds $f(\frac{1}{n}, \frac{1}{n}) > 0$ and $f(\frac{1}{n}, 0) < 0$). Therefore $(0,0)$ is not a local extremum.

c) For all $(x,y,z) \in \mathbb{R}^3$ it holds $\frac{\partial f}{\partial x}(x,y,z) = yz^2 + y$, $\frac{\partial f}{\partial y}(x,y,z) = xz^2 + x$ and $\frac{\partial f}{\partial z}(x,y,z) = 2z(1+xy)$. The stationary points of $f$ are the solutions of the system

$$\begin{cases} y(z^2+1) = 0 \\ x(z^2+1) = 0 \\ 2z(1+xy) = 0. \end{cases}$$

From the first two equations it follows that $y = 0$ and $x = 0$, respectively. By replacing these values in the last equation, we reach the conclusion that $z = 0$. Hence $(0,0,0)$ is the only stationary point of the function.

For all $(x,y,z) \in \mathbb{R}^3$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 0, \quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = 2(1+xy),$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = z^2 + 1, \quad \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 2yz, \quad \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 2xz,$$

thus

$$H_f(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since Sylvester's criterion cannot be applied to the matrix $H(f)(0,0,0)$, we study its nature by other means. Its associated quadratic form is the map $\Phi \colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\Phi(h_1,h_2,h_3) = 2h_1h_2 + 2(h_3)^2.$$

Since $\Phi(-1,1,0) = -2 < 0$ and $\Phi(0,0,1) = 2 > 0$, it follows that this quadratic form (therefore its corresponding matrix) is indefinite. Thus $(0,0,0)$ is not a local extremum point of the function.

d) For all $(x,y) \in \mathbb{R}^2$ it holds $\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15$ and $\frac{\partial f}{\partial y}(x,y) = 6xy - 12$. The stationary points of $f$ are the solutions of the system

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0. \end{cases}$$

It follows that $x^2+y^2=5$ and $xy=2$. As $x^2+y^2=(x+y)^2-2xy$ it follows that $(x+y)^2=9$, which means that $x+y\in\{-3,3\}$. Therefore $x$ and $y$ are the solutions of one of the equations $t^2+3t+2=0$ or $t^2-3t+2=0$. Thus the stationary point of the function are $(-2,-1)$, $(-1,-2)$, $(1,2)$ and $(2,1)$.

For all $(x,y)\in\mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y)=6x,\quad \frac{\partial^2 f}{\partial x\partial y}(x,y)=6y,\quad \frac{\partial^2 f}{\partial y^2}(x,y)=6x.$$

For the four stationary points of the function we get

$$H_f(-2,-1)=6\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix},\quad H_f(-1,-2)=6\begin{pmatrix}-1 & -2\\ -2 & -1\end{pmatrix},$$

$$H_f(1,2)=6\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\quad H_f(2,1)=6\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}.$$

The matrix $H_f(-2,-1)$ is negative definite, $H_f(2,1)$ is positive definite, while $H_f(-1,-2)$ and $H_f(1,2)$ are indefinite. Thus $(-2,-1)$ is a local maximum point of $f$, and $(2,1)$ is a local minimum, while $(-1,-2)$ and $(1,2)$ are not local extrema. Moreover $f(-2,-1)=28$ and $f(2,1)=-28$.

HOMEWORK:

**(H 25)**

a) For all $(x,y,z)\in\mathbb{R}^3$ it holds $\frac{\partial f}{\partial x}(x,y,z)=4x-y+2z$, $\frac{\partial f}{\partial y}(x,y,z)=-x-1+3y^2$ and $\frac{\partial f}{\partial z}(x,y,z)=2x+2z$. The stationary points of $f$ are the solutions of the system

$$\begin{cases}4x-y+2z=0\\ -x-1+3y^2=0\\ 2x+2z=0.\end{cases}$$

From the last equation it follows that $x=-z$. By replacing this in the first equation, we get $y=2x$, equality which, together with the second equation, provides us with $12x^2-x-1=0$. Thus $x\in\{-\frac{1}{4},\frac{1}{3}\}$. It follows that $(-\frac{1}{4},-\frac{1}{2},\frac{1}{4})$ and $(\frac{1}{3},\frac{2}{3},-\frac{1}{3})$ are the stationary points of the function.

For all $(x,y,z)\in\mathbb{R}^3$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y,z)=4,\quad \frac{\partial^2 f}{\partial y^2}(x,y,z)=6y,\quad \frac{\partial^2 f}{\partial z^2}(x,y,z)=2,$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y,z)=-1,\quad \frac{\partial^2 f}{\partial x\partial z}(x,y,z)=2,\quad \frac{\partial^2 f}{\partial y\partial z}(x,y,z)=0,$$

thus

$$H_f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)=\begin{pmatrix}4 & -1 & 2\\ -1 & -3 & 0\\ 2 & 0 & 2\end{pmatrix},\quad H_f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)=\begin{pmatrix}4 & -1 & 2\\ -1 & 4 & 0\\ 2 & 0 & 2\end{pmatrix}.$$

The matrix $H_f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)$ is positive definite, thus $(\frac{1}{3},\frac{2}{3},-\frac{1}{3})$ is a local minimum point. Moreover

$$f\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)=-\frac{13}{27}.$$

Since Sylvester's criterion cannot be applied to the matrix $H_f\left(-\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right)$, we will study its nature by other means. The quadratic from associated to this matrix is the map $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$\Phi(h_1,h_2,h_3) = 4(h_1)^2 - 2h_1h_2 + 4h_1h_3 - 3(h_2)^2 + 2(h_3)^2.$$

Since $\Phi(0,1,0) = -3 < 0$ and $\Phi(0,0,1) = 2 > 0$, it follows that this quadratic form (and its corresponding matrix) is indefinite. Therefore, $(-\frac{1}{4},-\frac{1}{2},\frac{1}{4})$ is not a local extremum.

b) For all $(x,y) \in \mathbb{R}^2$ it holds $\frac{\partial f}{\partial x}(x,y) = 2x - 3y^2(1-x)^2$ and $\frac{\partial f}{\partial y}(x,y) = 2y(1-x)^3$. The stationary points of $f$ are the solutions of the system

$$\begin{cases} 2x - 3y^2(1-x)^2 = 0 \\ 2y(1-x)^3 = 0. \end{cases}$$

From the last equation we get $y = 0$ or $x = 1$. If $y = 0$, then from the first equation of the system we get $x = 0$. We notice that $x = 1$ does not obey the first equation of the system, thus it is excluded. In conclusion $(0,0)$ is the only stationary point of the function.

For all $(x,y) \in \mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 + 6y^2(1-x), \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -6y(1-x)^2, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2(1-x)^3,$$

thus

$$H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

It follows that $H(f)(0,0)$ is positive definite, thus $(0,0)$ is a local minimum point. The corresponding extreme value is $f(0,0) = 0$.

**(H 26)**

a) Recall from the solution to (H 23) that

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2. \end{cases}$$

We study the second-order partial differentiability of $f$ with respect to $(x,y)$ at $0_2$:

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \lim_{y \to 0} \frac{\frac{y^5}{y^4}}{y} = 1.$$

We study the second-order partial differentiability of $f$ with respect to $(y,x)$ at $0_2$:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = 0.$$

We get that

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \begin{cases} \frac{y^2(5y^2-3x^2)(x^2+y^2)-4y^4(y^2-x^2)}{(x^2+y^2)^3}, & (x,y) \neq 0_2 \\ 1, \ (x,y) = 0_2 \end{cases}$$

and

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \begin{cases} \frac{y^2(9x^2+y^2)(x^2+y^2)-4x^2y^2(3x^2+y^2)}{(x^2+y^2)^3}, & (x,y) \neq 0_2 \\ 0, \ (x,y) = 0_2. \end{cases}$$

b) Since

$$\lim_{n\to\infty}\frac{\partial^2 f}{\partial y\partial x}\left(\frac{1}{n},0\right)=0\neq\frac{\partial^2 f}{\partial y\partial x}(0,0)\text{ and }\lim_{n\to\infty}\frac{\partial^2 f}{\partial x\partial y}\left(0,\frac{1}{n}\right)=1\neq\frac{\partial^2 f}{\partial x\partial y}(0,0),$$

**Th3** in lecture no. 9 implies that the functions $\frac{\partial^2 f}{\partial y\partial x}$ and $\frac{\partial^2 f}{\partial x\partial y}$ are not continuous at $0_2$.

**(H 27)**

Let $(x,y)\in M$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y)=\frac{2x(x+y)-(x^2+y^2)}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2}$$

and

$$\frac{\partial f}{\partial y}(x,y)=\frac{2y(x+y)-(x^2+y^2)}{(x+y)^2}=\frac{y^2+2xy-x^2}{(x+y)^2}.$$

Therefore we get

$$\begin{aligned}x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)&=x\frac{x^2+2xy-y^2}{(x+y)^2}+y\frac{y^2+2xy-x^2}{(x+y)^2}\\&=\frac{x^3+2x^2y-xy^2+y^3+2xy^2-x^2y}{(x+y)^2}\\&=\frac{x^3+x^2y+y^3+xy^2}{(x+y)^2}=f(x,y).\end{aligned}$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.11

# Analysis for CS

GROUPWORK:

**(G 28)**

Study the improper integrability of the following functions on their domains and, in case they are improperly integrable, compute the corresponding improper integrals.

a) $f\colon (-1,1) \to \mathbb{R}$, $f(x) = \frac{1}{\sqrt{1-x^2}}$, b) $f\colon [1,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x(1+x)}$,

c) $f\colon (0,1] \to \mathbb{R}$, $f(x) = \ln x$, d) $f\colon [0,1) \to \mathbb{R}$, $f(x) = \frac{\arcsin x}{\sqrt{1-x^2}}$,

e) $f\colon (0,1] \to \mathbb{R}$, $f(x) = \frac{\ln x}{\sqrt{x}}$, f) $f\colon [e,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x(\ln x)^3}$,

g) $f\colon \left(\frac{1+\sqrt{3}}{2}, 2\right] \to \mathbb{R}$, $f(x) = \frac{1}{x\sqrt{2x^2-2x-1}}$, h) $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \frac{\pi}{2} - \operatorname{arctg} x$.

HOMEWORK:

**(H 28) (To be delivered in the next exercise-class)**

Study the improper integrability of the following functions on their domains and, in case they are improperly integrable, compute the corresponding improper integrals.

a) $f\colon (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{4\sqrt{x}+\sqrt{x^3}}$, b) $f\colon (1,2] \to \mathbb{R}$, $f(x) = \frac{1}{x\ln x}$.

**(H 29) (To be delivered in the next exercise-class)**

Determine all local extrema, their type (minima or maxima) and the corresponding extreme values of the function $f\colon (0,\pi) \times (0,\pi) \to \mathbb{R}$ defined by $f(x,y) = \sin x + \sin y + \sin(x+y)$.

**(H 30) (Train your brain)**

Using the formula of Leibniz-Newton for definite integrals and the definition of improper integrals, prove the formula of Leibniz-Newton for improper integrals on intervals $[a,b)$, where $-\infty < a < b \leq \infty$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.11

# Analysis for CS

**(G 28)**

In all cases we use the formula of Leibniz-Newton for improper integrals.

a) The function $F\colon (-1,1) \to \mathbb{R}$, defined by $F(x) = \arcsin x$, is an antiderivative of $f$. Since

$$\lim_{\substack{x\to -1\\ x>-1}} \arcsin x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{\substack{x\to 1\\ x<1}} \arcsin x = \frac{\pi}{2},$$

it follows that $f$ is improperly integrable on $(-1,1)$ and $\int_{-1+}^{1-} \frac{1}{\sqrt{1-x^2}} dx = \pi$.

b) We first determine an antiderivative $F$ of the function $f$ on $[1,\infty)$. Since

$$\int \frac{1}{x(1+x)} dx = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx = \ln x - \ln(1+x) + \mathcal{C},$$

it follows that the function $F\colon [1,\infty) \to \mathbb{R}$, defined by $F(x) = \ln \frac{x}{1+x}$, is an antiderivative of $f$. From

$$\lim_{x\to\infty} \ln \frac{x}{1+x} = 0,$$

it follows that $f$ is improperly integrable on $[1,\infty)$ and $\int_1^\infty \frac{1}{x(1+x)} dx = \ln 2$.

c) We first determine an antiderivative $F$ of the function $f$ on $(0,1]$. Since

$$\int \ln x dx = \int (x)' \ln x dx = x \ln x - \int 1 dx = x \ln x - x + \mathcal{C},$$

it follows that the function $F\colon (0,1] \to \mathbb{R}$, defined by $F(x) = x\ln x - x$, is an antiderivative of $f$. Since

$$\lim_{\substack{x\to 0\\ x>0}} (x\ln x - x) = -\lim_{y\to\infty} \frac{\ln y}{y} = 0,$$

it follows that $f$ is improperly integrable on $(0,1]$ and $\int_{0+}^{1} \ln x dx = -1$.

d) We first determine an antiderivative $F$ of the function $f$ on $[0,1)$. Since

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \int \arcsin x (\arcsin x)' dx = \frac{1}{2}(\arcsin x)^2 + \mathcal{C},$$

it follows that the function $F\colon [0,1) \to \mathbb{R}$, defined by $F(x) = \frac{1}{2}(\arcsin x)^2$, is an antiderivative of $f$. From

$$\lim_{\substack{x\to 1\\ x<1}} (\arcsin x)^2 = \left(\frac{\pi}{2}\right)^2,$$

it follows that $f$ is improperly integrable on $[0,1)$ and $\int_0^{1-} \frac{\arcsin x}{\sqrt{1-x^2}}dx = \frac{(\pi)^2}{8}$.

e) We first determine an antiderivative $F$ of the function $f$ on $(0,1]$. Since

$$\int \frac{\ln x}{\sqrt{x}}dx = 2\int (\sqrt{x})' \ln x dx = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\ln x - 4\sqrt{x} + \mathcal{C},$$

it follows that the function $F\colon (0,1] \to \mathbb{R}$, defined by $F(x) = 2\sqrt{x}\ln x - 4\sqrt{x}$, is an antiderivative of $f$. As

$$\lim_{\substack{x\to 0\\ x>0}} \left(2\sqrt{x}\ln x - 4\sqrt{x}\right) = -2\lim_{y\to\infty} \frac{\ln y}{\sqrt{y}} = 0,$$

it follows that $f$ is improperly integrable on $(0,1]$ and $\int_{0+}^1 \frac{\ln x}{\sqrt{x}}dx = -4$.

f) We first determine an antiderivative $F$ of the function $f$ on $[e,\infty)$. Since

$$\int \frac{1}{x(\ln x)^3}dx = \int \frac{(\ln x)'}{(\ln x)^3}dx = -\frac{1}{2(\ln x)^2} + \mathcal{C},$$

it follows that the function $F\colon [e,\infty[\to \mathbb{R}$, defined by $F(x) = -\frac{1}{2(\ln x)^2}$, is an antiderivative of $f$. As

$$\lim_{x\to\infty} -\frac{1}{2(\ln x)^2} = 0,$$

it follows that $f$ is improperly integrable on $[e,\infty)$ and $\int_e^{\infty} \frac{1}{x(\ln x)^3}dx = \frac{1}{2}$.

g) We notice that the roots of the equation $2x^2 - 2x - 1 = 0$ are $x_1 = \frac{1-\sqrt{3}}{2}$ and $x_2 = \frac{1+\sqrt{3}}{2}$. We first determine an antiderivative $F$ of the function $f$ on $(x_2, 2]$. Since

$$\int \frac{1}{x\sqrt{2x^2-2x-1}}dx = \int \frac{1}{x^2\sqrt{2-\frac{2}{x}-\frac{1}{x^2}}}dx = -\int \frac{\left(1+\frac{1}{x}\right)'}{\sqrt{3-\left(1+\frac{1}{x}\right)^2}}dx = -\arcsin\frac{1+\frac{1}{x}}{\sqrt{3}} + \mathcal{C},$$

it follows that the function $F\colon ]x_2, 2] \to \mathbb{R}$, defined by $F(x) = -\arcsin\frac{1+\frac{1}{x}}{\sqrt{3}}$, is an antiderivative of $f$. As

$$\lim_{\substack{x\to x_2\\ x>x_2}} \arcsin\frac{1+\frac{1}{x}}{\sqrt{3}} = \arcsin 1 = \frac{\pi}{2} \quad\text{and}\quad \arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3},$$

it follows that $f$ is improperly integrable $(x_2, 2]$ and $\int_{x_2+}^2 \frac{1}{x\sqrt{2x^2-2x-1}}dx = \frac{\pi}{6}$.

h) We first determine an antiderivative $F$ of the function $f$ on $[0,\infty)$. Since

$$\int \left(\frac{\pi}{2} - \operatorname{arctg} x\right)dx = \frac{\pi}{2}x - \int (x)'\operatorname{arctg} x dx = \frac{\pi}{2}x - x\operatorname{arctg} x + \int \frac{x}{1+x^2}dx =$$
$$= x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \frac{1}{2}\ln(1+x^2) + \mathcal{C} = x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \ln(\sqrt{1+x^2}) + \mathcal{C},$$

it follows that the function the function $F\colon [0,\infty) \to \mathbb{R}$, defined by $F(x) = x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \ln(\sqrt{1+x^2})$, is an antiderivative of $f$. As

$$\lim_{x\to\infty} x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) = \lim_{\substack{y\to 0\\ y>0}} \frac{\frac{\pi}{2} - \operatorname{arctg}\frac{1}{y}}{y} = \lim_{\substack{y\to 0\\ y>0}} \frac{1}{y^2+1} = 1$$

(when computing the limit we used L'Hospital's rules). It follows that

$$\lim_{x\to\infty} \left(x\left(\frac{\pi}{2} - \operatorname{arctg} x\right) + \ln(\sqrt{1+x^2})\right) = \infty,$$

thus $f$ is not improperly integrable on $[0,\infty)$.

HOMEWORK:

**(H 28)**

a) We first determine an antiderivative $F$ of the function $f$ on $(0,\infty)$. Since

$$\int \frac{1}{4\sqrt{x}+\sqrt{x^3}}dx = \int \frac{1}{\sqrt{x}(4+x)}dx = \int \frac{1}{\sqrt{x}(4+(\sqrt{x})^2)}dx = 2\int \frac{(\sqrt{x})'}{4+(\sqrt{x})^2}dx = \operatorname{arctg}\frac{\sqrt{x}}{2}+\mathcal{C},$$

it follows that the function $F\colon (0,\infty)\to\mathbb{R}$, defined by $F(x)=\operatorname{arctg}\frac{\sqrt{x}}{2}$, is an antiderivative of $f$. As

$$\lim_{\substack{x\to 0\\ x>0}} \operatorname{arctg}\frac{\sqrt{x}}{2} = 0 \quad \text{and} \quad \lim_{x\to\infty} \operatorname{arctg}\frac{\sqrt{x}}{2} = \frac{\pi}{2},$$

it follows that $f$ is improperly integrable on $(0,\infty)$ and $\int_{0+}^{\infty} \frac{1}{4\sqrt{x}+\sqrt{x^3}}dx = \frac{\pi}{2}$.

b) We first determine an antiderivative $F$ of the function $f$ on $[e,\infty)$. Since

$$\int \frac{1}{x(\ln x)^3}dx = \int \frac{(\ln x)'}{(\ln x)^3}dx = -\frac{1}{2(\ln x)^2}+\mathcal{C},$$

it follows that the function $F\colon [e,\infty[\to\mathbb{R}$, defined by $F(x)=-\frac{1}{2(\ln x)^2}$, is an antiderivative of $f$. As

$$\lim_{x\to\infty} -\frac{1}{2(\ln x)^2} = 0,$$

it follows that $f$ is improperly integrable on $[e,\infty)$ and $\int_{e}^{\infty} \frac{1}{x(\ln x)^3}dx = \frac{1}{2}$.

**(H 29)**

Let $(x,y)\in(0,\pi)\times(0,\pi)$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y) = \cos x + \cos(x+y) \text{ and } \frac{\partial f}{\partial y}(x,y) = \cos y + \cos(x+y).$$

The stationary points of $f$ are the solutions of the system

$$\begin{cases} \cos x + \cos(x+y) = 0 \\ \cos y + \cos(x+y) = 0. \end{cases}$$

By subtracting the two equation we get $\cos x = \cos y$, and taking into account the domain of $f$, we obtain that $x=y$. By replacing this into the first equation we obtain

$$\cos x + \cos 2x = 0 \Longleftrightarrow \cos x + 2(\cos x)^2 - 1 = 0$$

We make the notation $\cos x = t$. Then we get the equation $2t^2+t-1=0$ which has the solutions $t_1=-1$ and $t_2=\frac{1}{2}$. Since $x\in(0,\pi)$, we have that $\cos x\neq -1$, thus $\cos x=\frac{1}{2}$. We conclude that $(\frac{\pi}{3},\frac{\pi}{3})$ is the only stationary point of the function $f$. We compute now the second-order partial derivatives. For all $(x,y)\in(0,\pi)\times(0,\pi)$ we get

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin x - \sin(x+y), \frac{\partial^2 f}{\partial y^2}(x,y) = -\sin y - \sin(x+y),$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -\sin(x+y) = \frac{\partial^2 f}{\partial x \partial y}(x,y),$$

thus

$$H_f(x,y) = \begin{pmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin x - \sin(x+y) \end{pmatrix} \text{ and}$$

$$H_f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\begin{pmatrix} -\sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \sqrt{3} \end{pmatrix}.$$

Since $\Delta_1=-\sqrt{3}<0$ and $\Delta_2=\frac{3}{2}>0$, it follows (from **P2** or **Th3** in Lecture 10) that $H_f(\frac{\pi}{3},\frac{\pi}{3})$ is negative definite, thus $(\frac{\pi}{3},\frac{\pi}{3})$ is a local maximum point of $f$. The corresponding extreme value is

$$f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2}.$$

**(H 30)**

1° According to the definition, $f$ is improperly integrable on $[a,b)$ if and only if

$$\exists\, \ell := \lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)dx \in \mathbb{R}.$$

Let $t\in(a,b)$. Then $t\in\mathbb{R}$, $f$ is continuous on $[a,t]$, $F$ is an antiderivative of $f$ on $[a,t]$, and thus, according to the formula of Leibniz-Newton for definite integrals

$$\int_a^t f(x)dx = F(x)\Big|_a^t = F(t)-F(a). \tag{1}$$

Thus $f$ is improperly integrable on $[a,b)$ if and only if $F$ has finite (left-hand) limit at $b$.

2° Using (1) and the definition of the improper integral, we obtain that the improper integral $I$ of $f$ on $[a,b)$ is

$$I=\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)dx = \lim_{\substack{t\to b\\ t<b}} F(t)-F(a).$$

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.12

# Analysis for CS

GROUPWORK:

**(G 29)**

Study the improper integrability of the following continuous functions, using the second comparison criteria for improper integrals.

a) $f\colon [1,\infty) \to \mathbb{R},\ f(x) = \frac{1}{x\sqrt{1+x^2}}$, b) $f\colon [0,\frac{\pi}{2}) \to \mathbb{R},\ f(x) = \frac{1}{\cos x}$,

c) $f\colon (0,\infty) \to \mathbb{R},\ f(x) = \left(\frac{\operatorname{arctg} x}{x}\right)^2$, d) $f\colon (1,\infty) \to \mathbb{R},\ f(x) = \frac{\ln x}{x\sqrt{x^2-1}}$,

e) $f\colon [0,1) \to \mathbb{R},\ f(x) = \frac{1}{\sqrt{(1-x^2)(1-a^2x^2)}}$, where $a \in (-1,1)$ is fixed.

**(G 30)**

Using the integral criterion, decide whether the following series are convergent or not:

a) $\displaystyle\sum_{n\geq 2} \frac{1}{n(\ln n)^2}$, b) $\displaystyle\sum_{n\geq 2} \frac{\ln n}{n^2}$, c) $\displaystyle\sum_{n\geq 1} \frac{1}{\sqrt{1+e^n}}$.

HINT for a) and b): Use the formula of Leibniz-Newton for improper integrals in order to study the improper integrability of the functions associated with the series given at a) and b).

HOMEWORK:

**(H 31) (To be delivered in the next exercise-class)**

Study the improper integrability of the following continuous functions, using the second comparison criteria for improper integrals.

a) $f\colon (0,\infty) \to \mathbb{R},\ f(x) = \frac{\operatorname{arctg} x}{x(1+x^2)}$,

b) $f\colon [x_0,\infty) \to \mathbb{R},\ f(x) = \frac{1}{\sqrt{x(x-a)(x-b)}}$, where $x_0 > a > b > 0$ are fixed.

**(H 32) (To be delivered in the next exercise-class)**

Determine all local extrema, their type (minima or maxima) and the corresponding extreme values of the function $f\colon \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R},\ f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$.

**(H 33) (Train your brain)**

Having the proof of **Th2** as a model, prove **Th4** in the 12th lecture.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.12

# Analysis for CS

**(G 29)**

a) We have that $f(x) = \frac{1}{x\sqrt{1+x^2}} > 0$ for all $x \geq 1$. Due to the fact that $L = \lim\limits_{x\to\infty} x^2 f(x) = 1 < \infty$ and $p = 2 > 1$, it follows that $f$ is improperly integrable on $[1, \infty)$.

b) The function $f$ is positive on $[0, \frac{\pi}{2})$. By applying L'Hospital's rules, we get

$$L = \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \left(\frac{\pi}{2} - x\right) f(x) = \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} \frac{1}{\sin x} = 1.$$

Since $p = 1 \leq 1$, we conclude that $f$ is not improperly integrable on $[0, \frac{\pi}{2})$.

c) We have that $f(x) > 0$ for all $x > 0$. Due to the fact that

$$\lim_{\substack{x\to 0\\ x>0}} x^0 f(x) = \lim_{\substack{x\to 0\\ x>0}} \left(\frac{\operatorname{arctg} x}{x}\right)^2 = 1,$$

we conclude that $f$ is improperly integrable on $(0, 1]$. The relations

$$L = \lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 \left(\frac{\operatorname{arctg} x}{x}\right)^2 = \frac{\pi^2}{4} < \infty$$

and the fact that $p = 2 > 1$ lead us to the conclusion that $f$ is improperly integrable on $[1, \infty)$. Hence $f$ is improperly integrabe on $(0, \infty)$.

d) The function $f$ is positive on its domain. From

$$\lim_{\substack{x\to 1\\ x>1}} \sqrt{x-1} f(x) = \lim_{\substack{x\to 1\\ x<1}} \frac{\sqrt{x-1}\ln x}{x\sqrt{x^2-1}} = 0$$

and $p = \frac{1}{2} < 1$ it follows that $f$ is improperly integrable on $(1, 2]$. Since

$$L = \lim_{x\to\infty} x^{\frac{3}{2}} f(x) = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2-1}} \cdot \frac{\ln x}{\sqrt{x}} = 0$$

and $p = \frac{3}{2} > 1$, it follows that $f$ is improperly integrable on $[2, \infty)$, thus $f$ is improperly integrable on $(1, \infty)$.

e) Note that $-1 < a < 1$ implies that $1 - a^2x^2 > 0$ for all $x \in [0, 1]$. As well, $f(x) > 0$ for all $x \in [0, 1)$. From

$$L = \lim_{\substack{x\to 1\\ x<1}} \sqrt{1-x} f(x) = \lim_{\substack{x\to 1\\ x<1}} \frac{\sqrt{1-x}}{\sqrt{(1-x^2)(1-a^2x^2)}} = \frac{1}{\sqrt{2}(1-a^2)} < \infty$$

and the fact that $p = \frac{1}{2} < 1$ it follows that $f$ is improperly integrable on $[0, 1)$.

**(G 30)**

a) We consider the function $f\colon [2,\infty) \to \mathbb{R}$, defined by $f(x) = \frac{1}{x(\ln x)^2}$, which is decreasing on its domain, and we notice that

$$\int \frac{1}{x(\ln x)^2} dx = \int (\ln x)^{-2} \cdot (\ln x)' dx = -\frac{1}{\ln x} + \mathcal{C}.$$

Thus $F\colon [2,\infty) \to \mathbb{R}$, defined by $F(x) = -\frac{1}{\ln x}$, is an antiderivative of $f$. Moreover, $\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty} -\dfrac{1}{\ln x} = 0$. Thus (by the formula of Leibniz-Newton for improper integrals) $f$ is improperly integrable on $[2,\infty)$. From the integral criterion it follows that the series $\sum\limits_{n\geq 2} \dfrac{1}{n(\ln n)^2}$ is convergent.

b) We consider the function $f\colon [2,\infty) \to \mathbb{R}$, defined by $f(x) = \frac{\ln x}{x^2}$, which is decreasing on its domain (this may be checked by observing that $f'$ is negative on$[2,\infty)$), and we notice that

$$\int \frac{\ln x}{x^2} dx = \int \ln x \cdot \left(-\frac{1}{x}\right)' dx = -\frac{1}{x} \cdot \ln x + \int \frac{1}{x^2} dx = -\frac{1}{x} \cdot \ln x - \frac{1}{x} + \mathcal{C}.$$

Thus $F\colon [2,\infty) \to \mathbb{R}$, defined by $F(x) = -\frac{1}{x} \cdot \ln x - \frac{1}{x}$, is an antiderivative of $f$. Moreover, by applying L'Hopital's rule we have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} -\frac{\ln x + 1}{x} = -\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

Thus $f$ is improperly integrable on $[2,\infty)$ (by the formula of Leibniz-Newton for improper integrals). From the integral criterion it follows that the series $\sum\limits_{n\geq 2} \dfrac{\ln n}{n^2}$ is convergent.

c) We consider the function $f\colon [1,\infty) \to \mathbb{R}$, defined by $f(x) = \frac{1}{\sqrt{1+e^x}}$, which is decreasing on its domain. Since

$$L = \lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} \sqrt{\frac{x^4}{1+e^x}} = 0$$

and $p = 2$, the second comparison criterion for improper integrals yields that $f$ is improperly integrable on $[1,\infty)$. Thus we conclude from the integral criterion that the series $\sum\limits_{n\geq 1} \dfrac{1}{\sqrt{1+e^n}}$ is convergent.

Homework:

**(H 31)**

a) The function $f$ is positive on its domain. Since

$$\lim_{\substack{x\to 0\\ x>0}} x^0 f(x) = \lim_{\substack{x\to 0\\ x>0}} \frac{\operatorname{arctg} x}{x(1+x^2)} = 1,$$

$f$ is improperly integrable on $(0,1]$. The equalities

$$L = \lim_{x\to\infty} x^3 f(x) = \lim_{x\to\infty} x^3 \frac{\operatorname{arctg} x}{x(1+x^2)} = \frac{\pi}{2}$$

and the fact that $p = 3 > 1$ imply that $f$ is improperly integrable on $[1,\infty)$ as well. Hence $f$ is improperly integrable on $(0,\infty)$.

b) The function $f$ is positive on its domain. Since

$$L = \lim_{x\to\infty} x^{\frac{3}{2}} f(x) = \lim_{x\to\infty} \frac{x^{\frac{3}{2}}}{\sqrt{x(x-a)(x-b)}} = 1$$

and $p = \frac{3}{2} > 1$, it follows that $f$ is improperly integrable on pe $(x_0, \infty)$.

**(H 32)**

We have for all $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$ that $\frac{\partial f}{\partial x}(x, y) = y - \frac{1}{x^2}$ and $\frac{\partial f}{\partial y}(x, y) = x - \frac{1}{y^2}$. Thus the stationary points of $f$ are the solutions of the system

$$\begin{cases} y = \frac{1}{x^2} \\ x = \frac{1}{y^2}. \end{cases}$$

It follows that $x^4 = x$. Since $x \in \mathbb{R}^*$, we conclude that $x = 1$. Hence $(1, 1)$ is the only stationary point of $f$. We have for all $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$ that

$$H_f(x, y) = \begin{pmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{pmatrix},$$

thus $H_f(1, 1)$ is positive definite. We conclude that $(1, 1)$ is a local minimum of $f$ and that $f(1, 1) = 3$ is the corresponding extreme value.

**(H 33)**

Without any loss of generality we may assume that $a > 0$.

1° Using the definition of the limit of a function at a point, there exists a real $c \geq a$ such that $x^p f(x) \leq L + 1$, for all $x \geq c$. Thus

$$0 \leq f(x) \leq \frac{L+1}{x^p}, \forall\, x \geq c.$$

We know from the exercise-class no. 11 that the function $x \in [a, \infty) \mapsto \frac{L+1}{x^p} \in \mathbb{R}$ is improperly integrable on $[a, \infty)$, since $p > 1$. Using assertion 1° of the first comparison criterion for improper integrals (see **Th1** in lecture no. 12), we conclude that $f$ is improperly integrable on $[a, \infty)$.

2° Fix a real number $r$ such that $0 < r < L$. Using once again the definition of the limit of a function at a point, there exists a positive real $c \geq a$ such that $r \geq x^p f(x)$, for all $x \geq c$. Thus

$$0 < \frac{r}{x^p} \leq f(x), \forall\, x \geq c.$$

We know from the exercise-class no. 11 that the function $x \in [a, \infty) \mapsto \frac{r}{x^p} \in \mathbb{R}$ is not improperly integrable on $[a, \infty)$, since $p \leq 1$. Using assertion 2° of the first comparison criterion for improper integrals (see **Th1** in lecture no. 12), we conclude that $f$ is not improperly integrable on $[a, \infty)$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.13

# Analysis for CS

GROUPWORK:

**(G 31) (Integration over compact intervals)**

Compute the following double and triple integrals:

a) $\int\int\int_A \frac{2z}{(x+y)^2} dxdydz$, where $A = [1,2] \times [2,3] \times [0,2]$,

b) $\int\int_A \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dxdy$, where $A = [0,\sqrt{3}] \times [0,1]$,

c) $\int\int\int_A \frac{x^2 z^3}{1+y^2} dxdydz$, where $A = [0,1] \times [0,1] \times [0,1]$.

**(G 32) (Integration over normal domains)**

Let

$$M := \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1,\ -x^2 \le y \le 1 + x^2\}.$$

a) Represent $M$ in a Cartesian coordinate system.

b) Compute $\int\int_M (x^2 - 2y) dxdy$.

c) Is $M$ a normal domain with respect to the $x$-axis?

HOMEWORK:

**(H 34) (Integration over compact intervals)**

Compute the following double and triple integrals:

a) $\int\int_A (xy + y^2) dxdy$, where $A = [0,1] \times [0,1]$,

b) $\int\int_A \min\{x,y\} dxdy$, where $A = [0,1] \times [0,2]$,

c) $\int\int\int_A \frac{1}{(x+y+z)^3} dxdydz$, where $A = [1,2] \times [1,2] \times [1,2]$.

**(H 35) (Integration over normal domains)**

Let $M$ be the subset of $\mathbb{R}^2$ bounded by the triangle with vertices $(0,0)$, $(1,0)$ and $(\frac{1}{2}, \frac{1}{2})$.

a) Represent $M$ in a Cartesian coordinate system.

b) Show that $M$ is a normal domain with respect to the $x$-axis.

c) Compute $\int\int_M (x^2 + y^2) dxdy$.

**(H 36) (Train your brain)**

Show that the function $f\colon [0,1] \times [0,1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0,1] \times [0,1] \cap \mathbb{Q} \times \mathbb{Q} \\ 0 \text{ else,} \end{cases}$$

is not Riemann integrable on $[0,1] \times [0,1]$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Solutions to Exercise Sheet no.13

# Analysis for CS

**(G 31)**

The functions to be integrated in exercises a)–c) are continuous. Thus we can apply assertion 2° of **Th1** in lecture no. 13 to compute the corresponding multiple integrals, which we denote by $I$.

a) We have that $I = \int_1^2 dx \int_2^3 dy \int_0^2 \frac{2z}{(x+y)^2} dz = \int_1^2 dx \int_2^3 \frac{4}{(x+y)^2} dy$. Since for every $x \in [1,2]$

$$\int_2^3 \frac{4}{(x+y)^2} dy = -\left.\frac{4}{x+y}\right|_2^3 = \frac{4}{x+2} - \frac{4}{x+3},$$

we finally obtain $I = \int_1^2 (\frac{4}{x+2} - \frac{4}{x+3}) dx = 4 \ln \left.\frac{x+2}{x+3}\right|_1^2 = 4 \ln \frac{16}{15}$.

b) We have that $I = \int_0^1 dy \int_0^{\sqrt{3}} \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dx$. Since for all $y \in [0,1]$

$$\int_0^{\sqrt{3}} \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dx = -\left.\frac{1}{(1+x^2+y^2)^{\frac{1}{2}}}\right|_0^{\sqrt{3}} = \frac{1}{\sqrt{1+y^2}} - \frac{1}{\sqrt{4+y^2}},$$

we get $I = \int_0^1 \frac{1}{\sqrt{1+y^2}} dy - \int_0^1 \frac{1}{\sqrt{4+y^2}} dy = \ln \left.\frac{y+\sqrt{1+y^2}}{y+\sqrt{4+y^2}}\right|_0^1 = \ln \frac{2(1+\sqrt{2})}{1+\sqrt{5}}$.

c) We have that $I = \int_0^1 dx \int_0^1 dy \int_0^1 \frac{x^2 z^3}{1+y^2} dz = \frac{1}{4} \int_0^1 dx \int_0^1 \frac{x^2}{1+y^2} dy$. Since for all $x \in [0,1]$

$$\int_0^1 \frac{x^2}{1+y^2} dy = x^2 \operatorname{arctg} y \big|_0^1 = \frac{\pi}{4} x^2,$$

we get $I = \frac{\pi}{16} \int_0^1 x^2 dx = \frac{\pi}{48}$.

**(G 32)**

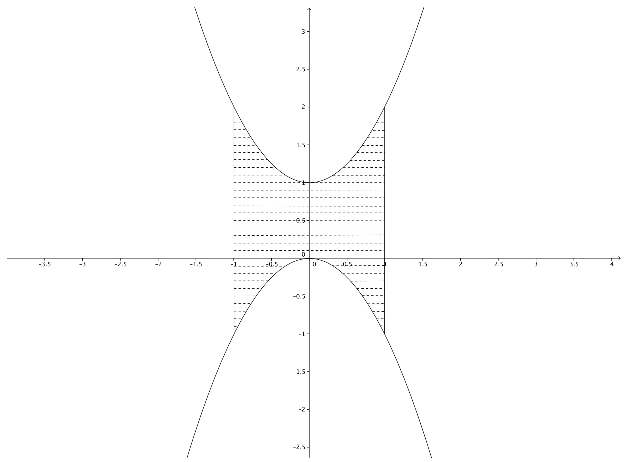
b) We denote by $I$ the double integral to be computed. Since $M$ is a normal domain with respect to the $y$-axis, we apply **Th1** in the exercise-class no. 13 to compute $I$. Hence $I = \int_{-1}^1 dx \int_{-x^2}^{1+x^2} (x^2 - 2y) dy$. For all $x \in [-1,1]$

$$\int_{-x^2}^{1+x^2} (x^2 - 2y) dy = (x^2 y - y^2)\big|_{-x^2}^{1+x^2} = 2x^4 - x^2 - 1,$$

thus

$$I = \int_{-1}^1 (2x^4 - x^2 - 1) dx = \left(\frac{2}{5}x^5 - \frac{1}{3}x^3 - x\right)\Big|_{-1}^1 = -\frac{28}{15}.$$

a)



c) We see from the above figure that not every parallel line to the $x$-axis intersects $M$ along a compact interval. Thus $M$ is not normal with respect to the $x$-axis.

HOMEWORK:

**(H 34)**

The functions to be integrated in exercises a)–c) are continuous. Thus we can apply assertion $2^\circ$ of **Th1** in lecture no. 13 to compute the corresponding multiple integrals, which we denote by $I$.

a) We have that $I = \int_0^1 dy \int_0^1 (xy + y^2)dx$. Since for every all $y \in [0, 1]$

$$\int_0^1 (xy + y^2)dx = \left(\frac{x^2y}{2} + y^2x\right)\Bigg|_0^1 = \frac{y}{2} + y^2,$$

we obtain $I = \int_0^1 (\frac{y}{2} + y^2)dy = \left(\frac{y^2}{4} + \frac{y^3}{3}\right)\Big|_0^1 = \frac{7}{12}$.

b) We have that $I = \int_0^1 dx \int_0^2 \min\{x, y\}dy$. Note that for all $x \in [0, 1]$

$$\min\{x, y\} = \begin{cases} y, & 0 \le y \le x \\ x, & x \le y \le 2, \end{cases}$$

thus $\int_0^2 \min\{x, y\}dy = \int_0^x ydy + \int_x^2 xdy = -\frac{x^2}{2} + 2x$. We finally get $I = \int_0^1 (-\frac{x^2}{2} + 2x)dx = \frac{5}{6}$.

c) We have that $I = \int_1^2 dx \int_1^2 dy \int_1^2 \frac{1}{(x+y+z)^3}dz$. For every $(x, y) \in [1, 2] \times [1, 2]$ we compute

$$\int_1^2 \frac{1}{(x+y+z)^3}dz = -\frac{1}{2(x+y+z)^2}\Bigg|_1^2 = -\frac{1}{2}\left(\frac{1}{(x+y+2)^2} - \frac{1}{(x+y+1)^2}\right).$$

For $x \in [1, 2]$ we then get

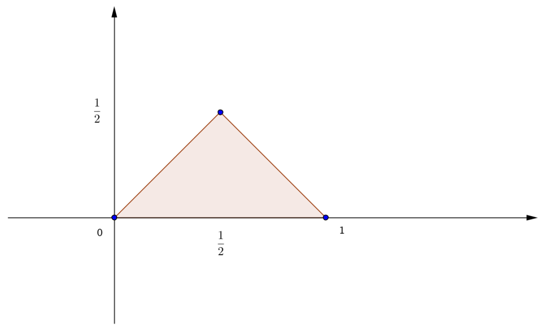
$$-\frac{1}{2}\int_1^2 \left(\frac{1}{(x+y+2)^2} - \frac{1}{(x+y+1)^2}\right)dy = \frac{1}{2}\left(\frac{1}{x+y+2} - \frac{1}{x+y+1}\right)\Bigg|_1^2.$$

Thus

$$I = \frac{1}{2}\int_1^2 \left(\frac{1}{x+4} - \frac{2}{x+3} + \frac{1}{x+2}\right)dx = \frac{1}{2}\ln\frac{(x+4)(x+2}{(x+3)^2}\Bigg|_1^2 = \frac{1}{2}\ln\frac{128}{125}.$$

**(H 35)**

a)




b) We see from the representation of $M$ performed at a) that this set is a normal domain with respect to both axes. When viewed as a normal domain with respect to the $x$-axis, then the boundary functions $\psi_1$, $\psi_2\colon [0, \frac{1}{2}] \to \mathbb{R}$ of $M$ are, respectively, given by

$$\psi_1(y) = y, \quad \psi_2(y) = 1 - y.$$

Thus

$$M = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le \frac{1}{2},\ y \le x \le 1 - y\right\}.$$

c) We denote by $I$ the integral to be computed. Using **Th2** in the exercise-class no. 13, we get that $I = \int_1^{\frac{1}{2}} dy \int_y^{1-y} (x^2 + y^2) dx$. Since for $y \in [0, \frac{1}{2}]$ we have that

$$\int_y^{1-y} (x^2 + y^2) dx = \left(\frac{1}{3}x^3 + xy^2\right)\Bigg|_y^{1-y} = \frac{1}{3}(1 - y)^3 + y^2 - \frac{7}{3}y^3,$$

we finally obtain that $I = \int_0^{\frac{1}{2}} \left(\frac{1}{3}(1 - y)^3 + y^2 - \frac{7}{3}y^3\right) dy = \frac{1}{12}$.

**(H 36)**

Let $(\tilde{\Delta}_k)_{k\in\mathbb{N}^*}$ be a partition of the interval $[0, 1]$ such that $\lim\limits_{k\to\infty} ||\tilde{\Delta}_k|| = 0$. (Take for instance, for every $k \in \mathbb{N}^*$, $\tilde{\Delta}_k$ to be the equidistant partition determined by the points $0$, $\frac{1}{k}$, $\frac{2}{k}, \ldots, \frac{k-1}{k}$, $1$. In this case $||\tilde{\Delta}_k|| = \frac{1}{k}$, for every $k \in \mathbb{N}^*$.) For every $k \in \mathbb{N}^*$ denote by $\Delta_k$ the partition of the square $[0, 1]\times[0, 1]$ determined by $\tilde{\Delta}_k$ and $\tilde{\Delta}_k$. Then clearly $\lim\limits_{k\to\infty} ||\Delta_k|| = 0$. For every $k \in \mathbb{N}^*$ let $\Delta_k := \{D_1^k, \ldots, D_{m_k}^k\}$. For $k \in \mathbb{N}^*$ consider now, for every $j \in \{1, \ldots, m_k\}$, a point $u_j^k \in D_j^k \cap \mathbb{Q}^2$ and a point $v^k \in D_j^k \setminus \mathbb{Q}^2$. Denote by $s_k := (u_1^k, \ldots, u_{m_k}^k)$ and by $s'_k := (v_1^k, \ldots, v_{m_k}^k)$. Then $s_k$, $s'_k \in S_{\Delta_k}$, for every $k \in \mathbb{N}^*$. We thus obtain the following values for the Riemann sums

$$S(f, \Delta_k, s_k) = 1, \quad S(f, \Delta_k, s'_k) = 0, \forall\ k \in \mathbb{N}^*.$$

Thus $\lim\limits_{k\to\infty} S(f, \Delta_k, s_k) \neq \lim\limits_{k\to\infty} S(f, \Delta_k, s'_k)$, so $f$ is not Riemann integrable on $[0, 1] \times [0, 1]$.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Winter semester 2013-2014

Exercise Sheet no.14

# Analysis for CS

GROUPWORK:

**(G 33) (Integration over normal domains)**

Let $\emptyset \neq M \subseteq \mathbb{R}^2$ be bounded and let $f\colon M \to \mathbb{R}$ be continuous. Represent $M$ in a Cartesian coordinate system and compute $I := \int\int_M f(x,y)dxdy$ in the following cases:

a) $M = \{(x,y) \in \mathbb{R}^2 \mid \ 0 \leq y \leq 1,\ -1 \leq x \leq y\}$, $f(x,y) = xy - y^3$,

b) $M =$ the domain in the first quadrant which lies between the line $y = x$ and the parabola $y = x^2$, $f(x,y) = xy$,

c) $M = \{(x,y) \in \mathbb{R}^2 \mid \ 0 \leq x \leq 1,\ 0 \leq y \leq x\}$, $f(x,y) = y + \sin(\pi x^2)$.

**(G 34) (Improper integrals)**

Using the formula of Leibniz-Newton for improper integrals, study the improper integrability of the following functions on their domains and, in case they are improperly integrable, compute the corresponding improper integrals.

a) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-2x}$,

b) $f\colon [2,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x(\ln x)^\alpha}$, where $\alpha \in \mathbb{R}$ is a parameter.

**(G 35) (Limits of real-valued functions of several variables)**

1) Show that, in each of the following cases, the function $f\colon \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$ does not have a limit at $0_2$:

a) $f(x,y) = \frac{y^2}{x^2+y^2}$, b) $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$.

2) Show that the function $g\colon \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, defined by $g(x,y) = \frac{xy^3}{x^2+y^2}$, has a limit at $0_2$ and determine this limit.

**(G 36) (Pythagoras' theorem in $\mathbb{R}^n$)**

Let $x$, $y \in \mathbb{R}^n$ be two orthogonal vectors, i.e., $\langle x, y\rangle = 0$. Prove that then the equality

$$||x+y||^2 = ||x||^2 + ||y||^2$$

holds true.

Dr. Brigitte Breckner
Dr. Anca Grad

"Babeş–Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

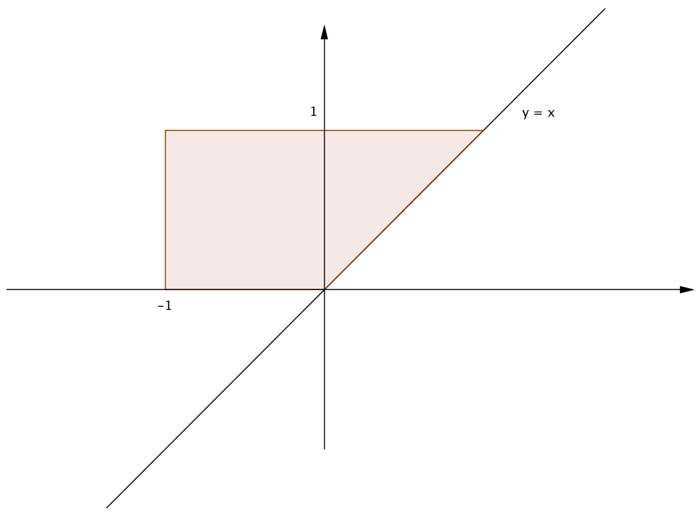
Winter semester 2013-2014

Solutions to Exercise Sheet no.14

# Analysis for CS

**(G 33)**

a)

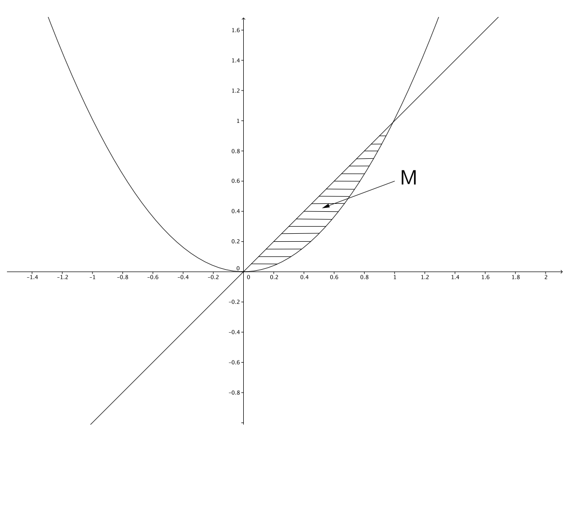



We regard $M$ as a normal domain with respect to the $x$-axis. Applying **Th2** in the exercise-class no. 13, we get that $I = \int_0^1 dy \int_{-1}^{y} (xy - y^3)dx$. Since for every $y \in [0, 1]$

$$\int_{-1}^{y} (xy - y^3)dx = \left(\frac{1}{2}x^2y - xy^3\right)\Bigg|_{-1}^{y} = -\frac{1}{2}y - \frac{1}{2}y^3 - y^4,$$

we finally obtain that $I = -\int_0^1 \left(\frac{1}{2}y + \frac{1}{2}y^3 + y^4\right) dy = -\frac{23}{40}$.

b)



We regard $M$ as a normal domain with respect to the $y$-axis, i.e.,

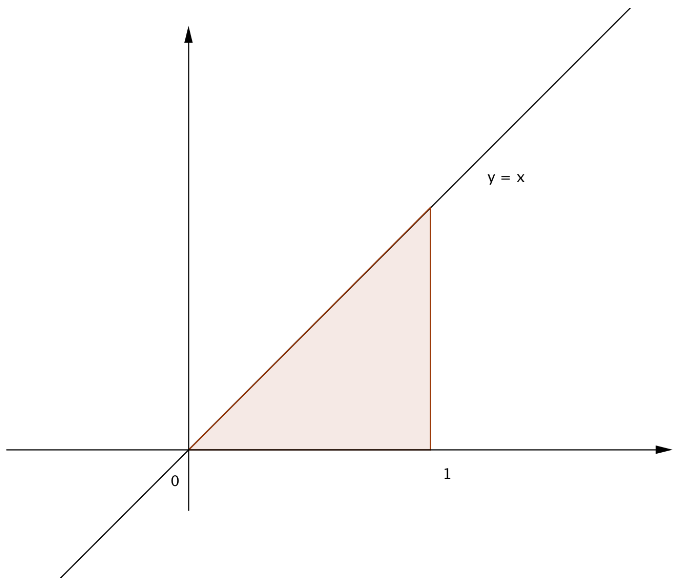
$$M = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le x\}.$$

Using **Th1** in the exercise-class no. 13, we get that $I = \int_0^1 dx \int_{x^2}^x xy dy$. Since for every $x \in [0, 1]$

$$\int_{x^2}^x xy dy = \frac{1}{2}xy^2\Big|_{x^2}^x = \frac{1}{2}x^3 - \frac{1}{2}x^5,$$

we finally obtain that $I = \int_0^1 \left(\frac{1}{2}x^3 - \frac{1}{2}x^5\right) dx = \frac{1}{24}$.

c)




We regard $M$ as a normal domain with respect to the $y$-axis. Applying **Th1** in the exercise-class no. 13, we get that $I = \int_0^1 dx \int_0^x (y + \sin \pi x^2) dy$. Since for every $x \in [0, 1]$

$$\int_0^x (y + \sin \pi x^2) dy = \left(\frac{1}{2}y^2 + y \sin \pi x^2\right)\Big|_0^x = \frac{1}{2}x^2 + x \sin \pi x^2,$$

we finally obtain that $I = \int_0^1 \left(\frac{1}{2}x^2 + x \sin \pi x^2\right) dx = \left(\frac{1}{6}x^3 - \frac{1}{2\pi} \cos \pi x^2\right)\Big|_0^1 = \frac{1}{6} + \frac{1}{\pi}$.

**(G 34)**

a) The function $F\colon \mathbb{R} \to \mathbb{R}$, defined by $F(x) = -e^{-x}$, is an antiderivative of $f$. Since $\lim_{x \to -\infty} F(x) = -\infty$, assertion 1° of **Th3** in the exercise-class no. 11 yields that $f$ is not improperly integrable on $\mathbb{R}$.

b) The function $F\colon [2, \infty) \to \mathbb{R}$, defined, for every $x \ge 2$, by

$$F(x) = \begin{cases} \frac{1}{1-\alpha}(\ln x)^{1-\alpha}, & \alpha \ne 1 \\ \ln(\ln x), & \alpha = 1, \end{cases}$$

is an antiderivative of $f$. Since

$$\lim_{x \to \infty} F(x) = \begin{cases} \infty, & 1 \ge \alpha \\ 0, & \alpha > 1, \end{cases}$$

assertion 1° of **Th2** in the exercise-class no. 11 implies that $f$ is improperly integrable on $[2, \infty)$ if and only if $\alpha > 1$. In this case, assertion 2° of the same theorem yields that $\int_2^\infty f(x)dx = \frac{1}{(\alpha-1)(\ln 2)^{\alpha-1}}$.

**(G 35)**

1) We apply in each cases **Th2** in lecture no. 9.

a) Since $\lim\limits_{n\to\infty} f\left(\dfrac{1}{n},0\right) = 0$ and $\lim\limits_{n\to\infty} f\left(0,\dfrac{1}{n}\right) = 1$, we conclude that $f$ doesn't have a limit at $0_2$.

b) Since $\lim\limits_{n\to\infty} f\left(\dfrac{1}{n},0\right) = 1$ and $\lim\limits_{n\to\infty} f\left(0,\dfrac{1}{n}\right) = -1$, we conclude that $f$ doesn't have a limit at $0_2$.

2) The inequality $|xy| \leq \frac{1}{2}(x^2+y^2)$ implies that

$$|g(xy)| \leq \frac{y^2}{2}, \forall\ (x,y) \in \mathbb{R}^2 \setminus \{0_2\}.$$

If $((x_n,y_n))_{n\in\mathbb{N}^*}$ is a sequence in $\mathbb{R}^2\setminus\{0_2\}$ converging to $0_2$, then the above inequality and the Sandwich-Theorem yield that $\lim\limits_{n\to\infty} g(x_n,y_n) = 0$. Applying **Th2** in lecture no. 9, we conclude that $\lim\limits_{(x,y)\to 0_2} g(x,y) = 0$.

**(G 36)**

Since (by the definition of the scalar product) $\langle x,y\rangle = \langle y,x\rangle$, we have that $\langle y,x\rangle = 0$. Using the definition of the Euclidean norm and the properties of the scalar product, we thus obtain

$$||x+y||^2 = \langle x+y, x+y\rangle = ||x||^2 + \langle x,y\rangle + \langle y,x\rangle + ||y||^2 = ||x||^2 + ||y||^2.$$