LECTURE

4

SERIES WITH NONNEGATIVE TERMS (II). SERIES WITH ARBITRARY TERMS

Series with nonnegative terms (II)

Theorem 4.1 (Kummer's Test) *Let $\sum_{n\geq 1} x_n$ be a series with positive terms.*

1° *If $\exists\,(c_n)_{n\in\mathbb{N}}$ in $(0,+\infty)$, $\exists\, r>0$ and $\exists\, n_0\in\mathbb{N}$, such that*

$$c_n\frac{x_n}{x_{n+1}}-c_{n+1}\geq r,\ \forall\, n\in\mathbb{N},\ n\geq n_0,$$

then the series $\sum\limits_{n\geq 1} x_n$ is convergent.

2° *If $\exists\,(c_n)_{n\in\mathbb{N}}$ in $(0,+\infty)$ and $\exists\, n_0\in\mathbb{N}$, such that*

$$\sum_{n=1}^{n}\frac{1}{c_n}=+\infty \quad \textit{and} \quad c_n\frac{x_n}{x_{n+1}}-c_{n+1}\leq 0,\ \forall\, n\in\mathbb{N},\ n\geq n_0,$$

then the series $\sum\limits_{n\geq 1} x_n$ is divergent.

Proof. 1° Since $c_nx_n-c_{n+1}x_{n+1}\geq rx_{n+1}$, $\forall n\geq n_0$, it follows that for any $n\geq n_0+1$,

$$\sum_{k=n_0}^{n-1}(c_kx_k-c_{k+1}x_{k+1})\geq r\sum_{k=n_0}^{n-1}x_{k+1}.$$

Denoting $s_n\coloneqq x_1+\ldots+x_n$, we deduce that $c_{n_0}x_{n_0}-c_nx_n\geq r\,(s_n-s_{n_0})$ and therefore

$$s_n\leq s_{n_0}+\frac{1}{r}\,(c_{n_0}x_{n_0}-c_nx_n)\leq s_{n_0}+\frac{c_{n_0}x_{n_0}}{r}.$$

Hence, the sequence of partial sums (s_n) is bounded, which means that the series $\sum\limits_{n\geq 1} x_n$ is convergent (by Lemma 3.13)

2° Since $c_n x_n \le c_{n+1} x_{n+1}$, $\forall n \ge n_0$, we have $c_{n_0} x_{n_0} \le c_n x_n$, $\forall n \ge n_0$. This yields

$$\frac{1}{c_n} \le \frac{1}{c_{n_0} x_{n_0}} x_n, \ \forall\, n \ge n_0.$$

Since the series $\sum_{n\ge 1} \frac{1}{c_n}$ is divergent, we conclude that the series $\sum_{n\ge 1} x_n$ is divergent as well, according to the Comparison Test (Theorem 3.18) □

Theorem 4.2 (Raabe-Duhamel's Test) *Let $\sum_{n\ge 1} x_n$ be a series with positive terms.*

1° *If $\exists\, q > 1$, $\exists\, n_0 \in \mathbb{N}$ such that $n\left(\frac{x_n}{x_{n+1}} - 1\right) \ge q$, $\forall\, n \ge n_0$, then $\sum_{n\ge 1} x_n$ is convergent.*

2° *If $\exists n_0 \in \mathbb{N}$ such that $n\left(\frac{x_n}{x_{n+1}} - 1\right) \le 1$, $\forall\, n \ge n_0$, then $\sum_{n\ge 1} x_n$ is divergent.*

3° *If the following limit exists*

$$R := \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) \in \overline{\mathbb{R}},$$

then we have

a) If $R > 1$, $\sum_{n\ge 1} x_n$ is convergent.

b) If $R < 1$, $\sum_{n\ge 1} x_n$ is divergent.

Proof. Follows from Kummer's Test (Theorem 4.1)for $c_n := n$ for all $n \in \mathbb{N}$. □

Example 4.3 *For any $a > 0$ consider the series*

$$\sum_{n\ge 1} \frac{n!}{a(a+1)\cdot \ldots \cdot (a+n)}.$$

This series is convergent for $a > 1$ and divergent for $a \in (0, 1]$.

Indeed, denoting $x_n := \frac{n!}{a(a+1)\cdot \ldots \cdot (a+n)}$, we have

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{a(a+1)\cdot \ldots \cdot (a+n+1)} \cdot \frac{a(a+1)\cdot \ldots \cdot (a+n)}{n!} = \frac{n+1}{a+n+1}.$$

Note that $D := \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$, hence the Ratio Test is inconclusive. However,

$$R := \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{a+n+1}{n+1} - 1\right) = \lim_{n\to\infty} n\frac{a}{n+1} = a,$$

which allows us to conclude, by Raabe-Duhamel's Test, that the given series is convergent if $a > 1$ and divergent if $a \in (0, 1)$.

Finally, for $a = 1$ the given series becomes $\sum_{n\ge 1} \frac{1}{n+1}$, which is divergent.

Theorem 4.4 (Bertrand's Test) *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms. If the following limits exists*

$$B := \lim_{n\to\infty} (\ln n)\left[n\left(\frac{x_n}{x_{n+1}} - 1\right) - 1\right] \in \overline{\mathbb{R}},$$

then we have

a) If $B > 1$, *then* $\sum\limits_{n\geq 1} x_n$ *is convergent.*

b) If $B < 1$, *then* $\sum\limits_{n\geq 1} x_n$ *is divergent.*

Proof. Follows from Kummer's Test (Theorem 4.1)for $c_n := n \cdot \ln n$, $n \in \mathbb{N}$, $n \geq 2$. □

Example 4.5 *The series* $\sum\limits_{n\geq 1}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2$ *is divergent.*

Indeed, denoting $x_n := \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 = \left[\frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot (2n)}\right]^2$ *we have*

$\frac{x_{n+1}}{x_n} = \left(\frac{2n+1}{2n+2}\right)^2$ *for all* $n \in \mathbb{N}$. *It is a simple exercise to check that*

$$\begin{aligned} D &:= \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1; \\ R &:= \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\left(\frac{2n+2}{2n+1}\right)^2 - 1\right] = \lim_{n\to\infty} \frac{4n^2+3n}{4n^2+4n+1} = 1, \end{aligned}$$

hence both the Ratio Test and the Raabe-Duhamel's Test are inconclusive.

On the other hand, we have

$$B := \lim_{n\to\infty} (\ln n)\left[n\left(\frac{x_n}{x_{n+1}} - 1\right) - 1\right] = \lim_{n\to\infty} (\ln n)\left(\frac{4n^2+3n}{4n^2+4n+1} - 1\right) = 0 < 1.$$

We conclude by Bertrand's Test that the given series is divergent.

Series with arbitrary terms

Theorem 4.6 (Abel-Dirichlet's Test) *Let* $\sum\limits_{n\geq 1} x_n$ *be a series of real numbers. Assume that there exist two sequences of real numbers,* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$, *satisfying the following three conditions:*

(i) $x_n = a_n \cdot b_n$, $\forall\, n \in \mathbb{N}$.

(ii) $\exists\, M > 0$ *s.t.* $-M \leq A_n := a_1 + \cdots + a_n \leq M$, $\forall\, n \in \mathbb{N}$, *i.e., the sequence* $(A_n)_{n\in\mathbb{N}}$ *is bounded.*

(iii) The sequence $(b_n)_{n\in\mathbb{N}}$ *is monotone and convergent to* 0.

Then the series $\sum\limits_{n\geq 1} x_n$ *is convergent.*

Proof. Without loss of generality we can assume in (iii) that (b_n) is decreasing. We will prove that $\sum\limits_{n\geq 1} x_n$ converges by using Cauchy's Criterion (Theorem 3.11)To this aim, consider an arbitrary $\varepsilon > 0$.

On the one hand, by (i), (ii) and the assumption that (b_n) is decreasing, we have

$$\begin{aligned} & |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| \\ = \ & |a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \ldots + a_{n+p}b_{n+p}| \\ = \ & |(A_{n+1} - A_n)\, b_{n+1} + (A_{n+2} - A_{n+1})\, b_{n+2} + \cdots + (A_{n+p} - A_{n+p-1})\, b_{n+p}| \\ = \ & |-A_n b_{n+1} + A_{n+1}\,(b_{n+1} - b_{n+2}) + \cdots + A_{n+p-1}\,(b_{n+p-1} - b_{n+p}) + A_{n+p}b_{n+p}| \\ \leq \ & |-A_n| \cdot |b_{n+1}| + |A_{n+1}| \cdot |b_{n+1} - b_{n+2}| + \cdots + |A_{n+p-1}| \cdot |b_{n+p-1} - b_{n+p}| + |A_{n+p}| \cdot |b_{n+p}| \\ = \ & |A_n| \cdot b_{n+1} + |A_{n+1}| \cdot (b_{n+1} - b_{n+2}) + \cdots + |A_{n+p-1}| \cdot (b_{n+p-1} - b_{n+p}) + |A_{n+p}| \cdot b_{n+p} \\ \leq \ & M\,[b_{n+1} + (b_{n+1} - b_{n+2}) + (b_{n+2} - b_{n+3}) + \ldots + (b_{n+p-1} - b_{n+p}) + b_{n+p}] \\ = \ & 2Mb_{n+1}, \ \forall\, n, p \in \mathbb{N}. \end{aligned}$$

On the other hand, since $\lim_{n\to\infty} b_n = 0$ by (iii), there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|b_n| < \frac{\varepsilon}{2M},\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon.$$

We conclude that $|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon,\ \forall\, p \in \mathbb{N}.$ □

Definition 4.7 *A series* $\sum_{n\geq 1} x_n$ *is called* alternating *if either*

$$x_1 \geq 0,\ x_2 \leq 0, x_3 \geq 0, \ldots (i.e.,\ x_n = (-1)^{n+1}|x_n| \textit{ for all } n \in \mathbb{N})$$

or

$$x_1 \leq 0,\ x_2 \geq 0, x_3 \leq 0, \ldots (i.e.,\ x_n = (-1)^n\,|x_n| \textit{ for all } n \in \mathbb{N}).$$

Theorem 4.8 (Leibniz's Criterion for Alternating Series) *Consider an alternating series* $\sum_{n\geq 1} x_n$. *If the sequence* $(|x_n|)_{n\in\mathbb{N}}$ *is decreasing, then the following assertions are equivalent:*

1° *The series* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *The sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* 0.

Proof. Assume that $x_n = (-1)^{n+1}|x_n|$ for all $n \in \mathbb{N}$. Then the conclusion follows by Abel-Dirichlet's Test for $a_n := (-1)^{n+1}$ and $b_n := |x_n|$. □

Definition 4.9 *A series of real numbers* $\sum_{n\geq 1} x_n$ *is called absolutely convergent if the series* $\sum_{n\geq 1} |x_n|$ *is convergent.*

Theorem 4.10 *If a series of real numbers* $\sum_{n\geq 1} x_n$ *is absolutely convergent, then it is also convergent.*

Proof. Let $\varepsilon > 0$. Since $\sum_{n\geq 1} |x_n|$ is convergent, there exists in view of the Cauchy's Criterion (Theorem 3.11) a number $n_\varepsilon \in \mathbb{N}$ such that

$$||x_{n+1}| + \cdots + |x_{n+p}|| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon,\ \forall\, p \in \mathbb{N}.$$

Noting that $|x_{n+1} + \cdots + x_{n+p}| \leq |x_{n+1}| + \cdots + |x_{n+p}| = ||x_{n+1}| + \cdots + |x_{n+p}||$, we infer

$$|x_{n+1} + \cdots + x_{n+p}| < \varepsilon,\ \forall\, n \in \mathbb{N},\ n \geq n_\varepsilon,\ \forall\, p \in \mathbb{N}.$$

By Cauchy's Criterion (Theorem 3.11)we conclude that $\sum_{n\geq 1} x_n$ is convergent. □

Definition 4.11 *A series of real numbers* $\sum_{n\geq 1} x_n$ *is called* semi-convergent *(or* conditionally convergent*) if it is convergent but not absolutely convergent.*

Remark 4.12 *A series* $\sum_{n\geq 1} x_n$ *with nonnegative terms is absolutely convergent if and only if it is convergent.*

Example 4.13 (The alternating generalized harmonic series) *Let $p \in \mathbb{R}$. The so-called* alternating generalized harmonic series

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^p}$$

is divergent for $p \in (-\infty, 0]$, semi-convergent for $p \in (0, 1]$ and absolutely convergent for $p \in (1, \infty)$.

In particular, for $p = 1$ we get the alternating harmonic series, *whose sum is*

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2.$$

Example 4.14 *The series $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n}}$ is absolutely convergent.*

Example 4.15 *The series $\displaystyle\sum_{n\geq 1} (-1)^{n+1} \sin\frac{1}{n}$ is semi-convergent.*

Example 4.16 *The series $\displaystyle\sum_{n\geq 1} (-1)^{n+1} \frac{n}{n+1}$ is divergent.*

Example 4.17 *The series $\displaystyle\sum_{n\geq 1} \cos(n\pi)$ is divergent.*

Theorem 4.18 (Cauchy) *If a series $\displaystyle\sum_{n\geq 1} x_n$ is absolutely convergent, then for any bijection (permutation) $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\displaystyle\sum_{n\geq 1} x_{\sigma(n)}$ is absolutely convergent and its sum coincides with the sum of the initial series, i.e., $\displaystyle\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} x_n$.*

Theorem 4.19 (Riemann) *If a series $\displaystyle\sum_{n\geq 1} x_n$ is semi-convergent, then for every $s \in \overline{\mathbb{R}}$ there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{\infty} x_{\sigma(n)} = s$.*

Example 4.20 *Consider the alternating harmonic series (see Example 4.13), whose sum is*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n+1}}{n} + ... = \ln 2.$$

If we permute its terms by alternating $p := 2$ positive terms followed by $q := 3$ negative terms we obtain

$$1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + ... = \sqrt{\frac{p}{q}} \ln 2.$$

Indeed, consider the Euler's constant $\gamma := \lim\limits_{n\to\infty} \gamma_n$ (see Exercise 2 of Seminar 3), where $\gamma_n := \dfrac{1}{n} + ... + \dfrac{1}{n} - \ln n$ for all $n \in \mathbb{N}$.

Denote by $(s_n)_{n\in\mathbb{N}}$ the sequence of partial sums of the permuted series. Then, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
s_{5k} &= \left(1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}\right)+\ldots+ \\
&\quad +\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{6k-4}-\frac{1}{6k-2}-\frac{1}{6k}\right) \\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{4k}-\ln 4k\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2k}-\ln 2k\right)- \\
&\quad -\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{3k}-\ln 3k\right)+\ln 4k-\frac{1}{2}\ln 2k-\frac{1}{2}\ln 3k \\
&= \gamma_{4k}-\frac{1}{2}\gamma_{2k}-\frac{1}{2}\gamma_{3k}+\ln\frac{4k}{\sqrt{6}k},
\end{aligned}$$

hence

$$\lim_{k\to\infty} s_{5k} = \gamma-\frac{1}{2}\gamma-\frac{1}{2}\gamma+\ln\frac{4}{\sqrt{6}} = \sqrt{\frac{2}{3}}\ln 2.$$

On the other hand, we also have

$$\begin{aligned}
s_{5k+1} &= s_{5k}+\frac{1}{4k+1}, \\
s_{5k+2} &= s_{5k}+\frac{1}{4k+1}+\frac{1}{4k+3}, \\
s_{5k+3} &= s_{5k}+\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{6k+2}, \\
s_{5k+4} &= s_{5k}+\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{6k+2}-\frac{1}{6k+4},
\end{aligned}$$

which show that $\lim\limits_{k\to\infty} s_{5k} = \lim\limits_{k\to\infty} s_{5k+1} = \lim\limits_{k\to\infty} s_{5k+2} = \lim\limits_{k\to\infty} s_{5k+3} = \lim\limits_{k\to\infty} s_{5k+4}$.

We conclude that

$$\lim_{n\to\infty} s_n = \sqrt{\frac{2}{3}}\ln 2.$$