BABEŞ-BOLYAI UNIVERSITY
Faculty of Mathematics and Computer Science

Fundamentals of Programming

Lecture 11 – Problem solving methods (II)

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Course content

Programming in the large

- Introduction in the software development process
- Procedural programming
- Modular programming
- Abstract data types
- Software development principles
- Testing and debugging

Programming in the small

- Recursion
- Complexity of algorithms
- Search and sorting algorithms
- **Problem solving methods**
  - Generate and test, Backtracking, Divide et impera
  - **Dynamic programming, Greedy**
- Recap

Last time

- Problem solving methods
  - Types
  - Techniques
    - Exact methods
    - Heuristic methods
  - Algorithms
    - Backtracking
    - Divide and conquer

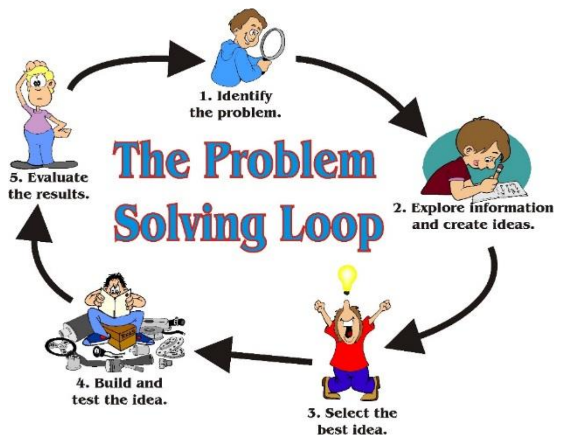
Today

- Exact methods
  - Dynamic programming

- Heuristics methods
  - Greedy algorithms

Recap: the problem solving loop

- Step in solving a problem
  - Problem definition
  - Problem analysis
  - Choose problem solving technique
    - **Search**
    - Knowledge representation
    - Abstraction



Problem solving by search

- Solving problems by search using standard methods
  - Exact methods
    - **Generate and test**
    - **Backtracking**
    - **Divide and conquer**
    - **Dynamic programming**
  - Heuristic methods
    - **Greedy method**

Dynamic Programming (DP)

- Basic idea:
  - Break the problem in overlapping sub-problems which are similar to the initial problem but are smaller in size
  - Solve the sub-problems
  - Compute the final solution by combining the sub-solutions
- Applicable in solving problems where:
  - Problems where one needs to find the best decisions one after another
  - The solution is the result of a sequence of decisions dec_1; dec_2; ...; dec_n.
  - The principle of optimality holds (whatever the initial state is remaining decisions must be optimal with regard the state following from the first decision)

DP: Mechanism

- Break the problem in nested sub-problems P(P1(P2(P3(...(Pn))...)

- Solve the most inner sub-problem P_n and store the partial result
- Solve the sub-problem P_{n-1} based on the solution found for sub-problem P_n and store the partial result
- Solve the sub-problem P_{n-2} based on the solution found for sub-problem P_{n-1} and store the partial result
- ....
- Solve the sub-problem P_1 based on the solution found for sub-problem P_2 and store the partial result
- Solve the problem P based on the solution found for sub-problem P_1 and store the **final** result

Dynamic Programming

- When DP can be used?
  - Problem P (optimization problem) with input data D can be solved by solving the same problem P but with input data d, where d < D
  - Solution is the result of a sequence of decisions dec1, dec2, ...
  - The problem can be divided in overlapping problems
  - The solutions of the sub-problems can be stored for future uses
  - The principle of optimality
- Features
  - Always gives the optimal solution
  - Polynomial run time

Dynamic Programming

- Notations
- We consider states s_0, s_1, ... s_n
  - s_0 is the initial state
  - s_n is the final state
  - States are obtained by successively applying the sequence of decisions dec_1, dec_2,..., dec_n (using the decision d_i we pass from state s_{i-1} to state s_i)

$$\begin{tikzcd}
s_0 \arrow[r, "dec_1"] & s_1 \arrow[r, "dec_2"] & s_2 \arrow[r] & \cdots \arrow[r] & s_{n-1} \arrow[r, "dec_n"] & s_n
\end{tikzcd}$$

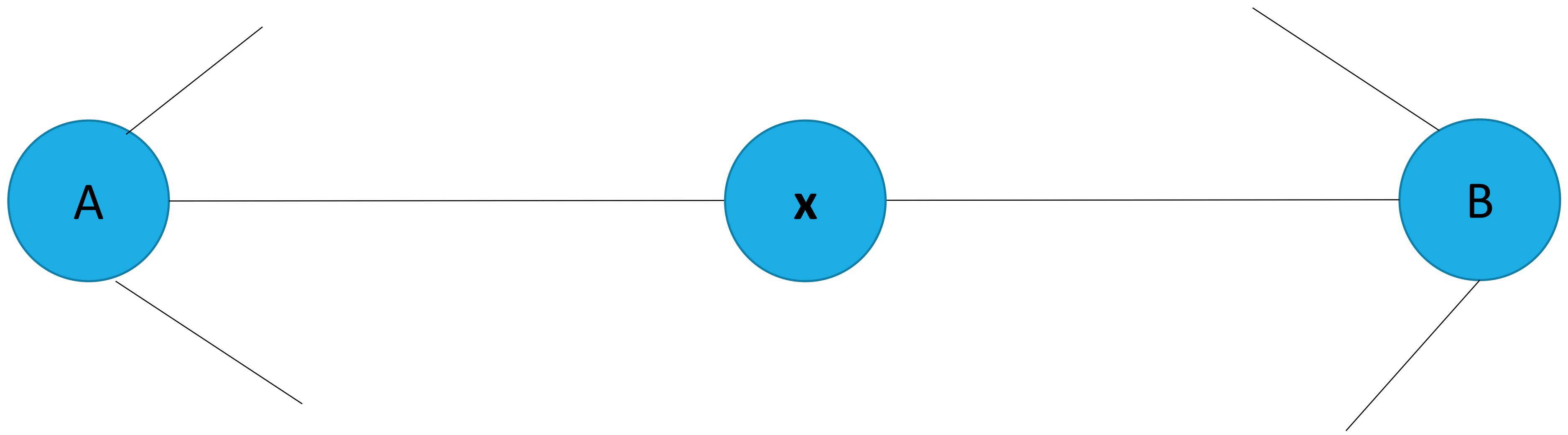
DP: Principle of optimality

- Principle of optimality
  - The general optimum implies the local / partial optimum
  - In an optimal sequence of decision, each decision is optimum
  - The principle does not hold true for any problem

- Formally, a sequence of decisions dec_1, dec_2,..., dec_n optimally leads from state s_0 to state s_n if at least one of the following conditions is satisfied:
  - dec_k, dec_{k+1},..., dec_n is a sequence of decisions that optimally leads from state s_{k-1} to state s_n for any k, $1 \leq k \leq n$ (**forward** method)
  - dec_1, dec_2,..., dec_k is a sequence of decisions that optimally leads from state s_0 to state s_k for any k, $1 \leq k \leq n$ (**backward** method)
  - dec_{k+1},..., dec_n and dec_1, dec_2,..., dec_k are two sequences of decisions that optimally lead from state s_{k-1} to state s_n and from state s_0 to state s_k for any k, $1 \leq k \leq n$ (**mixed** method)

Principle of optimality

- In solving a problem, we have to make a sequence of n decisions
- If this sequence is optimal then the last k decisions (1<k<n) must be optimal



DP: Algorithm

- Verify the principle of optimality
- Establish the structure of the solution
  - Break the problem in sub-problems
  - Overlapping sub-problems – break down the problem into sub-problems which are reused multiple times
- Memoization
  - Store the solutions to the sub-problems for later use
- Based on the principle of optimality, the value of the optimal solution is recursively defined
- The value of the optimal solution is computed in a bottom-up manner, starting from the smallest cases for which the value of the solution is known

DP: Example

- **Problem**: find the longest increasing subsequence from a list of integer numbers.

i	**1**	**2**	**3**	**4**	**5**
list	2	1	9	6	12

- Solution
  - For each i, calculate the length of the longest increasing subsequence that can be formed
  - In the end, select the element where the longest subsequence is formed

i	**1**	**2**	**3**	**4**	**5**
list	2	1	9	6	12
L	**3** 2,9,12 sau 2,6,12	**3** 1,9,12 sau 1,6,12	**2** 9,12	**2** 6,12	**1** 12

DP: Example

- **Problem**: find the longest increasing subsequence from a list of integer numbers.

i	1	2	3	4	5
list	2	1	9	6	12

- Step 1: The principle of optimality
  - The principle of optimality is verified in its forward variant
  - The longest subsequence that starts at position i has k elements => the subsequences that can be formed from it (with k-1, k-2,...elements) are increasing subsequences and have maximal length

DP: Example

- **Problem**: find the longest increasing subsequence from a list of integer numbers.

i	1	2	3	4	5
list	2	1	9	6	12

- Step 2: The structure of the optimal solution
  - Break the problem in sub-problems
  - Problem: determine the longest increasing subsequence
  - Sub-problem: determine the longest increasing subsequences that starts with list[i] for i =n, n-1, n-2,..., 1. These subsequences have the length at most 1,2,...,n.
  - Solution: the longest subsequence from the n subsequences.

DP: Example

i	1	2	3	4	5
list	2	1	9	6	12

- **Problem**: longest increasing subsequence

- Step 3: Determine the global optimum based on the partial optimums
  - Let L_i be the length of the longest subsequence that starts with list[i]
  - The increasing sub-sequences that start with list[i] are obtained by adding the element list[i] in front of an increasing subsequence that starts with list[j] if list[i] ≤ list[j]

$$L_i = 1 + \max_{j=i+1,n}\{L_j, list[i] \leq list[j]\}, for\ i = n-1, \dots, 1$$

  - $L_n = 1$

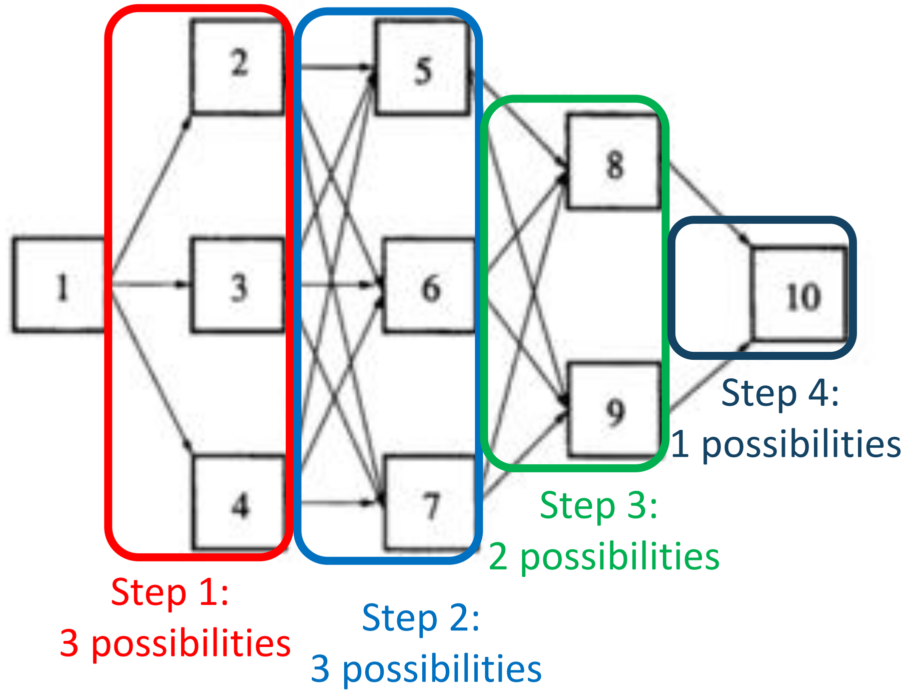
  - Optimal solution: $L_{max} = \max_{i=i,n}\{L_i\}$

```python
def long_seq(s):
    L = [0] * len(s)
    ind = [0] * len(s) #index of the succesor of list[i] in the long seq
    #compute vector L
    L[len(s) - 1] = 1
    ind[len(s) - 1] = -1
    for i in range(len(s) - 2, -1, -1):
        ind[i] = -1
        L[i] = 1
        for j in range(i+1, len(s)):
            if (s[i] <= s[j]):
                if (L[i] <= L[j] + 1):
                    L[i] = L[j] + 1
                    ind[i] = j

    #determine position max elem from L
    max_pos = 0
    for i in range(1, len(s)):
        if (L[i] > L[max_pos]):
            max_pos = i

    #construct the solution
    sol = []
    i = max_pos
    while (i != -1):
        sol.append(s[i])
        i = ind[i]
    return sol
```

```python
def test_long_seq():
    assert long_seq([2,1,9,6,12]) == [2, 6, 12]
    assert long_seq([0,-2,3,1,0,-1,2,5,-5,5,-8.10,7,-3,1]) == [0,0,2,5,5,7]
```

DP: Example 2

Stagecoach problem *(TSP with less roads due to hostile territory)*



Costs:

	2	3	4
1	2	4	3

	5	6	7
2	7	4	6
3	3	2	4
4	4	1	5

	8	9
5	1	4
6	6	3
7	3	3

	10
8	3
9	4

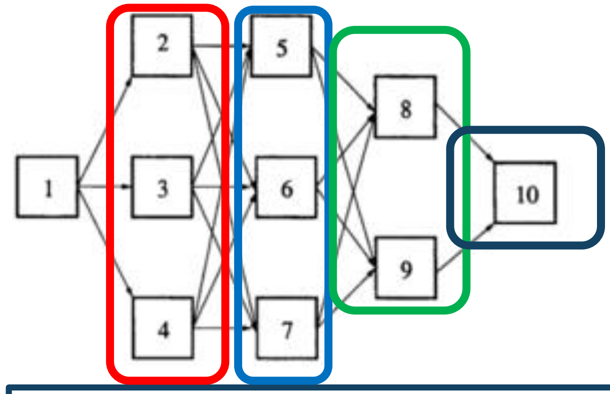
DP: Example 2

- There are **n** step
- Decision regarding state in step n is x_n

- From state s in step n choose x_n the next state: $f_n(s, x_n)$ is the cost of best decision for all remaining steps

- $x_n^*(s)$ is value of x_n that minim$izes\ f_n(s, x_n)$ and $f_n^*(s)$ is the minimum corresponding value

- The objective $is\ to\ find\ f_1^*(s)$
  - To do that, first we need to find $f_4^*(s)$, $f_3^*(s)$, $f_2^*(s)$



	2	3	4
1	2	4	3

	5	6	7
2	7	4	6
3	3	2	4
4	4	1	5

	8	9
5	1	4
6	6	3
7	3	3

	10
8	3
9	4

$f_4^*(s)$ = ?
For which x_4 is $f_4(s, x_4)$ min?
x_4 can be only 10 ...

s	$f_4^*(s)$	$x_4^*(s)$
8	3	10
9	4	10

$f_3^*(s)$ = ?
$f_3(s, x_3) = cost(s, x_3) + f_4^*(x_3)$
x_3 can be 8 or 9
s can be 5, 6 or 7

s	$f_3(s,8)$	$f_3(s,9)$	$f_3^*(s)$	$x_3^*(s)$
5	4	8	4	8
6	9	7	7	9
7	6	7	6	8

$f_2^*(s)$ = ?
$f_2(s, x_2) = cost(s, x_2) + f_3^*(x_2)$

s	$f_2(s,5)$	$f_2(s,6)$	$f_2(s,7)$	$f_2^*(s)$	$x_2^*(s)$
2	11	11	12	11	5 or 6
3	7	9	10	7	5
4	8	8	11	8	5 or 6

$f_1^*(s)$ = ?

s	$f_1(s,2)$	$f_1(s,3)$	$f_1(s,4)$	$f_1^*(s)$	$x_1^*(s)$
1	13	11	11	11	3 or 4

Solutions (cost 11):
1 -> 3 -> 5 -> 8 -> 10
1 -> 4 -> 5 -> 8 -> 10
1 -> 4 -> 6 -> 9 -> 10

Greedy method

- Basic idea
  - Break the problem in successive sub-problems similar to the initial problem but of smaller dimensions
  - Solve the sub-problems and determine the final solution by successively selecting the best sub-solutions
  - Global optimum = a sequence of local optimas

- Mechanism
  - Divide the problem in successive sub-problems P1, P2, ...Pn
  - Progress to the final solution by selecting at each step the best decision

Greedy method

- When to use Greedy?
  - Problem P (optimization)
  - Solution is the result of a successive selections of local optima
  - Problems with solution represented by subsets or chartesian products that achieve a certain optimum (minum or maximum) of an objective function
- Features
  - Can reach the optimal solution
  - Builds the solution step by step
  - Offers a single solution (unlike backtracking)
  - Polynomial run time
- Disadvantages: Short-sighted and non-recoverable

Greedy Algorithm

- Let *S* be a solution to the problem and *C* the set of local optima for each sub-problem (candidate elements of the solution)

```
def greedy(C):
    S = Φ
    while (not isSolution(S)) and (C≠Φ):
        el = selectMostPromissing(C)
        C.remove(el)
        if acceptable(el, S):
            S.append(el)
    if isSolution(S):
        return S
    else:
        return None
```

Greedy - Example of Problems

- Coins Problem
  - Consider a sum of money and a set of coins units. The problem is to establish a modality to pay the sum of money using a minimum number of coins.

- Knapsack Problem
  - Consider a set of objects, each having a value and weight, and a knapsack able to support a total weight of W. Place in the knapsack some of the objects, such that the total weight of the objects is not larger than W and the objects have max value.

- General Problem
  - Let us consider the given set C of candidates to the solution of a given problem P.
  - The objective is to provide a subset B to fulll certain conditions (called internal conditions) and to maximize (minimize) a certain objective function.

Greedy strategy

- Greedy algorithm finds the solution in an incremental way
- Greedy strategy
  - Successively incorporate elements that realize the local optimum
  - No second thoughts are allowed on already made decisions
- Generally, the required elements of a greedy strategy are:
  - A **candidate set** (from which a solution is created)
  - A **selection function** (selects the best candidate to be added to the solution)
  - A **feasibility function** (determines if a candidate can be used in a solution)
  - An **objective function** (assigns a value to a solution, or a partial solution)
  - A **solution function** (checks if a complete solution has been found)

Greedy: Coins Problem

- Problem: Find a way to pay a sum of money using a minimum number of coins (different values of coins are available).
  - Data: Sum = 80, Coins = [1, 5, 10, 25, 50]
  - Results: 80 = 50 + 25 + 5

  - Data: Sum = 10, Coins = [1, 2, 3, 4]
  - Results : 10 = 4 + 3 + 2 + 1

  - Data : Sum = 10, Coins = [2, 3, 4, 5]
  - Results : 10 = 5 + 3 + 2

Greedy: Example

- Solution
  - C – list of available coins
  - *isSolution(sol)*
    - If the sum of coins selected in sol is equal to the desired sum
  - *selectMostPromissing(C)*
    - Select the highest value coin in C
  - *acceptable(el,sol)*
    - If the sum of coins in sol + el is not over the desired sum

```python
def sum(l):
    s = 0
    for el in l:
        s = s + el
    return s

def isSolution(solution, limit):
    return sum(solution) == limit

def selectMostPromising(candidates):
    return max(candidates)

def acceptable(element, solution, limit):
    return sum(solution) + element <= limit

def greedy_coins(coins, sumOfMoney):
    sol = []
    while (not isSolution(sol, sumOfMoney)) and (coins != []):
        el = selectMostPromising(coins)
        coins.remove(el)
        if acceptable(el, sol, sumOfMoney):
            sol.append(el)
    if isSolution(sol, sumOfMoney):
        return sol
    else:
        return None
```

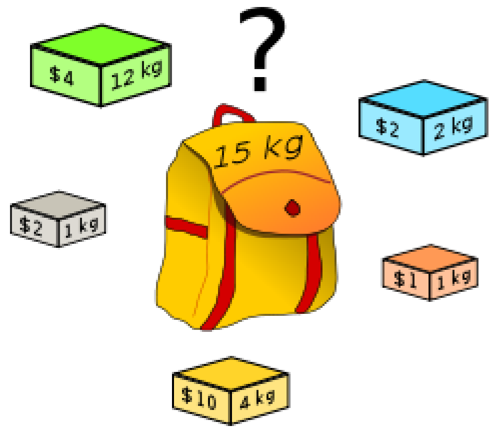
```python
def test_greedy_coins():
    assert greedy_coins([1, 5, 10, 25, 50], 80) == [50, 25, 5]
    assert greedy_coins([1, 2, 3, 4], 10) == [4, 3, 2, 1]
    #assert greedy_coins([1, 2, 3, 4, 5], 10) == [5, 3, 2]
    assert greedy_coins([2, 3, 4, 5], 10) == None

test_greedy_coins()
```

Knapsack problem

- Each object has a value (***v***) and a weight (***w***).
- Place objects of total maximum value without going over the total weight W allowed.

$$\text{maximize } \sum_{i=1}^{n} v_i x_i$$

$$\text{subject to } \sum_{i=1}^{n} w_i x_i \leq W \text{ and } x_i \in \{0,1\}.$$



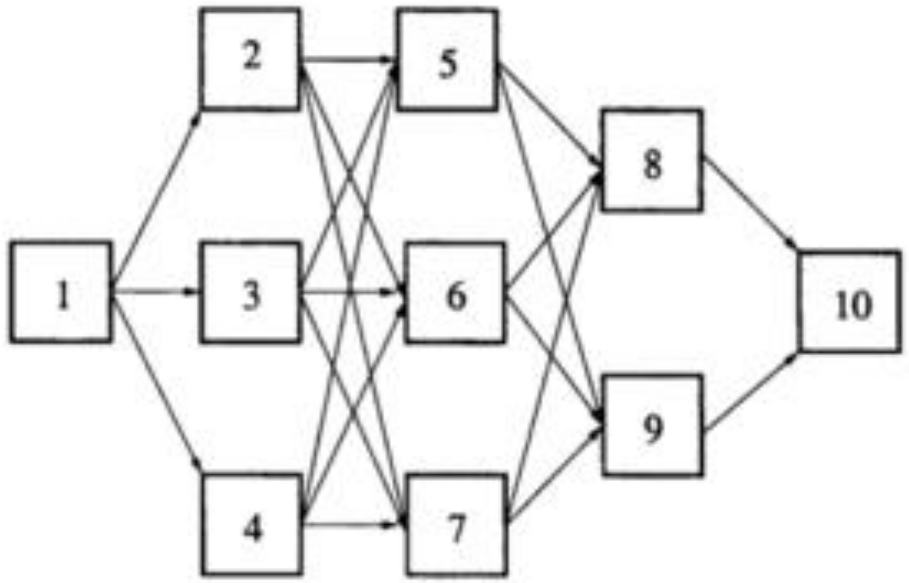
➢ *0/1 Knapsack*

➢ *Fractional Knapsack*

- Greedy Solution
  - Max value vs. Min weight vs. Max value/weight

Stagecoach problem

Stagecoach problem *(TSP with less roads due to hostile territory)*



	2	3	4
1	2	4	3

	5	6	7
2	7	4	6
3	3	2	4
4	4	1	5

	8	9
5	1	4
6	6	3
7	3	3

	10
8	3
9	4

DP Solutions (cost 11):
1 -> 3 -> 5 -> 8 -> 10
1 -> 4 -> 5 -> 8 -> 10
1 -> 4 -> 6 -> 9 -> 10

Greedy Solution :
- 1 -> 2 (cost 2)
- 2 -> 6 (cost 4)
- 6 -> 9 (cost 3)
- 9 -> 10 (cost 4)

=> **Cost 13** !!

Dynamic Programming vs Greedy

- Both techniques are applied in optimization problems

- DP is applicable to problems in which the general optimum implies partial optima
- Greedy is applicable to problems for which the general optimum is obtained from partial (local) optima

- DP always provides the optimal solution
- Greedy does not guarantee finding the optimal solution

Example

- Example: take the problem of finding the optimal path between two vertices i and j of a graph
- The principle of optimality is verified
  - If the path from i to j is optimal and it passes through node x, then the path from i to x is optimal and also the path from x to j is optimal.
- The fact that the general optimum implies partial optima does not mean that partial optima also implies the general optimum
  - if the paths i -> x and x -> j are optimal, there is no guarantee that the path from i to j that passes through x is also optimal
- Greedy , DP

Recap today

- Problem solving methods

- Dynamic programming
- Greedy

Next time

- Recap

Reading materials and useful links

1. The Python Programming Language - https://www.python.org/
2. The Python Standard Library - https://docs.python.org/3/library/index.html
3. The Python Tutorial - https://docs.python.org/3/tutorial/
4. M. Frentiu, H.F. Pop, Fundamentals of Programming, Cluj University Press, 2006.
5. MIT OpenCourseWare, Introduction to Computer Science and Programming in Python, https://ocw.mit.edu, 2016.
6. K. Beck, Test Driven Development: By Example. Addison-Wesley Longman, 2002. http://en.wikipedia.org/wiki/Test-driven_development
7. M. Fowler, Refactoring. Improving the Design of Existing Code, Addison-Wesley, 1999. http://refactoring.com/catalog/index.html

Bibliography

The content of this course has been prepared using the reading materials from previous slide, different sources from the Internet as well as lectures on Fundamentals of Programming held in previous years by:

- Prof. Dr. Laura Dioşan - www.cs.ubbcluj.ro/~lauras
- Conf. Dr. Istvan Czibula - www.cs.ubbcluj.ro/~istvanc
- Lect. Dr. Andreea Vescan - www.cs.ubbcluj.ro/~avescan
- Lect. Dr. Arthur Molnar - www.cs.ubbcluj.ro/~arthur