CHAPTER 4

Convex sets

Contents

4.1 Convex sets

Here we restrict to the case where $k = \mathbb{R}$.

Definition. A subset Y of an affine space X is called *convex* if for any $P, Q \in Y$ the segment

$$[PQ] = \Big\{(1-t)P + tQ : t \in [0,1]\Big\}$$

lies in Y.

Examples 4.1. We have

1. Any affine subspace of a real affine space is a convex set.
2. The interior of any triangle, trapezoid, regular Polygon, tetrahedron or parallelepiped.
3. The interior of a circle or the interior of any n-dimensional ball.
4. Closed and open half-spaces.

Remark 4.2. If $\mathcal{C}$ is a family of convex sets in X, then

$$\bigcap_{C \in \mathcal{C}} C \quad \text{is convex.}$$

Remark 4.3. If M is convex then so are its interior $\mathring{M}$ and its closure $\overline{M}$.

Remark 4.4. $Y \subseteq X$ is convex if and only if (why?)

$$\forall m \in \mathbb{N},\ \forall P_1,\dots,P_m \in X \text{ and } \forall \mu_1,\dots,\mu_m \in [0,1] \text{ with } \sum_{i=1}^{n} \mu_i = 1 \text{ we have } \operatorname{Bar}(P_1,\dots,P_m;\mu_1,\dots,\mu_m) \in Y. \tag{4.1}$$

Definition. The affine combination (4.1) with coefficients in $[0,1]$ is called *convex combination*.

4.2 Convex hulls

Definition. For a subset M of the affine space X, the *convex hull* of M is

$$\operatorname{conv}(M) = \bigcap \{Y : \text{ convex subset of } X \text{ containing } M\}.$$

Proposition 4.5. *For $M,N \subseteq X$ we have*

1. $M \subseteq \operatorname{conv}(M)$,
2. $\operatorname{conv}(M)$ *is a convex set,*
3. *if Y is a convex set containing M then* $\operatorname{conv}(M) \subseteq Y$,
4. *if $M \subseteq N$ then* $\operatorname{conv}(M) \subseteq \operatorname{conv}(N)$,
5. $\operatorname{conv}(M) = M$ *if and only if M is a convex set,*
6. $\operatorname{conv}(\operatorname{conv}(M)) = \operatorname{conv}(M)$,
7. $\operatorname{conv}(M) \subseteq \operatorname{aff}(M)$.

Theorem 4.6 (Carathéodory)**.** *For $M \subseteq X$*

$$\operatorname{conv}(M) = \left\{ \operatorname{Bar}(P_0,\dots,P_m;\mu_1,\dots,\mu_m) : \forall m \le n,\ \forall P_0,\dots,P_m \in X \text{ and } \forall \mu_0,\dots,\mu_m \in [0,1] \text{ with } \sum_{i=1}^{n} \mu_i = 1. \right\}$$

Proof. □

Theorem 4.7 (Radon)**.** *Let X be a real affine space of dimension n and $M \subseteq X$ a finite subset. If $|M| \ge n+2$, then there exists a partition $M = M_1 \cup M_2$ of M ($M_1 \cap M_2 = \emptyset$) such that*

$$\operatorname{conv}(M_1) \cap \operatorname{conv}(M_2) \neq \emptyset.$$

Proof. □

Theorem 4.8 (Helly)**.** *Let X be a real affine space of dimension n and $M_1,\dots,M_m$ convex subsets of X. If $m \ge n+1$ and the intersection of any $n+1$ of the sets is non-empty, then*

$$M_1 \cap \cdots \cap M_m \neq \emptyset.$$

Proof. □

4.3 Exercises

Exercise 1. Let A and B be two convex sets in $\mathbb{R}^n$. Show that $A+B$ is convex.

Exercise 2. For an affine space X we define the operator s by

$$s(M) = \Big\{tP + (1-t)Q : P, Q \in M, t \in [0,1]\Big\} \subseteq X$$

for any subset $M \subseteq X$.

1. For $X = \mathbb{R}^2$ and $M = \{(1,0), (-1,-1), (0,3)\}$ describe $s(M)$ and $s^2(M) = s(s(M))$.
2. What are $s(M)$, $s^2(M)$ and $s^3(M)$ for a the vertices M of a parallelepiped?
3. Show that the sequence
$$M \subseteq s(M) \subseteq s^2(M) \subseteq \dots$$
is stationary if X is finite dimensional.

Exercise 3. In $\mathbb{R}^2$ consider the points $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$. Show that a point $P(x_P, y_P)$ lies in the triangle ABC if and only if

$$S_{AB}(x_P, y_P)S_{AB}(x_C, y_C) > 0 \text{ and } S_{BC}(x_P, y_P)S_{BC}(x_A, y_A) > 0 \text{ and } S_{CA}(x_P, y_P)S_{CA}(x_B, y_B) > 0$$

where

$$S_{ij}(x,y) = \begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}.$$

Can you generalize this to more than three points?

Exercise 4 (Gauss-Lucas theorem)**.** Let $P(x)$ be a polynomial with complex coefficients and consider the isomorphism $\mathbb{C} \cong \mathbb{R}^2$. Show that the roots of P' lie in the convex hull of the roots of P.

Bibliography

[1] I.P. Popescu *Geometrie Afină si euclidiană* Timişoara, 1984

[2] P. Michele *Géométrie - notes de cours* Lausanne, 2016

[3] C. Pintea *Geometrie afină - note de curs* Cluj-Napoca, 2017

[4] F. Radó, B. Orbán, V. Groze, A. Vasiu, *Culegere de probleme de geometrie* Cluj-Napoca, 1979.