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Homework 9, Week 11

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$\boxed{Ex 1}$ a) $f:(-1,\infty)\to\mathbb{R}$, $f(x)=(1+x)^n$, $n\in\mathbb{R}$

$f'(x)=\left((1+x)^n\right)'=(1+x)'\, n\,(x+1)^{n-1}=n\,(x+1)^{n-1}$

$f''(x)=n\left((x+1)^{n-1}\right)'=n\cdot(x+1)'\cdot(n-1)\cdot(x+1)^{n-2}=$
$=n(n-1)(x+1)^{n-2}$

$f'''(x)=n(n-1)\left((x+1)^{n-2}\right)'=n(n-1)\cdot(n-2)(x+1)^{n-3}$

...

$f^{(m)}(x)=n(n-1)\dots(n-(m-1))(x+1)^{n-m}=$
$=n(n-1)\dots(n-m+1)(x+1)^{n-m}$

Let $k\in\mathbb{N}$, $k>1$, be fixed. We assume $P(k)$ is true and prove $P(k+1)$.

$P(k)$: $n(n-1)\dots(n-k+1)(x+1)^{n-k}=f^{(k)}(x)$

$P(k+1)$: $f^{(k+1)}(x)=n(n-1)\dots(n-(k+1)+1)(x+1)^{n-(k+1)}=$
$=n(n-1)\dots(n-k)(x+1)^{n-k-1}=n(n-1)\dots(n-k+1)(n-k)(x+1)^{n-k-1}$

$f^{(k+1)}(x)=\left(f^{(k)}(x)\right)'=\left(\underbrace{n(n-1)\dots(n-k+1)}_{\text{constants}}(x+1)^{n-k}\right)'=$

$=n(n-1)\dots(n-k+1)\left((x+1)^{n-k}\right)'=n(n-1)\dots(n-k+1)(n-k)(x+1)^{n-k-1}$
$=f^{(k)}(x)\Rightarrow P(k)$ is true, $\forall k\in\mathbb{N}$

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$\boxed{\text{Ex2}}$ a) $f:\mathbb{R}\setminus\{-\frac{b}{a}\}\to\mathbb{R},\ f(x)=\frac{1}{ax+b}$

$f'(x)=\left((ax+b)^{-1}\right)'=-(ax+b)^{-2}(ax+b)'=-a(ax+b)^{-2}$

$f''(x)=-a\left[(ax+b)^{-2}\right]'=-a\cdot(-2)(ax+b)^{-3}(ax+b)'=$

$=a^2(-1)(-2)(ax+b)^{-3}$

...

$f^{(m)}(x)=a^m(-1)(-2)\dots(-m)(ax+b)^{-(m+1)}$

$P(k):\ a^k(-1)(-2)\dots(-k)(ax+b)^{-k-1}=f^{(k)}(x)$

$P(k+1):\ a^{k+1}(-1)(-2)\dots(-k-1)(ax+b)^{-k-2}=f^{(k+1)}(x)$

$f^{(k+1)}(x)=\left(f^{(k)}(x)\right)'=a^k(-1)(-2)\dots(-k)(ax+b)'\cdot(-k-1)\cdot(ax+b)^{-k-2}$

$=a^{k+1}(-1)(-2)\dots(-k-1)(ax+b)^{-k-2}\Rightarrow P(k)$ true

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b) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin(ax+b)$

$f'(x)=(\sin(ax+b))'=(ax+b)'\cdot\cos(ax+b)=a\sin\left(ax+b+\frac{\pi}{2}\right)$

$f''(x)=a\left(ax+b+\frac{\pi}{2}\right)'\cos\left(ax+b+\frac{\pi}{2}\right)=a^2\sin\left(ax+b+2\cdot\frac{\pi}{2}\right)$

...

$f^{(m)}(x)=a^m\sin\left(ax+b+m\frac{\pi}{2}\right)$

$P(k):\ f^{(k)}(x)=a^k\sin\left(ax+b+k\frac{\pi}{2}\right)$

$P(k+1):\ f^{(k+1)}(x)=a^{k+1}\sin\left(ax+b+(k+1)\frac{\pi}{2}\right)$

$\left(f^{(k)}(x)\right)'=a^k\cdot\left(ax+b+k\frac{\pi}{2}\right)'\left(\cos ax+b+k\frac{\pi}{2}\right)=a^k\cdot a\cdot\sin\left(ax+b+(k+1)\frac{\pi}{2}\right)$

$=(P(k))'\Rightarrow P(k)$ true

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c) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\cos(ax+b)$; $\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$

$f'(x)=(ax+b)'\cdot(-\sin(ax+b))=-a\sin(ax+b)=$

$=-a\cos\left(ax+b+\frac{\pi}{2}\right)$

$f''(x)=-a\cdot\left(ax+b+\frac{\pi}{2}\right)'\cdot-\sin\left(ax+b+\frac{\pi}{2}\right)=$

$=+a^2(\cos ax+b+2\frac{\pi}{2})$

...

$f^{(m)}(x)=(-1)^m a^m\cos\left(ax+b+m\frac{\pi}{2}\right)$

$P(k):\ f^{(k)}(x)=(-1)^k\cdot a^k\cdot\cos\left(ax+b+k\frac{\pi}{2}\right)$

$P(k+1):\ f^{(k+1)}(x)=(-1)^{k+1}a^{k+1}\cos\left(ax+b+(k+1)\frac{\pi}{2}\right)$

$\left(f^{(k)}(x)\right)'=(-1)^k a^k\left(ax+b+k\frac{\pi}{2}\right)'\cdot\left(-\sin\left(ax+b+k\frac{\pi}{2}\right)\right)=$

$=(-1)^k a^k\cdot a\cdot(-1)\cdot\cos\left(ax+b+(k+1)\frac{\pi}{2}\right)=$

$=(-1)^{k+1}a^{k+1}\cos\left(ax+b+(k+1)\frac{\pi}{2}\right)\Rightarrow P(k)$ true

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d) $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{ax+b}$

$f'(x)=(ax+b)'e^{ax+b}=a\,e^{ax+b}$

$f''(x)=a(ax+b)'e^{ax+b}=a^2e^{ax+b}$

...

$f^{(m)}(x)=a^m e^{ax+b}$

$P(k):\ f^{(k)}(x)=a^k e^{ax+b}$

$P(k+1):\ f^{(k+1)}(x)=a^{k+1}e^{ax+b}$;

$\left(f^{(k)}(x)\right)'=a^k(ax+b)'e^{ax+b}=$

$=a^{k+1}e^{ax+b}\Rightarrow$

$\Rightarrow P(k)$ true

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Ex3

a) $(x^x)' = x^x \cdot \ln x \cdot x' + x \cdot x^{x-1} \cdot x' =$

$= x^x \ln x + x^x = \underline{x^x(\ln x + 1)}$

b) $f:(0,\infty) \to \mathbb{R}$, $f(x) = x^{\frac{1}{x}}$

$f'(x) = \left(x^{\frac{1}{x}}\right)' = \left(\frac{1}{x}\right)' \cdot \ln x \cdot x^{x^{-1}} + x' \cdot \frac{1}{x} \cdot x^{\frac{1}{x}-1} =$

$= (x^{-1})' \ln x \cdot x^{\frac{1}{x}} + \frac{1}{x} \cdot x^{\frac{1-x}{x}} =$

$= -1 \cdot x^{-2} \ln x \cdot x^{\frac{1}{x}} + \frac{1}{x} \cdot \frac{x^{\frac{1}{x}}}{x} =$

$= -\frac{1}{x^2} \ln x \cdot x^{\frac{1}{x}} + \frac{1}{x^2} \cdot x^{\frac{1}{x}} =$

$= \frac{x^{\frac{1}{x}}}{x^2}(1 - \ln x) = x^{\frac{1}{x}-\frac{x}{2}}(1 - \ln x) =$

$= \underline{x^{\frac{1-2x}{x}}(1 - \ln x)}$

c) $f:(0,\pi) \to \mathbb{R}$, $f(x) = \sin x^x$

$f'(x) = x' \cdot \ln(\sin x) \cdot \sin x^x + \cancel{x}(\sin x)' \cdot \sin x^{x-1} \cdot x =$

$= \cancel{\sin x^x} \ln(\sin x) + x \cos x \sin x^{x-1} = \underline{(\sin x)^x\left(\ln(\sin x) + \frac{x \cos x}{\sin x}\right)}$

$\sin(x^x)$ sau $(\sin x)^x$

$(\sin(x^x))' = (x^x)' \cdot \cos(x^x) = x^x(\ln x + 1)\cos(x^x)$

d) $f:(0,\infty) \to \mathbb{R}$, $f(x) = x^{\sin x}$

$f'(x) = \left(x^{\sin x}\right)' = (\sin x)' \cdot \ln x \cdot x^{\sin x} + x' \cdot \sin x \cdot x^{\sin x - 1} =$

$= x^{\sin x} \cos x \ln x + x^{\sin x} \cdot \frac{1}{x} \cdot \sin x = \underline{x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)}$