CHAPTER 7

Affine morphism - Projections

Contents

7.1 Projections

Definition. Let X be an affine space and $Y, Z \subseteq X$ two affine subspaces in general position such that

$$\dim Y + \dim Z = \dim X \quad \text{and} \quad \dim Y \cap Z \neq \emptyset.$$

By the dimension formula

$$\dim(Y \cap Z) = \dim Y + \dim Z - \underbrace{\dim \operatorname{aff}(Y \cup Z)}_{\dim X} = 0$$

since the two affine subspaces are in general position. We say Y, Z are *complementary (affine) subspaces*.

Remark 7.1. Notice that the condition on dimensions implies that Y and Z intersect in at most one point, and, since we assume that $\dim Y \cap Z \neq \emptyset$, it follows that Y and Z intersect in exactly one point.

Remark 7.2. Notice also that for any point $P \in X$ there is a unique Z' parallel to Z of dimension $\dim Z$ which contains P. Indeed, for some point $A \in Z$

$$Z = A + D(Z) \quad \text{and we can choose} \quad Z' = P + D(Z)$$

and the uniqueness follows since $\dim Z = \dim Z'$. In other words, Z' is a translate of Z, i.e. $Z' = t_{P-A}(Z)$.

Definition. Given two complementary affine subspaces Y, Z in X. The projection on Y along Z is

$$\mathrm{Pr}_{Y,Z} : X \to Y$$

defined by, $\mathrm{Pr}_{Y,Z}(P) = P'$ where P' is the unique point $Y \cap Z'$ (for Z' the unique affine subspace parallel to Z and containing P).

Proposition 7.3. *The map $\mathrm{Pr}_{Y,Z}$ is an affine morphism. Since Y is an affine subspace of X, we have $\mathrm{Pr}_{Y,Z} \in \mathrm{End}_{\mathrm{aff}}(X)$.*

Proof. □

7.2 Factorizations of affine morphisms

Proposition 7.4. *Let $f : X \to X'$ be an affine morphism. Then there are complementary affine subspace $Y, Z \subseteq X$ and an affine subspace $Y' \subseteq X'$ such that*

$$f = i \circ \varphi \circ \mathrm{Pr}_{Y,Z}$$

Where φ is an isomorphism and $i : Y' \to X'$ is the inclusion map.

Remark 7.5. In the above statement $\mathrm{Pr}_{Y,Z}$ can be an isomorphism. This happens when $Y = X$ and Z a point.

Proof of the proposition. □

7.3 Projections in dimension 3

7.3.1 Intersection of a line with a plane

In a 3-dimensional affine space X, we consider a line given by a point $P \in X$ and a direction vector $v \in D(X)$

$$l = P + \langle v \rangle = \{P + tv : t \in k\}.$$

Further we consider a (hyper)plane described by an affine map $\varphi : X \to k$

$$\pi = \varphi^{-1}(0) = \{Q \in X : \varphi(Q) = 0\}.$$

The intersection of the line l with the plane π is given by those scalars $t \in k$ such that

$$\varphi(P + tv) = 0 \Leftrightarrow \varphi(P) + \operatorname{lin}\varphi(tv) = 0 \Leftrightarrow \varphi(P) + t \operatorname{lin}\varphi(v) = 0.$$

Remark 7.6. If v is in the kernel of $\operatorname{lin}\varphi$ then $v \in D(X)$. In this case l is parallel to π and it intersects π if and only if $\varphi(P) = 0$, i.e. if and only if it is contained in π.

We assume that $l \nparallel \pi$. Then there is a unique intersection point given by the parameter

$$t = -\frac{\varphi(P)}{\operatorname{lin}\varphi(v)}$$

so the intersection point is

$$P' = P - \frac{\varphi(P)}{\operatorname{lin}\varphi(v)} v. \tag{7.1}$$

7.3.2 Projection on a plane along a line (cartesian coordinates)

We keep the above setting for the line l and the plane π (i.e. $l \nparallel \pi$). In addition we explicitate the description of the projecton

$$\mathrm{Pr}_{\pi,l} : X \to \pi$$

in coordinates. If a cartesian coordinate system is chosen, then we may view X as k^3 (3-dim vector space over k). Then

$$P = P(x_P, y_P, z_P), \quad v = v(v_x, v_y, v_z), \quad \varphi(x,y,z) = Ax + By + Cz + D.$$

The projection of the point P is P' as in (7.1). Form which we get

$$\begin{cases} x_{P'} = x_P - v_x \mu \\ y_{P'} = y_P - v_y \mu \\ z_{P'} = z_P - v_z \mu \end{cases} \quad \text{where} \quad \mu = \frac{Ax_P + By_P + Cz_P + D}{Av_x + Bv_y + Cv_z}.$$

7.3.3 Projection on a line along a plane (cartesian coordinates)

As in the previous subsection we have

$$l = Q + \langle v \rangle \quad \nparallel \quad \pi : \{P : \phi(P) = 0\}.$$

Here we are interested in the explicit description of

$$\mathrm{Pr}_{l,\pi} : X \to l$$

As before, in a cartesian coordinate system, we have

$$P = P(x_P, y_P, z_P), \quad Q = Q(x_Q, y_Q, z_Q), \quad v = v(v_x, v_y, v_z), \quad \varphi(x,y,z) = Ax + By + Cz + D.$$

Moreover, any π' parallel to π has the form

$$\pi' : \varphi'(x,y,z) = Ax + By + Cz + D' = 0$$

for some $D' \in k$. Moreover, the plaine π' parallel to π and containing P is obtained with

$$D' = -\operatorname{lin}\varphi(P) \quad \Leftrightarrow \quad \varphi'(P) = 0.$$

So

$$M \in \pi' \quad \Leftrightarrow \quad Ax_M + By_M + Cz_M \underbrace{-Ax_P - By_P - Cz_P}_{D' = -(\operatorname{lin}\varphi)(P)} = 0 \quad \Leftrightarrow \quad (\operatorname{lin}\varphi)(M) - (\operatorname{lin}\varphi)(P) = 0$$

It follows that the plane π' parallel to π and containing P is $\psi^{-1}(0)$ where

$$\psi(M) = (\operatorname{lin}\varphi)(M) - (\operatorname{lin}\varphi)(P).$$

Then, the projection of the point P on the plane π parallel to l is P'. With (7.1) we get

$$\begin{cases} x_{P'} = x_Q - v_x\mu \\ y_{P'} = y_Q - v_y\mu \\ z_{P'} = z_Q - v_z\mu \end{cases} \text{where } \mu = \frac{\psi(Q)}{(\operatorname{lin}\psi)(v)} = \frac{(\operatorname{lin}\varphi)(Q) - (\operatorname{lin}\varphi)(P)}{(\operatorname{lin}\varphi)(v)} = \frac{A(x_Q - x_P) + B(y_Q - y_P) + C(z_Q - z_P)}{Av_x + Bv_y + Cv_z}.$$

Here we intersect π' with l.

7.4 Reflections

A *Reflection* can be defined in the general setup of two complementary affine subspaces Y, Z of an affine space X if the ground field k has characteristic different from 2. Namely, the *reflection of a point $P \in X$ in Y along Z* is the point $\operatorname{Ref}_{Y,Z}(P) := P'$ given by

$$\operatorname{Pr}_{Y,Z}(P) = \frac{P + P'}{2}. \tag{7.2}$$

In other words, $\operatorname{Ref}_{Y,Z}(P)$ is defined by the property that $\operatorname{Pr}_{Y,Z}(P)$ is the midpoint of P and $\operatorname{Ref}_{Y,Z}(P)$.

With (7.1) we have

$$P - \frac{\varphi(P)}{\operatorname{lin}\varphi(v)}v = \frac{P + P'}{2} \qquad \underbrace{\Leftrightarrow}_{\text{in cartesian coords}} \qquad 2P - 2\frac{\varphi(P)}{\operatorname{lin}\varphi(v)}v = P + P'.$$

or

$$\operatorname{Ref}_{Y,Z}(P) = P - 2\frac{\varphi(P)}{\operatorname{lin}\varphi(v)}v \tag{7.3}$$

7.5 Reflections in dimension 3

Similar to section 7.3.1, in a 3-dimensional affine space X, we consider a line given by a point $P \in X$ and a direction vector $v \in D(X)$

$$l = P + \langle v \rangle = \{P + tv : t \in k\}.$$

Further we consider a (hyper)plane described by an affine map $\varphi : X \to k$

$$\pi = \ker\varphi = \{\varphi(Q) : Q \in X\}.$$

in a cartesian coordinate system, we have

$$P = P(x_P, y_P, z_P), \quad v = v(v_x, v_y, v_z), \quad \varphi(x,y,z) = Ax + By + Cz + D.$$

With (7.3) we obtain

$$\begin{cases} x_{P'} = x_P - 2\mu v_x \\ y_{P'} = y_P - 2\mu v_y \\ z_{P'} = z_P - 2\mu v_z \end{cases} \quad \text{where} \quad \mu = \frac{Ax_P + By_P + Cz_P + D}{Av_x + Bv_y + Cv_z}.$$

7.5.1 Reflection in a line along a plane (cartesian coordinates)

With (7.3) we obtain

$$\begin{cases} x_{P'} = 2x_A - x_P - 2\mu v_x \\ y_{P'} = 2y_A - y_P - 2\mu v_y \\ z_{P'} = 2z_A - z_P - 2\mu v_z \end{cases} \text{where } \mu = \frac{\psi(A)}{(\operatorname{lin}\psi)(v)} = \frac{(\operatorname{lin}\varphi)(A) - (\operatorname{lin}\varphi)(P)}{(\operatorname{lin}\varphi)(v)} = \frac{A(x_A - x_P) + B(y_A - y_P) + C(z_A - z_P)}{Av_x + Bv_y + Cv_z}.$$

7.6 Exercises

Exercise 1. Give the equations of the line passing through the point $M(1,0,7)$, parallel to the plane $\pi : 3x - y + 2z - 15 = 0$ and intersecting the line

$$l : \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

Exercise 2. Give the equations of the projection of the line

$$l : 2x - y + z - 1 = 0 \cap x + y - z + 1 = 0$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction of $\overrightarrow{u}(1,1,-2)$. Write the equations of the line obtained by reflecting l in the plane π parallel to the direction of $\overrightarrow{u}$.

Exercise 3. Write the equation of the plane determined by the line

$$l : x - 2y + 3z = 0 \cap 2x + z - 3 = 0$$

and the point $A(-1,2,6)$.

Exercise 4. Show that two parallel lines are mapped onto parallel lines by the reflection $\operatorname{Ref}_{\pi,d}$ where

$$\pi : Ax + By + Cz + D = 0, \quad l : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and $\pi \nparallel d$.

Exercise 5. For the plane and line

$$\pi : Ax + By + Cz + D = 0, \quad l : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

with $\pi \nparallel l$, show that the matrix (with respect to the basis giving the above equations) of $\operatorname{lin}\operatorname{Pr}_{\pi,l}$ is

$$[\operatorname{lin}\operatorname{Pr}_{\pi,l}] = \frac{1}{Ap + Bq + Cr}\begin{bmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{bmatrix}.$$

and that the matrix (with respect to the basis giving the above equations) of $\operatorname{lin}\operatorname{Ref}_{\pi,l}$ is

$$[\operatorname{lin}\operatorname{Ref}_{\pi,l}] = \frac{1}{Ap + Bq + Cr}\begin{bmatrix} -Ap + Bq + Cr & -2Bp & -2Cp \\ -2Aq & Ap - Bq + Cr & -2Cq \\ -2Ar & -2Br & Ap + Bq - Cr \end{bmatrix}.$$

Exercise 6. Find the linear parts of the projections and reflections in dimension 2 (see previous exercise).

Exercise 7. For the plane and line

$$\pi : Ax + By + Cz + D = 0, \quad l : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

with $\pi \nparallel l$, show that

1. $\operatorname{Pr}_{\pi,l} \circ \operatorname{Pr}_{\pi,l} = \operatorname{Pr}_{\pi,l}$ and
2. $\operatorname{Ref}_{\pi,l} \circ \operatorname{Ref}_{\pi,l} = \operatorname{Id}$.

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