B.10. 2018

lecture 7

Def: $\rho = (A, A, R)$ is an <u>equivalence relation</u> if ρ is <u>reflexive</u>, <u>transitive</u>, <u>symmetric</u>

Def: $\Pi \subseteq P(A) \setminus \{\emptyset\}$ is a <u>partition of A</u> if $\begin{cases} \bigcup_{B \in \Pi} B = A \\ B, B' \in \Pi, B \neq B' \Rightarrow B \cap B' = \emptyset \end{cases}$

$\Leftrightarrow \forall x \in A \quad \exists! B \in \Pi$ s.t. $x \in B$

Prop 1. If ρ is an equivalence on A, then the <u>quotient</u> <u>(factor)</u> <u>set</u>

$\boxed{A/\rho \overset{def}{=} \{\rho\langle x\rangle \mid x \in A\}}$ where $\boxed{\rho\langle x\rangle = \{y \in A \mid x \rho y\}}$

$\rho\langle x\rangle = [x]_\rho$ the class of x modulo ρ

is a partition of ρ.

Prop 2 If Π is a partition of A, then the relation $\rho_\Pi = (A, A, R_\Pi)$ on A, where for $x, y \in A$

$\boxed{x \rho_\Pi y \overset{def.}{\Longleftrightarrow} \exists B \in \Pi \text{ s.t. } x, y \in \Pi}$ is an equivalence rel on A

Moreover: $\rho_{A/\rho} = \rho$ and $A/\rho_\Pi = \Pi$.

Def 1. Let $f: A \to B$ be a function. The <u>kernel of f</u> is the following rel on A:

$\forall x, y \in A$ $\quad \boxed{x \ker f \, y \overset{def.}{\Longleftrightarrow} f(x) = f(y)}$

Prop 1. 1) $\ker f$ is an equivalence rel on A
2) We have $A/_{\ker f} = \{f^{-1}(b) \mid b \in \operatorname{Im} f\}$

proof. (1) (R) $\forall x \in A$ $\quad x \ker f\, x \Leftrightarrow f(x) = f(x)$ true.

(T) Let $x, y, z \in A$ s.t. $x \ker f\, y$ and $y \ker f\, z$.
Then $f(x) = f(y)$ and $f(y) = f(z) \Rightarrow f(x) = f(z)$
$\Rightarrow x \ker f\, z$.

(S) Let $x, y \in A$ s.t. $x \ker f\, y$. Then: $f(x) = f(y) \Rightarrow f(y) = f(x)$
$\Rightarrow y \ker f\, x$.

(2) We know that $A/_{\ker f} = \{(\ker f)\langle x\rangle \mid x \in A\}$.

We only need to prove that for $x \in A$, if $b = f(x) \in \operatorname{Im} f$, then $(\ker f)\langle x\rangle = f^{-1}(b)$

Indeed, let $y \in A$. We have: $y \in (\ker f)\langle x\rangle \Longleftrightarrow$
$\Longleftrightarrow x \ker f\, y \Leftrightarrow f(x) = f(y) \Longleftrightarrow f(y) = b \Longleftrightarrow y \in f^{-1}(b)$.

Def 2. Let ρ be an equivalen relation on A.
The function $p_\rho : A \to A/\rho \qquad p_\rho(x) = \rho\langle x\rangle$
(every elem. it's send to its class)
is called the canonical projection

Prop 2. Let ρ be an equivalen real. on A. Then
1) the can projection p_ρ is surjective.
2) $\ker p_\rho = \rho$.

proof 1) Let $\rho\langle x\rangle \in A/\rho$ where $x \in A$

Then: $\rho\langle x\rangle = p_\rho\langle x\rangle \in \operatorname{Im} p_\rho$, hence p_ρ is surjective.

2) Both are relations on A.

$\forall x, y \in A$ we have:

$$x \ker p_\rho\, y \Leftrightarrow p_\rho(x) = p_\rho(y) \Leftrightarrow \rho\langle x\rangle = \rho\langle y\rangle$$

$\Leftrightarrow x \rho y$. Hence $\ker p_\rho = \rho$.

The 1^{st} Factorization theorem

Let $f: A \to B$ be a function.

Then $\exists!$ bijective function $\bar{f}: A/\ker f \to \operatorname{Im} f$

s.t. the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p_{\ker f}"'] & B \\ A/\ker f \arrow[r, dashed, "\bar{f}"'] & \operatorname{Im} f \arrow[u, hook, "\iota\ (\text{iota})"'] \end{tikzcd}$$

i.e. $f = \iota \circ \bar{f} \circ p_{\ker}$

(where the canonical $p_{\ker f}$ is surj)

and the canonical inclusion

$\iota: \operatorname{Im} f \to B$, $\iota(b) = b$

is injective.)

From the proof we get:

$\boxed{\bar{f}((\ker f)\langle x\rangle) = f(x)}$

proof (!) We assume that $\bar{f}$ exists and has the claimed property, and we show that $\bar{f}$ is unique:

Let $(\ker f)\langle x\rangle \in A/\ker f$, where $x \in A$.

By the commutativity of the diagram, we have:

$$f(x) = (\iota \circ \bar{f} \circ p_{\ker f})(x) = \iota(\bar{f}(p_{\ker f}(x))) =$$

$= \bar{f}((\ker f)\langle x\rangle)$ so $\forall\, x \in A$,

$\boxed{f(\ker f)\langle x\rangle}$ is uniquely determined.

$f(x)$",

$(\Rightarrow)$ Let $\underbrace{\bar{f} : A/\ker f \longrightarrow \operatorname{Im} f, \quad \bar{f}((\ker f)\langle x\rangle) = f(x)}$ . ....

• We prove that the definition of $\bar{f}$ is correct, i.e. it does not depend on the choice of representatives. Indeed, let $x' \in (\ker f)\langle x\rangle$ be another representative in the class of x.

Then $x \ker f\, x'$, so $f(x) = f(x')$ hence $\bar{f}((\ker f)\langle x'\rangle) = f(x') = f(x)$.

• We show that $\bar{f}$ is injective:

Let $(\ker f)\langle x\rangle, (\ker f)\langle x'\rangle \in A/\ker f$ s.t.

$\bar{f}((\ker f)\langle x\rangle) = \bar{f}((\ker f)\langle x'\rangle) \Rightarrow f(x) = f(x')$

$\Rightarrow x \ker f\, x' \Rightarrow (\ker f)\langle x\rangle = (\ker f)\langle x'\rangle$

Hence $\bar{f}$ is injective.

• We show that $\bar{f}$ is surjective:

Let $b \in \operatorname{Im} f$, so $\exists\, x \in A$ s.t. $f(x) = b$. Then $b = f(x) = \bar{f}((\ker f)\langle x\rangle)$.

This shows that $\operatorname{Im} \bar{f} = \operatorname{Im} f$, hence $\bar{f}$ is surj.

Hence $\bar{f}$ is bijective.

• We show that the diagram is commutative:

Let $x \in A$. We have:

$$(\iota \circ \bar{f} \circ p_{\ker f})(x) = \iota(\bar{f}(p_{\ker f}(x))) =$$
$$= \bar{f}((\ker f)\langle x \rangle) = f(x), \text{ hence } \iota \circ \bar{f} \circ p_{\ker f} = f$$

Exercise.

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Prove that the function $f : A \to B$ is injective $\Leftrightarrow \ker f = 1_A$ (the kernel of f is the equality relation on A)

proof. Note that since $\ker f$ is reflexive, we always have that $1_A \subseteq \ker f$.

(i.e. $x = x' \Rightarrow x \ker f\, x'$)

"$\Rightarrow$" Assume f is injective

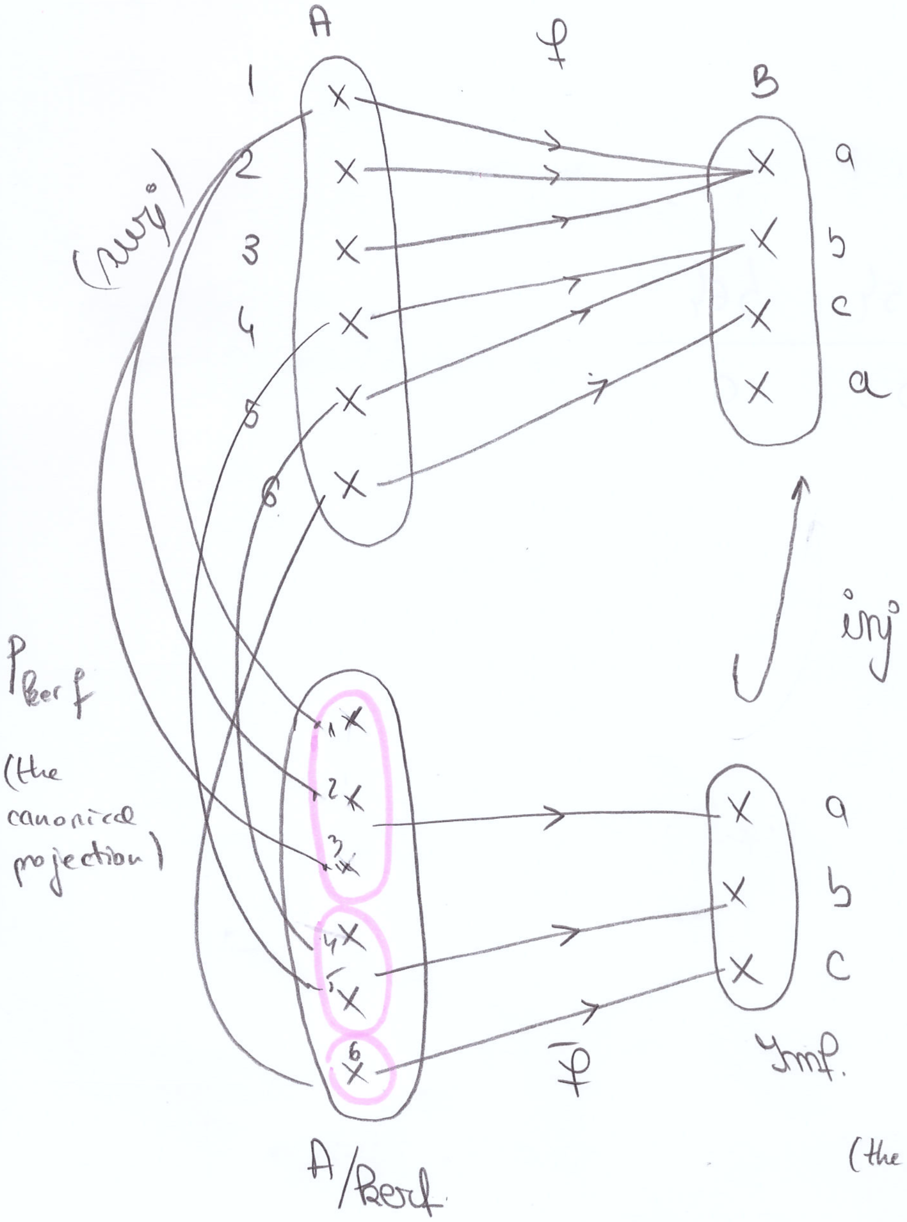
Let $x, x' \in A$ s.t. $x \ker f\, x'$

The $f(x) = f(x') \overset{f \text{ inj}}{\Longrightarrow} x = x'$

Hence $\ker f \subseteq 1_A$.

We apply the 1^{st} factorisation theorem to the function f given below:

Example



$Im f = \{a,b,c\} \subseteq B$ the image of f.

$f^{-1}(a) = \{1,2,3\}$

$f^{-1}(b) = \{4,5\}$

$f^{-1}(c) = \{6\}$

$\iota : Im f \to B$
(iota) $\iota(y) = y$

(the canonical inclusion)

$A/Ker f = \{\{1,2,3\},\{4,5\},\{6\}\}$
(the quotient set)

$Ker f = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(4,4),(5,5),(4,5),(5,4),(6,6)\}$
(the kernel of f)

In this example, we write down all the functions using tables:

$A \ni x$	1	2	3	4	5	6
$Im f \ni f(x)$	a	a	a	b	b	c
$A/Ker f \ni P_{Ker f}(x)$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{4,5\}$	$\{4,5\}$	$\{6\}$

$\operatorname{Im} f \ni y$	a	b	c
$B \ni ?(y)$	a	b	c

$A/\ker f \ni \ker f\langle x\rangle$	$\{1,2,3\}$	$\{4,5\}$	$\{6\}$
$\operatorname{Im} f \ni \bar{f}((\ker f)\langle x\rangle)$	a	b	c