Prof. Dr. Dorin Andrica
Asist. Drd. Tudor Micu
1st Semester, 2018-2019

Geometry 1 (Analytic Geometry)

Exercise Sheet 4-5

Exercise 1. Consider the triangle $\triangle ABC$ and the midpoint A' of the side $[BC]$. Show that $4AA'^2 - BC^2 = 4\overline{AB} \cdot \overline{AC}$.

Exercise 2. For a tetrahedron $ABCD$:

$$\cos(\widehat{\overline{AB}, \overline{CD}}) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2AB \cdot CD}$$

(the 3D version of the **cosine theorem**)

Exercise 3. Let $ABCD$ be a tetrahedron and G_A the center of mass of the BCD side. Then the following equality holds:

$$9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2)$$

(the 3D version of the **median line theorem**)

Exercise 4. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ and $A'B'$, respectively, are concurrent. Then the perpendicular lines through A', B', C' on BC, CA and AB, respectively are also concurrent.
(Steiner's theorem on **orthologic triangles**)

Remark. The result remains true if, instead of the vertices of a triangle, the points A', B' and C' are the feet of the perpendiculars from A, B, C to a line d. The point of intersection of the perpendiculars from A', B' and C' to BC, AC and AB, respectively, is called the **orthopole**. Try to prove this!

Exercise 5. If two pairs of opposite edges of a tetrahedron ABCD are perpendicular, $AB \perp CD$ and $AD \perp BC$, show that:

(a) The third pair of opposite edges also consists of perpendicular lines, $AC \perp BD$;

(b) $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$;

(c) The heights of the tetrahedron, the perpendicular lines through the vertices on the faces opposite to them, are concurrent. (such a tetrahedron is called **orthocentric**)

Remark. Using exercise 2 here makes everything almost too easy. Try to solve this exercise without it. The proofs here will be similar to the proof of 2, in any case.

Exercise 6. Using the dot product, prove the **Cauchy–Bunyakovsky–Schwarz** inequality: if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Exercise 7. Let $\overline{a}$, $\overline{b}$, $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if $\overline{a} \times \overline{b} = \overline{b} \times \overline{c} = \overline{c} \times \overline{a}$.

Exercise 8. Find the area of the plane triangle having the vertices $A(1, 0, 1)$, $B(0, 2, 3)$, $C(2, 1, 0)$.

Exercise 9. Let $\overline{a}, \overline{b}, \overline{c}$ be vectors in $\mathcal{V}_3$. Prove the following formulae:

1. $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{a} \cdot \overline{b}) \cdot \overline{c} = \begin{vmatrix} \overline{b} & \overline{c} \\ \overline{a} \cdot \overline{b} & \overline{a} \cdot \overline{c} \end{vmatrix}$;

2. $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a} = \begin{vmatrix} \overline{b} & \overline{a} \\ \overline{b} \cdot \overline{c} & \overline{a} \cdot \overline{c} \end{vmatrix}$

   (the **double cross product** rules);

3. $(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{a} \cdot \overline{d} \\ \overline{b} \cdot \overline{c} & \overline{b} \cdot \overline{d} \end{vmatrix}$ (**Laplace's** formula);

4. $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = (\overline{a}, \overline{c}, \overline{d}) \cdot \overline{b} - (\overline{b}, \overline{c}, \overline{d}) \cdot \overline{a} = (\overline{a}, \overline{b}, \overline{d}) \cdot \overline{c} - (\overline{a}, \overline{b}, \overline{c}) \cdot \overline{d}$;

5. $(\overline{a} \times \overline{b}, \overline{b} \times \overline{c}, \overline{c} \times \overline{a}) = (\overline{a}, \overline{b}, \overline{c})^2$

Exercise 10. Let $\overline{u}$, $\overline{v}$, $\overline{w}$ be noncoplanar vectors in $\mathcal{V}_3$. Their **reciprocal vectors** are defined to be

$$\overline{u}' = \frac{\overline{v} \times \overline{w}}{(\overline{u}, \overline{v}, \overline{w})}, \ \overline{v}' = \frac{\overline{w} \times \overline{u}}{(\overline{u}, \overline{v}, \overline{w})}, \ \overline{w}' = \frac{\overline{u} \times \overline{v}}{(\overline{u}, \overline{v}, \overline{w})}$$

1. Find the reciprocal vectors of $\overline{i}$, $\overline{j}$ and $\overline{k}$;

2. If $\overline{a} = x\overline{u} + y\overline{v} + z\overline{w}$, prove that

$$x = \overline{a} \cdot \overline{u}', \ y = \overline{a} \cdot \overline{v}', \ z = \overline{a} \cdot \overline{w}'$$

3. Show that the mutual vectors of $\overline{u}'$, $\overline{v}'$ and $\overline{w}'$ are respectively $\overline{u}$, $\overline{v}$ and $\overline{w}$.

Exercise 11. Show that the sum of some outer-pointing vectors perpendicular to the faces of a tetrahedron $ABCD$, whose lengths are proportional to the areas of the faces is the zero vector. In other words, if $\overline{v_A}, \overline{v_B}, \overline{v_C}, \overline{v_D}$ are perpendicular to (BCD), (ACD), (ABD) and (ABC), they point outwards and

$$\frac{|\overline{v_A}|}{S_{BCD}} = \frac{|\overline{v_B}|}{S_{ACD}} = \frac{|\overline{v_C}|}{S_{ABD}} = \frac{|\overline{v_D}|}{S_{ABC}}$$

then $\overline{v_A} + \overline{v_B} + \overline{v_C} + \overline{v_D} = \overline{0}$.