28.02.2018
SIMION Julian
MIE 1. Sem. 2

GEOMETRY $\boxed{\text{SEMINAR 1}}$

① Proof that for an abelian group, the notions of left and right actions coincide.

X set, $G \curvearrowright X$ $\quad \odot : G \times X \to X$

Properties:

1. $e \odot x \quad \forall x \in X$
2. $a \odot (b \odot x) = (a \cdot b) \odot x$

G abelian

We define $x \circleddash g := g \odot x \quad \forall g \in G, \ \forall x \in X$

1'). $x \circleddash e \stackrel{?}{=} x$

$\overset{\text{def}}{=} e \odot x \stackrel{1}{=} x$

2'). $\underbrace{(x \circleddash a)}_{a \odot x} \circleddash b = x \circleddash (ab) \overset{\text{def}}{=} (a \cdot b) \odot x$

$b \odot (a \odot x) \stackrel{2}{=} (b \cdot a) \odot x \overset{G \text{ abelian}}{=} (a \cdot b) \odot x$

⑤ $X = \mathbb{A}(\mathbb{R}^4) \quad \delta(X) = \mathbb{R}^4$

$\alpha = \langle (1,1,1,1), (0,1,0,1) \rangle_{\mathbb{R}} + (2,4,1,2)$ for a plane $\longrightarrow$ vector space spanned

$\beta = \langle (1,1,-1,1) \rangle + (2,3,-1,1)$ for a line

$\alpha \cap \beta$

$\langle v, w \rangle_{\mathbb{R}} = \{ t \cdot v + s \cdot w : t, s \in \mathbb{R} \}$ $\longrightarrow$ we can write $t(1,1,1,1)$ like $t\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$\alpha = \{ \underbrace{t\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + s\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \begin{pmatrix}2\\4\\1\\2\end{pmatrix}}_{} : t, s \in \mathbb{R} \}$ $\longrightarrow$ denote it by $\alpha(t,s)$

$\beta = \{ \underbrace{t'\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + \begin{pmatrix}2\\3\\-1\\1\end{pmatrix}}_{} : t' \in \mathbb{R} \}$ $\longrightarrow \beta(t')$

$\exists ?\ t, s, t' \in \mathbb{R}$ s.t. $\alpha(t,s) = \beta(t')$

$$t\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + s\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \begin{pmatrix}2\\4\\1\\2\end{pmatrix} = t'\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + \begin{pmatrix}2\\3\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}t+2\\t+s+4\\t+1\\t+s+2\end{pmatrix} = \begin{pmatrix}t'+2\\t'+3\\-t'-1\\t'+1\end{pmatrix}$$

$t+2 = t'+2 \Rightarrow t = t'$

$t+s+4 = t'+3 \Leftrightarrow t'+s+4 = t'+3$

$s+4 = 3$

$s = 1$

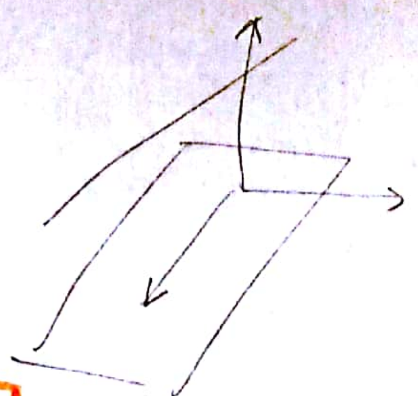
$t+1 = -t'-1 \Leftrightarrow t'+1 = -t'-1$

$2t' = -2 \Rightarrow t' = -1 \Rightarrow t = -1$

$\Big\} \Rightarrow \alpha(-1,1) = \beta(-1) \in \alpha \cap \beta$

$$\begin{pmatrix}-1+2\\-1-1+4\\-1+1\\-1-1+2\end{pmatrix}=\begin{pmatrix}-1+2\\-1+3\\1-1\\-1+1\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\0\\0\end{pmatrix}=\begin{pmatrix}1\\2\\0\\0\end{pmatrix}\in\alpha\cap\beta$$

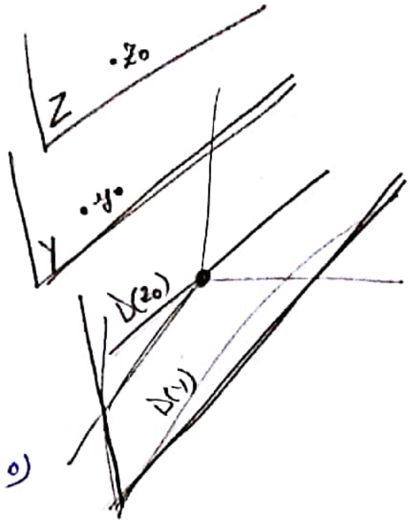


Def X affine space $\quad Y, Z \subseteq X$ aff. subspaces

$$Y=\mathcal{D}(Y)+y_0$$
$$Z=\underbrace{\mathcal{D}(Z)}_{\underset{v.s.}{\leq}\mathcal{D}(X)}+z_0$$

$$Y \parallel Z \overset{def}{\Longleftrightarrow} \mathcal{D}(Y)\subseteq\mathcal{D}(Z) \text{ or } \mathcal{D}(Z)\subseteq\mathcal{D}(Y)$$

aff. = affine
v.s. = vector space



⑥ $X=\mathbb{R}^4 \quad \alpha=(2,1,2,1)$

$\beta=\langle(1,1,1,1)\rangle+(1,3,0,0)$

$\gamma=\langle(2,1,3,-1),(1,0,2,-2)\rangle+(1,0,1,0)$

$\delta=\langle(1,0,0,0),(0,1,0,0),(0,0,0,1)\rangle+(0,0,0,0)$

Which one of the following is true?

1). $\alpha\in\beta$
2). $\alpha\in\gamma$
3). $\alpha\in\delta$
4). $\beta\parallel\gamma$
5). $\beta\parallel\delta$
6). $\gamma\parallel\delta$
7). $\beta\subseteq\gamma$
8). $\gamma\subseteq\delta$

1. $\exists^{?}\, t\in\mathbb{R}$ s.t. $\overset{\alpha\in\beta\Leftrightarrow}{\alpha}=t\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\3\\0\\0\end{pmatrix}\Leftrightarrow\begin{pmatrix}2\\1\\2\\1\end{pmatrix}=t\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\3\\0\\0\end{pmatrix}$

$\Leftrightarrow t\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\3\\0\\0\end{pmatrix}-\begin{pmatrix}2\\1\\2\\1\end{pmatrix}=0_{\mathbb{R}^4}\Leftrightarrow\begin{pmatrix}t-1\\t+2\\t-2\\t-1\end{pmatrix}=0_{\mathbb{R}^4}\Leftrightarrow\begin{cases}t-1=0\Rightarrow t=1\\t+2=0\Rightarrow t=-2\\t-2=0\\t-1=0\end{cases}$ ↯ $\Rightarrow\alpha\notin\beta$

2. $\exists^{?}\, t,s\in\mathbb{R}:\ \underset{\begin{pmatrix}2\\1\\2\\1\end{pmatrix}}{\alpha}=t\begin{pmatrix}2\\1\\3\\-1\end{pmatrix}+s\begin{pmatrix}1\\0\\2\\-2\end{pmatrix}+\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\Leftrightarrow\begin{cases}2t+s+1=2\\t=1\\3t+2s+1=2\\-t-2s=1\end{cases}\Rightarrow\begin{cases}s=1-2=-1\\t=1\\3+2s=1\\-1-2s=1\end{cases}\Rightarrow\begin{cases}s=-1\\t=1\\3-2=1\\-1+2=1\end{cases}\Rightarrow$

$\Rightarrow$ s. comp. $\Rightarrow\alpha\in\gamma$

$V = \langle v_i : i \in I \rangle$
$W = \langle w_j : j \in J \rangle$
if $v_i \in W$
$\Rightarrow V \subseteq W$

4. $\beta \parallel \gamma \Leftrightarrow D(\beta) \subseteq D(\gamma)$

$D(\beta) = \langle (1,1,1,1) \rangle$

$D(\gamma) = \langle (2,1,3,-1), (1,0,2,-2) \rangle$

$\Leftrightarrow \exists^? s,t \in \mathbb{R} : s\begin{pmatrix}2\\1\\3\\-1\end{pmatrix} + t\begin{pmatrix}1\\0\\2\\-2\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \Leftrightarrow \begin{cases} 2s+t=1 \\ s=1 \\ 3s+2t=1 \\ -s-2t=1 \end{cases} \Rightarrow \begin{cases} 2+t=1 \\ s=1 \\ 3+2t=1 \\ -1-2t=1 \end{cases} \Rightarrow \begin{cases} t=-1 \\ s=1 \\ t=-1 \ (A) \\ t=-1 \ (A) \end{cases} \Rightarrow \beta \parallel \gamma$

5. $\beta \parallel \delta \Leftrightarrow D(\beta) \subseteq D(\delta)$

$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = s\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + p\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \Leftrightarrow \begin{cases} 1=s \\ 1=t \\ 1=p \\ 1=0 \end{cases}$ $\lightning$ so $\beta \nparallel \delta$

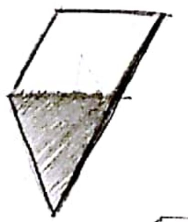
6. luăm pe rând primul vector din γ (adică $(2,1,3,-1)$) cu δ și verificăm ca la 5 și după luăm al doilea vector din γ (adică $(1,0,2,-2)$) cu δ și verificăm ca la 5.

7. $\beta \subseteq \gamma$

$A(1,3,0,0) \in^? \gamma$ (if yes, since $\beta \parallel \gamma$, we get $\beta \subseteq \gamma$)

$t\begin{pmatrix}2\\1\\3\\-1\end{pmatrix} + s\begin{pmatrix}1\\0\\2\\-2\end{pmatrix} + \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\0\\0\end{pmatrix} \Leftrightarrow \begin{cases} 2t+s+1=1 \\ t=3 \\ 3t+2s+1=0 \\ -t-2s=0 \end{cases} \Rightarrow t=3 \Rightarrow \begin{matrix} s=-6 \\ s=-5 \end{matrix} \Rightarrow \lightning \Rightarrow A \notin \gamma \Rightarrow \beta \nsubseteq \gamma$

8. $\gamma \nparallel \delta \Rightarrow \gamma \nsubseteq \delta$



⑧ Let K be a finite field. $X = K^m$, $m \in \mathbb{R}$. Determine:

a). the number of points in an affine subspace Y of X | $q^{\dim Y}$, where $K = \mathbb{F}_q$

b). the number of lines passing through a given point $x \in X$

$\mathbb{F}_q$ = the field with q elements where $q = p^m$
where p is a prime number, $m \in \mathbb{N}$

$m=1 \quad \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

$X = \mathbb{F}_q^m \Rightarrow |X| = q^m$

$T_\square(P) : G = \mathbb{F}_q^m \longrightarrow X$
$\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad D(X)$

$|X| = |D(X)| = q^m$

$Y \subseteq X$

$T_\square(P)\big|_{D(Y)} : D(Y) \longrightarrow Y \quad \Big\} \Rightarrow |Y| = q^d$
$\quad\quad\quad\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad \mathbb{F}_q^d$

map

the nr. of lines through a point = number of 1-dimensional vector space of $\delta(x) = \mathbb{F}_2^n$.

$$= \frac{\text{number of non-zero vectors of } \delta(x)}{\text{number of non-zero scalars}} = \frac{2^n - 1}{2 - 1}$$

dacă 2 vectori generează aceeași linie (dreaptă), atunci unul dintre ei e produsul celuilalt cu un scalar.

each vector will generate a line

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10 which of the following are affine subspaces?

1). $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 - 2 = 0\}$

2). $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \overbrace{(x_1 + x_2, 2x_2 + x_3, x_3 - 2x_1)}^{*} \in A\}$

3). $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3 = 0\}$

4). $D = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2 - 2x_3 + x_4 = 0\}$

1. $\mathbb{R}^3 \xrightarrow{\phi} \overset{?}{\square}$

$\cup\!\!|$

$A \overset{?}{=} \phi^{-1}(N)$

~~$A = \phi^{-1}(\ldots)$~~
~~↳ aff. subspace of $\mathbb{R}$~~

$\phi(x_1, x_2, x_3) = 2x_1 - x_2 + x_3 - 2$

$\phi(x_1, x_2, x_3) = 2x_1 - x_2 + x_3$ lin

$A = \phi^{-1}(0)$ $A \overset{?}{=} \phi^{-1}(2)$ ↳ aff. subspace of $\mathbb{R}$

linearity is checked by $\phi(\alpha v + \beta w) \overset{?}{=} \alpha\phi(v) + \beta\phi(w)$

$\alpha, \beta \in K$ vectors

-2 is a term that destroys linearity

$\phi(0+0) = \phi(0) + \phi(0) = -4$

$\phi(0)$

-2

2. $B = \psi^{-1}(A)$, $\psi : \mathbb{R}^3 \to \mathbb{R}^3$

$\mathbb{R}^3 \xrightarrow{\phi} \overset{?}{\boxed{\mathbb{R}}}$

$\cup\!\!|$

$A \overset{?}{=} \phi^{-1}(N)$

$\psi(x_1, x_2, x_3) = *$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \psi(x_1, x_2, x_3)$$

3. $(x_1 - x_2 + x_3)^2 = 0$

$C = \psi_3^{-1}$

$\psi : \mathbb{R}^3 \to \mathbb{R}$

$\psi(x_1, x_2, x_3) = x_1 - x_2 + x_3$

4. $D \overset{?}{=} \psi^{-1}(0)$

$\psi : \mathbb{R}^4 \to \mathbb{R}$

$\psi(x_1, x_2, x_3, x_4) = x_1^4 + x_2 - 2x_3 + x_4$

not linear

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$\delta \in O_{\mathbb{R}_4}$

It is aff $\Leftrightarrow$ it is vector space of $\mathbb{R}_4$

$\vec{v} = (x_1, x_2 x_3, x_4)$

$\vec{w} = (y_1, y_2, y_3, y_4)$

$\delta \ni \vec{v} + \vec{w} = (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4)$

$\vec{v} + \vec{w} \in \delta$ iff $(x_1+y_1)^4 + x_4+y_4 = 0$

we know $x_1^4 + x_4 = 0$

$y_1^4 + y_4 = 0$

$\forall x_1, y_1 \in \mathbb{R}_4 \ldots \ldots = 0$

Consultații
Vineri $15^{00} - 17^{00}$

11) $a_1, \ldots, a_4 \in \mathbb{R}$

$b_1, \ldots, b_m \in \mathbb{R}$

$I \subseteq \mathbb{R}$

$(g: I \to \mathbb{R}) \in \mathcal{C}^\infty(I)$

Which of the following are aff? ⊛⊛

1) $A = \{ f \in \mathcal{C}^\infty(I) : \overbrace{a_m f^{(m)} + a_{m-1} f^{(m-1)} + \ldots + a_1 f' + g}^{} = 0 \}$

2) $B = \{ f \in \mathcal{C}^\infty(I) : b_m f^{(m)} + b_{m-1} f^{(m-1)} + \ldots + b_1 f' \in A \}$

3) $C = \{ f \in \mathcal{C}^\infty(I) : f^3 - 5f^2 + 6f = 0 \}$

$g, f \in \mathcal{C}^\infty(I) \ni \alpha g + \beta f : \underset{\in}{\overset{I}{}} x \mapsto \alpha \cdot g(x) + \beta f(x)$

$\alpha, \beta \in \mathbb{R}$

$A: \phi(f) = $ ⊛⊛

$\phi^{-1}(-g) = A$

$B = \phi^{-1}(A)$

$C: \quad f(f-2)(f-3) = 0$

$\Rightarrow f = 0$

$f = 2$

$f = 3$

$C \ni O_{\mathcal{C}^\infty(I)}$

$\hookrightarrow$ C aff $\Leftrightarrow$ C v.s. of $\mathcal{C}^\infty(I)$

v.s. $\downarrow$ vector space

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18 $S=\{A_1,\dots,A_p\}\subseteq X$

$$\mathrm{Bar}(S)=\mathrm{Bar}(A_1,\dots,A_p)=\sum_{j=1}^{p}\frac{1}{p}A_j$$

1. $S_i = S\setminus\{A_i\}$. Show that:

$$\mathrm{Bar}(S)=\mathrm{Bar}(\mathrm{Bar}(S_1),\dots,\mathrm{Bar}(S_p))$$

2. $S'=\{A_1,\dots,A_q\}$ $\quad S''=\{A_{q+1},\dots,A_p\}$

$$\mathrm{Bar}(S)=\frac{q}{p}\,\mathrm{Bar}(S')+\frac{p-q}{p}\,\mathrm{Bar}(S'')$$

3. $p=2,3,4$

1. $$\sum_{j=1}^{p}\frac{1}{p}A_j \overset{?}{=} \mathrm{Bar}\left(\sum_{j=2}^{p}\frac{1}{p-1}A_j,\ \sum_{\substack{j=1\\ j\neq 2}}^{p}\frac{1}{p-1}A_j,\ \dots,\ \sum_{j=1}^{p-1}\frac{1}{p-1}A_j\right)$$

$$=\sum_{i=1}^{p}\frac{1}{p}\left(\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{p-1}A_j\right)$$

$$=\sum_{j=1}^{p}\frac{1}{p}\,\frac{1}{p-1}\,(p-1)A_j.$$

$- - - .\ A_2 \quad A_3 - - - A_p$

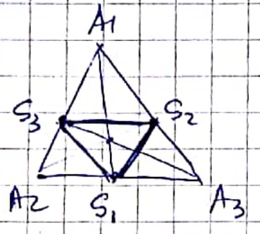
$A_1 \ \_\_\_\_ \ A_3 - - - A_p$

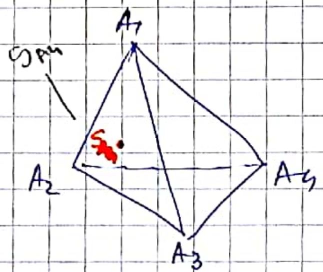
$A_1 \quad A_2 \quad A_3 - - - A_p$

2. $$\sum_{i=1}^{p}\frac{1}{p}A_i=\frac{q}{p}\sum_{i=1}^{q}\frac{1}{q}A_i+\frac{p-q}{p}\sum_{j=q+1}^{p}\frac{1}{p-q}A_i$$

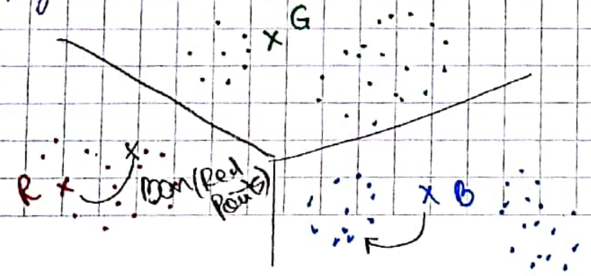
$$=\sum_{i=1}^{q}\frac{q}{p}\cdot\frac{1}{q}A_i+\sum_{j=q+1}^{p}\frac{p-q}{p}\cdot\frac{1}{p-q}\cdot A_i$$

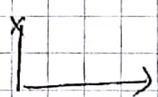
$$=\sum_{i=1}^{p}\frac{1}{p}A_i$$





Algoritmul k-means





means apartments/price
OR
activity at seminar/ grades

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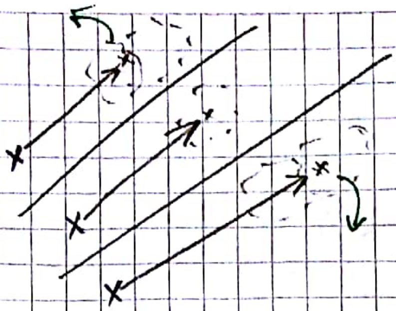
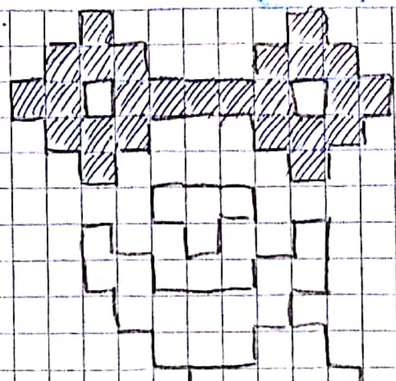




15 X aff $\quad M \subseteq X \quad l(M) = \{ tP + (1-t)Q : \forall P, Q \in M, \forall t \in K \}$

1. $X = \mathbb{R}^2 \quad M = \{ \underset{A}{(1,0)}, \underset{B}{(0,2)}, \underset{C}{(0,0)} \}$

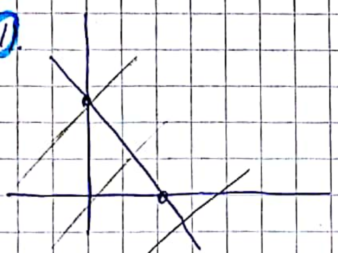
   $l(M) = ?$

   $l^2(M) = l(l(M)) = ?$

2. $X = \mathbb{R}^3 \quad M = \{ (1,0,0), (0,1,0), (0,0,1), (0,0,0) \}$ $\quad l^3(M)$

3. Show that $M \subseteq l(M) \subseteq l^2(M) \subseteq \ldots$ is stationary if X is finite dimension

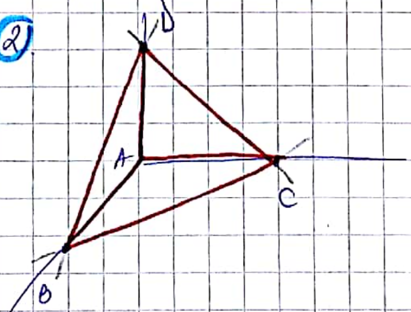
1.



$l(M) = AB \cup AC \cup BC$

$l^2(M) = \mathbb{R}^2$

2.



$l^2(M) = ABC \cup ABD \cup \ldots$

$l^3(M) = \mathbb{R}^3$

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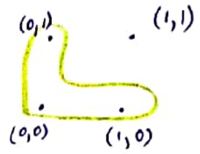
$K=\mathbb{F}_2=\{0,1\}$, $M=\{(0,0),(1,0),(0,1)\}$
$\cap|$
$A(\mathbb{F}_2^2)$

- $\ell(M)=M$
- M not aff.



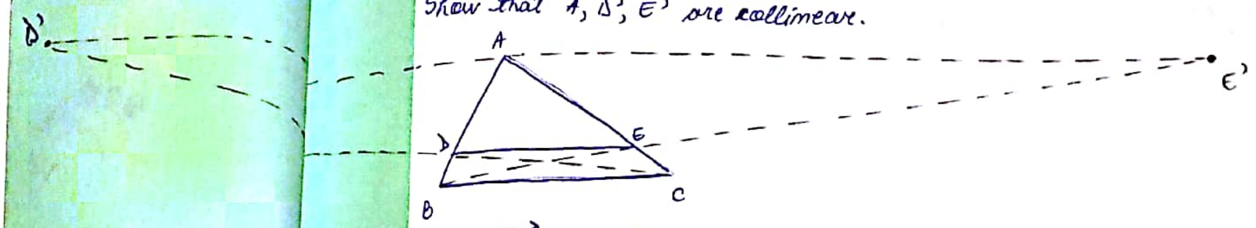
$(1-t)P+tQ$

Ex. 4 $\Delta ABC \subseteq X$ real aff. space

$D\in[AB]$ $\quad E\in[AC]$ s.t. $\frac{|AD|}{|AB|}=\frac{|AE|}{|AC|}=\frac{3}{4}$

$D', E' \in X:\ \overrightarrow{EE'}=3\overrightarrow{BE} \qquad \overrightarrow{DD'}=3\overrightarrow{CD}$

Show that A, D', E' are collinear.



$\overrightarrow{AE'}=\alpha\overrightarrow{AC}+\beta\overrightarrow{AB}$

$\hookrightarrow \overrightarrow{AE'}=\overrightarrow{AE}+\overrightarrow{EE'}=\frac{3}{4}\overrightarrow{AC}+3\left(-\frac{3}{4}\overrightarrow{CA}-\overrightarrow{AB}\right)=\frac{3}{4}\overrightarrow{AC}-\frac{9}{4}\overrightarrow{CA}-3\overrightarrow{AB}=3\overrightarrow{AC}-3\overrightarrow{AB}$ (1)

$\overrightarrow{AE}=\frac{3}{4}\overrightarrow{AC}$, $\quad \overrightarrow{EE'}=3\overrightarrow{BE}=-(\overrightarrow{EA}+\overrightarrow{AB})=-\frac{3}{4}\overrightarrow{CA}-\overrightarrow{AB}$

$\overrightarrow{AD'}=\overrightarrow{AC}+\overrightarrow{CD'}=\overrightarrow{AC}+4\left(\overrightarrow{CA}+\frac{3}{4}\overrightarrow{AB}\right)=\overrightarrow{AC}+4\overrightarrow{CA}+3\overrightarrow{AB}=3\overrightarrow{AB}-3\overrightarrow{AC}$ (2)

$\overrightarrow{CD'}=4\overrightarrow{CD}=4(\overrightarrow{CA}+\overrightarrow{AD})$, $\quad \overrightarrow{AD}=\frac{3}{4}\overrightarrow{AB}$

(1),(2) $\Rightarrow AE'=-AD' \Rightarrow A, E', D'$ – colinear

II. $D=\frac{1}{4}B+\frac{3}{4}A$

$E=\frac{1}{4}C+\frac{3}{4}A$

$E'=B+4(E-B)$

$D'=C+4(D-C)$

$E'=B+4\left(\frac{1}{4}C+\frac{3}{4}A-B\right)=-3B+C+3A$

$D'=C+4\left(\frac{1}{4}B+\frac{3}{4}A-C\right)=-3C+B+3A$

după ce obținem E' și D' vom arăta că vectorii $\overrightarrow{AD'}$ și $\overrightarrow{AE'}$ coliniari

$D(AD')$

$D''(AE')$

Ex 5 ΔABC

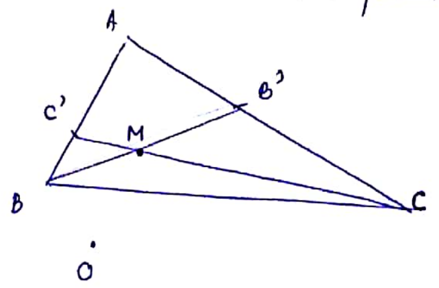
$C' \in AB \qquad B' \in AC$

$\vec{AC'} = \lambda \vec{BC'} \qquad \vec{AB'} = \mu \vec{CB'}$

$BB' \cap CC' = M$, $O \in X$

$$\vec{OM} = \frac{\vec{OA} - \lambda \vec{OB} - \mu \vec{OC}}{1-\lambda-\mu} \Leftrightarrow M = \frac{A - \lambda B - \mu C}{1-\lambda-\mu}$$

vector form $\qquad$ affine form



$C' = A + (C' - A)$

$\lambda (C' - B) \Rightarrow (1-\lambda) C' = A - \lambda B \Rightarrow C' = \frac{A - \lambda B}{1-\lambda}$

$\left.\begin{array}{l} B' = A + (B' - A) \\ \quad \mu (B' - C) \end{array}\right\} \Rightarrow (1-\mu) B' = A - \mu C \Rightarrow B' = \frac{A - \mu C}{1-\mu}$

$M = (1-t)B + tB' \Rightarrow B' = \frac{M - (1-t)B}{t}$

$M = (1-p)C + pC' \Rightarrow C' = \frac{M - (1-p)C}{p}$

$C' = \frac{A - \lambda B}{1-\lambda} = \frac{M - (1-p)C}{p} \Rightarrow p(A - \lambda B) = (1-\lambda)[M - (1-p)C] \Rightarrow$

$\Rightarrow M - (1-p)C = \frac{p(A - \lambda B)}{1-\lambda}$

$\Rightarrow M = \frac{p(A - \lambda B)}{1-\lambda} + (1-p)C \quad (2)$

(1) $M = (1-t)B + \frac{t(A - \mu C)}{1-\mu}$

Coeficient

$A \Rightarrow \frac{p}{1-\lambda} = \frac{t}{1-\mu} \Rightarrow p(1-\mu) = t(1-\lambda) \Rightarrow t = \frac{p(1-\mu)}{1-\lambda}$

$B \Rightarrow \frac{-p\lambda}{1-\lambda} = 1 - t \Rightarrow \frac{-p\lambda}{1-\lambda} = 1 - \frac{p(1-\mu)}{1-\lambda} \Rightarrow (*)$

$C \Rightarrow 1 - p = \frac{-\mu t}{1-\mu}$

$(*) \Rightarrow -p\lambda = 1 - \lambda - p(1-\mu) \Rightarrow$

$\Rightarrow -p\lambda + p - p\mu = 1 - \lambda \Rightarrow$

$\Rightarrow p(-\lambda + 1 - \mu) = 1 - \lambda \Rightarrow p = \frac{1-\lambda}{1-\lambda-\mu}$

$$M = \left(1 - \frac{1-\lambda}{1-\lambda-\mu}\right)C + \frac{A - \lambda B}{1-\lambda-\mu} = \frac{(\cancel{1} - \cancel{\lambda} - \mu - \cancel{1} + \cancel{\lambda})C + A - \lambda B}{1-\lambda-\mu} = \frac{A - \lambda B - \mu C}{1-\lambda-\mu}$$

Ex 6

ΔABC

G = centroid (centru de greutate) → intersecția medianelor

H = orthocenter

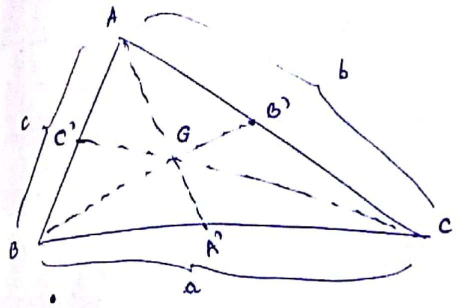
I = incenter → intersecția bisectoarelor de $\sphericalangle$

$P \in X$

a). $\overrightarrow{PG} = \dfrac{\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}}{3} = \dfrac{A+B+C}{3}$

b). $\overrightarrow{PI} = \dfrac{a\,\overrightarrow{PA} + b\,\overrightarrow{PB} + c\,\overrightarrow{PC}}{a+b+c}$

c). $\overrightarrow{PH} = \dfrac{(\tan\hat{A})\overrightarrow{PA} + (\tan\hat{B})\overrightarrow{PB} + (\tan\hat{C})\overrightarrow{PC}}{\tan\hat{A} + \tan\hat{B} + \tan\hat{C}}$



$P^{\bullet}$ → $P \in X$ — affine way of writing; am fost la tablă => continuare pagină

$\overrightarrow{PG} = \dfrac{\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}}{3} = \dfrac{A+B+C}{3}$

$\overrightarrow{PI} = \dfrac{a\cdot\overrightarrow{PA} + b\cdot\overrightarrow{PB} + c\cdot\overrightarrow{PC}}{a+b+c}$; where a, b, c are lengths of the sides of Δ.

$\overrightarrow{PH} = \dfrac{(\tan\hat{A})\overrightarrow{PA} + (\tan\hat{B})\overrightarrow{PB} + (\tan\hat{C})\overrightarrow{PC}}{\tan\hat{A} + \tan\hat{B} + \tan\hat{C}}$

(a) We show that $\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$, for $\lambda = -1$

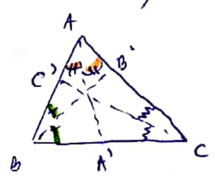
$\overrightarrow{AC'} := \frac{1}{2}\overrightarrow{AB}$ $\overrightarrow{BC'} = -\frac{1}{2}\overrightarrow{AB}$

Similarly for $\mu = -1$ (previous exercise)

(b). $\overrightarrow{PI}$ = fracția dată, simplificată prin a.

$= \dfrac{\overrightarrow{PA} + \frac{b}{a}\overrightarrow{PB} + \frac{c}{a}\overrightarrow{PC}}{\frac{a+b+c}{a}}$

$\lambda = -\frac{b}{a}$ $\mu = -\frac{c}{a}$



$\overrightarrow{AC'} = \lambda\,\overrightarrow{BC'}$

$\overrightarrow{AB'} = \mu\,\overrightarrow{CB'}$

$\dfrac{\|\overrightarrow{AC'}\|}{\|\overrightarrow{BC'}\|} = \dfrac{AC}{BC} = \dfrac{b}{a}$

$AC' = -\|\overrightarrow{AC'}\| \dfrac{\overrightarrow{BC'}}{\|\overrightarrow{BC'}\|} = -\dfrac{b}{a}\cdot\overrightarrow{BC'} \Rightarrow \lambda = -\dfrac{b}{a}$

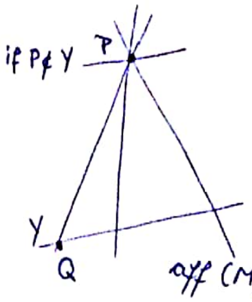
$\dfrac{\|\overrightarrow{AB'}\|}{\|\overrightarrow{BC'}\|} = \dfrac{\|\overrightarrow{AB}\|}{\|\overrightarrow{BC}\|} = \dfrac{c}{a} \Rightarrow \overrightarrow{AB'} = -\|\overrightarrow{AB'}\|\cdot\dfrac{\overrightarrow{CB'}}{\|\overrightarrow{CB'}\|} = -\dfrac{c}{a}\,\overrightarrow{CB'} \Rightarrow \mu = -\dfrac{c}{a}$.

Ex. $Y \underset{\text{aff.}}{\subseteq} X$, $P \in X$

Is $M = \bigcup_{Q \in Y} \text{aff}\{Q,P\}$ affine? NO since $P + D(Y) \notin M$

$$\text{aff}\{Q,P\} = \begin{cases} PQ & , P \neq Q \\ P & , P = Q \end{cases}$$

if $P \notin Y$



$\longleftarrow M = K^2 \setminus \{P + D(Y)\}$

$\text{aff}(M) = K^2$

23 4 dimensional affine space X

Show that if two hyperplanes intersect non-trivially, then there is a plane in the intersection.

H_1, H_2 $\quad H_1 \cap H_2 \neq \emptyset$

$$\dim H_1 \cap H_2 = \underbrace{\underbrace{\dim H_1 + \dim H_2}_{6} - \underbrace{\dim \text{aff } H_1 \cup H_2}_{\leq 4}}_{\geq 2}$$

$\boxed{? \geq 2}$

24 α, β planes in a 4-dim. aff. space.

Give the relative positions of α, β. (considering dim. aff. $(\alpha \cap \beta)$).

<u>Case 1</u>: $\alpha \cap \beta \neq \emptyset \Leftrightarrow \underset{\leq 4}{\dim \alpha \cap \beta} = \underbrace{\dim \alpha + \dim \beta}_{4} - \underbrace{\dim \text{aff } \alpha \cup \beta}_{\leq 4}$

a). $\dim \text{aff } \alpha \cup \beta = 2 \Leftrightarrow \alpha = \beta$ ✓

b). $\dim \text{aff } \alpha \cup \beta = 3 \Leftrightarrow \alpha \cap \beta$ is a line

c). $\dim \text{aff } \alpha \cup \beta = 4 \Leftrightarrow \alpha \cap \beta$ is a point

<u>Case 2</u>: $\alpha \cap \beta = \emptyset$

a). $\dim(D(Y) + D(Z)) = 2 \Leftrightarrow D(Y) = D(Z) \Leftrightarrow$

$\Leftrightarrow \dim \text{aff } (Y \cup Z) = 3$

$\Updownarrow$

$\boxed{Y \parallel Z}$

b). $\dim(D(Y) + D(Z)) = 3 \Leftrightarrow \dim(D(Y) \cap D(Z)) = 1 \Leftrightarrow$

$\Leftrightarrow \dim \text{aff } (Y \cup Z) = 4$

$\underset{\leq 4}{\dim(Y \cup Z)} = \underbrace{\dim(D(Y) + D(Z))}_{\leq 4} + 1$

$D(Y) = D(Z)$ if dim

(22), (26) $\dim X = n$

$H \subseteq X$ hyperplane

$Y \subseteq X$ $\dim Y = d$

a). $\forall P \in X$ $\exists$ unique $Z \underset{aff}{\subseteq} X$, $Z \ni P$, $\dim Z = d$ s.t. $Z \parallel Y$

b).. Show that exactly one of the following holds:

$$\begin{cases} \dim(H \cap Y) = d-1 \\ H \parallel Y \end{cases}$$

(a). We show that $\exists$ $Z \underset{aff}{\subseteq} X$.

Y aff $\Rightarrow$ $Y = y + \vec{\delta}(Y)$

$Z = P + \delta(Y)$, $\forall P \in X$

$$\left.\begin{array}{l} Z_1 = P + \delta(Z_1) \parallel Y \\ Z_2 = P + \delta(Z_2) \parallel Y \end{array}\right\} \Rightarrow \dim(\delta(Z_1)) = \dim(\delta(Z_2)) = \dim(\delta(Y))$$

$\Rightarrow \delta(Z_1) = \delta(Z_2) = \delta(Y) \Rightarrow Z_1 = Z_2$

(b). $\left.\begin{array}{l} Y \subseteq X \\ H \subseteq X \end{array}\right\} \xrightarrow{prop} [Y \cup H = \emptyset \Rightarrow Y \parallel H]$

Assume $Y \cap H \neq \emptyset$ $(\dim(Y \cap H) \neq 0)$

$\Rightarrow \exists$ $T \in Y \cap H$

$Y \cap H = T + (\delta(Y) \cap \delta(H))$

Case I: $\delta(Y) \subseteq \delta(H) \Rightarrow Y \parallel H$

Case II: $\delta(Y) \not\subseteq \delta(H) \Rightarrow \dim(Y \cap H) = \underset{d}{\underset{\parallel}{\dim Y}} + \underset{n-1}{\underset{\parallel}{\dim H}} - \underset{n}{\underset{\parallel}{\dim \operatorname{aff}(Y \cup H)}}$

$\Rightarrow \dim(Y \cap H) = d-1$

Möbius: $P_1, P_2, P_3 \subseteq AB$

$$(AB \mid P_1P_2)(AB \mid P_2P_3)(AB \mid P_3P_1) = 1$$

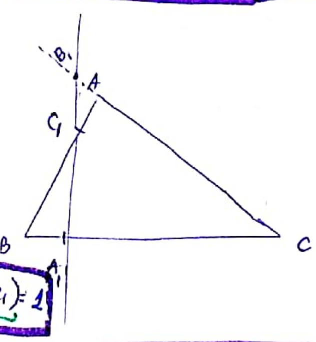
Menelaus' theorem:

$\triangle ABC \subseteq$ aff space

$A_1 \in BC$

$B_1 \in CA$ distinct from A, B, C

$C_1 \in AB$



$$\boxed{A_1, B_1, C_1 \text{ colinear} \Leftrightarrow \underline{(BC \mid A_1)}\,\underline{(CA \mid B_1)}\,\underline{(AB \mid C_1)} = 1}$$

$(BC \mid A_1) = \dfrac{a}{1-a}$

$(CA \mid B_1) = \dfrac{b}{1-b}$

$(AB \mid C_1) = \dfrac{c}{1-c}$

proof: $A_1 = \operatorname{Bar}(B, C; 1-a, a) = (1-a)B + aC = B + a \cdot \overrightarrow{BC}$

$B_1 = \operatorname{Bar}(C, A; 1-b, b) = (1-b)C + bA = C + b \cdot \overrightarrow{CA}$

$C_1 = \operatorname{Bar}(A, B; 1-c, c) = (1-c)A + cB = A + c \cdot \overrightarrow{AB}$

$\dfrac{a}{1-a} \cdot \dfrac{b}{1-b} \cdot \dfrac{c}{1-c} \neq -1 \Rightarrow$ $A_1 = B + a\,\overrightarrow{BC}$

$B_1 = C + b\,\overrightarrow{CA}$

$C_1 = A + c\,\overrightarrow{AB}$

$B_1 = B + \vec{BC} + (\vec{CB} + \vec{BA})b = B(1-b)\vec{BC} + b\vec{BA}$

$C_1 = B + \vec{BA} + c\vec{AB} = B + (1-c)\vec{BA}$

$\langle A_1 - B_1 \rangle = \langle A_1 - C_1 \rangle \Leftrightarrow A_1, B_1, C_1$ collinear

$\Leftrightarrow \langle CB + a\vec{BC} - B - (1-b)\vec{BC} - b\vec{BA} \rangle =$

$= \langle (a-1+b)\vec{BC} - b\vec{BA} \rangle$

$\langle B + a\vec{BC} - B - (1-c)\vec{BA} \rangle = \langle a\vec{BC} + (c-1)\vec{BA} \rangle$

$\begin{vmatrix} a+b-1 & -b \\ a & c-1 \end{vmatrix} = 0 \Leftrightarrow (a+b-1)(c-1)+ab=0$

$ac+bc-c-a-b+1+ab=0$

$ab+ac+bc-(a+b+c)+1=0$

$\Leftrightarrow abc=(1-a)(1-b)(1-c)$

$\Leftrightarrow abc=(1-b-a+ab)(1-c)=(1-b-a+ab-c+cb-abc+ca)$

$0=ab+bc+ac-(a+b+c)+1$

10 $\mathbb{R}^m$ $(m\geq 2)$ $L = P + \langle (v_1,\dots,v_m) \rangle$

$H: \alpha_1 x_1 + \dots + \alpha_m x_m + \beta = 0$

Show that $L \parallel H \Leftrightarrow \alpha_1 v_1 + \dots + \alpha_m v_m = 0$

$\hookrightarrow \Leftrightarrow (v_1,\dots,v_m) \in \delta(H)$ so if $v = \vec{OP}$ then we are done

9 $\mathbb{R}^4$

$Y: \begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 = 0 \end{cases}$

$Z: \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 = 1 \\ x_2 + x_3 - 3x_4 = -1 \\ x_1 - x_2 + 3x_4 = 3 \end{cases}$

1. determine parametric eq. and dim Y, Z
2. show that $Y \parallel Z$

10 fix $O \in H$ $\quad O = (O_1,\dots,O_m) \Rightarrow \alpha_1 O_1 + \dots + \alpha_m O_m = 0$

$\forall P \in X$ $\quad \vec{OP}(P_1 - O_1, \dots, P_m - O_m)$

$\alpha_1(P_1 - O_1) + \dots + \alpha_m(P_m - O_m) = \sum_{i=1}^{m} \alpha_i P_i$

$\vec{OP} \in \delta(H) \Leftrightarrow P \in H \Leftrightarrow \sum_{i=1}^{m} \alpha_i P_i = 0$

{29} $\mathbb{R}^2 \ni A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)$

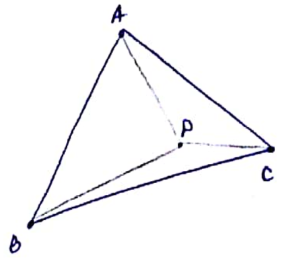
Show that $P(x_P, y_P)$ lies in ΔABC inside

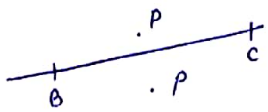
$\iff S_{AB}(x_P, y_P) \cdot S_{AB}(x_C, y_C) > 0$ and

$S_{BC}(x_P, y_P) \cdot S_{BC}(x_A, y_A) > 0$ and

$S_{CA}(x_P, y_P) \cdot S_{CA}(x_B, y_B) > 0$

where $S_{ij}(x, y) = \begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}$ $i, j \in \{A, B, C, P\}$





$ax + by = c$

$ax + by - c \leq 0$

$-c \geq 0$

$(ax_A + by_A - c)(ax_P + by_P - c) > 0$

$\alpha \cdot \beta > 0$

$ax + by - c = 0$ $\quad L_{BC} : S_{BC}(x, y) = 0$

$ax + by - c \leq 0$

$-c \geq 0$

$(ax_A + by_A - c)(ax_P + by_P - c) > 0$

$\iff S_{BC}(x_A, y_A) \cdot S_{BC}(x_P, y_P) > 0$

O (Gauss-Lucas Theorem)

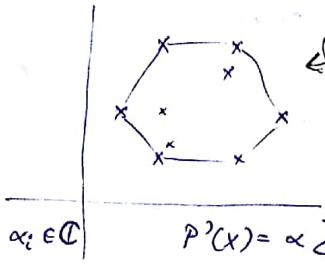
$P \in \mathbb{C}[x]$

Show that the roots of P' lie in the convex hull of the roots of P.

particular case $P = (x - 3i - j)^m$

$\mathbb{C} \cong \mathbb{R}^2$

$(a + ib) \longleftrightarrow (a, b)$



$P(X) = \alpha \prod_{i=1}^{m} (X - \alpha_i)$ $\quad \alpha_i \in \mathbb{C}$ $\quad P'(x) = \alpha \sum_{i=1}^{m} \prod_{\substack{j=1 \\ j \neq i}}^{m} (x - \alpha_j)$

Take z root of $P'(x)$

1. z is said to be a root of $P(x) \Rightarrow z \in \text{conv}(\alpha_1 \dots \alpha_m)$
2. else $P(z) \neq 0$

$$0 = \frac{P'(z)}{P(z)} = \sum_{i=1}^{m} \frac{1}{z - \alpha_i} = \sum_{i=1}^{m} \frac{\bar{z} - \bar{\alpha_i}}{|z - \alpha_i|^2}$$

Conjugating $\iff$
$$\underbrace{z \sum_{i=1}^{m} \frac{1}{|z - \alpha_i|^2}}_{S_{z,P}} = \sum_{i=1}^{m} \frac{\alpha_i}{|z - \alpha_i|^2}$$

$$z = \sum_{i=1}^{m} \underbrace{\frac{\frac{1}{|z - \alpha_i|^2}}{S_{z,P}}}_{\in \mathbb{R}^+} \alpha_i$$

$\Rightarrow z$ is a convex combination of α_i

$z = \sum \mu_i \cdot \alpha_i \quad \sum \mu_i = 1$

$\Rightarrow z \in \text{conv}(\alpha_1 \dots \alpha_m) \quad \mu \geq 0$

$$\sum_{i=1}^{m} \frac{\frac{1}{|z - \alpha_i|^2}}{S_{z,P}} = \frac{\sum \frac{1}{|\ \ |^2}}{\sum \frac{1}{|\ \ |^2}} = 1$$

$S_{z,P} > 0$

$\frac{1}{|z - \alpha_i|^2} > 0$

{27} $A, B \subseteq \mathbb{R}^n$ two convex sets

show that $A+B$ is convex

$= \{a+b : a \in A,\ b \in B\}$ Minkowski sum

$A+B$ convex $\Leftrightarrow \forall P, Q \in A+B \quad [P,Q] = \{(1-t)P + tQ : t \in [0,1]\} \subseteq A+B$

$P \in A+B \Leftrightarrow P = P_A + P_B$

$Q \in A+B \Leftrightarrow Q = Q_A + Q_B$

$(1-t)(P_A+P_B) + t(Q_A+Q_B) \overset{?}{\subseteq} A+B \qquad \overrightarrow{P_A Q_A}$

$P_A + P_B - tP_A - tP_B + tQ_A + tQ_B = P_A + P_B + t(Q_A - P_A) + t(Q_B - P_B) =$

$= \underbrace{t\overrightarrow{P_A Q_A} + P_A}_{\cap} + \underbrace{t\overrightarrow{P_B Q_B} + P_B}_{\cap}$

$= [P_A, Q_A] + [P_B, Q_B]$

{9} $Y, Z \subseteq \mathbb{R}^4$

$$Y: \begin{cases} x_1 + x_3 - 2 = 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 = 0 \end{cases}$$

$$Z: \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 = 1 \\ x_2 + x_3 - 3x_4 = -1 \\ x_1 - x_2 + 3x_4 = 3 \end{cases}$$

1. dim $Y = ?$ dim $Z = ?$
2. parametric eq. of Y and Z
3. show that $Y \parallel Z$

2. $$Y: \begin{cases} x_1 = 2 - x_3 \\ x_2 = 2(2-x_3) + x_3 + 3x_4 - 1 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_3 + 2 \\ x_2 = -x_3 + 3x_4 + 3 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$$Z: \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & -1 & 0 & 3 & 3 \end{array}\right) \qquad \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} \quad \mathbb{R}^4 \to \mathbb{R}^3$$

dim $Z = 4 - \text{rank } A = 4 - 3 = 1$

$$Z: \begin{cases} x_1 = -x_2 - 2x_3 + 3x_4 + 1 \\ x_2 = x_2 \\ x_3 = -x_2 + 3x_4 - 1 \\ x_4 = \dfrac{3 - x_1 + x_2}{3} \end{cases}$$

$x_1 = -x_2 - 2(-x_2 + 3x_4 - 1) + 3x_4 + 1$

$x_1 = -x_2 + 2x_2 - 6x_4 + 2 + 3x_4 + 1$

$x_1 = x_2 - 3x_4 + 3$

$x_1 = x_2 - 3 + x_1 - x_2 + 3 \Rightarrow x_1 = x_1$

BUT

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + x_2 \Big($$

$\to$ det $Z = 0$

— continue next time —

 04. 04. 2018.

9. $Y, Z \subseteq \mathbb{R}^2$ affine spaces.

$$Y: \begin{cases} x_1+x_3-2=0 \\ 2x_1-x_2+x_3+3x_4-1=0 \end{cases} \qquad Z: \begin{cases} x_1+x_2+2x_3-3x_4=1 \\ x_2+x_3-3x_4=-1 \\ x_1-x_2+3x_4=3 \end{cases}$$

Parametric equations

- $Y = \left\{ \begin{pmatrix} -1-3t+s \\ s \\ 3+3t-s \\ t \end{pmatrix} \middle| \, t, s \in \mathbb{R} \right\}$

- $A = \left(\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & -1 & 0 & 3 & 3 \end{array} \right) \underset{r_1 - r_2}{\overset{r_3 - r_1,\ r_3 + 2r_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \begin{cases} x_1+x_3=2 \\ x_2+x_3-3x_4 \end{cases}$

$$\Rightarrow \begin{cases} x_3 = 2-x_1 \\ x_2 = -3+x_1+3x_4 \\ x_1 = s \\ x_4 = t \end{cases} \Rightarrow Z = \left\{ \begin{pmatrix} s \\ -3+s+3t \\ 2-s \\ t \end{pmatrix} \middle| \, t, s \in \mathbb{R} \right\}$$

$\Rightarrow \dim Y = \dim Z = 2$

$$Y \parallel Z \Leftrightarrow D(Y) \subseteq D(Z) \overset{\dim Y = \dim Z}{\Longleftrightarrow} D(Y) = D(Z)$$

or

$D(Z) \subseteq D(Y)$

$$D(Y) = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\rangle, \quad D(Z) = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

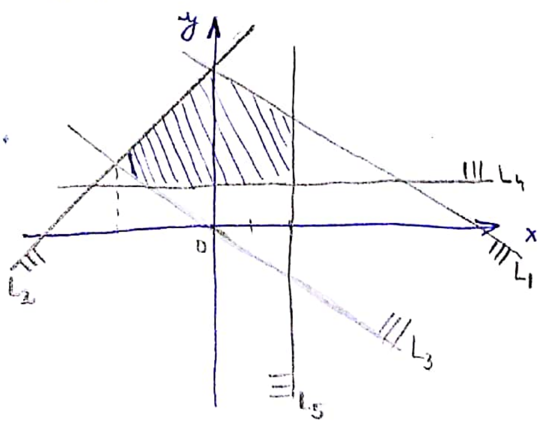
Let $S = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \Rightarrow \text{rank } S = 2$

$$T = \begin{pmatrix} -3 & 0 & 3 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \underset{r_1+3r_2}{\sim} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{pmatrix} \Rightarrow \text{rank } T = 2.$$

$\Rightarrow D(Y) = D(Z) \Rightarrow Y \parallel Z$

32.
$$\begin{aligned} 2x+3y &\le 12 \\ -2x+3y &\le 12 \\ -x-y &\le 0 \\ -y &\le -1 \\ x &\le 2 \end{aligned}$$

Use Fourier-Motzkin to eliminate the variables.



$2x+3y-12=0 \Leftrightarrow \frac{x}{6}+\frac{y}{4}=1$

$x=3, y=2$
$x=0, y=4$

$-2x+3y=12 \Leftrightarrow -\frac{x}{6}+\frac{y}{4}=1$

$x=0 \Rightarrow y=4$
$x=-6, \ y=0$

$P(-1,0) + 1-0 \le 0$ false

$y \le 4-\frac{2}{3}x$

$y \le 4+\frac{2}{3}x$

$y \ge -x$

$y \ge 1$

$$\max\{-x, 1\} \le y \le \min\left\{4-\tfrac{2}{3}x, 4+\tfrac{2}{3}x\right\}$$

$$\begin{cases} -x \le 4 - \frac{2}{3}x \\ -x \le 4 + \frac{2}{3}x \\ 1 \le 4 - \frac{2}{3}x \\ 1 \le 4 + \frac{2}{3}x \end{cases} \iff \begin{cases} -\frac{1}{3}x \le 4 \\ -\frac{5}{3}x \le 4 \\ \frac{2}{3}x \le 3 \\ -\frac{2}{3}x \le 3 \end{cases} \iff \begin{cases} x \ge -12 \\ x \ge -\frac{12}{5} \\ x \le \frac{9}{2} \\ x \ge -\frac{9}{2} \\ x \le 2 \end{cases}$$

$$x \in \left[-\frac{12}{5}, 2\right]$$

37 What are the Voronoi cells of $\{(n, m) : n, m \in \mathbb{N}\} \subseteq \mathbb{R}^2$