24.10.2017

COURSE 4

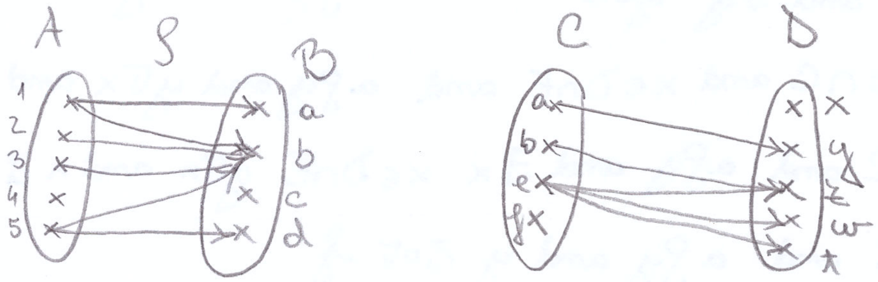
relation: $\rho = (A, B, R)$, where $R \subseteq A \times B$

A = dom ρ, B = codom ρ, R = graph ρ

composition: $\sigma = (C, D, S)$, where $S \subseteq C \times D$

$$\underbrace{\sigma}_{\text{the second}} \circ \underbrace{\rho}_{\text{the first}} = (\underbrace{A}_{\text{dom}\rho}, \underbrace{D}_{\text{codom}\sigma}, S \circ R)$$

where if $(a,d) \in A \times D$, then $\boxed{a\, \sigma \circ \rho\, d \overset{\text{def}}{\iff} \exists x \in B \cap C \text{ s.t. } a \rho x \text{ and } x \sigma d}$



Ex

$S \circ R = \{(1,y), (1,z), (2,z), (3,z), (5,z)\}$

Theorem 1) The identity relation is neutral element w.r.t. composition

$$\rho \circ 1_A = 1_B \circ \rho = \rho$$

2) The composition is associative $\rho = (A, B, R)$, $\sigma = (C, D, S)$, $\tau = (E, F, T)$ then $\tau \circ (\sigma \circ \rho) = (\tau \circ \sigma) \circ \rho$

Proof 1) $\rho \circ 1_A = (A, B, R \circ \Delta_A)$

$1_B \circ \rho = (A, B, \Delta_B \circ R)$

hence, all three relations have the same domain and codomain.

Let $(a,b) \in A \times B$ ... We have • $a\, \rho \circ 1_A\, b \iff \exists x \in A$ s.t. $a 1_A x$ and $x \rho b \iff \exists x \in A$ s.t. $a = x$ and $x \rho b$

$\iff a \rho b$

- $a\, 1_B \circ \rho\, b \iff \exists x \in B$ s.t. $a \rho x$ and $x\, 1_B\, b$
$\iff \exists x \in B$ s.t. $a \rho x$ and $x = b$
$\iff a \rho b$

2) $\tau \circ \sigma = (C, F, T \circ S)$

$\tau \circ (\sigma \circ \rho) = (A, F, T \circ (S \circ R))$

$(\tau \circ \sigma) \circ \rho = (A, F, (T \circ S) \circ R)$

hence both relations have domain A and codomain F

Let $(a, f) \in A \times F$

$a\, \tau \circ (\sigma \circ \rho)\, f \iff (\exists x)\, x \in D \cap E$ and $a\, \sigma \circ \rho\, x$ and $x \tau f \iff$

$\iff \exists x\; x \in D \cap E$ and $\exists y\; y \in B \cap C$ and $a \rho y$ and $y \sigma x$ and $x \tau f$

$\iff \exists y\, \exists x\; y \in B \cap C$ and $x \in D \cap E$ and $a \rho y$ and $y \sigma x$ and $x \tau f$

$\iff \exists y\; y \in B \cap C$ and $a \rho y$ and $\underbrace{\exists x\; x \in D \cap E\; y \sigma x \text{ and } x \tau f}$

$\iff \exists y\; y \in B \cap C$ and $a \rho y$ and $y\, \tau \circ \sigma\, f$

$\iff a (\tau \circ \sigma) \circ \rho\, f$

TAUTOLOGIES used above:

- $P \wedge Q \iff Q \wedge P$
- $\exists x (P \wedge Q(x)) \iff P \wedge \exists x Q(x)$
  if P does not depend on x

The section of a relation w.r.t. a subset (the image of a subset w.r.t. a relation)

Def Let $\rho = (A, B, R)$, let $X \subseteq A$

Then the section

$$\boxed{\rho(X) \stackrel{def}{=} \{ b \in B \mid \exists x \in X \text{ s.t. } x \rho b \}}$$

Example:

Let $X = \{3, 4, 5\}$

$\rho(X) = \{b, d\}$

$\rho(\{1\}) = \{a, b\}$

Part. case if $X=\{x\}$

then we denote $\rho\langle x\rangle=\rho(\{x\})=\{b\in B \mid x\rho b\}$

If $Y\subseteq B$

$\rho^{-1}(Y)=\overleftarrow{\rho}(Y)=\{a\in A \mid \exists y\in Y \text{ s.t. } y\rho^{-1}a \ (a\rho y)\}$

example $\rho^{-1}(\{a,b\})=\{1,2,3,5\}$

$\rho^{-1}(\{d\})=\rho^{-1}\langle d\rangle=\{5\}$

$\rho^{-1}\langle c\rangle=\emptyset$

$\rho^{-1}\langle b\rangle=\{1,2,3,5\}$

Theorem Let $\rho=(A,B,R)$ and $\sigma=(C,D,S)$ be relations, and let $X\subseteq A$. Then $(\sigma\circ\rho)(X)=\sigma(\rho(X)\cap C)$.

In particular, if $\rho(X)\subseteq C$ then we have $(\sigma\circ\rho)(X)=\sigma(\rho(X))$

Proof Both sets are subsets of D. So let $d\in D$. We have:

$d\in(\sigma\circ\rho)(X) \iff \exists x \ x\in X$ and $x\,\sigma\circ\rho\, d$

$\iff \exists x \ x\in X$ and $\exists y \ y\in B\cap C$ and $x\rho y$ and $y\sigma d$

$\iff \exists y \ y\in B\cap C$ and $\underbrace{\exists x \ x\in X \text{ and } x\rho y}$ and $y\sigma d$

$\iff \exists y \ \underbrace{y\in B\cap C \text{ and } y\in\rho(X) \text{ and } y\sigma d}_{\rho(X)\subseteq B}$

$\iff \exists y \ y\in\rho(X)\cap C$ and $y\sigma d$

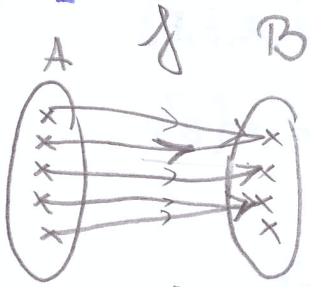
$\iff d\in\sigma(\rho(X)\cap C)$

Functions (functional relations)

Def The relation (A,B,F) is called a **function** if $\forall a \in A$

$$|f\langle a\rangle| = 1$$

Not $f\langle a\rangle = \{f(a)\}$



function

$f : A \to B$, $a \underset{\text{maps to}}{\mapsto} f(a)$

$$\begin{cases} A \xrightarrow{f} B \\ a \mapsto f(a) \end{cases}$$

Remark 1) The **inverse relation** $f^{-1} = (B, A, F^{-1})$ is not, in general a function

2) The **equality relation** $1_A = (A, A, \Delta_A)$ on A is a function, called the identical function of A

$$1_A(a) = a$$

3) The functions $f = (A,B,F)$ and $g = (C,D,G)$ are **equal** $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} A = C \\ B = D \\ F = G \end{cases} \Leftrightarrow \begin{cases} A = C \\ B = D \\ f(a) = g(a)\ \forall a \in A \end{cases}$$

$F = \{(a, f(a)) \mid a \in A\}$

4) **The composed relation**

$g \circ f = (A, D, G \circ F)$ is a function $\Leftrightarrow$ $f(B) = C$; $\boxed{(g \circ f)(a) = g(f(a))}$ part. case of the composition

5) **The identical function** is neutral element w.r.t. „$\circ$"

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[rr, bend right, "f \circ 1_A"'] & A \arrow[r, "f"] \arrow[rr, bend right, "1_B \circ f"'] & B \arrow[r, "1_B"] & B \end{tikzcd}$$

we have $1_A \circ f = f \circ 1_B = f$

6) **The composition of functions is associative**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend right=50, "h\circ(g\circ f)"'] \arrow[rrr, bend right=70, "(h\circ g)\circ f"'] & B \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

7) Let $A=\emptyset \Rightarrow A\times B=\emptyset$

$\emptyset=(\emptyset, B, \emptyset)$ is a function

8) Let $A\neq\emptyset$, $B=\emptyset \Rightarrow A\times B=\emptyset$

$\emptyset=(\underset{\neq\emptyset}{A}, \emptyset, \emptyset)$ <u>not</u> a function

9) **Image and counterimage**

Let $f: A\to B$

- If $X\subseteq A$, then $f(X)=\{f(x) \mid x\in X\}$
  i.e. $b\in f(X) \Leftrightarrow \exists x\in X$ s.t. $b=f(x)$

example if $X=\{1,2,3\}$
$f(X)=\{a,b\}$

- If $Y\subseteq B$, then $f^{-1}(Y)=\{a\in A \mid f(a)\in Y\}$

e.g. Let $Y=\{c,d\}$, $f^{-1}(Y)=\{4,5\}$
$f^{-1}(a)=\{1,2\}$ $\quad f^{-1}$ not a function

Commutative diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[ur, "h"'] & \end{tikzcd}$$

commutative $\Leftrightarrow f=h\circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

comm. $\Leftrightarrow k\circ f=h\circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[r] & D \arrow[u, "k"'] \end{tikzcd}$$ comm. $\iff f = k \circ h \circ g$

HW: $\xrightarrow{ex} 43$