CARDINAL NUMBERS

We explain what is the number of elements of a set

Def: The sets A and B are equipotent (not $A \sim B$) if $\exists$ a bijective function $f: A \to B$.
(have the same power)

Lemma: The equipotence relation "$\sim$" is an equivalence relation

proof: (R) $A \sim A$ because the identity map $1_A : A \to A$, $1_A(a) = a, \forall a \in A$ is bijective.

(T) Assume $A \sim B$ and $B \sim C$.
Then $\exists\ f: A \to B$ and $g: B \to C$ bijective functions.
We know that the map $g \circ f : A \to C$ is also bij., hence $A \sim C$.

(S) Assume that $A \sim B$. Then $\exists\ f: A \to B$ bijective. We know that $f^{-1}: B \to A$ is also bijective, hence $B \sim A$.

Def: The cardinal number of the set A is the equipotence class of A, i.e.
$|A| \overset{def}{=} \{B \mid A \sim B\}$

Remark: 1) This is the "naive" definition given by Georg Cantor ~ 1870

2) $A \sim B \iff |A| = |B|$ (have the same cardinality)

Operations with cardinal numbers:

1) Addition: Idea: if $A \cap B = \emptyset$, then $|A| + |B| \overset{def}{=} |A \cup B|$

In general, let $(\alpha_i)_{i \in I}$ be a family of cardinal numbers, where $\alpha_i = |A_i|$, so the set A is a representative for α_i. The problem is that the sets A_i, $i \in I$ may not be pairwise disjoint.

We replace the family $(A_i)_{i\in I}$ with another family $(A'_i)_{i \in I}$, where $A'_i \overset{def}{=} A_i \times \{i\} = \{(a_i, i) \mid a_i \in A_i\}$. Obviously $A_i \sim A'_i$ and $A'_i \cap A'_j = \emptyset$ if $i \neq j$

Def: $\sum_{i \in I} \alpha_i \overset{def}{=} \left| \bigcup_{i \in I} A'_i \right| = \left| \bigcup_{i \in I} A_i \times \{i\} \right|$

Remark: The definition does not depend on the choice of representatives

2) Multiplication: Idea: 3×5 

Let $(\alpha_i)_{i\in I}$ be a family of cardinal numbers

Def: $\prod_{i\in I} \alpha_i = \left| \prod_{i\in I} A_i \right|$, where $\prod_{i\in I} A_i := \{(a_i)_{i\in I} \mid a_i \in A_i \ \forall i \in I\}$ is the generalized cartesian product of the family $(A_i)_{i\in I}$ $(\alpha_i = |A_i|)$.

Remark: The definition does not depend on the choice of representatives

3) Exponentiation:

Let $\alpha = |A|$ and $\beta = |B|$

Def: $\beta^\alpha = |B^A| = |\mathrm{Hom}(A,B)|$ where $B^A = \mathrm{Hom}(A,B) = \{f \mid f: A \to B\}$ is the set of functions from A to B.

Remark: The definition does not depend on the choice of representatives.

Theorem (the properties of the operations):

1) Addition and multiplication are commutative
2) Addition and multiplication are associative
3) The multiplication is distributive w.r.t. addition

$$\left(\sum_{i\in I} \alpha_i\right) \cdot \left(\sum_{j\in J} \beta_j\right) = \sum_{(i,j)\in I\times J} \alpha_i \beta_j$$

4) $\beta^{\sum_{i\in I} \alpha_i} = \prod_{i\in I} \beta^{\alpha_i}$

5) $\left(\prod_{i\in I} \beta_i\right)^\alpha = \prod_{i\in I} \beta_i^\alpha$

6) $\gamma^{\alpha\beta} = (\gamma^\beta)^\alpha$

proof: 1), 2) (we skip them)

3) Let $\alpha_i = |A_i|$, $\beta_j = |B_j|$

$$\left(\sum \alpha_i\right)\left(\sum \beta_j\right) = \left| \bigcup_{i\in I} (A_i \times \{i\}) \times \bigcup_{j\in J} (B_j \times \{j\}) \right|$$

$$= \{((a_i, i), (b_j, j)) \mid a_i \in A_i, i \in I, b_j \in B_j, j \in J\}$$

$$\sum_{(i,j)\in I\times J} \alpha_i \beta_j = \left| \bigcup_{(i,j)\in I\times J} (A_i \times B_j) \times \{(i,j)\} \right|$$

$= \{((a_i, b_j), (i,j)) \mid a_i \in A_i,\ b_j \in B_j,\ (i,j) \in I \times J\}$

We need a bijective function

$$\bigcup_{i\in I} (A_i \times \{i\}) \times \bigcup_{j \in J} (B_j \times \{j\}) \xrightarrow{\varphi} \bigcup_{(i,j)\in I\times J} (A_i \times B_j) \times \{(i,j)\}$$

Obviously $((a_i, i), (b_j, j)) \xrightarrow{\varphi} ((a_i, b_j), (i,j))$ is a bijection, because it has an obvious inverse.

proof: 4), 5) - use the "universal property of the direct sum and direct product"

6) Let $\alpha = |A|$, $\beta = |B|$, $\gamma = |C|$

Then $\gamma^{\alpha\beta} = |\mathrm{Hom}(A \times B, C)|$

$(\gamma^{\beta})^{\alpha} = |\mathrm{Hom}(A, \mathrm{Hom}(B, C))|$

In order to prove the equality, we must find bijective functions:

$$\mathrm{Hom}(A\times B, C) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(A, \mathrm{Hom}(B,C)) \quad \text{s.t. } \psi = \varphi^{-1}$$

f

- Let $f : A \times B \to C$. We want to define $\varphi(f) : A \to \mathrm{Hom}(B,C)$,

  so $\varphi(f)(a) : B \to C$

  so $\varphi(f)(a)(b) \in C$

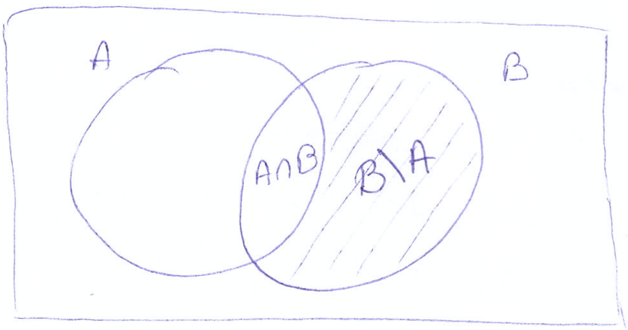
We define $\boxed{\varphi(f)(a)(b) \overset{def}{=} f(a,b)}$

- Let $g : A \to \mathrm{Hom}(B,C)$, so $g(a) : B \to C$, so $g(a)(b) \in C$, $\forall a \in A, b \in B$

We want to define $\psi(g) : A \times B \to C$: so let $\boxed{\psi(g)(a,b) \overset{def}{=} g(a)(b).}$

The above definitions show that $\varphi(f) = g \Leftrightarrow \psi(g) = f$, hence $\psi = \varphi^{-1}$, so φ is bijective.

Prop: $|A| + |B| = |A \cup B| + |A \cap B|$



$A \cup B = A \cup (B \setminus A)$ (disjoint)

hence $|A \cup B| = |A| + |B \setminus A|$

$\Rightarrow |A\cup B| + |A \cap B| = |A| + \underbrace{|B\setminus A| + |A \cap B|}_{(*)}$

But we have $B = (B \setminus A) \cup (A \cap B)$, (disjoint union)

hence $|B| = |B \setminus A| + |A \cap B|$

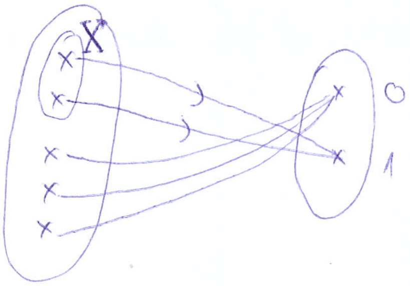
$(*) = |A| + |B|$

Theorem (Cantor): $|\mathcal{P}(A)| = 2^{|A|}$

Proof: $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

and $2^{|A|} = |\mathrm{Hom}(A, \{0,1\})|$ so we must find bijective functions, φ, ψ where $\psi = \varphi^{-1}$

$$\mathcal{P}(A) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(A, \{0,1\})$$



Let $\begin{cases} \chi_X : A \to \{0,1\} \\ \chi_X(a) \overset{def}{=} \begin{cases} 1, & \text{if } a \in X \\ 0, & \text{if } a \notin X \end{cases} \end{cases}$ be the characteristic function of the subset X of A

Let $\varphi(X) \overset{def}{=} \chi_X \in \mathrm{Hom}(A, \{0,1\})$

Let $\psi(\chi) \overset{def}{=} \chi^{-1}(1) = \{a \in A \mid \chi(a) = 1\}$ for any $\chi : A \to \{0,1\}$.

We have $(\psi \circ \varphi)(X) = \psi(\varphi(X)) = \chi_X^{-1}(1) = X = 1(X)$;

$(\varphi \circ \psi)(\chi) = \varphi(\psi(\chi)) = \varphi(\chi^{-1}(1)) = \chi_{\chi^{-1}(1)} = \chi = 1(\chi)$

Ordering cardinal numbers

Def: Let $\alpha = |A|$, $\beta = |B|$

Then $\alpha \le \beta \overset{def}{\iff} \exists f : A \to B$ an injective function

Rem: the definition does not depend on the choice of representatives

Lemma: The relation "$\le$" is an order relation

proof: (R) $\alpha \le \alpha$ because $1_A : A \to A$ is injective

(T) Assume $\alpha \le \beta$ and $\beta \le \gamma$. Let $f : A \to B$ and $g : B \to C$ inj.
We know that $g \circ f : A \to C$ is also injective, hence $\alpha \le \gamma$

(A) Assume $\alpha \le \beta$ and $\beta \le \alpha$. Let $f : A \to B$ and $g : B \to A$ be injective functions

We must show that $\exists\, h : A \to B$ bijective (not easy!). It is a consequence of the Cantor-Bernstein-Schröder theorem.

Theorem (Cantor): $\alpha < 2^{\alpha}$

proof: Let $\alpha = |A|$, so $2^{\alpha} = |\mathcal{P}(A)|$

We have an injective function $f: A \to \mathcal{P}(A)$, $f(a) = \{a\}$, hence $|A| \leq |\mathcal{P}(A)|$

Assume, by contradiction, that $\exists$ a bijective function $\varphi: A \to \mathcal{P}(A)$

Let $X = \{a \in A \mid a \notin \varphi(a)\} \in \mathcal{P}(A)$

By assumption, $\exists\, x \in A$ s.t. $\varphi(x) = X$

Case 1: Assume $x \in X$. Then $x \in \varphi(x) \Rightarrow x$ does not satisfy the condition in the def of $X \Rightarrow x \notin X$ contradiction

Case 2: Assume $x \notin X$. $\Rightarrow x \notin \varphi(x) \Rightarrow x$ satisfies the condition in the def of $X \Rightarrow x \in X$ contradiction

Hence, we do not have bijective functions $A \to \mathcal{P}(A)$.