COURSE 2

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Propositional logic

- formulas (syntax)
- interpretations (we assign truth values to formulas) (semantics)
- decision problem (to decide whether a formula is a tautology, contradictiction, or is satisfiable)

Methods to solve it:

- truth tables
- normal forms
- formal deduction

Completeness theorem (Frege & Lukasiewicz)

$A_1, \dots, A_m \models B \iff A_1, \dots, A_m \vdash B$ (B is formally deducible from formulas: $A_1, \dots, A_m$)

(B is a consequence of the formula $A_1, \dots, A_m$)

i.e. $A_1 \wedge \dots \wedge A_m \to B$ is a tautology

FIRST ORDER (predicate) LOGIC

- In addition to what we have in propositional logic, we formalise other expressions from natural language, by introducing variables and quantifiers $\exists$, $\forall$

existential universal

- First order – we quantize only variables $\exists x, \forall x$ (elements of set)

- Second order – we also quantize sets $\exists M$ $\forall M$

⋮

Naively: a predicate is more than a proposition, it is a kind of "open" sentence, i.e. depending on variables:

e.x. $x+y=1$

such that if we give values to the variables, then we get a sentence

Def: A first order language $\mathcal{L}$ consists of the following data:

a) Symbols

1) parantheses (,)
2) connectives : $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$
3) quantifiers $\forall$, $\exists$ (for all, exists)
4) the symbol of equality $=$
5) constants : $a, b, \dots, a_1, a_2, \dots$
6) variables : $x, y, \dots, x_1, x_2, \dots$
7) function symbols : $f, g, \dots, f_1, f_2, \dots$
8) predicate symbols $P, Q, \dots, P_1, P_2, \dots$

We are also given a natural number $n \in \mathbb{N}$ called arity (number of variables)

b) Terms (expressions) (these are defined recursively/inductively)

1) constants are terms
2) variable are terms

} atomic expressions

3) if f is an n-ary function symbol and $t_1, \dots, t_n$ are terms then $f(t_1, \dots, t_n)$ is a term
4) no other sequence of symbols is an expression

c) Formulas

atomic formulas:

1) If P is an n-ary predicate symbol, and $t_1, \dots, t_n$ are terms, then $P(t_1, \dots, t_n)$ is a formula
2) If t_1, t_2 are terms, then $t_1 = t_2$ is a formula

3) If A, B are formulas, then $\neg A, (A \wedge B), (A \vee B), (A \to B)$, $(A \leftrightarrow B)$ are also formulas

4) If A is a formula which depends on the variable x, then $\exists x, \forall x A$ are also formulas

5) no other sequence of symbols is a formula

$\boxed{\text{Remark}}$ In the formula $\exists x\, A(x,y)$ we say that y is a free variable, x is a linked (bound) variable $\exists x \forall y\, A(x,y)$ - it has no free variables, it is called a closed formula.

! We will see that the closed formulas can be regarded as sentences (they will always have truth values)

Example:

$x + y = 1$

var. eq. const.

$x+y$ is an expression

$\left.\begin{array}{l} \forall x \exists y\, (x+y=1) \quad \text{true} \\ \exists y \forall x\, (x+y=1) \quad \text{false} \end{array}\right\}$ formulas (closed)

Consider above the set of real numbers $\mathbb{R}$

The structure of a formula. Interpretation

Def. A structure of a first order language consists of the following data:

1) a set M
2) to each constant a we associate a fixed element $\bar{a} \in M$
3) to each function symbol f (n-ary) we associate a function $\bar{f} : M^n \to M$

$$M^n = M \times M \times \ldots \times M = \{(x_1, \ldots, x_n) \mid x_i \in M\}$$

4) To each predicate symbol P (n-ary) we associate a subset $\bar{P} \subseteq M^n$
5) To the equality symbol $=$ we associate the equality relation on M, i.e. the subset $\{(x,x) \mid x \in M\} \subseteq M^2$

We assign values from the set M to the expressions, and truth values to the formulas. For this, we need:

Def: An interpretation of the first order language $\mathcal{L}$ corresponding to a given structure M is a function $s: V \to M$.

the set of variables

Def We assign values to terms:

1) the value of the constant a is $\tilde{a} \in M$
2) the value of the variable x is $s(x) \in M$
3) If the values $\tilde{t}_1, \dots, \tilde{t}_n \in M$ of the terms $t_1, \dots, t_n$ are given, then the value of $f(t_1, \dots, t_n)$ is $\tilde{f}(\tilde{t}_1, \dots, \tilde{t}_n) \in M$

Def We assign truth values to formulas:

1) the value $P(t_1, \dots, t_m)$ is $1 \Leftrightarrow (\tilde{t}_1, \dots, \tilde{t}_m) \in \tilde{P} \subseteq M^m$
2) the truth value of the formula $t_1 = t_2$ is $1 \Leftrightarrow$ $\Leftrightarrow \tilde{t}_1 = \tilde{t}_2$ in M
3) the truth values of $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$ are the same as in the propositional logic
4) The value of $\exists x A(x)$ is $1 \Leftrightarrow$ there exists an element $x \in M$ such that $A(x)$ is true.

The value of $\forall x A(x)$ is $1 \Leftrightarrow$ for any element $x \in M$ the value of $A(x)$ is 1

Def - a formula A is a tautology if it is true for any structure and interpretation

- a formula A is a contradiction if it is false for any struct. and interpretation

— || — satisfiable

Example: $(\forall x \forall y A(x,y) \Leftrightarrow \forall y \forall x A(x,y))$ is a tautology

$\exists x \forall y A(x,y) \Rightarrow \exists y \forall x A(x,y)$ taut.

$\Leftarrow$ is not true in general

Homework
$\to$ ex. 18