Continuous Functions

Exercise 1: By using the definition, prove that the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$$

is continous.

Exercise 2:
a) By using the characterization theorem with ε and δ prove that the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ -x & : x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases}$$

is continuous at 0.

b)By using the defintion, prove that the function f does not have other continuity points except for 0, therefore $\forall x \in \mathbb{R}\backslash\{0\}$, f is discontinous at x.

Exercise 3: Study the continuity of the functions:

a) $f : (-\infty, 0] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in (-\infty, 0) \\ 7 & : x = 0, \end{cases}$$

b)$f : [-1, 2] \cup \{4\} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x + 3 & : x \in [-1, 2] \\ 0 & : x = 4. \end{cases}$$

Exercise 4: Study the continuity of the functions:

a)$f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{\sin x^2}{|x|} & : x \neq 0 \\ 0 & : x = 0. \end{cases} .$$

b)$f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{x^{-1}} & : x \in (0, \infty) \\ 0 & : x = 0, \\ x^2 + 2x + \sin x & : x \in (-\infty, 0). \end{cases} .$$

c)$f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x & : x \in \mathbb{Q} \\ \cos x & : x \in \mathbb{R}\backslash\mathbb{Q}, \end{cases} .$$

d) $f : [-2,1] \cup \{3\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \cos(\pi x) & : x \in [-2,0] \\ 1 + \sin x & : x = (0,1] \\ 2 & : x = 3. \end{cases}$$

Exercise 5: Depending on the value of the real parameter $a \in \mathbb{R}$, discuss the continuity of the following funcions:

a) $f : [1,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt{a^2 - 2ax + x^2} & : x \in [1,2] \\ \\ 3a + 2x & : x \in (2,3]. \end{cases}$$

b) $f : (0,\pi) \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{3x} & : x \in (0,1] \\ \\ a\dfrac{\sin(x-1)}{x^2 - 5x + 4} & : x \in (1,\pi). \end{cases}$$

Exercise6: Let $0 < a < b \in \mathbb{R}$ and $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$, defined by:

$$f(x) = \left(\frac{b^x - a^x}{x(b-a)}\right)^{\frac{1}{x-1}}, \forall x \in \mathbb{R}\backslash\{0,1\}.$$

- a) Prove that the function f is contnuous.
- b) Prove that there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = f(x), \forall x \in \mathbb{R}\backslash\{0,1\}$.
- c) Compute $\lim\limits_{x\to-\infty} F(x)$ şi $\lim\limits_{x\to\infty} F(x)$.

Theoretical aspects

Hypotheses regarding continuous functions

$$\begin{cases} \emptyset \neq D \subseteq \mathbb{R} \\ f : D \to \mathbb{R} \\ x_0 \in D. \end{cases}$$

Definition:

The function f is **continuous** at x_0 if

$$\forall (x_n) \subseteq D \quad \textit{with} \quad \lim_{n\to\infty} x_n = x_0 \Longrightarrow \lim_{n\to\infty} f(x_n) = x_0.$$

The characterization theorem with neighborhoods:

f este continuă în x_0 dacă şi numai dacă

$$\forall V \in \mathcal{V}(f(x_0)), \exists U \in \mathcal{V}(x_0) \quad astfel \quad incat \quad f(x) \in V,$$

The characterization theorem with ε and δ:

f is continuous at x_0 if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \quad such \quad that \quad \forall x \in D \quad cu \quad |x - x_0|, \quad itholds \quad |f(x) - f(x_0)| < \varepsilon.$$

The theorem regarding the connection between continuous functions and limits of functions:

If $x_0 \in D \cap D' = D \backslash IzoD$, then the following statements are true:

1. f is continous at $x_0 \Longrightarrow \exists \lim\limits_{x \to x_0} f(x) = f(x_0 - 0) = f(x_0 + 0) = f(x_0)$.

2. If $\begin{cases} \exists f(x_0 - 0) \\ \exists f(x_0 + 0) \\ f(x_0 - 0) = f(x_0 + 0) = f(x_0) \end{cases} \Longrightarrow f$ is continuous x_0.

Remark: It is easy to prove, by using the definition, that all the elementary functions are continuous of their maximum definition domain. Therefore, when you are not explicitly asked to prove that a certain function is continuous, you may simply write the explanation above.