LECTURE

5

LIMITS, CONTINUITY AND DIFFERENTIATION OF REAL FUNCTIONS OF ONE VARIABLE

Limits of functions

Definition 5.1 *Let $A \subseteq \mathbb{R}$ be a set. An element $c \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be a* cluster point *(or* accumulation point*) of A if*

$$\forall V \in \mathcal{V}(c), \quad V \cap (A \setminus \{c\}) \neq \emptyset.$$

The set of all accumulation points of A is denoted by

$$A' := \{c \in \overline{\mathbb{R}} \mid \forall V \in \mathcal{V}(c),\ V \cap A \setminus \{c\} \neq \emptyset\}$$

and it is called the derived set *of A. The elements of*

$$A \setminus A' = \{a \in A \mid \exists V \in \mathcal{V}(a) \ \ s.t.\ \ V \cap A = \{a\}\}$$

are called isolated points *of A.*

Theorem 5.2 (Sequential characterization of accumulation points) *Let $A \subseteq \mathbb{R}$ be a set. For any $c \in \overline{\mathbb{R}}$ the following assertions are equivalent:*

1° *c is an accumulation point of A, i.e., $c \in A'$.*

2° *there exists a sequence (x_n) in $A \setminus \{c\}$ such that $\lim\limits_{n\to\infty} x_n = c$.*

Proof. $2^\circ \Rightarrow 1^\circ$: Assume that (x_n) is a sequence in $A \setminus \{c\}$ such that $\lim\limits_{n\to\infty} x_n = c$. Then, for any $V \in \mathcal{V}(a)$, $\exists n_V \in \mathbb{N}$ such that $x_n \in V$ for all $n \in \mathbb{N}$, $n \geq n_V$. In particular, we have $x_{n_V} \in V \cap (A \setminus \{c\})$, hence $V \cap (A \setminus \{c\}) \neq \emptyset$. It follows that $c \in A'$.

$1^\circ \Rightarrow 2^\circ$: Assume that $c \in A'$. We distinguish three cases.

Case 1: $c \in \mathbb{R}$. In this case, for every $n \in \mathbb{N}$ we have $\left(c - \frac{1}{n}, c + \frac{1}{n}\right) \in \mathcal{V}(c)$, hence there exists $x_n \in \left(c - \frac{1}{n}, c + \frac{1}{n}\right) \cap (A \setminus \{c\})$. The sequence (x_n) converges to c, since $|x_n - c| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Case 2: $c = -\infty$. In this case, for every $n \in \mathbb{N}$ we have $(-\infty, -n) \in \mathcal{V}(c)$, hence there exists $x_n \in (-\infty, -n) \cap (A \setminus \{c\}) = (-\infty, -n) \cap A$. Since $x_n < -n$ for all $n \in \mathbb{N}$, we infer that $\lim\limits_{n\to\infty} x_n = -\infty = c$.

Case 3: $c = +\infty$. For every $n \in \mathbb{N}$ we have $(n, \infty) \in \mathcal{V}(c)$, hence there exists $x_n \in (n, \infty) \cap (A \setminus \{c\})$. Since $x_n > n$ for all $n \in \mathbb{N}$, we infer that $\lim\limits_{n\to\infty} x_n = +\infty = c$. $\square$

Example 5.3 *a) Let $A = \{0, 1\}$. We have $A' = \emptyset$, hence 0 and 1 are isolated points of A. Notice that finite sets have no accumulation points!*
b) Let $A = (0, 1)$. Then $A' = [0, 1]$, hence A has no isolated points. Notice that 0 and 1 are accumulation points of A, but do not belong to A.
c) Let $A = (0, 1) \cup (1, 2)$, In this case we have $A' = [0, 2]$. The set A has no isolated points.
d) Let $A = \mathbb{N}$. Then $A' = \{+\infty\}$ and all points of A are isolated.
e) Let $A = [0, 1] \cap \mathbb{Q}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, it follows that $A' = [0, 1]$. The set A has no isolated points.
f) Let $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. In this case, $A' = \{0\}$ and all points of A are isolated.

Definition 5.4 *Let $f : A \to \mathbb{R}$ be a function, defined on a nonempty set $A \subseteq \mathbb{R}$, and let $c \in A'$. We say that f* has a limit at c *if there exists $\ell \in \overline{\mathbb{R}}$ such that*

$$\forall V \in \mathcal{V}(\ell),\ \exists U \in \mathcal{V}(c) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap (A \setminus \{c\}). \tag{5.1}$$

Remark 5.5 *If f has a limit at c, then there exists a unique $\ell \in \overline{\mathbb{R}}$ satisfying (5.1). In this case ℓ is called the* limit of f at c *and we write*

$$\lim_{x \to c} f(x) = \ell$$

or

$$f(x) \to \ell \text{ as } x \to c,$$

which reads as follows: "f has the limit ℓ at c" or "$f(x)$ approaches L as x approaches c*".*

Proof (of the uniqueness of ℓ): Suppose by the contrary that there exist $\ell_1, \ell_2 \in \overline{\mathbb{R}}$, $\ell_1 \neq \ell_2$ such that (5.1) holds for $\ell = \ell_1$ as well as for $\ell = \ell_2$. Choose $V_1 \in \mathcal{V}(\ell_1)$ and $V_2 \in \mathcal{V}(\ell_2)$ such that $V_1 \cap V_2 = \emptyset$. Then, on the one hand, there exists $U_1 \in \mathcal{V}(c)$ such that $f(x) \in V_1$ for all $x \in U_1 \cap (A \setminus \{c\})$. On the other hand, there exists $U_2 \in \mathcal{V}(c)$ such that $f(x) \in V_2$ for all $x \in U_2 \cap (A \setminus \{c\})$. Since $U_1 \cap U_2 \in \mathcal{V}(c)$ and $c \in A'$, we can find some $x_0 \in (U_1 \cap U_2) \cap (A \setminus \{c\})$. This yields $f(x_0) \in V_1 \cap V_2$, a contradiction. □

Remark 5.6 *f has no limit at c if $\forall \ell \in \overline{\mathbb{R}}$, $\exists V \in \mathcal{V}(\ell)$, $\forall U \in \mathcal{V}(c)$, $\exists x \in U \cap (A \setminus \{c\})$, $f(x) \notin V$.*

Theorem 5.7 (Characterization of limits in terms of ε-δ) *Let $f : A \to \mathbb{R}$ be a function defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in A'$.*

1° *If $c \in \mathbb{R}$, then*
   (i) $\lim\limits_{x \to c} f(x) = \ell \in \mathbb{R} \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ 0 < |x - c| < \delta,\ |f(x) - \ell| < \varepsilon.$
   (ii) $\lim\limits_{x \to c} f(x) = -\infty \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ 0 < |x - c| < \delta,\ f(x) < -\varepsilon.$
   (iii) $\lim\limits_{x \to c} f(x) = +\infty \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ 0 < |x - c| < \delta,\ f(x) > \varepsilon.$

2° *If $c = -\infty$, then*
   (i) $\lim\limits_{x \to -\infty} f(x) = \ell \in \mathbb{R} \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ x < -\delta,\ |f(x) - \ell| < \varepsilon.$
   (ii) $\lim\limits_{x \to -\infty} f(x) = -\infty \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ x < -\delta,\ f(x) < -\varepsilon.$
   (iii) $\lim\limits_{x \to -\infty} f(x) = +\infty \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ x < -\delta,\ f(x) > \varepsilon.$

3° *If $c = \infty$, then*
   (i) $\lim\limits_{x \to \infty} f(x) = \ell \in \mathbb{R} \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ x > \delta,\ |f(x) - \ell| < \varepsilon.$
   (ii) $\lim\limits_{x \to \infty} f(x) = -\infty \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ x > \delta,\ f(x) < -\varepsilon.$
   (iii) $\lim\limits_{x \to \infty} f(x) = +\infty \Leftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ x > \delta,\ f(x) > \varepsilon.$

Theorem 5.8 (Heine's sequential characterization of limits) *Let $f : A \to \mathbb{R}$ be a function defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in A'$. For any $\ell \in \overline{\mathbb{R}}$ the following assertions are equivalent:*

1° $\lim\limits_{x\to c} f(x) = \ell$.

2° *for every sequence (x_n) in $A \setminus \{c\}$ with $\lim\limits_{n\to\infty} x_n = c$ we have $\lim\limits_{n\to\infty} f(x_n) = \ell$.*

Example 5.9 *Function $f : \mathbb{R}^* \to \mathbb{R}$, defined by $f(x) := \sin \dfrac{1}{x}$, has no limit at 0. Indeed, there exist two sequences (x_n) and (x'_n) in $\mathbb{R}^*$, $x_n := \dfrac{1}{2\pi n}$ and $x'_n := \dfrac{1}{2\pi n + \frac{\pi}{2}}$, such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x'_n = 0$, but $\lim\limits_{n\to\infty} f(x_n) = 0 \neq \lim\limits_{n\to\infty} f(x'_n) = 1$.*

Remark 5.10 *Function $f : A \to \mathbb{R}$ has a limit at $c \in A'$ if and only if for every sequence (x_n) in $A \setminus \{c\}$ with $\lim\limits_{n\to\infty} x_n = c$ the sequence $(f(x_n))$ has a limit in $\overline{\mathbb{R}}$.*

Example 5.11 *The "sign" function $sgn : \mathbb{R} \to \mathbb{R}$, defined by $sgn(x) := \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0 \end{cases}$ has no limit at $c := 0$. Indeed, defining the sequence (x_n) in $\mathbb{R} \setminus \{0\}$ by $x_n := \dfrac{(-1)^n}{n}$, we have $\lim\limits_{n\to\infty} x_n = 0$, but the sequence $(sgn(x_n))$, i.e., $((-1)^n)$, has no limit.*

By means of Heine's Theorem 5.8 we can derive several important results for limits of functions from corresponding ones known for limits of sequences. To illustrate this, we present a few results.

Theorem 5.12 *Let $f, g : A \to \mathbb{R}$ be two functions defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in A'$. Suppose that there is $U \in \mathcal{V}(c)$ such that $f(x) \leq g(x)$, $\forall x \in U \cap (A \setminus \{c\})$.*

(i) If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist, then $\lim\limits_{x\to c} f(x) \leq \lim\limits_{x\to c} g(x)$.

(ii) If $\lim\limits_{x\to c} f(x) = +\infty$, then $\lim\limits_{x\to c} g(x) = +\infty$.

(iii) If $\lim\limits_{x\to c} g(x) = -\infty$, then $\lim\limits_{x\to c} f(x) = -\infty$.

Remark 5.13 *Strict inequalities are not preserved under the limiting process. For instance, we have $\dfrac{x+1}{x} > 1, \forall x > 0$, but $\lim\limits_{x\to\infty} \dfrac{x+1}{x} = 1$.*

Theorem 5.14 (Squeeze Theorem for functions) *Let $f, g, h : A \to \mathbb{R}$ be three functions defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in A'$. If there is $U \in \mathcal{V}(c)$ such that $f(x) \leq g(x) \leq h(x)$, $\forall\, x \in U \cap (A \setminus \{c\})$ and $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} h(x) = \ell \in \mathbb{R}$, then $\lim_{x\to c} g(x) = \ell$.*

Proof. Let (x_n) be a sequence in $A \setminus \{c\}$ such that $\lim\limits_{n\to\infty} x_n = c$. Thus, $\exists n_0 \in \mathbb{N}$ such that $f(x_n) \leq g(x_n) \leq h(x_n)$, $\forall\, n \geq n_0$. On the other hand, by Theorem 5.8, $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} h(x_n) = \ell$. Applying the Squeeze Theorem for sequences, it follows that $\lim\limits_{n\to\infty} g(x_n) = \ell$. Using again Theorem 5.8, we obtain that $\lim\limits_{x\to c} g(x) = \ell$. $\square$

Theorem 5.15 *Let $f : A \to \mathbb{R}$ be a function defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in A'$. For any $\ell \in \mathbb{R}$ we have $\lim\limits_{x\to c} f(x) = \ell \Longleftrightarrow \lim\limits_{x\to c} |f(x) - \ell| = 0$.*

Definition 5.16 *Let $f : A \to \mathbb{R}$ be a function defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$.*

If c is an accumulation point of $A \cap (-\infty, c)$ and the restriction $f|_{A\cap(-\infty,c)}$ has a limit at c, then we call this limit the left-hand limit of f at c and we denote it by

$$\lim_{\substack{x\to c\\ x<c}} f(x) \quad \text{or} \quad \lim_{x\to c^-} f(x) \quad \text{or} \quad \lim_{x\nearrow c} f(x).$$

Similarly, if c is an accumulation point of $A \cap (c, \infty)$ and $f|_{A\cap(,\infty)}$ has a limit at c, then we call this limit the right-hand limit of f at c and we denote it by

$$\lim_{\substack{x\to c\\ x>c}} f(x) \quad or \quad \lim_{x\to c^+} f(x) \quad or \quad \lim_{x\searrow c} f(x).$$

These two limits are called one-sided limits of f at c.

Theorem 5.17 (Characterization of limits using one-sided limits) *Let $f : A \to \mathbb{R}$ be a function defined on a nonempty set $A \subseteq \mathbb{R}$, let $\ell \in \overline{\mathbb{R}}$ and let $c \in \mathbb{R}$ be an accumulation point of both the sets $A \cap (-\infty, c)$ and $A \cap (c, +\infty)$. Then*

$$\lim_{x\to c} f(x) = \ell \Longleftrightarrow \lim_{\substack{x\to c\\ x<c}} f(x) = \lim_{\substack{x\to c\\ x>c}} f(x) = \ell.$$

Remark 5.18 *When c is an accumulation point of both the sets $A \cap (-\infty, c)$ and $A \cap (c, +\infty)$, the usual limit $\lim\limits_{x\to c} f(x)$ is also called the* two-sided limit *of f at c.*

Continuous functions

Definition 5.19 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. We say that f is continuous at c if*

$$\forall V \in \mathcal{V}\left(f(c)\right), \exists U \in \mathcal{V}(c) \text{ such that } \forall x \in U \cap A \text{ we have } f(x) \in V.$$

In this case we call c a continuity point of f.

If f fails to be continuous at c, then we say that f is discontinuous at c and that c is a discontinuity point of f.

If B is a subset of A, we say that f is continuous on B if it is continuous at every point of B. In particular, if f is continuous on A, we simply say that f is continuous.

Remark 5.20 *(i) An important difference between the notions of limit and continuity is that the point c is now assumed to belong to A (but not necessarily to A') so that $f(c)$ makes sense.*
(ii) If $c \in A \cap A'$, then f is continuous at c if and only if $\lim\limits_{x\to c} f(x) = f(c)$.
(iii) If c is an isolated point of A, then $\exists U \in \mathcal{V}(c)$ such that $U \cap A = \{c\}$. Thus, f is continuous at c.

Theorem 5.21 (Characterizations of continuity) *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A$. The following assertions are equivalent:*

1° *f is continuous at c.*
1° *$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in A, |x - c| < \delta, |f(x) - f(c)| < \varepsilon$.*
1° *For every sequence (x_n) in A with $\lim\limits_{n\to\infty} x_n = c$ we have $\lim\limits_{n\to\infty} f(x_n) = f(c)$.*

Remark 5.22 *Sums, products, quotients and compositions of continuous functions (when defined) are continuous.*

Definition 5.23 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, let $c \in A$ be a discontinuity point of f. We say that c is a discontinuity point of the first kind of f (or that f has a discontinuity of the first kind at c) if the one-sided limits of f at c both exist and are finite. The discontinuities that are not of the first kind are called discontinuities of the second kind.*

Remark 5.24 *If c is a discontinuity of the first kind, then it is either a jump discontinuity when the one-sided limits are distinct or a removable discontinuity if the one-sided limits coincide (but they are not equal to $f(c)$).*

Example 5.25 *(i) The function* $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

has a discontinuity of the second kind at 0.
(ii) The function $sgn(x)$ *has a jump discontinuity at* 0.
(iii) The function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0 \end{cases}$$

has a removable discontinuity at 0.

Definition 5.26 *Let* $A \subseteq \mathbb{R}$. *A function* $f : A \to \mathbb{R}$ *is said to be bounded on* A *if the set* $f(A)$ *is bounded, i.e., there exists* $M > 0$ *such that* $\forall x \in A$, $|f(x)| \leq M$. *We say that* f *attains its maximum if there exists* $\overline{x} \in A$ *such that* $\forall x \in A$, $f(x) \leq f(\overline{x})$. *Likewise, we say that* f *attains its minimum if there exists* $\underline{x} \in A$ *such that* $\forall x \in A$, $f(\underline{x}) \leq f(x)$. *In this case* $\overline{x}$ *is called a maximum point for* f *and* $\underline{x}$ *is called a minimum point for* f.

Theorem 5.27 (Weierstrass' theorem on extrema of continuous functions) *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$. *Then* f *is bounded and it attains both its maximum and minimum on* $[a, b]$.

Remark 5.28 *(i) The function* f *can be unbounded if*

- *the interval is unbounded:* $f : [0, +\infty) \to \mathbb{R}$, $f(x) = x$.
- *the interval is not closed:* $f : (0, 1] \to \mathbb{R}$, $f(x) = 1/x$.
- f *is not continuous:* $f : [0, 1] \to \mathbb{R}$, $f(x) = \begin{cases} 1/x, & \text{if } x \in (0, 1], \\ 0, & \text{if } x = 0. \end{cases}$

(ii) A maximum (minimum) point is not necessarily unique.

Theorem 5.29 (Intermediate Value Theorem, Bolzano-Darboux) *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function, where* $a, b \in \mathbb{R}$, $a < b$, *and let* $v \in \mathbb{R}$. *If* $\min\{f(a), f(b)\} < v < \max\{f(a), f(b)\}$, *then there exists (at least one)* $c \in (a, b)$ *such that* $f(c) = y$.

Remark 5.30 *(i) Location of Roots (Bolzano Theorem): If* $f : [a, b] \to \mathbb{R}$ *is continuous and* $f(a) \cdot f(b) < 0$, *then* $\exists c \in (a, b)$ *such that* $f(c) = 0$.
(ii) If I *is an interval and* $f : I \to \mathbb{R}$ *is continuous, then for every interval* $J \subseteq I$ *the set* $f(J)$ *is an interval (possibly degenerated into a singleton).*
(iii) If $f : [a, b] \to \mathbb{R}$ *is continuous, then* $f([a, b])$ *is a compact interval (possibly degenerated into a singleton).*

Differentiation of functions

Definition 5.31 *Let* $A \subseteq \mathbb{R}$ *and* $c \in A \cap A'$. *A function* $f : A \to \mathbb{R}$ *has a derivative at* c *if the limit*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (in $\overline{\mathbb{R}}$*). In this case, the above limit is called the derivative of* f *at* c *and is denoted by*

$$f'(c).$$

If f *has a finite derivative at* c, *then* f *is said to be differentiable at* c.

If B *is a subset of* A, *we say that* f *is differentiable on* B *if it is differentiable at every point of* B. *In this case, the function* $f' : B \to \mathbb{R}$, $x \in B \mapsto f'(x) \in \mathbb{R}$ *is called the derivative of* f *on* B. *In particular, if* f *is differentiable on* A, *then we simply say that* f *is differentiable.*

Example 5.32 *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. At any $c \in \mathbb{R}$,*

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c}(x + c) = 2c.$$

Thus, f is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R}$, $f'(x) = 2x$.

Theorem 5.33 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A \cap A'$. If f is differentiable at c, then f is also continuous at c.*

Proof. The conclusion follows by

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$$

for all $x \in A$, $x \neq c$. □

Remark 5.34 *1) A function can have a derivative at a point without being continuous at that point: $sgn(x)$ has a derivative at 0, $sgn'(0) = +\infty$, but it is not continuous at 0.*

2) A function can be continuous at some point without being differentiable at that point: $f(x) = |x|$ is continuous at every $x \in \mathbb{R}$, but has no derivative at 0.

3) There exist functions that are continuous on $\mathbb{R}$, but nowhere-differentiable. For instance, let $d(x)$ be the distance from $x \in \mathbb{R}$ to the integer closest to x, that is, $d(x) = |x|$ for $x \in [-1/2, 1/2]$, extended on $\mathbb{R}$ by periodicity. Then, the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \sum_{n=0}^{\infty} \frac{d(2^n x)}{2^n}, \ \forall x \in \mathbb{R}$$

is continuous on $\mathbb{R}$, but nowhere differentiable.

Definition 5.35 *Let $f : A \to \mathbb{R}$ be a function defined on a nonempty set $A \subseteq \mathbb{R}$ and let $c \in A$. If c is an accumulation point of $A \cap (-\infty, c)$, then we say that f* has a left-hand derivative at *c if the following left-hand limit exists*

$$f_l'(c) := \lim_{\substack{x \to c \\ x < c}} \frac{f(x) - f(c)}{x - c} \in \overline{\mathbb{R}}.$$

In this case, $f_l'(c)$ is called the left-hand derivative of f at c. *If $f_l'(c) \in \mathbb{R}$ (i.e., f has a finite left-hand derivative at c), then f is said to be* left-hand differentiable at c.

Similarly, when c is an accumulation point of $A \cap (c, \infty)$, we say that f has a right-hand derivative at c *if the following right-hand limit exists*

$$f_r'(c) := \lim_{\substack{x \to c \\ x > c}} \frac{f(x) - f(c)}{x - c} \in \overline{\mathbb{R}}$$

and we say that f is right-hand differentiable at c *whenever $f_r'(c) \in \mathbb{R}$.*

Remark 5.36 *If $A = [a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, then the differentiability of $f : A \to \mathbb{R}$ at a is actually the right-hand differentiability of f at a while the differentiability of f at b is actually the left-hand differentiability of f at b.*

Theorem 5.37 (Calculus Rules) *Let $f, g : A \to \mathbb{R}$ be two functions, defined on a nonempty set $A \subseteq \mathbb{R}$, that are differentiable at $c \in A \cap A'$.*

1° *If $\alpha \in \mathbb{R}$, then function αf is differentiable at c and*

$$(\alpha f)'(c) = \alpha f'(c).$$

2° *Function $f + g$ is differentiable at c and*

$$(f + g)'(c) = f'(c) + g'(c).$$

3° *Function fg is differentiable at c and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

4° *If $g(c) \neq 0$, then function f/g (defined on some neighborhood of c) is differentiable at c and*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Theorem 5.38 (Chain rule) *Let $I, J \subseteq \mathbb{R}$ be intervals, $c \in I$, $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, such that $f(I) \subseteq J$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f : I \to \mathbb{R}$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Theorem 5.39 (Inverse Function Theorem) *Let $I, J \subseteq \mathbb{R}$ be intervals, $c \in I$ and let $f : I \to J$ be an invertible function (i.e., f is a bijection). If f is differentiable at c, $f'(c) \neq 0$ and $f^{-1} : J \to I$ is continuous at $f(c)$, then f^{-1} is differentiable at $f(c)$ and*

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}.$$

Local extrema and derivatives

Definition 5.40 *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. We say that f attains a local maximum (local minimum) at $c \in A$ if there exists $V \in \mathcal{V}(c)$ such that c is a maximum point (minimum point) for $f|_{A \cap V}$. We say that f attains a local extremum at $c \in A$ if it attains either a local maximum or a local minimum at c.*

Theorem 5.41 (Fermat) *Let $a, b \in \mathbb{R}$, $a < b$ and $f : (a, b) \to \mathbb{R}$. If $c \in (a, b)$, f has a derivative at c and f attains a local extremum at c, then $f'(c) = 0$.*

Remark 5.42 *(i) The above result may fail if one does not assume that f has a derivative at c (take $f : (-1, 1) \to \mathbb{R}$, $f(x) = |x|$, $c = 0$) or if the open interval is replaced by a closed one (take $f : [0, 1] \to \mathbb{R}$, $f(x) = x$. Then f attains a minimum at $c = 0$, but $f'(0) = 1$).*
(ii) If $f'(c) = 0$, it does not follow that f attains a local extremum at c (take $f : (-1, 1) \to \mathbb{R}$, $f(x) = x^3$, $c = 0$).

Theorem 5.43 (Darboux) *Let $a, b \in \overline{\mathbb{R}}$, $a < b$, and let $f : (a, b) \to \mathbb{R}$ be differentiable. Then, for any $x_1, x_2 \in (a, b)$ and $v \in \mathbb{R}$ such that*

$$x_1 < x_2 \quad \textit{and} \quad \min\{f'(x_1), f'(x_2)\} < y < \max\{f'(x_1), f'(x_2)\},$$

there exists $x \in (x_1, x_2)$ such that $f'(x) = y$.

Remark 5.44 *The derivative of a differentiable function is not always continuous. For instance, consider the function* $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then f *is differentiable on* $\mathbb{R}$, *but* f' *is not continuous at* 0.

Definition 5.45 *A function* $f : I \to \mathbb{R}$, *defined on an interval* $I \subseteq \mathbb{R}$, *is called continuously differentiable if it is differentiable and its derivative* $f' : I \to \mathbb{R}$ *is continuous.*

Theorem 5.46 (Rolle) *Let* $a, b \in \mathbb{R}$, $a < b$ *and* $f : [a, b] \to \mathbb{R}$. *If* f *is continuous on* $[a, b]$, *differentiable on* (a, b) *and* $f(a) = f(b)$, *then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$.

Theorem 5.47 (Lagrange's Mean Value Theorem) *Let* $a, b \in \mathbb{R}$, $a < b$ *and* $f : [a, b] \to \mathbb{R}$. *If* f *is continuous on* $[a, b]$ *and differentiable on* (a, b), *then there exists* $c \in (a, b)$ *such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Theorem 5.48 (Cauchy's Generalized Mean Value Theorem) *Let* $a, b \in \mathbb{R}$, $a < b$. *If* $f, g : [a, b] \to \mathbb{R}$ *are continuous on* $[a, b]$ *and differentiable on* (a, b), *then there is* $c \in (a, b)$ *such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Theorem 5.49 (L'Hôpital's rule for right-hand limits) *Let* $a, b \in \overline{\mathbb{R}}$ *with* $a < b$ *and let* $f, g : (a, b) \to \mathbb{R}$ *be two functions satisfying the following conditions:*

1) f *and* g *are differentiable on* (a, b).
2) $\forall x \in (a, b)$, $g'(x) \neq 0$.
3) $\lim\limits_{\substack{x \to a \\ x > a}} f(x) = \lim\limits_{\substack{x \to a \\ x > a}} g(x) \in \{-\infty, 0, +\infty\}$.
4) $\lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f'(x)}{g'(x)} = \ell \in \overline{\mathbb{R}}$.

Then $\lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f(x)}{g(x)} = \ell$.

Theorem 5.50 (L'Hôpital's rule for left-hand limits) *Let* $a, b \in \overline{\mathbb{R}}$ *with* $a < b$ *and let* $f, g : (a, b) \to \mathbb{R}$ *be two functions satisfying the following conditions:*

1) f *and* g *are differentiable on* (a, b).
2) $\forall x \in (a, b)$, $g'(x) \neq 0$.
3) $\lim\limits_{\substack{x \to b \\ x < b}} f(x) = \lim\limits_{\substack{x \to b \\ x < b}} g(x) \in \{-\infty, 0, +\infty\}$.
4) $\lim\limits_{\substack{x \to b \\ x < b}} \dfrac{f'(x)}{g'(x)} = \ell \in \overline{\mathbb{R}}$.

Then $\lim\limits_{\substack{x \to b \\ x < b}} \dfrac{f(x)}{g(x)} = \ell$.

Theorem 5.51 (L'Hôpital's rule for two-sided limits) *Let* $a, b \in \overline{\mathbb{R}}$ *with* $a < b$ *and let* $c \in (a, b)$. *Let* $f, g : (a, b) \to \mathbb{R}$ *be two functions satisfying the following conditions:*

1) f *and* g *are differentiable on* $(a, b) \setminus \{c\}$.
2) $g'(x) \neq 0$ *for all* $x \in (a, b) \setminus \{c\}$.
3) $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x) \in \{-\infty, 0, +\infty\}$.
4) $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)} = \ell \in \overline{\mathbb{R}}$.

Then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \ell$.

Example 5.52 *(i)* $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}.$

(ii) $\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=1.$

(iii) $\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0.$

(iv) $\lim_{\substack{x\to 0\\ x>0}}\frac{\ln x}{\ln(\sin x)}=\lim_{\substack{x\to 0\\ x>0}}\frac{1/x}{(\cos x)/(\sin x)}=\lim_{\substack{x\to 0\\ x>0}}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)=1.$