BABEŞ-BOLYAI UNIVERSITY
Faculty of Mathematics and Computer Science

Algorithms and Programming

Lecture 8 – Recursion. Computational complexity

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Course content

Programming in the large

- Introduction in the software development process
- Procedural programming
- Modular programming
- Abstract data types
- Software development principles
- Testing and debugging

Programming in the small

- **Recursion**
- **Complexity of algorithms**
- Search and sorting algorithms
- Backtracking and other problem solving methods

- Recap

Last time

- Testing
  - Black box testing
  - White box testing
  - Examples
- Design patterns
  - GRASP
  - Information Expert
  - Pure Fabrication (Repository)
  - Controller

Today

- Recursion
  - Basic concept
  - Mechanism
  - Examples
- Computational complexity
  - Why?
  - Examples
  - Analyzing the efficiency of a program

Recursion

- What is recursion?
  - A way to solve a problem by reducing it to simpler versions of itself
  - A programming technique where a function calls itself
- Basic concepts
  - Recursive element – an element that is defined by itself
  - Recursive algorithm – an algorithm that calls itself
  - Note: condition to stop recursion
- Recursion can be:
  - **Direct** – a function calls itself (`f` calls `f`)
  - **Indirect** – a function `f` calls a function `g`, function `g` calls `f`

Example: factorial

- n! = n*(n-1)*(n-2)*...*2*1
- n! = n*(n-1)! = n*(n-1)*(n-2)! =...

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n * (n-1)! & \text{otherwise} \end{cases}$$

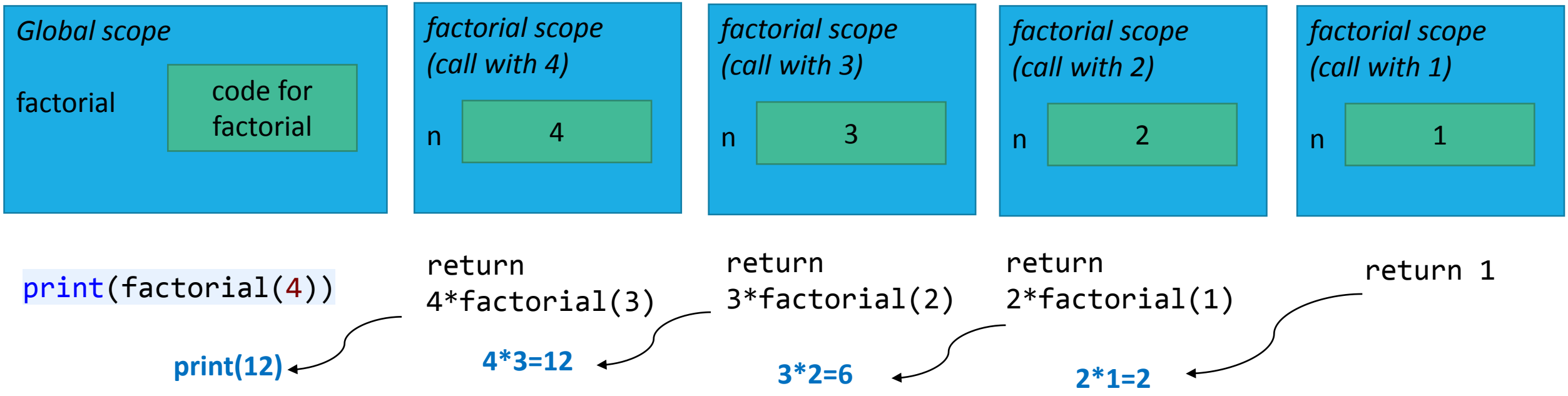
```
def factorial(n):
    '''
    Computes n! - the factorial for the given positive integer
    input: n - positive integer
    output: an integer 1 * 2 * 3 * ... * n
    '''
    # base case
    if n == 1:
        return 1
    else:
        # Recursive step: progresses toward the base case
        return n * factorial(n - 1)
```

```
def testFactorial():
    assert factorial(5) == 120
    assert factorial(4) == 24
    assert factorial(1) == 1
    assert factorial(0) == 1

testFactorial()
```

Example: factorial

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)

print(factorial(4))
```

- Function scope



Recursion: mechanism

- Main idea of developing a recursive algorithm for a problem of size n
  - **Base case**
    - How to stop recursion
    - Identify the base case solution (for n=1)
  - **Inductive step**
    - Break the problem into a simpler version of the same problem plus some other steps
    - e.g. A smaller problem (of size n-1) and some simple computations
    - e.g. Two smaller problems (of size n1, n2 such that n1+n2=n-1) and simple computations
- How recursion works
  - On each method invocation a new symbol table is created: it contains all the parameters and the local variables defined in the function
  - The symbol tables are stored in a stack: when a function is returning, the current symbol table is removed from the stack

Inductive reasoning

- How do you know that your recursive algorithm works?
- Mathematical induction
  - To prove a statement is true for all values of n:
    - Prove that the statement is true for smallest value of n (n=0 or n=1)
    - Assume that the statement is true for n, then prove that it is also true for n+1
- Same logic applies here:
  - ***Base case***: n=1, correctly returns 1
  - ***Recursive case***: assume that the answer is correct for problem size smaller than k, then for k+1 the answer will be (k+1)*k! = (k+1)!
  - By induction, the result returned is correct

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

Example: sum and product

Note: Inductive step

```
def sumList(lst):
    '''
    Compute the sum of the elements in the list
    input: lst - the list
    output: The sum of the elements
    '''
    # base case
    if len(lst) == 0:
        return 0
    else: # lst has at least one element
        return lst[0] + sumList(lst[1:])

def testSumList():
    assert sumList([]) == 0
    assert sumList([0]) == 0
    assert sumList([1, 2, 6]) == 9
    assert sumList([-1, 4, -100, 50]) == -47
    assert sumList([1, 2, 3, 4, 5, 6]) == 21

testSumList()
```

```
def product(lst):
    '''
    Computes the product of elements from a list
    input: a list of integers
    output: product of elements
    '''
    if (len(lst) == 0):
        return 1
    else:
        middle = len(lst) // 2
        return product(lst[0:middle]) * lst[middle] *
               product(lst[middle+1:])

def testProduct():
    assert product([1,2,3,4]) == 24
    assert product([]) == 1
    assert product([1,2,3,4,5]) == 120
    assert product([2]) == 2

testProduct()
```

Example: palindrome

Note: symbol table

```
>>> isPalindrom("noon")
49018080    {'s': 'noon'}
49171216    {'s': 'oo'}
48909600    {'s': ''}
True

>>> isPalindrom("redivider")
49020384    {'s': 'redivider'}
49018080    {'s': 'edivide'}
49171216    {'s': 'divid'}
48909600    {'s': 'ivi'}
48973552    {'s': 'v'}
True
```

```
def isPalindrom(s):
    '''
    Checks if a string is palindrom
    Input: a string
    Res: true, if str is palindrom and false, otherwise
    '''
    dico = locals()
    print(id(dico), " ", dico)

    if len(s) <= 1:
        return True
    else:
        return s[0]==s[-1] and isPalindrom(s[1:-1])

def testIsPalindrom():
    assert isPalindrom("abcba") == True
    assert isPalindrom("abccba") == True
    assert isPalindrom("abcdba") == False

testIsPalindrom()
```

"Able was I, ere I saw Elba" (Napoleon)
"Are we not drawn onward, we few, drawn onward to new era?" (Anne Michaels)

Example: belongs
Note: symbol table

```
>>> belongs(2, [4, 2, 3, 5])
49171168    {'l': [4, 2, 3, 5], 'el': 2}
48909648    {'l': [2, 3, 5], 'el': 2}
True
```

```
>>> belongs(7, [4, 2, 3, 5])
48909600    {'l': [4, 2, 3, 5], 'el': 7}
49171168    {'l': [2, 3, 5], 'el': 7}
48909648    {'l': [3, 5], 'el': 7}
49018080    {'l': [5], 'el': 7}
49171600    {'l': [], 'el': 7}
False
```

```
def belongs(el, l):
    '''
    Checks if an element belongs to a list
    Input: an integer and a list of integers
    Output: true, if el is in list
            and false, otherwise
    '''
    dico = locals()
    print(id(dico), " ", dico)

    if (l == []):
        return False
    else:
        if (el == l[0]):
            return True
        else:
            return belongs(el, l[1:])

def testBelongs():
    assert belongs(5, []) == False
    assert belongs(5, [5,2,6,3]) == True
    assert belongs(5, [1,2,5,4,3]) == True
    assert belongs(5, [6,2,5]) == True
    assert belongs(5, [1,2,3]) == False

testBelongs()
```

Iteration vs. Recursion

```
def factorial_iter(n):
    res = 1
    for i in range(1,n+1):
        res *= i
    return res
```

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

✓Recursive code is simple and intuitive

- However:
  - May be efficient from programming perspective, but
  - May not be efficient from the computer perspective
  - Why?
    - Large memory needed for in-depth recursion
    - Each function self call creates a new symbol table

Computational complexity

- What is it?
  - Study the efficiency of an algorithm from a mathematical perspective
- Why?
  - Problem: search for a number in a list
  - Solution:
    - Iterative search
    - Recursive search – one sub-problem
    - Recursive search – two sub-problems
  - **Which solution is better?**
    - **Algorithm efficiency**
    - Compare algorithms based on:
      - **Time needed for computations**
      - **Extra memory needed for temporary data**
    - Run time depends on:
      - Entry data (structure and size)
      - Changes from one run to another (due to hardware and software environment)
      - Hardware

Example: Fibonacci numbers

$F_n = F_{n-1} + F_{n-2}$ and $F_0 = F_1 = 1$

```
def fibonacci_recursiv(n):
    '''
    Computes the Fibonacci number
    data: a positive integer
    res: fibonacci number of n
    '''
    if n == 0 or n == 1:
        return 1
    return fibonacci_recursiv(n-1) +
           fibonacci_recursiv(n-2)
```

```
def fibonacci_iterativ(n):
    ''' ...
    '''
    s1 = 1
    s2 = 1
    fibo = 0
    for i in range(2, n + 1):
        fibo = s1 + s2
        s1 = s2
        s2 = fibo
    return fibo
```

```
def timeFibonacci(n):
    import time
    start_time = time.time()
    print("computing Fibbonacci_iterativ(", n, ") = ",
                                        fibonacci_iterativ(n))
    end_time = time.time()
    print(" takes ", end_time - start_time, " seconds")
    start_time = time.time()
    print("computing Fibbonacci_recursiv(", n, ") = ",
                                         fibonacci_recursiv(n))
    end_time = time.time()
    print(" takes ", end_time - start_time, " seconds")
```

```
>>> timeFibonacci(23)
computing Fibbonacci_iterativ( 23 ) =  46368
 takes  0.07200431823730469  seconds
computing Fibbonacci_recursiv( 23 ) =  46368
 takes  0.13400769233703613  seconds
```

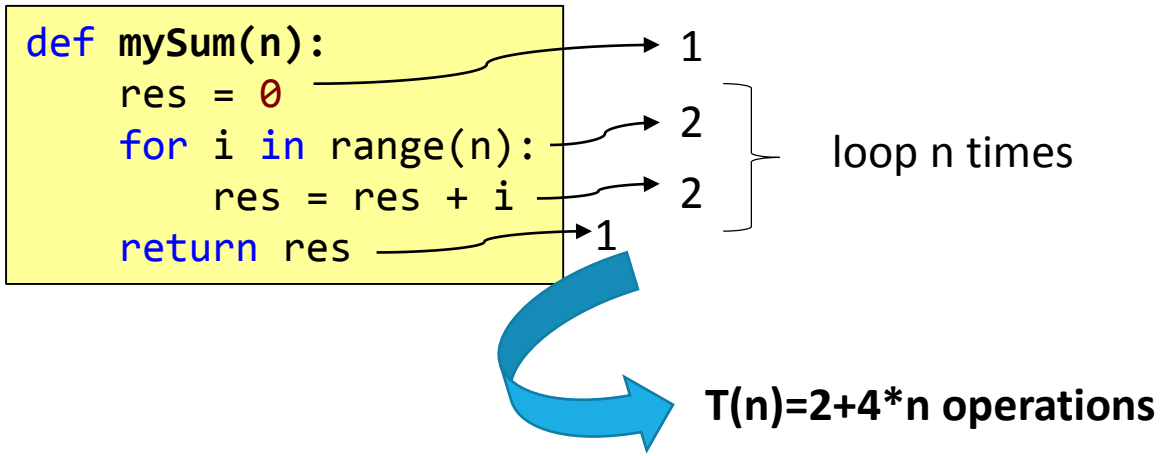
```
>>> timeFibonacci(33)
computing Fibbonacci_iterativ( 33 ) =  5702887
 takes  0.07800436019897461  seconds
computing Fibbonacci_recursiv( 33 ) =  5702887
 takes  11.261644124984741  seconds
```

Algorithm efficiency

- To analyse the efficiency of an algorithm (function):
  - The amount of resources used
  - **Time** and space efficiency of a program
  - Measure efficiency
    - **Asymptotic analysis**
      - Can provide the efficiency for all possible input data
      - Can not provide exact execution times
    - Empirical analysis
      - Can not predict the performance of the algorithm for all possible input data
      - Can determine the execution time for specific set of inputs
  - Run time is studied in connection with the size of the input data

Run time complexity

- Complexity in time
- Running time of an algorithm:
  - It is not a fixed number
  - It is a function *T(n)* that depends on the size *n* of the input data
  - Measures the basic steps the algorithm makes
- Example
  - Steps that take constant time:
    - Mathematical operations
    - Assignments
    - Comparisons
    - Accessing objects in memory
    - Return statement
  - Count the number of operations executed as a function of the input size



An example

```
def searchElement(l, e):
    for i in l:
        if e == i:
            return True
    return False
```

- e is the first element in the list – best case
  - Minimum running time over all possible inputs
- e is not in the list – worst case
  - Maximum running time over all possible inputs
- e is found after about half of the list is searched – average case
  - Average running time over all possible inputs

Run time complexity

- **Best case (BC)**
  - For the entry data leading to minimum running time of the algorithm
  - *Algorithm complexity*: $BC(A) = \min_{I \in D_A} E_A(I)$
  - Gives a lower bound to the running time
- **Worst case (WC)**
  - For the entry data leading to maximum running time of the algorithm
  - *Algorithm complexity*: $WC(A) = \max_{I \in D_A} E_A(I)$
  - Gives an upper bound to the running time
- **Average case (AC)**
  - Average running time of the algorithm
  - *Algorithm complexity*: $AC(A) = \sum_{I \in D_A} P_A(I) E_A(I)$
  - Offers a prediction of the running time

where

A – algorithm

D_A – domain of the input data

I – an instance of input data

$E_A(I)$ – number of operations performed by algorithm A having input data I

$P_A(I)$ – probability that algorithm A receives input data I

Run time complexity: Examples

```
def sumOfFirstNumbers(n):
    '''
    computes the sum of first n natural numbers
    data: a natural number
    res: the sum of first n numbers
    '''
    s = 0
    for i in range(1, n + 1):
        s = s + i
    return s
```

$$T(n) = \sum_{i=1}^{n} 1 = n$$

Case	T(n)
Best case	$\sum_{i=1}^{n} 1$
Worst case	$\sum_{i=1}^{n} 1$
Average case	$\sum_{i=1}^{n} 1$

Run time complexity: Examples

```
def search(el, list):
    '''
    checks if an element belongs to a list
    data: an integer and a list of integers
    res: true, if elements belongs to list
    false, otherwise
    '''
    for i in range(0, len(list)):
        if (list[i] == el):
            return True
    return False
```

Case	len(list)=n	T(n)
Best case	el=list[0]	1
Worst case	el=list[n-1] el is not in list	$\sum_{i=0}^{n-1} 1 = n$ $\sum_{i=0}^{n-1} 1 + 1 = n + 1$
Average case	el=list[0] el=list[1] el=list[2] … el=list[n-1] el is not in list	1 2 3 … n n+1 _____ (1+2+3+…+n+(n+1))/(n+1)=(n+2)/2

Big Oh notation

- O(n) measure
  - How the running time grows depending on the input data size
  - Expression for the number of operations -> asymptotic behavior as the problem gets bigger
- Exact steps vs

Big Oh or

O() notation

```
def factorial_iter(n):
    i = 1
    res = 1
    while i <= n:
        res = res * i
        i = i -1
    return res
```

- Exact steps T(n) = 1 + 1 + n*5 + 1
- Worst case asymptotic complexity O(n)
  - Ignore additive constants
  - Ignore multiplicative constants

O()

- Drop constants and multiplicative factors
- Focus on dominant terms (consider the leading term)

O(n) $T(n) = 2n + 2$

O(n²) $T(n) = n^2 + 2n + 2$

O(n³) $T(n) = n^3 + 1000n + 3^{1000}$

O(n) $T(n) = n + \log n$

O(n log n) $T(n) = n \log n + 100n$

O(2ⁿ) $T(n) = n^2 + 2^n$

$T(n) = n^3 + 100n^2 + 10n \log_2 n + 2\sqrt{n} + 27^{100}$

```
for i in range(n):
    print(i)
for i in range(n):
    for j in range(n):
        print(i," ",j)
```

O(n)

O(n)*O(n)=O(n*n)=O(n²)

O(n)+O(n²)=O(n+n²)=O(n²)

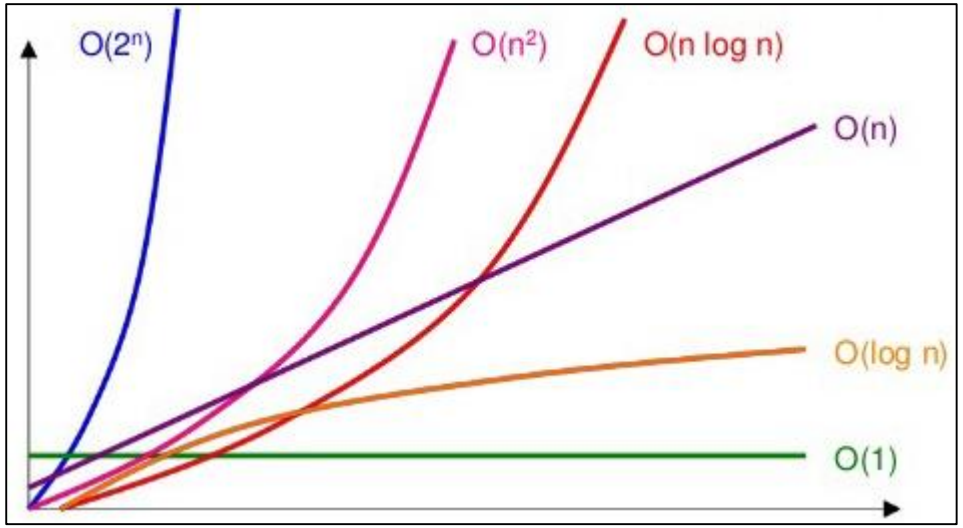
Run time complexity: theoretical aspects

- For a function $f: N \to R$ and $T: N \to N$, how to decide the complexity class of T?
- $T(n) \in O\big(f(n)\big)$ if there exist 2 positive and independent constants c and n_0 such that $0 \leq T(n) \leq c * f(n), \forall n \geq n_0$
- Examples
  - O(1) = 1, 5, 500
  - O(n) = n, 2n + 1, 5n – 200
  - O(n^2) = n^2, n^2 + 5, $2n^2$ + 3n - 1
- If $T(n) \in O\big(f(n)\big)$ then $\lim_{n \to \infty} \frac{T(n)}{f(n)} < \infty$ (the limit is constant)

Complexity classes

O(1)	Constant running time	e.g. 1, 47, 100	Add an element to a list
O(log n)	Logarithmic running time	e.g. 10 + log n	Find an element in a sorted list
O(n)	Linear running time	e.g. n, 3n, 10n+100	Find an entry in an unsorted list
O(n log n)	Log-linear running time	e.g. n + n log n	Sort a list (MergeSort, QuickSort)
O(n^c), c is constant	Polynomial running time	e.g. n^2+1, n^3+n^2+5n	Shortest path between two nodes
O(c^n), c is constant	Exponential running time	e.g. 2^n+1, 3^n	Traveling Salesman Problem (TSP)

O(n^2) - quadratic time
O(n^3) - cubic time



Recap today

- Recursion
  - Basic concept
  - Mechanism
  - Examples
- Computational complexity
  - Examples
  - The efficiency of a program
  - Time and space complexity

Next time

- Algorithms
  - Sort
  - Search

Reading materials and useful links

1. The Python Programming Language - https://www.python.org/
2. The Python Standard Library - https://docs.python.org/3/library/index.html
3. The Python Tutorial - https://docs.python.org/3/tutorial/
4. M. Frentiu, H.F. Pop, Fundamentals of Programming, Cluj University Press, 2006.
5. MIT OpenCourseWare, Introduction to Computer Science and Programming in Python, https://ocw.mit.edu, 2016.
6. K. Beck, Test Driven Development: By Example. Addison-Wesley Longman, 2002. http://en.wikipedia.org/wiki/Test-driven_development
7. M. Fowler, Refactoring. Improving the Design of Existing Code, Addison-Wesley, 1999. http://refactoring.com/catalog/index.html

Bibliography

The content of this course has been prepared using the reading materials from previous slide, different sources from the Internet as well as lectures on Fundamentals of Programming held in previous years by:

- Prof. Dr. Laura Dioşan - www.cs.ubbcluj.ro/~lauras
- Conf. Dr. Istvan Czibula - www.cs.ubbcluj.ro/~istvanc
- Lect. Dr. Andreea Vescan - www.cs.ubbcluj.ro/~avescan
- Lect. Dr. Arthur Molnar - www.cs.ubbcluj.ro/~arthur