Ex 1

a) $\lim\limits_{x\to\infty} x\cos^2\left(\frac{x+2}{x}\right) = \lim\limits_{x\to\infty} x\cdot\cos^2(1) = \infty$

$\frac{x+2}{x} \downarrow 1$

$\lim\limits_{x\to\infty} \cos^2\frac{x+2}{x} = \lim\limits_{u\to 1}(\cos(u))^2 = \cos^2 1$

~~b) $\lim\limits_{x\to\infty}\frac{x}{x^2+1} = \lim\limits_{x\to\infty}\frac{x}{x(x+\frac{1}{x})} = \frac{1}{x} = 0$~~

b) $\lim\limits_{x\to 1} = \frac{x}{x^2+1} = \frac{1}{1+1} = \frac{1}{2}$

c) $\lim\limits_{x\to-\infty}\frac{x^2+5}{x^3} = \lim\limits_{x\to-\infty}\frac{\cancel{x^2}(1+\frac{5}{x^2})}{\cancel{x^2}\cdot x} = \lim\limits_{x\to-\infty}\frac{1}{x} = 0$

d) $\lim\limits_{x\to+\infty}\frac{(x+2)(2x+1)}{x^2+3x+5} = \lim\limits_{x\to\infty}\frac{2x^2+5x+2}{x^2+3x+5} = 2$

e) $\lim\limits_{x\to 1}\frac{x^2-1}{x^3-1} = \lim\limits_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x^2+x+1)} = \lim\limits_{x\to 1}\frac{x+1}{x^2+x+1} = \frac{2}{3}$

f) $\lim\limits_{x\to 2}\left(\frac{1}{2-x} - \frac{2x}{4-x^2}\right) = \lim\limits_{x\to 2}\frac{2+x-2x}{4-x^2} = \lim\limits_{x\to 2}\frac{\cancel{2-x}}{\cancel{(2-x)}(2+x)} = \frac{1}{4}$

g) $\lim\limits_{x\to 1}\frac{1+x+x^2+\dots+x^n-(n+1)}{x-1}$, $n\in\mathbb{N}$ $= \lim\limits_{x\to 1}\frac{x^n+x^{n-1}+\dots+x-n}{x-1} =$

$= \lim\limits_{x\to 1}\frac{\cancel{(x-1)}(x^{n-1}+2x^{n-2}+\dots+(n-1)x+n)}{\cancel{x-1}} = 1+2+\dots+(n-1)+n = \frac{n\cdot(n+1)}{2}$

h) $\lim\limits_{x\to 1}\frac{x+x^2+\dots+x^n-n}{x+x^2+\dots+x^m-m}$, $\forall\, n,m\in\mathbb{N}$ $= \lim\limits_{x\to 1}\frac{\cancel{(x-1)}(x^{n-1}+2x^{n-2}+\dots+(n-1)x+n)}{\cancel{(x-1)}(x^{m-1}+2x^{m-2}+\dots+(m-1)x+m)} =$

$= \frac{\frac{n(n+1)}{2}}{\frac{m(m+1)}{2}} = \frac{n(n+1)}{m(m+1)}$

i) $\lim_{x\to 27} \frac{x-27}{\sqrt[3]{x}-3} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 27} \frac{1}{\frac{1}{3x^{\frac{2}{3}}}} = \lim_{x\to 27} 3x^{\frac{2}{3}} = 3\cdot\sqrt[3]{(3^3)^2} =$

$= 3\cdot\sqrt[3]{3^6} = 3\cdot 9 = 27$

j) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 1} \frac{\frac{1}{3x^{\frac{2}{3}}}}{\frac{1}{4x^{\frac{3}{4}}}} = \frac{4\cdot 1^{\frac{3}{4}}}{3\cdot 1^{\frac{2}{3}}} = \frac{4}{3}$

k) $\lim_{x\to\infty} \left(\sqrt[3]{ax^3+x^2+bx+c} - (bx+c)\right), \forall a,b,c>0$

$a^3-b^3=(a-b)(a^2+ab+b^2)$
$a-b=(a^{\frac{1}{3}}-b^{\frac{1}{3}})(a^{\frac{2}{3}}+ab^{\frac{1}{3}}+b^{\frac{2}{3}})$
$a^2-b^2=(a+b)(a-b)$
amp

$\lim_{x\to\infty} \frac{ax^3+x^2+bx+c-(bx+c)^3}{\sqrt[3]{(ax^3+x^2+bx+c)^2}+\sqrt[3]{ax^3\ldots}\cdot(bx+c)+(bx+c)^2} =$

$\lim_{x\to\infty} \frac{ax^3-bx^3+\ldots}{x^2\ldots+x^2\ldots+b^2x^2\ldots} = \cancel{\begin{cases} a\in(0,1), & 0 \\ a>1, & +\infty \end{cases}} = +\infty$

Ex 2

a) $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}} = \lim_{x\to\infty} e^{\frac{5x+1}{2x+4}\cdot\ln\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{5x+1}{2x+4}\ln\frac{1}{x}} =$

$= e^{\frac{5}{2}\cdot-\infty} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$

! b) $\lim_{x\to 0} \left(\frac{3\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}}$

c) $\lim_{x\to 0} (1+\cos x)^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x^2}\cdot\ln(1+\cos x)} = e^{\overbrace{\lim_{x\to 0}\frac{1}{x^2}}^{\infty}\cdot\overbrace{\lim_{x\to 0}\ln(1+\cos x)}^{\ln 2}} =$

$= e^{\infty\cdot\ln 2} = \infty$

d) $\lim_{x\to 0} (e^x-x+1)^{\frac{1}{1-\cos x}} = e^{\lim_{x\to 0}\frac{1}{1-\cos x}\cdot\ln e^x-x+1} = e^{\lim_{x\to 0}\frac{1}{1-\cos x}\cdot\lim_{x\to 0}\ln e^x-x+1}$

$= e^{\infty\cdot\ln 2} = e^{\infty} = +\infty$

e) $\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} = e^{\lim_{x\to 0} \frac{1}{x} \cdot \ln(1+\sin x)} = e^{\lim_{x\to 0} \frac{\ln(1+\sin x)}{x}} \underset{(*)}{=} e^1 = e$

(*) $\lim_{x\to 0} \frac{\ln(1+\sin x)}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \frac{\frac{\cos(x)}{\sin(x)+1}}{1} = \frac{\cos 0}{1+\sin 0} = 1$

f) $\lim_{x\to\infty} \left(\frac{x+7}{x}\right)^x = \lim_{x\to\infty} \left(1+\frac{\cancel{x}-7-\cancel{x}}{x}\right)^x = \lim_{x\to\infty} \Bigg[\underbrace{\left(1+\frac{-7}{x}\right)^{\frac{x}{-7}}}_{e}\Bigg]^{\frac{-7}{x}\cdot x} = e^{-7} = \frac{1}{e^7}$

Ex 3

a) $\lim_{n\to\infty} \left[\lim_{x\to\infty} (1+\sin^2 x+\sin^2 2x+\dots+\sin^2 nx)^{\frac{1}{n^3x^2}}\right]$

b) $\lim_{n\to\infty} \left[\lim_{x\to 0} (1+\ln(1+x)+\ln(1+2x)+\dots+\ln(1+nx))^{\frac{1}{n^2x}}\right]$

Ex 4

a) $\lim_{x\to 0} \frac{e^{2x}-1}{3x} = \frac{1}{3}\lim_{x\to 0} \frac{e^{2x}-1}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \frac{1}{3}\lim_{x\to 0} \frac{2e^{2x}}{1} = \frac{1}{3}\cdot 2 = \frac{2}{3}$

b) $\lim_{x\to 0} \frac{e^{\overset{?}{x}}-\cos x}{3x} = \frac{1}{3}\lim_{x\to 0} \frac{e^{2x}-\cos x}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \frac{1}{3}\lim_{x\to 0} (2e^{2x}+\sin x) =$

$= \frac{1}{3}\cdot(2+0) = \frac{2}{3}$