LECTURE

2

SEQUENCES OF REAL NUMBERS

Definition 2.1 *By a* sequence of real numbers (*or sequence in* $\mathbb{R}$) *we mean any function of type* $f : \mathbb{N} \to \mathbb{R}$. *Denoting* $x_n := f(n)$ *for all* $n \in \mathbb{N}$ *we may use one of the following notations to represent the sequence* f:

$$(x_n)_{n\in\mathbb{N}}, \ (x_n)_{n\geq 1} \ \text{ or simply } \ (x_n).$$

Remark 2.2 *Any function* $g : \mathbb{N} \cap [m, \infty) \to \mathbb{R}$ *with* $m \in \mathbb{Z}$ *can be seen as a sequence, too. Indeed,* g *can be identified with* $f : \mathbb{N} \to \mathbb{R}$, *given by*

$$f(n) := g(m + n - 1), \ \forall n \in \mathbb{N}.$$

In this case instead of $(x_n)_{n\in\mathbb{N}}$, *with* $x_n := f(n)$ *for all* $n \in \mathbb{N}$, *we write*

$$(y_n)_{n\geq m}, \ \text{ where } \ y_n := g(n), \ \forall n \in \mathbb{Z}, \ n \geq m.$$

Definition 2.3 *A sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$ *is said to be bounded below (bounded above; bounded; unbounded) if the set* $\{x_n \mid n \in \mathbb{N}\}$ *is bounded below (bounded above; bounded; unbounded).*

Remark 2.4 *For any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *we have:*

$$
\begin{aligned}
(x_n)_{n\in\mathbb{N}} \text{ is bounded below} &\iff \exists\, a \in \mathbb{R} \text{ s.t. } x_n \geq a, \ \forall n \in \mathbb{N}; \\
(x_n)_{n\in\mathbb{N}} \text{ is bounded above} &\iff \exists\, a \in \mathbb{R} \text{ s.t. } x_n \leq a, \ \forall n \in \mathbb{N}; \\
(x_n)_{n\in\mathbb{N}} \text{ is bounded} &\iff \exists\, a \in \mathbb{R} \text{ s.t. } |x_n| \leq a, \ \forall n \in \mathbb{N}; \\
(x_n)_{n\in\mathbb{N}} \text{ is unbounded} &\iff \forall\, a \in \mathbb{R}, \ \exists\, n \in \mathbb{N} \text{ s.t. } |x_n| > a.
\end{aligned}
$$

Definition 2.5 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is called*
- *increasing if* $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$;
- *decreasing if* $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$;
- *strictly increasing if* $x_{n+1} > x_n$, $\forall n \in \mathbb{N}$;
- *strictly decreasing if* $x_{n+1} < x_n$, $\forall n \in \mathbb{N}$;
- *monotonic (or monotone) if it is increasing or decreasing;*
- *strictly monotonic (or strictly monotone) if it is strictly increasing or strictly decreasing.*

Remark 2.6 *Every increasing sequence is bounded below. Similarly, every decreasing sequence is bounded above.*

Definition 2.7 *We say that a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ has a limit (in $\overline{\mathbb{R}}$) if there exists $\ell \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty; +\infty\}$ such that*

$$\forall V \in \mathcal{V}(\ell), \ \exists n_V \in \mathbb{N} \ \ s.t. \ \ x_n \in V, \ \forall n \in \mathbb{N}, \ n \geq n_V. \tag{2.1}$$

Remark 2.8 *For any $\ell_1, \ell_2 \in \overline{\mathbb{R}}$, $\ell_1 \neq \ell_2$, there exist $V_1 \in \mathcal{V}(\ell_1)$ and $V_2 \in \mathcal{V}(\ell_2)$ s.t. $V_1 \cap V_2 = \emptyset$. Hence, whenever $(x_n)_{n\in\mathbb{N}}$ has a limit, there is a unique $\ell \in \overline{\mathbb{R}}$ satisfying (2.1).*

Definition 2.9 *If a sequence $(x_n)_{n\in\mathbb{N}}$ has a limit, then the unique $\ell \in \overline{\mathbb{R}}$ that satisfies (2.1) is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. In this case we denote $\lim_{n\to\infty} x_n := \ell$ or $x_n \to \ell$ and we say that $(x_n)_{n\in\mathbb{N}}$ tends to ℓ.*

Proposition 2.10 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that has a limit. We have*
$$\lim_{n\to\infty} x_n = \ell \in \mathbb{R} \iff \forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N} \ \ s.t. \ \ |x_n - \ell| < \varepsilon, \ \forall n \in \mathbb{N}, \ n \geq n_\varepsilon;$$
$$\lim_{n\to\infty} x_n = -\infty \iff \forall a \in \mathbb{R}, \ \exists n_a \in \mathbb{N} \ \ s.t. \ \ x_n < a, \ \forall n \in \mathbb{N}, \ n \geq n_a;$$
$$\lim_{n\to\infty} x_n = +\infty \iff \forall a \in \mathbb{R}, \ \exists n_a \in \mathbb{N} \ \ s.t. \ \ x_n > a, \ \forall n \in \mathbb{N}, \ n \geq n_a.$$

Definition 2.11 *A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is said to be*
- convergent, *if it has a finite limit, i.e., $\lim_{n\to\infty} x_n \in \mathbb{R}$; in this case we say that $(x_n)_{n\in\mathbb{N}}$ converges to the real number $\lim_{n\to\infty} x_n$;*
- divergent, *if it is not convergent.*

Proposition 2.12 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $\ell \in \mathbb{R}$. The following assertions are equivalent:*

1° *The sequence $(x_n)_{n\in\mathbb{N}}$ converges to ℓ, i.e., $\lim_{n\to\infty} x_n = \ell$.*

2° *The sequence $(|x_n - \ell|)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} |x_n - \ell| = 0$.*

3° *There exists a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying the following two conditions:*

(i) *$|x_n - \ell| \leq a_n$ for all $n \in \mathbb{N}$;*

(ii) *$(a_n)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} a_n = 0$.*

Corollary 2.13 *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers. If $(a_n)_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} (a_n \cdot b_n) = 0$.*

Proposition 2.14 *Let $(x_n), (y_n)$ be sequences in $\mathbb{R}$ such that $\forall n \in \mathbb{N}, \ x_n \leq y_n$.*

(i) *If (x_n) and (y_n) converge, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.*

(ii) *If $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} y_n = +\infty$.*

(iii) *If $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.*

Theorem 2.15 (Squeeze Theorem) *Let $(x_n), (y_n), (z_n)$ be sequences in $\mathbb{R}$ such that*

$$x_n \leq y_n \leq z_n, \ \forall n \in \mathbb{N}.$$

Suppose that (x_n) and (z_n) are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell \in \mathbb{R}$. Then (y_n) is also convergent and $\lim_{n\to\infty} y_n = \ell$.

Corollary 2.16 (Cantor's Theorem on Nested Intervals) *Consider a nested sequence of compact intervals, i.e.,*

$$I_n := [a_n, b_n] \subseteq \mathbb{R}, \quad s.t. \ \ a_n \leq a_{n+1} < b_{n+1} \leq b_n, \ \forall n \in \mathbb{N}.$$

If $\lim_{n\to\infty} (b_n - a_n) = 0$, then there exists $x \in \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n = \{x\}$.

Proposition 2.17 *Every convergent sequence of real numbers is bounded.*

Notice that there are bounded sequences that are not convergent.

Theorem 2.18 (Counterpart of the Weierstrass' Theorem) *Let $(x_n)_{n \in \mathbb{N}}$ be a monotonic (i.e., increasing or decreasing) sequence of real numbers. The following assertions hold:*

$1°$ *$(x_n)_{n \in \mathbb{N}}$ has a limit in $\overline{\mathbb{R}}$.*

$2°$ *If $(x_n)_{n \in \mathbb{N}}$ is increasing, then $\lim\limits_{n \to \infty} x_n = \sup\limits_{n \in \mathbb{N}} x_n$, hence*
(x_n) *is convergent if and only if it is bounded above.*

$2°$ *If $(x_n)_{n \in \mathbb{N}}$ is decreasing, then $\lim\limits_{n \to \infty} x_n = \inf\limits_{n \in \mathbb{N}} x_n$, hence*
(x_n) *is convergent if and only if it is bounded below.*

Theorem 2.19 (Toeplitz) *Consider an "infinite triangular matrix" of real numbers*

$$
\begin{array}{lllll}
c_{1,1} & & & & \\
c_{2,1} & c_{2,2} & & & \\
\cdots & \cdots & \cdots & & \\
c_{n,1} & c_{n,2} & c_{n,3} & \cdots & c_{n,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

which satisfies the following three conditions:

(i) $c_{n,k} \geq 0, \; \forall \, n \in \mathbb{N}, \; \forall \, k \in \{1, 2, ..., n\}$;

(ii) $\sum\limits_{k=1}^{n} c_{n,k} = 1, \; \forall n \in \mathbb{N}$;

(iii) $\lim\limits_{n \to \infty} c_{n,k} \to 0, \; \forall \, k \in \mathbb{N}$.

If $(x_n)_{n \in \mathbb{N}}$ is a sequence of real numbers that has a limit $\ell \in \overline{\mathbb{R}}$ then the sequence $(y_n)_{n \in \mathbb{N}}$, given by

$$ y_n = c_{n,1} x_1 + c_{n,2} x_2 + ... + c_{n,n} x_n, \; \forall \, n \in \mathbb{N} $$

has the same limit ℓ.

Theorem 2.20 (Stolz-Cesàro) *Let $(a_n), (b_n)$ be sequences in $\mathbb{R}$ such that*

(i) (b_n) *is strictly increasing and* $\lim\limits_{n \to \infty} b_n = +\infty$,

(ii) $\lim\limits_{n \to \infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = L \in \overline{\mathbb{R}}$.

Then $\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = L$.

Proof. We can apply Toeplitz's Theorem, letting $a_0 = b_0 := 0$,

$$ x_n := \frac{a_n - a_{n-1}}{b_n - b_{n-1}}, \; \forall \, n \in \mathbb{N} \quad \text{and} \quad c_{n,k} := \frac{b_k - b_{k-1}}{b_n}, \; \forall \, k \in \{1, 2, ..., n\}. $$

Since $y_n = \frac{a_n}{b_n}$, the conclusion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 2.21 *Let (x_n) be a sequence in $\mathbb{R}$. The following assertions hold:*

$1°$ *If $\lim\limits_{n \to \infty} x_n = \ell \in \overline{\mathbb{R}}$, then $\lim\limits_{n \to \infty} \dfrac{x_1 + x_2 + ... + x_n}{n} = \ell$.*

$2°$ *If $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} x_n = \ell \in \overline{\mathbb{R}}$, then $\lim\limits_{n \to \infty} \sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n} = \ell$.*

$3°$ *If $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} \dfrac{x_{n+1}}{x_n} = L \in \overline{\mathbb{R}}$, then $\lim\limits_{n \to \infty} \sqrt[n]{x_n} = L$.*

Definition 2.22 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequences of real numbers. A sequence $(y_k)_{k\in\mathbb{N}}$ is said to be a subsequence of $(x_n)_{n\in\mathbb{N}}$ if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers (i.e., $n_k \in \mathbb{N}$ and $n_k < n_{k+1}$ for all $k \in \mathbb{N}$) such that*

$$y_k = x_{n_k}, \ \forall\, k \in \mathbb{N}.$$

Proposition 2.23 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ that has a limit $x = \lim\limits_{n\to\infty} x_n \in \overline{\mathbb{R}}$. Then any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ has the same limit, i.e., $\lim\limits_{k\to\infty} x_{n_k} = x$.*

Theorem 2.24 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Theorem 2.25 (Cauchy's criterion for convergence of sequences) *For any sequence $(x_n)_{n\in\mathbb{N}}$ the following assertions are equivalent:*
- 1° *$(x_n)_{n\in\mathbb{N}}$ is convergent.*
- 2° *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, with $m \geq n_\varepsilon$ and $n \geq n_\varepsilon$, we have $|x_m - x_n| < \varepsilon$.*
- 3° *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ and any $p \in \mathbb{N}$ we have $|x_{n+p} - x_n| < \varepsilon$.*

Corollary 2.26 (Sufficient condition for convergence of sequences) *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying the following two conditions:*
- 1° *$|x_{n+p} - x_n| \leq a_n$ for all $n, p \in \mathbb{N}$;*
- 2° *$(a_n)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} a_n = 0$.*

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent.

Limit Laws

$x + \infty = \infty + x = \infty, \ \forall x \in \mathbb{R},$

$x + (-\infty) = (-\infty) + x = -\infty, \ \forall x \in \mathbb{R},$

$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty,$

$x \cdot \infty = \infty \cdot x = \begin{cases} \infty, & \text{if } x \in (0, \infty) \\ -\infty, & \text{if } x \in (-\infty, 0), \end{cases}$

$x \cdot (-\infty) = (-\infty) \cdot x = \begin{cases} -\infty, & \text{if } x \in (0, \infty) \\ \infty, & \text{if } x \in (-\infty, 0), \end{cases}$

$\infty \cdot \infty = \infty, \quad (-\infty) \cdot (-\infty) = \infty, \quad \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty,$

$\frac{x}{\infty} = \frac{x}{-\infty} = 0, \ \forall x \in \mathbb{R},$

$\frac{1}{0+} = \infty, \quad \frac{1}{0-} = -\infty,$

$x^\infty = \begin{cases} \infty, & \text{if } x \in (1, \infty) \\ 0, & \text{if } x \in [0, 1), \end{cases}$

$x^{-\infty} = \begin{cases} 0, & \text{if } x \in (1, \infty) \\ \infty, & \text{if } x \in (0, 1), \end{cases}$

$(\infty)^x = \begin{cases} \infty, & \text{if } x \in (0, \infty) \\ 0, & \text{if } x \in (-\infty, 0), \end{cases}$

$\infty^\infty = \infty, \quad \infty^{-\infty} = 0.$

The following limits are undefined

$\infty + (-\infty), \quad (-\infty) + \infty,$

$0 \cdot \infty, \quad \infty \cdot 0, \quad 0 \cdot (-\infty), \quad (-\infty) \cdot 0,$

$\frac{\infty}{\infty}, \quad \frac{-\infty}{-\infty}, \quad \frac{\infty}{-\infty}, \quad \frac{-\infty}{\infty},$

$1^\infty, \quad 0^0, \quad \infty^0, \quad 1^{-\infty}.$