CURS 3

20.10.2017

Abbreviations (from Latin)

i.e. = id est = that is; in other words

e.g. = exempli gratia = for example

Test 1: week 7

Saturday, Nov. 18, 9:00

Lecture room 6/II

1 hour : 2 questions

Propositional logic : decision problem — truth tables / normal forms

First order logic: apply tautologies in order to solve a problem

SETS

"Naively" – a set is a collection of well-determined and unique (abstract) objects

– the notions of
- element
- set
- belong

are not defined – they are primary notions

$x \in A$

element, belongs, set

Set theory was initiated by Georg Cantor in ~1870

- inclusion of sets $A \subseteq B$ (is a subset of) $\Leftrightarrow (\forall x \in A \Rightarrow x \in B)$
- equality of sets $A = B \Leftrightarrow (\forall x \;\; x \in A \Leftrightarrow x \in B) \Leftrightarrow A \subseteq B$ and $B \subseteq A$
- sets can be given in two ways:
  - by enumerating the elements : $A = \{a, b, c, d\}$
  - by giving a property (a predicate) $A = \{x \mid P(x)\}$

    In this situation, if $B = \{x \mid Q(x)\}$

    we have: $A \subseteq B \Leftrightarrow (\forall x \;\; P(x) \Rightarrow Q(x))$

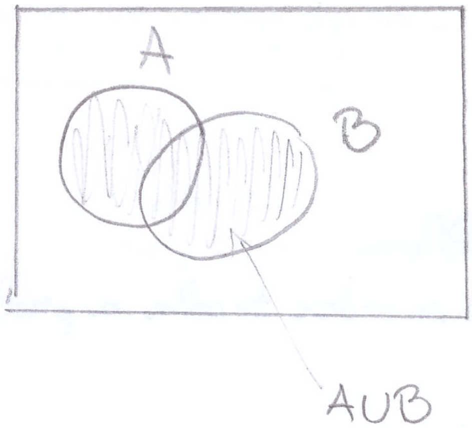
    $A = B \Leftrightarrow (\forall x \;\; P(x) \Leftrightarrow Q(x))$

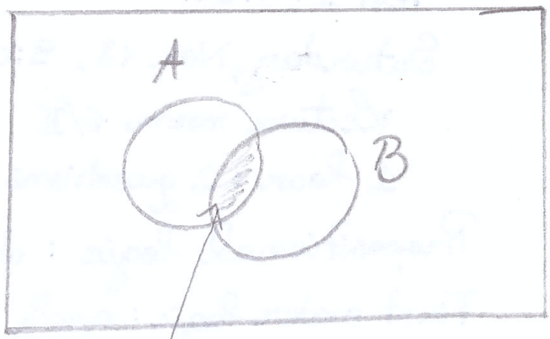
Operations with sets:

Union $A \cup B \overset{def}{=} \{x \mid x \in A$ or $x \in B\}$ ($\vee$)

Intersection $A \cap B \overset{def}{=} \{x \mid x \in A$ and $x \in B\}$ ($\wedge$)

Euler-Venn diagrams



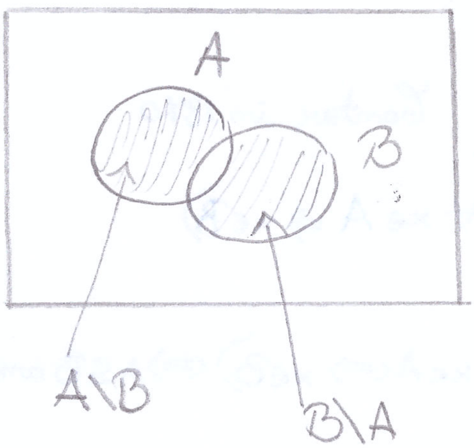


$\neg(x\in B)$ $A\cap B$

Difference $A\setminus B \overset{def}{=} \{x \mid x\in A \text{ and } x\notin B\} = A\cap CB$

Complement of a set $CA \overset{def}{=} \{x \mid x\notin A\}$

$x\in CA \iff x\notin A$



Symmetric difference $A\Delta B \overset{def}{=} (A\setminus B)\cup(B\setminus A)$

$= \{x \mid x\in A \text{ xor } x\in B\}$

exclusive or

$=(A\cup B)\setminus(A\cap B)$

Ordered pair of elements

We want to define the pair (a,b) with the following property: $(a,b)=(c,d) \iff a=c$ and $b=d$

$\boxed{\text{Def}}$ $(a,b)=\{\{a\},\{a,b\}\}$

Property $(a,b)=(c,d) \Leftrightarrow a=c$ and $b=d$

Proof „$\Leftarrow$" Assume that $a=c$ and $b=d$. Then, obviously,

$$(a,b)=\{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}=(c,d)$$

„$\Rightarrow$" Assume that $(a,b)=(c,d) \Rightarrow \{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}$

So we have $\{a\}=\{c\}$ or $\{a\}=\{c,d\}$.

Case 1 $\{a\}=\{c\} \Rightarrow a=c \Rightarrow \{\{a\},\{a,b\}\}=\{\{a\},\{a,d\}\}$.

Case 1.1. if $a=b$ then $(a,b)=\{\{a\}\} \Rightarrow$ we must also have $a=d \Rightarrow (c,d)=\{\{a\}\}$.

Case 1.2. if $a\neq b$ then we must have $a\neq d \Rightarrow$
$\Rightarrow \{a,b\}=\{a,d\} \Rightarrow b=d$

so $(c,d)=\{\{a\},\{a,b\}\}=(a,b)$.

Case 2 Assume $\{a\}=\{c,d\} \Rightarrow c=d=a \Rightarrow (c,d)=\{\{a\}\}$

So we must have $a=b$, so $(a,b)=\{\{a\}\}$.

The catesian product

René Descartes

$$A\times B=\{(a,b)\mid a\in A \text{ and } b\in B\}$$

$$A\times B\times C=(A\times B)\times C=\{((a,b),c)\mid (a,b)\in A\times B,\ c\in C\}$$

By allowing any kind of collections, the set theory of Cantor's leads to contradictions (paradoxes).

Russell's paradox (Bertrand Russell)

Let R be the set of all sets which do not contain themselves as elements.

i.e. $R=\{X \mid X \text{ set}, X\notin X\}$

There are two cases: $R\in R$ or $R\notin R$

Case 1 Assume $R \in R$, so R does not satisfy the condition in the definition of R $\Rightarrow R \notin R$ contradiction ↯

Case 2 Assume $R \notin R$, so R satisfies the condition in the definition of $R \Rightarrow R \in R$ contradiction.

Such paradoxes justify the need for axiomatic set theory

eg: NBG $\rightarrow$ von Neumann-Bernays-Gödel
ZF $\rightarrow$ Zermelo-Fraenkel

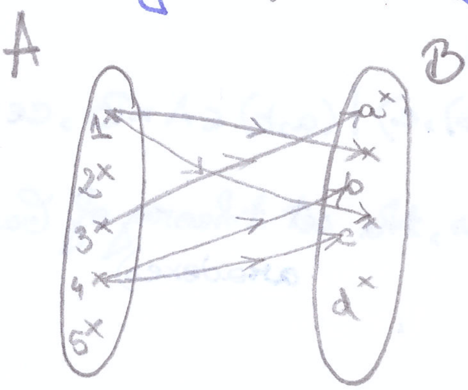
RELATIONS

Def A (binary) relation (correspondence) between the sets A and B is a triple $\rho = (A, B, R)$, where $R \subseteq A \times B$

A – domain, B – codomain, R – graph

Not. $(a,b) \in R \iff a \rho b$

We may use oriented graphs to depict relations:



$R = \{(1,b), (1,c), (3,a), (4,b), (4,c)\}$

• equality of relations Let $\rho = (A,B,R)$ and $\sigma = (C,D,S)$, where $S \subseteq C \times D$

Then $\rho = \sigma \iff \begin{cases} A = C & \text{(have the same domain)} \\ B = D & \text{(— " — codomain)} \\ R = S & \text{(— " — graph)} \end{cases}$

• inclusion $\rho \subseteq \sigma \Leftrightarrow \begin{cases} A=C \\ B=D \\ R\subseteq S \end{cases}$

Examples

1) $\emptyset = (A, B, \emptyset)$ empty relation

2) The universal relation is $(A, B, A\times B)$

3) $\rho = (A, B, R)$ is homogeneous if $A=B$ and $R\subseteq A\times A$

4) equality (diagonal) relation on a set : $1_A = (A, A, \Delta_A)$

where $\Delta_A = \{(a,a')\in A\times A' \mid a=a'\} = \{(a,a) \mid a\in A\}$

so $a\, 1_A\, a' \Leftrightarrow a=a' \quad \forall (a,a')\in A\times A$

Operations with relations

1) Union $\rho = (A,B,R)$, $\rho' = (A,B,R')$.

Then $\rho\cup\rho' \overset{def}{=} (A,B, R\cup R')$

i.e. $\forall (a,b)\in A\times B$ we have $a\,\rho\cup\rho'\,b \overset{def}{\Longleftrightarrow} a\rho b$ or $a\rho' b$

2) Intersection $\rho\cap\rho' = (A,B, R\cap R')$

so $a\,\rho\cap\rho'\,b \overset{def}{\Longleftrightarrow} a\rho b$ and $a\rho' b$

3) Inverse of a relation if $\rho = (A,B,R)$, then $\rho^{-1} = (B,A,R^{-1})$

where $R^{-1} = \{(b,a)\in B\times A \mid a\rho b\}$, so $a\rho b \Leftrightarrow b\rho^{-1}a$

4) Composition Let $\rho = (A,B,R)$, $\sigma = (C,D,S)$, where

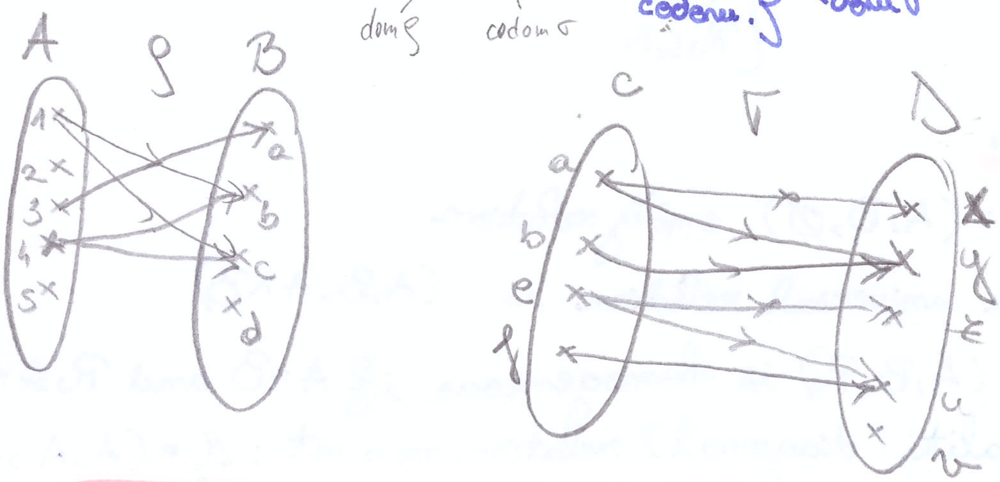
$S\subseteq C\times D$

$\sigma\circ\rho = (A, D, S\circ R)$, where $S\circ R \subseteq A\times F$

the second   the first   dom. ρ codom. σ

$S \circ R \overset{def}{=} \{(a,d) \in A \times D \mid \exists\, x \in B \cap C$ s.t. $a \rho x$ and $x \sigma d\}$

dom ρ — codom σ — codom. ρ — dom σ



In this case:

$S \circ R = \{(1,y),(3,x),(3,y),(4,y)\}$

$$\boxed{a\, \sigma \circ \rho\, d \overset{def}{\iff} \exists\, x\ \ x \in B \cap C \text{ s.t. } a \rho x \text{ and } x \sigma d}$$

Prop. 1) the equality rel. is neutral element w.r.t. „∘"

2) „∘" is associative