CHAPTER 11

Quaternions

Contents

11.1 Algebraic considerations

Definition. Denote the standard basis of $\mathbb{R}^4$ by $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and consider the bilinear form

$$\_\cdot\_ : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$$

given on the basis vectors by

$$\begin{array}{c|cccc} & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \hline 1 & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{i} & \mathbf{i} & -1 & \mathbf{k} & -\mathbf{j} \\ \mathbf{j} & \mathbf{j} & -\mathbf{k} & -1 & \mathbf{i} \\ \mathbf{k} & \mathbf{k} & \mathbf{j} & -\mathbf{i} & -1 \end{array}.$$

We denote $\mathbb{R}^4$ with the above multiplication by $\mathbb{H}$. The elements of $\mathbb{H}$ are called *quaternions*. The product is the *Hamilton product*.

Remark 11.1. From the definition we observe

1. The multiplication map on arbitrary quaternions $p = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $q = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ is
$$\begin{aligned} pq &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\mathbf{i} \\ &+ (a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1)\mathbf{j} + (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)\mathbf{k} \end{aligned} \tag{11.1}$$

2. Direct calculations show that $\mathbb{H}$ is an algebra, usually called *quaternion algebra*.

3. $\mathbb{H}$ is not commutative, $\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$.

4. $\mathbb{R} \cdot 1$ is a subfield of $\mathbb{H}$ so we just write $\mathbb{R}$ for it.

5. $\mathbb{C} = \mathbb{R} \cdot 1 + \mathbb{R} \cdot \mathbf{i}$ is a subfield of $\mathbb{H}$.

Definition. For a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$, a is the *real part* $\mathfrak{Re}(q)$ *of* q and $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ the *imaginary part* $\mathfrak{Im}(q)$ *of* q. We say that q is *real* if it equals its real part. We say that q is *purely imaginary* if it equals its imaginary part.

Proposition 11.2. *A quaternion is real if and only if it commutes with all quaternions, i.e. the center of* $\mathbb{H}$ *is* $\mathbb{R}$.

Proof. □

Proposition 11.3. *A quaternion is purely imaginary if and only if its square is real and non-pozitive.*

Proof. □

Definition. For a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$, the *conjugate of* q is

$$\overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} = \mathfrak{Re}(q) - \mathfrak{Im}(q) \in \mathbb{H}.$$

Proposition 11.4. *For* $p, q \in \mathbb{H}$ *and* $a \in \mathbb{R}$ *we have*

1. $\overline{p+q} = \overline{p} + \overline{q}$
2. $\overline{ap} = a\overline{p}$
3. $\overline{\overline{p}} = p$
4. $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$
5. $p \in \mathbb{R} \Leftrightarrow \overline{p} = p$
6. *p is purely imaginary* $\Leftrightarrow \overline{p} = -p$
7. $\mathfrak{Re}(p) = \frac{1}{2}(p + \overline{p})$
8. $\mathfrak{Im}(p) = \frac{1}{2}(p - \overline{p})$

Proof. □

11.2 Quaternions and $\mathbb{E}^4$

By construction $\mathbb{H}$ is $\mathbb{R}^4$ as real vector space, so we may view it as a 4-dimensional real affine space. If in addition we consier the 4-dimensional Euclidean structure we may identify $\mathbb{H}$ with $\mathbb{E}^4$. In particular, we may consider the standard scalar product $\langle \_, \_ \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \cong \mathbb{R}^4$.

Proposition 11.5 (Compare this with the similar statements for $\mathbb{C} \cong \mathbb{E}^2$)**.** *For $p, q \in \mathbb{H}$ we have*

1. $\langle p, q \rangle = \frac{1}{2}(\overline{p}q + \overline{q}p)$
2. $\langle p, p \rangle = \overline{p}p$
3. $||p|| = \sqrt{\overline{p}p}$

If in addition p and q are purely imaginary, we have

4. $\langle p, q \rangle = -\frac{1}{2}(pq + qp) = -\mathfrak{Re}(pq)$
5. $\langle p, p \rangle = -p^2$
6. $||p|| = \sqrt{-p^2}$
7. $\langle p, q \rangle = 0 \Leftrightarrow pq = -qp$.

Proof. □

Definition. With our identification $||q|| = (\overline{q}\,q)^{\frac{1}{2}}$ is the *norm* of the quaternion q. If $||q|| = 1$ we say that q is a *unit quaternion*.

Proposition 11.6. *For any $p, q \in \mathbb{H}$ we have*

$$||pq|| = ||p|| \cdot ||q||.$$

In particular, left and right multiplication by unit quaternions are isometries.

Proof. □

Proposition 11.7. *$\mathbb{H}$ is a skew field. The inverse of $q \in \mathbb{H} \setminus \{0\}$ is*

$$q^{-1} = \frac{\overline{q}}{||q||^2}.$$

Proof. □

11.3 Quaternions and rotations in $\mathbb{E}^3$

We identified $\mathbb{H}$ with $\mathbb{E}^4$. Next we view $\mathbb{E}^3$ as a subspace of $\mathbb{H}$ identifying it with purely imaginary quaterions $\mathfrak{Im}(\mathbb{H}) = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$.

Proposition 11.8. *Let q_1, q_2 be two quaternions with $a_i = \mathfrak{Re}\, q_i$, $v_i = \mathfrak{Im}\, q_i$. Making use of the scalar product and the vector product in $\mathbb{E}^3$ me have*

$$q_1 q_2 = (a_1 + v_1)(a_2 + v_2) = a_1 a_2 - \langle v_1, v_2 \rangle + a_2 v_1 + a_1 v_2 + v_1 \times v_2. \tag{11.2}$$

Proof. □

Proposition 11.9. *Let $v = v_i\mathbf{i} + v_j\mathbf{j} + v_k\mathbf{k} \in D(\mathbb{E}^3) \cong \mathfrak{Im}(\mathbb{H})$ be a unit quaternion and $p \in \mathbb{E}^3 \cong \mathfrak{Im}(\mathbb{H})$ a point. The rotation of p around the axis $\mathbb{R}v$ by an angle θ is given by*

$$p' = qpq^{-1}$$

where

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) v$$

Proof. □

11.4 Exercises

Exercise 1. Show that the conjugation map on quaternions restricted to $\mathbb{E}^3$ is an isometry. Does it preserve orientation?

Exercise 2. Check that the properties of the conjugation map given in Proposition 11.4 hold.

Exercise 3. Let $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{i}$. Consider the map

$$x \mapsto qxq^{-1} + \mathbf{i} + \mathbf{j}$$

restricted to $\mathbb{E}^3 \cong \mathfrak{Im}\mathbb{H}$.

1. Why is it a helical displacement?
2. Find its pace and Chasles' decomposition for this map.

Exercise 4. (hard, see last semester) Show that $(a \times b) \times c = \langle a, c \rangle b - \langle b, c \rangle a$.

Exercise 5. Let q be one of the quaternions

$$\mathbf{i}, \quad \mathbf{k}, \quad \cos(\alpha) + \sin(\alpha)\mathbf{j}, \quad \cos(\alpha)\mathbf{j} + \sin(\alpha)\mathbf{k}$$

Determine The matrix of the isometry obtained by

1. left and right multiplication with q,
2. conjugating with q, i.e. $x \mapsto qxq^{-1}$.

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