LECTURE

7

RIEMANN INTEGRALS. IMPROPER INTEGRALS

Riemann integrals

In what follows we assume that $a, b \in \mathbb{R}$, $a < b$.

Definition 7.1 *A partition of $[a, b]$ is a finite ordered set $P = (x_0, x_1, \ldots, x_n)$ of numbers s.t.*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

By a subinterval *of P we mean any interval $[x_{i-1}, x_i]$ with $i \in \{1, \ldots, n\}$.*
The norm *of P is the length of the largest subinterval of P, i.e.,*

$$\|P\| := \max\{x_i - x_{i-1} \mid i = \overline{1, n}\}.$$

If $\xi := (\xi_1, \ldots, \xi_n)$ is an ordered set of real numbers such that

$$\xi_i \in [x_{i-1}, x_i],\ \forall\, i \in \{1, \ldots, n\},$$

then (P, ξ) is called a tagged partition of $[a, b]$.

Definition 7.2 *Let $f : [a, b] \to \mathbb{R}$ be a function. By the the Riemann sum of f with respect to a tagged partition (P, ξ) of $[a, b]$, we mean*

$$\sigma(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Definition 7.3 *A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable on $[a, b]$ if there exists $I \in \mathbb{R}$ satisfying the following condition:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } |\sigma(f, P, \xi) - I| < \varepsilon, \forall (P, \xi) \text{ tagged partition with } \|P\| < \delta.$$

The family of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.

Remark 7.4 *(i) If $f \in \mathcal{R}[a,b]$, then $I \in \mathbb{R}$ satisfying the required condition in Definition 7.3 is uniquely determined and called the Riemann integral (or definite integral) of f on $[a,b]$. We denote*

$$\int_a^b f(x)dx = \int_a^b f := I.$$

(ii) If $f : [a,b] \to \mathbb{R}_+$ and $f \in \mathcal{R}[a,b]$, then $\mathcal{A} = \displaystyle\int_a^b f$ is the area of the set

$$(\text{hypo} f) \cap (\mathbb{R} \times \mathbb{R}_+) = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b],\ 0 \le y \le f(x)\}$$

located under the graph of f above the axis $0x$.
(iii) If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a,b]$.
(iv) If $f : [a,b] \to \mathbb{R}$ is monotone, then $f \in \mathcal{R}[a,b]$.
(v) If $f \in \mathcal{R}[a,b]$, then f is bounded.

Theorem 7.5 *For any $f, g \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$ we have:*

(i) $f + g \in \mathcal{R}[a,b]$ *and* $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g.$

(ii) $(\alpha f) \in \mathcal{R}[a,b]$ *and* $\displaystyle\int_a^b (\alpha f) = \alpha \int_a^b f.$

(iii) $(f \cdot g) \in \mathcal{R}[a,b]$.

(iv) $|f| \in \mathcal{R}[a,b]$.

(v) If $f \le g$, *then* $\displaystyle\int_a^b f \le \int_a^b g.$

Theorem 7.6 *Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then*

$$f \in \mathcal{R}[a,b] \iff f|_{[a,c]} \in \mathcal{R}[a,c] \text{ and } f|_{[c,b]} \in \mathcal{R}[c,b].$$

In this case, $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f.$

Theorem 7.7 (First Fundamental Theorem of Calculus) *Let $f \in \mathcal{R}[a,b]$. Define the function $F : [a,b] \to \mathbb{R}$ for all $t \in [a,b]$ by*

$$F(t) := \int_a^t f.$$

Then F is continuous. Moreover, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Theorem 7.8 (Second Fundamental Theorem of Calculus) *If $f \in \mathcal{R}[a,b]$ and $F : [a,b] \to \mathbb{R}$ is an antiderivative of f (that is, $F'(x) = f(x)$, $\forall\, x \in [a,b]$), then the* Leibniz-Newton Formula *holds:*

$$\int_a^b f = F(b) - F(a).$$

Improper integrals

Remark 7.9 *Consider the function $f : [0, 1) \to \mathbb{R}$,*

$$f(x) := \frac{1}{\sqrt{1-x^2}}.$$

Note that $x = 1$ is a vertical asymptote of f and hence the question of how one could define the area under the graph of f arises. To this end, let $t \in [0, 1)$ and $f|_{[0,t]}$. Then

$$\mathcal{A}_t = \int_0^t \frac{1}{\sqrt{1-x^2}} dx = \arcsin t$$

is the area under the graph of $f|_{[0,t]}$. One can now define the area under the graph of f as

$$\mathcal{A} = \lim_{\substack{t\to 1\\ t<1}} \mathcal{A}_t = \frac{\pi}{2}.$$

In a similar way one treats the problem for the function $f : [1, +\infty) \to \mathbb{R}$,

$$f(x) := \frac{1}{x^2}.$$

For $t \in [1, +\infty)$, $\mathcal{A}_t = \displaystyle\int_1^t \frac{1}{x^2} dx = 1 - \frac{1}{t}$ and so $\mathcal{A} = \displaystyle\lim_{t\to\infty} \mathcal{A}_t = 1$.

Definition 7.10 *Let $f : I \to \mathbb{R}$ be a function defined on an interval $I \subseteq \mathbb{R}$. We say that f is locally Riemann integrable on I if for all $a, b \in I$ with $a < b$ the function $f|_{[a,b]}$ is Riemann integrable on $[a, b]$.*

Remark 7.11 *(i) If $f \in \mathcal{R}[a, b]$, then f is locally Riemann integrable on $[a, b]$.*
(ii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is locally Riemann integrable on $\mathbb{R}$.

Definition 7.12 *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b) \to \mathbb{R}$ be a function, which is locally Riemann integrable on $[a, b)$. If the following limit exists in $\overline{\mathbb{R}}$, then it is called the improper integral of f on $[a, b)$:*

$$\int_a^b f(x)dx := \int_a^{b-0} f(x)dx := \lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)dx.$$

We say that the improper integral $\displaystyle\int_a^{b-0} f(x)dx$ is convergent *if it is finite; in this case, f is said to be* improperly integrable on $[a, b)$. *Otherwise, we say that the improper integral $\displaystyle\int_a^{b-0} f(x)dx$ is* divergent.

Definition 7.13 *Let $a \in \mathbb{R}$ and let $f : [a, +\infty) \to \mathbb{R}$ be a function, which is locally Riemann integrable on $[a, +\infty)$. If the following limit exists in $\overline{\mathbb{R}}$, then it is called the improper integral of f on $[a, +\infty)$:*

$$\int_a^{+\infty} f(x)dx := \lim_{t\to+\infty} \int_a^t f(x)dx.$$

We say that the improper integral $\displaystyle\int_a^{+\infty} f(x)dx$ is convergent *if it is finite; in this case, f is said to be* improperly integrable on $[a, +\infty)$. *Otherwise, we say that the improper integral $\displaystyle\int_a^{+\infty} f(x)dx$ is* divergent.

Definition 7.14 *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a, b] \to \mathbb{R}$ be a function, which is locally Riemann integrable on $(a, b]$. If the following limit exists in $\overline{\mathbb{R}}$, then it is called the improper integral of f on $(a, b]$:*

$$\int_a^b f(x)dx := \int_{a+0}^b f(x)dx := \lim_{\substack{t \to a \\ t > a}} \int_t^b f(x)dx.$$

We say that the improper integral $\int_{a+0}^b f(x)dx$ is convergent *if it is finite; in this case, f is said to be* improperly integrable on $(a, b]$. *Otherwise, we say that the improper integral $\int_{a+0}^b f(x)dx$ is* divergent.

Definition 7.15 *Let $b \in \mathbb{R}$ and let $f : (-\infty, b] \to \mathbb{R}$ be a function, which is locally Riemann integrable on $(-\infty, b]$. If the following limit exists in $\overline{\mathbb{R}}$, then it is called the improper integral of f on $(-\infty, b]$:*

$$\int_{-\infty}^b f(x)dx := \lim_{t \to -\infty} \int_t^b f(x)dx.$$

We say that the improper integral $\int_{-\infty}^b f(x)dx$ is convergent *if it is finite; in this case, f is said to be* improperly integrable on $(-\infty, b]$. *Otherwise, we say that the improper integral $\int_{-\infty}^b f(x)dx$ is* divergent.

Definition 7.16 *Let $a,\ b \in \overline{\mathbb{R}}$ with $a < b$ and let $f : (a, b) \to \mathbb{R}$ be a function, which is locally Riemann integrable on (a, b). If there exists $c \in (a, b)$ such that both improper integrals $\int_a^c f(x)dx$ and $\int_c^b$ are convergent (i.e., $f|_{(a,c]}$ and $f|_{[c,b)}$ are improperly integrable), then the improper integral of f on (a, b) is defined as:*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Remark 7.17 *There exists a close connection between the improper integrals on intervals of type $[a, +\infty)$ and the series of real numbers.*

Theorem 7.18 (Cauchy's Integral Test for Convergence of Series) *Let $f : [m, +\infty) \to [0, +\infty)$ be a decreasing function, where $m \in \mathbb{N}$. Then the improper integral $\int_m^{+\infty} f(x)dx$ is convergent if and only if the series $\sum_{n \geq m} f(n)$ is convergent.*

Example 7.19 (The generalized harmonic series) *For $\sum_{n \geq 1} \frac{1}{n^\alpha}$ with $\alpha > 0$, let us define the function $f : [1, +\infty) \to [0, +\infty)$, $f(x) = \frac{1}{x^\alpha}$. According to the Integral Test we recover the known fact that the generalized harmonic series converges for $\alpha > 1$ and diverges for $0 < \alpha \leq 1$.*

Theorem 7.20 (Comparison Test for Improper Integrals) *Let $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ with $a < b$ and let $f, g : [a, b) \to \mathbb{R}$ be locally Riemann integrable functions, such that*

$$\exists c \in [a, b) \text{ s.t. } \forall x \in [c, b),\ 0 \leq f(x) \leq g(x). \tag{7.1}$$

Then the following assertions hold true:

1° *If the improper integral* $\displaystyle\int_a^b g(x)dx$ *is convergent,*

then the improper integral $\displaystyle\int_a^b f(x)dx$ *is convergent.*

2° *If the improper integral* $\displaystyle\int_a^b f(x)dx$ *is divergent,*

then the improper integral $\displaystyle\int_a^b g(x)dx$ *g is divergent.*

Remark 7.21 *If f and g in the above theorem are nonnegative locally Riemann integrable functions on $[a,b)$ satisfying the following condition*

$$\exists \alpha, \beta > 0, \exists c \in [a,b) \ s.t. \ \forall x \in [c,b), \ \alpha g(x) \le f(x) \le \beta g(x),$$

then the improper integrals $\displaystyle\int_a^b f(x)dx$ *and* $\displaystyle\int_a^b g(x)dx$ *have the same nature.*

Corollary 7.22 *Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a,b) \to [0,+\infty)$ be a locally Riemann integrable function on $[a,b)$ and $p \in \mathbb{R}$ such that the following limit exists in $\overline{\mathbb{R}}$:*

$$L := \lim_{\substack{x \to b \\ x < b}} (b-x)^p f(x).$$

Then the following assertions hold true:

1° *If $p < 1$ and $L < +\infty$, then the improper integral* $\displaystyle\int_a^{b-0} f(x)dx$ *is convergent.*

2° *If $p \ge 1$ and $L > 0$, then the improper integral* $\displaystyle\int_a^{b-0} f(x)dx$ *is divergent.*

Proof. 1° By definition of L, there exists $c \in [a,b)$ such that

$$\forall x \in [c,b), \ (b-x)^p f(x) < L+1.$$

Thus,

$$\forall x \in [c,b), \ 0 \le f(x) < \frac{L+1}{(b-x)^p}.$$

Take $g : [a,b) \to \mathbb{R}$, $g(x) = \dfrac{L+1}{(b-x)^p}$. Since $p < 1$, the improper integral $\displaystyle\int_a^{b-0} g(x)dx$ is convergent. By Theorem 7.20 (1°) it follows that the improper integral $\displaystyle\int_a^{b-0} f(x)dx$ is convergent.

2° Let $r \in (0, L)$. By definition of L, there exists $c \in [a,b)$ such that

$$\forall x \in [c,b), \ r < (b-x)^p f(x).$$

Thus, we have

$$\forall x \in [c,b), \ 0 < \frac{r}{(b-x)^p} < f(x).$$

Take $h : [a,b) \to \mathbb{R}$, $h(x) = \dfrac{r}{(b-x)^p}$. Since $p \ge 1$, the improper integral $\displaystyle\int_a^{b-0} h(x)dx$ is divergent. Applying Theorem 7.20 (2°), we conclude that the improper integral $\displaystyle\int_a^{b-0} f(x)dx$ is divergent. $\square$

Corollary 7.23 *Let $a, b \in \mathbb{R}$ with $a < b$, $f : (a, b] \to [0, +\infty)$ be a locally Riemann integrable function on $[a, b)$ and $p \in \mathbb{R}$ such that the following limit exists in $\overline{\mathbb{R}}$:*

$$L := \lim_{\substack{x \to a \\ x > a}} (x - a)^p f(x).$$

Then the following assertions hold true:

1° *If $p < 1$ and $L < +\infty$, then the improper integral $\displaystyle\int_{a+0}^{b} f(x)dx$ is convergent.*

2° *If $p \geq 1$ and $L > 0$, then the improper integral $\displaystyle\int_{a+0}^{b} f(x)dx$ is divergent.*

Corollary 7.24 *Let $a \in \mathbb{R}$, $f : [a, +\infty) \to [0, +\infty)$ be a locally Riemann integrable function on $[a, +\infty)$ and $p \in \mathbb{R}$ such that the following limit exists in $\overline{\mathbb{R}}$:*

$$L := \lim_{x \to \infty} x^p f(x).$$

Then the following assertions hold true:

1° *If $p > 1$ and $L < +\infty$, then the improper integral $\displaystyle\int_{a}^{+\infty} f(x)dx$ is convergent.*

2° *If $p \leq 1$ and $L > 0$, then the improper integral $\displaystyle\int_{a}^{+\infty} f(x)dx$ is divergent.*