CHAPTER 10

Affine transformations in the Euclidean spaces

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10.1 Affine transformations in dimension 3

Theorem 10.1 (Euler). *All displacements in* $\mathrm{AGL}(\mathbb{E}^3)$ *which fix a point are either the identity or a rotation through an axis containing that point.*

Proof. □

Remark 10.2. Recall that the *trace* of a matrix is the sum of the diagonal entries of the matrix. It is a linear algebra fact that the trace of a matrix does not change when performing a base change. In other words the trace is a property of the corresponding linear map.

The next proposition follows directly from the proof of the above theorem.

Proposition 10.3. *The angle* θ *of a rotation* $A \in \mathrm{SO}(3)$ *is given by the equation*

$$\mathrm{trace}(A) = 1 + 2\cos(\theta).$$

Remark 10.4. Rotations around the coordinate axes are standard examples

$$[\mathrm{Rot}_x(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$[\mathrm{Rot}_x(\theta)] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\mathrm{Rot}_x(\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The composition of these maps is $\mathrm{Rot}_x(\psi) \circ \mathrm{Rot}_y(\theta) \circ \mathrm{Rot}_z(\phi) =$

$$\begin{bmatrix} \cos\psi\cos\theta\cos\psi - \sin\psi\sin\phi & -\cos\psi\cos\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta \\ \sin\psi\cos\theta\cos\phi + \cos\psi\sin\phi & -\sin\psi\cos\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}.$$

All orientation preserving rotations are of this form, i.e. any matrix in SO(3) can be written in this form for some $\psi \in [0, 2\pi[$, $\theta \in [0, \pi[$ and $\phi \in [0, 2\pi[$. The angles ψ, θ and ϕ are called *Euler's angles*. We will see later, in Chapter **??**, that there is a better way of dealing with rotations in $\mathbb{E}^3$.

Proposition 10.5 (Euler-Rodrigues)**.** *Let* $\mathbf{v} \in D(\mathbb{E}^3)$ *be a unit vector and* $\theta \in \mathbb{R}$. *The rotation of angle* θ *and axis* $\mathbb{R}\mathbf{v}$ *(here* $\|\mathbf{v}\| = 1$*) is given by*

$$\mathrm{Rot}_{\mathbf{v},\theta}(x) = \cos(\theta)x + \sin(\theta)(\mathbf{v} \times x) + (1 - \cos(\theta))\langle \mathbf{v}, x\rangle \mathbf{v}. \tag{10.1}$$

Proof. □

10.2 Classification of isometries in dimension 3

Definition. A *helical displacement* is the composition of a rotation and a translation in the direction of the rotation axis.

Example 10.6. We can construct obvious examples if we use rotations around the coordinate axes:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \delta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{for} \quad \delta > 0.$$

Theorem 10.7 (Chasles)**.** *If* φ *is a displacement of* $\mathbb{E}^3$ *then exactly one of the following holds*

1. φ *is the identity*
2. φ *is a translation*
3. φ *is a rotation*
4. φ *is a helical displacement*

Proof. □

Remark 10.8. If $\varphi : x \mapsto Ax + b$ is a helical displacement, then b need not be parallel to the direction of the axis of the helical displacement.

Definition. The decomposition

$$\varphi : x \mapsto (Ax + b_2) + b_1 \tag{10.2}$$

with $Ax + b_2$ a rotation and b_1 parallel to the axis of rotation is called *Chasles' decomposition of φ*.

The unique line left invariant by a rotation or helical displacement is called *axis*. In case of a helical displacement (10.3), the points on the axis are translated by

$$\delta_\varphi = \|b_1\|$$

which we call *the pace of φ*.

Proposition 10.9. *Let $x \to Ax + b$ be a rotation or a helical displacement. The axis of this transformation is*

$$\left\{a(t) = \frac{1}{2}b + \frac{\tau}{2}\mathbf{v} \times b + t\mathbf{v} : t \in \mathbb{R}\right\}$$

where $\mathbf{v}$ is an eigenvector of unit length of A for the eigenvalue 1 and $\tau = \cot(\frac{\theta}{2})$ where the angle θ is given by $\operatorname{trace}(A) = 1 + 2\cos(\theta)$.

Proof. □

Theorem 10.10. *Let φ be an indirect isometry of $\mathbb{E}^3$. Then φ leaves a plane Π invariant. Moreover, exactly one of the following holds*

1. *φ is a reflection in Π*
2. *φ is a gliding reflection in Π, i.e. a reflection in the plane Π composed with a translation parallel to Π*
3. *φ is a reflection in Π composed with a rotation with axis orthogonal to Π.*

Proof. □

10.3 Homogeneous coordinates

An affine map $\varphi : x \mapsto Ax + v$ can be represented as follows

$$\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}.$$

For this to make sense, we view $\mathbb{A}(\mathbb{R}^n)$ as a hyperplane in $\mathbb{A}(\mathbb{R}^{n+1})$ with

$$\mathbb{R}^n \ni \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$$

The matrix

$$\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix}$$

is then called *the homogeneous matrix of the affine map* φ w.r.t. the standard basis.

10.4 Exercises

Exercise 1. Show that

$$\langle v \times w, v \times w \rangle = \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle \langle w, v \rangle.$$

Exercise 2. Show that if $A \in \mathrm{O}(3)$ has determinant -1 then it has -1 as eigenvalue.

Exercise 3. Let $A \in \mathrm{SO}(3)$ and $x, y \in \mathbb{R}^3$. Show that

$$A(x \times y) = (Ax) \times (Ay).$$

Show by means of examples that if $A \notin \mathrm{SO}(3)$ then the above equality doesn't hold.

Exercise 4. Calculate the matrix of a rotation with angle $\theta = \frac{\pi}{4}$ around the axis passing through the origin and having direction $(1,0,1)$. Verify that the matrix is orthogonal.

Exercise 5. Let $\varphi, \psi \in \mathrm{AGL}(\mathbb{R}^3)$ be the rotations obtained by composing a rotation by $\frac{\pi}{4}$ around the z-axis with a rotation by $\frac{\pi}{6}$ around the y axis in the two possible ways. Find the axes of these isometries.

Exercise 6. Verify that the matrices

$$A = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{9}{11} & -\frac{2}{11} & \frac{6}{11} \\ \frac{6}{11} & -\frac{6}{11} & \frac{7}{11} \\ \frac{2}{11} & \frac{9}{11} & \frac{6}{11} \end{bmatrix}$$

are in $\mathrm{SO}(3)$ and find their angles of rotation.

Exercise 7. Find Chasles' decomposition for $x \mapsto Ax + b$ with A as in the previous exercise and

$$b = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

In particular, indicate the angle and the pace.

Exercise 8. Consider the displacement of $\mathbb{E}^2$ given by the homogeneous matrix

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1+\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1-\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

what is its center?

Exercise 9. For two affine maps $\varphi : x \mapsto Ax + b$ and $\psi : x \mapsto Cx + d$ give $\varphi \circ \psi$ in homogeneous coordinates. Assume that φ is an affine transformation and give φ^{-1} in homogeneous coordinates as well as $\varphi \circ \psi \circ \varphi^{-1}$ (the *conjugate of ψ by φ*).

Exercise 10. For an affine transformation φ, let $[\varphi]_h$ denote the homogeneous matrix of φ with respect to the standard basis.

$$\text{Is} \quad \mathrm{AGL}(\mathbb{R}^n) \ni \varphi \mapsto [\varphi]_h \in \mathrm{GL}_{n+1}(\mathbb{R}) \quad \text{a homomorphism of groups?}$$

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