LECTURE

3

SERIES OF REAL NUMBERS. SERIES WITH NONNEGATIVE TERMS (I)

Definition 3.1 *To any given sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers we attach another sequence,* $(s_n)_{n\in\mathbb{N}}$*, defined for all* $n \in \mathbb{N}$ *by*

$$s_n := x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k \, .$$

The couple $\big((x_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}\big)$ *is called* series *and it is denoted by*

$$\sum_{n\geq 1} x_n \, .$$

For any $n \in \mathbb{N}$*, the number* s_n *is called the* partial sum of the series up to rank n*. If the sequence* $(s_n)_{n\in\mathbb{N}}$ *of partial sums converges (resp. diverges), we say that* the series $\sum_{n\geq 1} x_n$ is convergent (resp. divergent)*. If the sequence* $(s_n)_{n\in\mathbb{N}}$ *of partial sums has a limit we say that* the series has a sum*; in this case, the sum of the series is denoted by*

$$\sum_{n=1}^{\infty} x_n := \lim_{n\to\infty} s_n \, .$$

Remark 3.2 *If* $(x_n)_{n\geq m}$ *is a sequence of real numbers (where* $m \in \mathbb{Z}$*), then we consider a series of type*

$$\sum_{n\geq m} x_n.$$

It is easy to check that, for any $p \in \mathbb{N}$*, the series* $\sum\limits_{n\geq m} x_n$ *has a sum (in* $\overline{\mathbb{R}}$*) if and only if the series* $\sum\limits_{n\geq m+p} x_n$ *has a sum (in* $\overline{\mathbb{R}}$*) and, in this case, we have*

$$\sum_{n=m}^{\infty} x_n = x_m + x_{m+1} + \ldots x_{m+p-1} + \sum_{n=m+p}^{\infty} x_n.$$

Example 3.3 (The geometric series) *For any number $q \in \mathbb{R}$, consider the so-called geometric series*

$$\sum_{n\geq 0} q^n$$

where, by convention, $q^0 = 1$ even if $q = 0$. We distinguish three cases:

- *If $q \in (-\infty, -1]$, then the geometric series has no sum, hence it is divergent;*
- *If $q \in (-1, 1)$, i.e., $|q| < 1$, then the geometric series is convergent and has the sum*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q};$$

- *If $q \in [1, \infty)$, then the geometric series has the sum $\displaystyle\sum_{n=0}^{\infty} q^n = +\infty$, hence it is divergent.*

Indeed, the sequence of partial sums of the geometric series is given by

$$s_n := 1 + q + \ldots + q^n = \begin{cases} \frac{1-q^{n+1}}{1-q}, & \text{if } q \neq 1, \\ n+1, & \text{if } q = 1. \end{cases}$$

Therefore, if $|q| < 1$, then $\displaystyle\lim_{n\to\infty} s_n = \frac{1}{1-q}$. If $q \leq -1$, the sequence (s_n) has no limit, hence it diverges. Finally, when $q \geq 1$, the sequence (s_n) diverges while $\displaystyle\lim_{n\to\infty} s_n = +\infty$.

Example 3.4 (The harmonic series) *The so-called* harmonic series

$$\sum_{n\geq 1} \frac{1}{n}$$

is divergent and has the sum

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Indeed, denoting the partial sums by $s_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{2^n} + \ldots + \frac{1}{2^n}\right) = 1 + \frac{n}{2} \end{aligned}$$

hence $\sup\limits_{n\in\mathbb{N}} s_n \geq \sup\limits_{n\in\mathbb{N}} s_{2^n} = +\infty$. On the other hand, we have $s_n < s_{n+1}$ for all $n \in \mathbb{N}$.
By Theorem 2.18 (Weierstrass), we infer that $\displaystyle\lim_{n\to\infty} s_n = +\infty$.

Example 3.5 (Euler's number as a sum of a series) *The series*

$$\sum_{n\geq 0} \frac{1}{n!}$$

is convergent and its sum is the Euler's number, i.e.,

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Indeed, let $s_n := 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$, $n \in \mathbb{N}$. Recall that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (see Exercise 3 of Seminar 2). By Newton's Binomial Formula,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots + \frac{n(n-1)\cdot\ldots\cdot 1}{n!}\cdot\frac{1}{n^n}\\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n}\right) \le s_n.\end{aligned}$$

Now, consider an arbitrary given $n \in \mathbb{N}^*$. Then, for any $m \ge n$, we have

$$\begin{aligned}\left(1+\frac{1}{m}\right)^m &= 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{m}\right)+\\ &\quad+\ldots+\frac{1}{m!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{m-1}{m}\right)\\ &\ge 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{m}\right).\end{aligned}$$

Letting $m \to \infty$, we have that $e \ge s_n$. Thus, $\forall n \in \mathbb{N}^*$, $\left(1+\frac{1}{n}\right)^n \le s_n \le e$. Letting $n \to \infty$, we obtain that $\lim\limits_{n\to\infty} s_n = e$, so $\sum\limits_{n\ge 1}\frac{1}{n!}$ is convergent and $\sum\limits_{n=1}^{\infty}\frac{1}{n!} = e$.

Example 3.6 (Telescoping series) *Given a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, we say that*

$$\sum_{n\ge 1}(x_n - x_{n+1})$$

is a telescoping series. *This series is convergent if and only if the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent. More precisely, we have*

$$\sum_{n=1}^{\infty}(x_n - x_{n+1}) = x_1 - \lim_{n\to\infty} x_n.$$

For instance, consider the series

$$\sum_{n\ge 1}\frac{1}{n(n+1)}.$$

It is easily seen that

$$\frac{1}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},\ \forall n \in \mathbb{N},$$

hence we have a telescopic series. Denoting its partial sums by

$$s_n := \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)},\ n \in \mathbb{N},$$

it follows that $s_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$.

Thus, $\lim\limits_{n\to\infty} s_n = 1$, so $\sum\limits_{n\ge 1}\frac{1}{n(n+1)}$ is convergent and its sum is

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1.$$

Proposition 3.7 *Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be convergent series and let $c \in \mathbb{R}$. Then, the following assertions hold:*

a) The series $\sum_{n\geq 1}(x_n + y_n)$ is convergent and

$$\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

b) The series $\sum_{n\geq 1}(c\,x_n)$ is convergent and

$$\sum_{n=1}^{\infty}(c\,x_n) = c\sum_{n=1}^{\infty} x_n.$$

Proposition 3.8 (The n^{th} Term Test – necessary condition for convergence) *If a series of real numbers $\sum_{n\geq 1} x_n$ converges, then its general term converges to zero, i.e., $\lim_{n\to\infty} x_n = 0$.*

Remark 3.9 *The condition $\lim\limits_{n\to\infty} x_n = 0$ is not sufficient for the convergence of a series $\sum_{n\geq 1} x_n$. For instance, the harmonic series is divergent while its general term converges to zero (see Example 3.4).*

Corollary 3.10 (Sufficient conditions for divergence of series) *A series $\sum_{n\geq 1} x_n$ is divergent whenever*

(i) the sequence (x_n) is divergent

or

(ii) the sequence (x_n) converges and $\lim_{n\to\infty} x_n \neq 0$.

Theorem 3.11 (Cauchy's Criterion for convergence of series) *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The following assertions are equivalent:*

1° The series $\sum_{n\geq 1} x_n$ is convergent.

2° For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ and $p \in \mathbb{N}$.

Corollary 3.12 (Sufficient condition for convergence of series) *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying the following two conditions:*

1° $|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| \leq a_n$ for all $n, p \in \mathbb{N}$;

2° $(a_n)_{n\in\mathbb{N}}$ converges to zero, i.e., $\lim_{n\to\infty} a_n = 0$.

Then the series $\sum_{n\geq 1} x_n$ is convergent.

Series with nonnegative terms (I)

Lemma 3.13 (Convergence of series vs boundedness of their partial sums) *Let $\sum_{n\geq 1} x_n$ be a series with nonnegative terms (i.e., $x_n \geq 0$ for all $n \in \mathbb{N}$) and let $(s_n)_{n\in\mathbb{N}}$ be the sequence of its partial sums. Then the series $\sum_{n\geq 1} x_n$ has a sum in $\mathbb{R} \cup \{+\infty\}$, namely*

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n.$$

Moreover, the following assertions are equivalent:

1° The series $\sum_{n\geq 1} x_n$ converges.

2° The sequence $(s_n)_{n\in\mathbb{N}}$ is bounded.

Proof. For any $n \in \mathbb{N}$ we have $x_{n+1} \geq 0$, hence $s_{n+1} = s_n + x_{n+1} \geq s_n$. Therefore the sequence (s_n) is increasing. By Theorem 2.18 (Weierstrass) it follows that (s_n) has a limit in $\overline{\mathbb{R}}$. More precisely, (s_n) is convergent if and only if it is bounded. $\square$

Remark 3.14 *If a series $\sum_{n\geq 1} x_n$ is convergent, then (in view of Propositions 2.17 and 3.8) the sequence (s_n) must be bounded, but this condition is not equivalent to the convergence of $\sum_{n\geq 1} x_n$. For instance, consider the series*

$$\sum_{n\geq 1}(-1)^n.$$

The sequence of partial sums of this series is given by

$$s_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Obviously, the sequence (s_n) is bounded, but does not converge (because it possesses two subsequences converging to different limits). Therefore the series $\sum_{n\geq 1}(-1)^n$ is divergent.

Theorem 3.15 (Cauchy's Condensation Criterion) *Let $\sum\limits_{n\geq 1} x_n$ be a series with nonnegative terms. If the sequence $(x_n)_{n\in\mathbb{N}}$ is decreasing, then the following assertions are equivalent:*

1° *The given series, $\sum\limits_{n\geq 1} x_n$, converges.*

2° *The series $\sum\limits_{n\geq 0} 2^n \cdot x_{2^n}$ converges.*

Example 3.16 (The generalized harmonic series) *For every number $p \in \mathbb{R}$ consider the so-called* generalized harmonic series

$$\sum_{n\geq 1} \frac{1}{n^p}.$$

This series is convergent if and only if $p > 1$.

Indeed, denote $x_n := \frac{1}{n^p}$ for all $n \in \mathbb{N}$. If $p \leq 0$, then we clearly have $\lim_{n\to\infty} x_n \neq 0$, hence the series $\sum_{n\geq 1} x_n$ diverges according to Corollary 3.10. Assume now that $p > 0$. Then the sequence (x_n) is decreasing and has positive terms. In this case, according to Cauchy's condensation criterion, the series $\sum_{n\geq 1} x_n$ converges if and only if the series $\sum\limits_{n\geq 0} 2^n \cdot x_{2^n}$ converges. The latter series actually is a geometric series, since for every $n \in \mathbb{N} \cup \{0\}$ we have $2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{2^{np}} = (2^{1-p})^n$. In view of Example 3.3 we deduce that the series $\sum_{n\geq 1} x_n$ converges if and only if $2^{1-p} < 1$, i.e., $p > 1$.

Remark 3.17 *For $p = 1$ we recover the classical harmonic series (see Example 3.4), which is divergent and has the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Actually, the generalized harmonic series has a sum in $\overline{\mathbb{R}}$ for every $p \in \mathbb{R}$. More precisely, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = +\infty \text{ if } p \in (-\infty, 1] \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} =: \zeta(p) \in (1, +\infty) \text{ if } p \in (1, +\infty)$$

where $\zeta : (1, \infty) \to (1, +\infty)$ represents the Riemann zeta function. Notice that ζ is strictly decreasing. In particular, for $p \in \{2, 3, 4\}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645, \ \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202 \ (\textit{Apéry's constant}), \ \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.082\,.$$

Theorem 3.18 (Comparison Test) *Let* $\sum\limits_{n\geq 1} x_n$ *and* $\sum\limits_{n\geq 1} y_n$ *be series with nonnegative terms. If there is* $n_0 \in \mathbb{N}$ *s.t.*

$$x_n \leq y_n \text{ for all } n \geq n_0,$$

then the following assertions hold:

(i) If $\sum\limits_{n\geq 1} y_n$ *is convergent, then* $\sum\limits_{n\geq 1} x_n$ *is convergent.*

(ii) If $\sum\limits_{n\geq 1} x_n$ *is divergent, then* $\sum\limits_{n\geq 1} y_n$ *is divergent.*

Proof. (i) Without loss of generality assume that $n_0 = 1$. Consider the partial sums

$$s_n := x_1 + x_2 + \ldots + x_n \quad \text{and} \quad \tilde{s}_n := y_1 + y_2 + \ldots + y_n, \ \forall\, n \in \mathbb{N}.$$

Since $\sum_{n\geq 1} y_n$ is convergent, it follows that $(\tilde{s}_n)$ is bounded (by Lemma 3.13), hence $\exists M > 0$ such that $\tilde{s}_n \leq M$, $\forall\, n \in \mathbb{N}$. Then $s_n \leq \tilde{s}_n \leq M$, $\forall\, n \in \mathbb{N}$. Thus, (s_n) is bounded and therefore $\sum_{n\geq 1} x_n$ is convergent (by Lemma 3.13).

Assertion (ii) is an equivalent counterpart of (i). $\square$

Corollary 3.19 (Comparison Test in practical form) *Let* $\sum\limits_{n\geq 1} x_n$ *be a series with nonnegative terms and let* $\sum\limits_{n\geq 1} y_n$ *be a series with positive terms, such that the following limit exists:*

$$\ell := \lim_{n\to\infty} \frac{x_n}{y_n} \in [0, +\infty) \cup \{+\infty\}.$$

The following assertions hold:

1° *If* $\ell \in (0, +\infty)$*, then the series* $\sum\limits_{n\geq 1} x_n$ *and* $\sum\limits_{n\geq 1} y_n$ *have the same nature, i.e., they are both convergent or both divergent.*

2° *If* $\ell = 0$*, then*

a) If $\sum\limits_{n\geq 1} y_n$ *converges, then* $\sum\limits_{n\geq 1} x_n$ *converges.*

b) If $\sum\limits_{n\geq 1} x_n$ *diverges, then* $\sum\limits_{n\geq 1} y_n$ *diverges.*

3° *If* $\ell = +\infty$*, then*

a) If $\sum\limits_{n\geq 1} x_n$ *converges, then* $\sum\limits_{n\geq 1} y_n$ *converges.*

b) If $\sum\limits_{n\geq 1} y_n$ *diverges, then* $\sum\limits_{n\geq 1} x_n$ *diverges.*

Example 3.20 *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms and let* $p \in \mathbb{R}$*. Assume that the following limit exists*

$$\ell := \lim_{n\to\infty} (n^p \cdot x_n) \in [0, \infty) \cup \{+\infty\}.$$

Applying the Comparison Test in practical form (Corollary 3.19) for the given series and the generalized harmonic series $\sum_{n\geq 1} y_n := \sum_{n\geq 1} \frac{1}{n^p}$*, we deduce that (see Exercise 3.16):*

1° *If* $0 \leq \ell < \infty$ *and* $p > 1$*, then* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *If* $0 < \ell \leq \infty$ *and* $p \leq 1$*, then* $\sum_{n\geq 1} x_n$ *is divergent.*

Corollary 3.21 *Let* $\sum\limits_{n\geq 1} x_n$ *and* $\sum\limits_{n\geq 1} y_n$ *be series with positive terms. If there is* $n_0 \in \mathbb{N}$ *s.t.*

$$\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n} \text{ for all } n \geq n_0,$$

then the following assertions hold:

a) If $\sum_{n\geq 1} y_n$ *converges, then* $\sum_{n\geq 1} x_n$ *converges.*

b) If $\sum_{n\geq 1} x_n$ *diverges, then* $\sum_{n\geq 1} y_n$ *diverges.*

Example 3.22 *The following series is divergent:*

$$\sum_{n\geq 1} (2-\sqrt{e})\cdot(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e}).$$

Indeed, letting $y_n := (2-\sqrt{e})\cdot(2-\sqrt[3]{e})\cdot\ldots\cdot(2-\sqrt[n]{e})$ *and taking into account that* $e < \left(1+\frac{1}{n}\right)^{n+1}$ *for all* $n \in \mathbb{N}$ *(see Exercise 3 of Seminar 2), we infer*

$$\frac{y_{n+1}}{y_n} = 2 - \sqrt[n+1]{e} > 1 - \frac{1}{n} = \frac{n-1}{n} = \frac{y_{n+1}}{y_n}$$

where $y_n := \frac{1}{n-1}$ *for all* $n \geq 2$. *Since the harmonic series* $\sum_{n\geq 2} y_n$ *diverges, we deduce by Corollary 3.21 that the given series diverges, too.*

Theorem 3.23 (d'Alembert's Ratio Test) *Let* $\sum_{n\geq 1} x_n$ *be a series with positive terms. The following assertions hold:*

1° *If* $\exists q \in (0,1), \exists n_0 \in \mathbb{N}$ *s.t.* $\dfrac{x_{n+1}}{x_n} \leq q,\ \forall\, n \geq n_0$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *If* $\exists n_0 \in \mathbb{N}$ *s.t.* $\dfrac{x_{n+1}}{x_n} \geq 1,\ \forall\, n \geq n_0,$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

3° *If the following limit exists*

$$D := \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0,+\infty)\cup\{+\infty\},$$

then we have

a) If $D < 1$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

b) If $D > 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

Example 3.24 *The series* $\sum_{n\geq 1} \dfrac{(n!)^2}{(2n)!}$ *is convergent. Indeed, since*

$$D := \lim_{n\to\infty} \frac{[(n+1)!]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{n+1}{2(2n+1)} = \frac{1}{4} < 1,$$

it follows by de Ratio Test (Theorem 3.23) that the given series is convergent.

Theorem 3.25 (Cauchy's Root Test) *Let* $\sum_{n\geq 1} x_n$ *be a series with nonnegative terms.*

1° *If* $\exists q \in [0,1), \exists n_0 \in \mathbb{N}$ *s.t.* $\sqrt[n]{x_n} \leq q,\ \forall\, n \geq n_0$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

2° *If* $\exists n_0 \in \mathbb{N}$ *s.t.* $\sqrt[n]{x_n} \geq 1,\ \forall\, n \geq n_0$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

3° *If the following limit exists*

$$C = \lim_{n\to\infty} \sqrt[n]{x_n} \in [0,+\infty)\cup\{+\infty\},$$

then we have

a) If $C < 1$, *then* $\sum_{n\geq 1} x_n$ *is convergent.*

b) If $C > 1$, *then* $\sum_{n\geq 1} x_n$ *is divergent.*

Example 3.26 *The series* $\sum_{n\geq 1} \dfrac{n^p}{2^n}$ *is convergent for every* $p \in \mathbb{R}$. *Indeed, since*

$$C := \lim_{n\to\infty} \sqrt[n]{\frac{n^p}{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^p}{2} = \frac{1}{2} < 1,$$

it follows by de Root Test (Theorem 3.25) that the given series is convergent.