CHAPTER 9

Affine transformations and complex numbers

Contents

9.1 $\mathbb{C}$ and $\mathbb{E}^2$

The field of complex numbers is a 2-dimensional real vector space with basis $(1, \mathbf{i})$. Choosing an orthonormal basis $(\mathbf{u}, \mathbf{v})$ for the Euclidean plane $\mathbb{E}^2$ we may identify it with $\mathbb{R}^2$

$$\mathbb{C} \leftrightarrow \mathbb{R}^2 \leftrightarrow \mathbb{E}^2 \quad \text{with} \quad a + bi \leftrightarrow (a, b) \leftrightarrow a\mathbf{u} + b\mathbf{v}.$$

Using this identification and the additional algebraic structure of $\mathbb{C}$ we obtain a more flexible framework for 2-dimensional geometry.

9.1.1 Lines

Proposition 9.1. *The equation of a (real) line in $\mathbb{C}$ is*

$$\overline{\alpha}\,\overline{z} + \alpha z + \beta = 0$$

with $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{R}$.

Proof. □

9.1.2 Scalar product and oriented area

From the identification $\mathbb{C} \cong \mathbb{E}^2$ and $\mathbb{C} \cong D(\mathbb{E}^2)$ it is easy to see that the scalar product can be expressed as

$$\langle z_1, z_2 \rangle = \frac{1}{2}(\overline{z_1} z_2 + z_1 \overline{z_2})$$

and the distance between two points is

$$d(z_1, z_2) = |z_1 - z_2| = \sqrt{\langle z_1 - z_2, z_1 - z_2 \rangle}.$$

Because of the identification, all known properties for the scalar product and the distance hold.

Clearly, since we are in dimension 2 we don't have a well defined vector product. However we do have the notion of oriented area. If we view $\mathbb{E}^2$ in $\mathbb{E}^3$ then, the norm of the vector product

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \sin \measuredangle(\mathbf{a}, \mathbf{b})$$

is the oriented area of the parallelogram spanned by the vectors $\mathbf{a}, \mathbf{b} \in D(\mathbb{E}^2)$.

In $\mathbb{C}$, consider now the product

$$z_1 \wedge z_2 = \frac{1}{2}(\overline{z_1} z_2 - z_1 \overline{z_2})$$

Then

$$z_1 \wedge z_2 = \varepsilon \mathbf{i} |z_1| \cdot |z_2| \cdot \sin \measuredangle(z_1, z_2) = \varepsilon \mathbf{i} \cdot \|z_1 \times z_2\|.$$

Where the last equality is to be understood in $\mathbb{E}^3$, and where

$$\varepsilon = \begin{cases} 1 & \text{if the basis } (z_1, z_2) \text{ is right oriented,} \\ -1 & \text{if the basis } (z_1, z_2) \text{ is left oriented.} \end{cases}$$

A relation between the two products is given by

$$\langle z_1, z_2 \rangle^2 + |z_1 \wedge z_2|^2 = |z_1|^2 \cdot |z_2|^2.$$

9.2 Affine transformations in dimension 2

9.2.1 Translations, Reflections, Rotations

It is clear what translations are, namely maps of the form

$$z \mapsto z + c$$

for some $c \in \mathbb{C}$. They are affine maps and they are isometries.

Next we consider *orthogonal reflections*. We have seen reflections along a certain affine subspace. Here we restrict to reflections in a line l along the orthogonal direction to l and denote them (as before) by Ref_l.

Proposition 9.2. *For the line $l : \overline{\alpha}\,\overline{z} + \alpha z + \beta = 0$ the orthogonal reflection $\mathrm{Ref}_l : \mathbb{C} \to \mathbb{C}$ is given by*

$$\mathrm{Ref}_l(z) = -\frac{\overline{\alpha}}{\alpha}\overline{z} - \frac{\beta}{\alpha}.$$

Proof. □

Next, Rotations. These are particularly easy to describe due to the geometric interpretation of multiplication in $\mathbb{C}$. The *rotation of angle θ around z_0* is

$$\mathrm{Rot}_{z_0,\theta} : \mathbb{C} \to \mathbb{C}, \quad \text{given by} \quad \mathrm{Rot}_{z_0,\theta}(z) = \varepsilon(z - z_0) + z_0$$

where $\varepsilon = \cos\theta + \mathbf{i}\sin\theta$, so $\theta = \arg\varepsilon$. Notice that for a map

$$f : z \mapsto az + b, \quad \text{with} \quad |a| = 1 \text{ and } a \neq 1,$$

the center of rotation is

$$z_0 = \frac{b}{1-a}.$$

Proposition 9.3. *The composition of two rotations is a rotation or a translation.*

Proof. □

Remark 9.4. Compositions of rotations with distinct centers is not commutative.

Retruning to isometries in general we can give a shorter proof of the following result.

Proposition 9.5. *An isometry $f : \mathbb{C} \to \mathbb{C}$ is of the form*

$$f(x) = az + b \quad or \quad f(x) = a\overline{z} + b$$

where $a, b \in \mathbb{C}$ and $|a| = 1$.

Proof. □

9.2.2 Chasles' classification

For a fixed point $z_0 \in \mathbb{C}$ and a $k \in \mathbb{R}^*$, the *homothety with center z_0 and dilation factor k* is

$$H_{z_0,k} : \mathbb{C} \to \mathbb{C}, \quad H_{z_0,k}(z) = k(z - z_0) + z_0.$$

Homotheties are also called *homogeneous dilations* since they are dilations of with the same dilation factor in all directions.

Remark 9.6. When $z_0 = \mathbf{0}$, the transformation is just multiplication by a non-zero scalar in subfield $\mathbb{R}$ of $\mathbb{C}$.

Clearly if $k = 1$ then $H_{z_0,1} = \mathrm{Id}_{\mathbb{C}}$. If $k = -1$, then $H_{z_0,-1}(z) = 2z_0 - z$ is the reflection in the point z_0.

Consider the maps $f, g : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = az + b \quad g(z) = a\overline{z} + b \tag{9.1}$$

for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. We have seen in proposition 6.5 that if $|a| = 1$, then these are isometries. In general ($a \in \mathbb{C}^*$) we can factor these transformations into products of translations, rotations, homotheties and reflections.

Theorem 9.7. *For f in (6.1) the following statements hold*

1. *If $z_0 \in \mathbb{C}$ is a fixed by f then*
$$f = H_{z_0,|a|} \circ \mathrm{Rot}_{z_0,\arg a}.$$
2. *If $z_0 \in \mathbb{C}$ is not fixed by f then*
$$f = T_{v_0 - z_0} \circ H_{z_0,|a|} \circ \mathrm{Rot}_{z_0,\arg a} = H_{z',|a|} \circ \mathrm{Rot}_{z_0,\arg a}$$
were $v_0 = f(z_0)$ and $z_0' = \frac{v_0 - |a|z_0}{1-|a|}$.

Remark 9.8. A similar decomposition can be given for g in (6.1) (see [1, Theorem 3.8.24]).

Proof of theorem 6.7. □

9.3 Exercises

Exercise 1. Find the fixed points of the transformation $f : \mathbb{C} \to \mathbb{C}$ given by

1. $f(z) = 2\mathbf{i}\overline{z} + 3 + \mathbf{i}$
2. $f(z) = \left(\frac{1}{2} + \mathbf{i}\frac{\sqrt{3}}{2}\right)\overline{z} - \frac{\sqrt{3}}{3} + \mathbf{i}$
3. $\left(\frac{1}{3} + \mathbf{i}\frac{2\sqrt{2}}{3}\right)\overline{z} + \mathbf{i}$
4. $f(z) = \overline{z} + 2 - \mathbf{i}$
5. $f(z) = \overline{z} + 9\mathbf{i}$
6. $f(z) = \overline{z}$

Exercise 2. Consider the lines

$$l_1 : \overline{\alpha}_1\overline{z} + \alpha_1 z + \beta_1 = 0 \quad \text{and} \quad l_2 : \overline{\alpha}_2\overline{z} + \alpha_2 z + \beta_2 = 0.$$

Show that

1. $l_1 \parallel l_2$ if and only if $\frac{\overline{\alpha}_1}{\alpha_1} = \frac{\overline{\alpha}_2}{\alpha_2}$,
2. $l_1 \perp l_2$ if and only if $\frac{\overline{\alpha}_1}{\alpha_1} + \frac{\overline{\alpha}_2}{\alpha_2} = 0$.

Exercise 3. For a line $l : \overline{\alpha}\,\overline{z} + \alpha z + \beta = 0$ show that the line passing through $z_0 \in \mathbb{C}$ and orthogonal to l is

$$z - z_0 = \frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z}_0).$$

Exercise 4. For a line $l : \overline{\alpha}\,\overline{z} + \alpha z + \beta = 0$ show that the orthogonal relfection $\mathrm{Ref}_l : \mathbb{C} \to \mathbb{C}$ is given by

$$\mathrm{Ref}_l(z) = -\frac{\overline{\alpha}}{\alpha}\overline{z} - \frac{\beta}{\alpha}.$$

Do this using the previous exercise and separately with the formulas for reflections deduced previously.

Further use this form of Ref_l to calculate $\mathrm{Ref}_l^2 = \mathrm{Ref}_l \circ \mathrm{Ref}_l$.

Exercise 5. Show that the lines invariant under homotheties $H_{z_0,k}$ are the lines passing through z_0.

Exercise 6. Consider the homothety $H_{z_0,k}$. Show that the image of a line l which doesn't pass through z_0 is parallel to l and doesn't pass through z_0.

Exercise 7. Show that the set of homotheties with center z_0 is a group isomorphic to $(\mathbb{R}^*, \cdot)$.

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