ex 172 CURS 14 19.01.2018

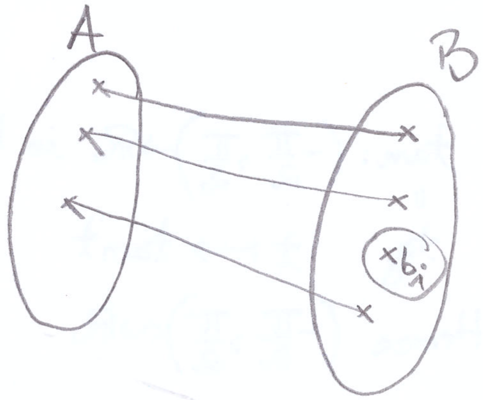
Let $A=\{a_1,\dots,a_m\}$ be a set with k elements $B=\{b_1,\dots,b_m\}$ a set with n elements. Find the number of surjective functions $f:A\to B$. Assume $m,k\in\mathbb{N}^*$

Sol. We know the number of all the functions $f:A\to B$ is $|Hom(A,B)|=m^k$

The idea is to find the number of non-surjective functions

$f:A\to B$ is not surj. $\iff Imf\neq B \iff \exists i\in\{1,\dots,m\}$ s.t. $b_i\notin Imf \iff \exists i\in\{1,\dots,m\}$ s.t. $f\in A_i \iff \exists f\in\bigcup_{i=1}^m A_i$

Not. $A_i:=\{f:A\to B \mid b_i\notin Imf\}$



Hence $|Hom_{non-surj}(A,B)|=\left|\bigcup_{i=1}^m A_i\right|$

We apply the inclusion-exclusion principle

$$\left|\bigcup_{i=1}^m A_i\right|=\underbrace{\sum_{i=1}^m |A_i|}_{(m\text{ terms})}-\underbrace{\sum_{1\le i<j\le m}|A_i\cap A_j|}_{(C_m^2\text{-terms})}+\underbrace{\sum_{1\le i<j<k\le m}|A_i\cap A_j\cap A_k|}_{(C_m^3\text{-terms})}-\dots+$$

$$+(-1)^{m+1}\sum_{1\le i_1<\dots<i_m\le m}\underbrace{|A_{i_1}\cap\dots\cap A_{i_m}|}_{(C_m^m\text{-terms})}+\dots+(-1)^{m+1}\left|\bigcap_{i=1}^m A_i\right|$$

$$|A_i|=|\{f:A\to B \mid b_i\notin Imf\}|=|\{f\mid f:A\to B\setminus\{b_i\}\}|=(m-1)^k$$

$$|A_i\cap A_j|=|\{f:A\to B\mid b_i,b_j\notin Imf\}|=|\{f:A\to B\setminus\{b_i,b_j\}\}|=(m-2)^k$$

$$|A_{i_1}\cap\dots\cap A_{i_m}|=|\{f:A\to B\setminus\{b_{i_1},\dots,b_{i_m}\}\}|=(m-m)^k$$

Hence $Hom_{surj}(A,B) = m^k - \left|\bigcup_{i=1} A_i\right| = m^k - \left(C_m^1(m-1)^k - C_m^2(m-2)^k + C_m^3(m-3)^k - \ldots\right.$
$\left. + (-1)^{m+1} C_m^m (m-m)^k + \ldots + (-1)^{m+1} \cdot 0^k\right) =$

$$= \boxed{m^k - C_m^1(m-1)^k + C_m^2(m-2)^k - C_m^3(m-3)^k + \ldots + (-1)^m C_m^m (m-m)^k + \ldots + (-1)^{m-1} C_m^{m-1} \cdot 1^k}$$

Remark This formula is for $k \le m$

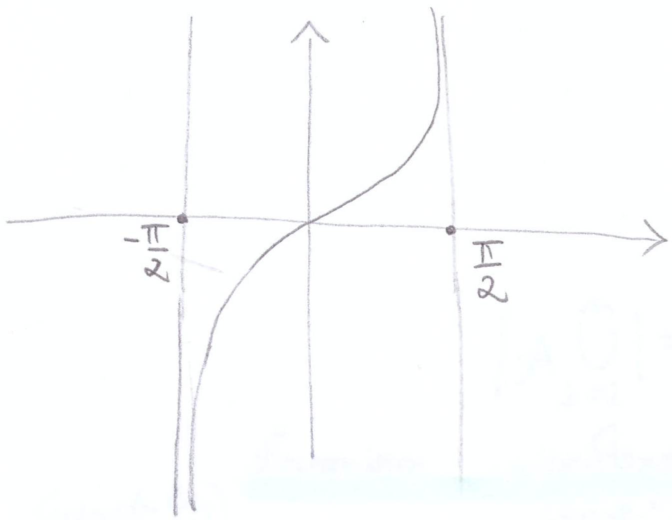
For $k > m$ the number of surjective functions is 0.

ex 163 a) Prove $\aleph_0 + \aleph_0 = \aleph_0$, $\aleph_0 \cdot \aleph_0 = \aleph_0$

Ex 165 $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$

$c = |\mathbb{R}| = 2^{\aleph_0}$

164 Prove that $\mathbb{R} \sim (a,b)$

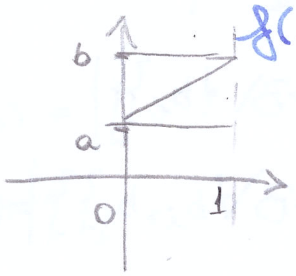


$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is bijective
$\shortparallel$
tg $\quad t \longmapsto \tan t$

Hence $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \sim \mathbb{R}$

We show that any two bounded intervals have the same cardinality.

Let $f : (0,1) \to (a,b)$ is bijective
$f(t) = a + (b-a)t$



ex 120 (refers to Theorem 7.2.4)

Prove that in the set $\mathbb{Z}$ of integers we have

$a<b,\ c>0 \Rightarrow ac<bc$ (i.e. the rel „<" is compatible with „·")

Sol

Let $a=\widehat{(m,n)}$, $b=\widehat{(p,q)}$, $c=\widehat{(r,s)}$ (where $m,n,p,q,r,s\in\mathbb{N}$)

Hyp: $a<b \Leftrightarrow \widehat{(m,n)}<\widehat{(p,q)} \overset{def}{\Longleftrightarrow} m+q<n+p$

$c>0 \Leftrightarrow \widehat{(r,s)}>\widehat{(0,0)} \Leftrightarrow r>s$

Remark which simplifies the calculations:

$c=\widehat{(r,s)}=\widehat{(r-s,0)}$

So, because the definitions do no depend on the choice of representatives, we may use $c=\widehat{(r-s,0)}$

So we may just take $s=0$, $c=\widehat{(r,0)}$, where $r\in\mathbb{N}^*$.

We have to prove $ac\overset{?}{<}bc \Leftrightarrow \widehat{(m,n)}\widehat{(r,0)}<\widehat{(p,q)}\widehat{(r,0)} \Leftrightarrow$

$\Leftrightarrow \widehat{(mr+n0,\ m\cdot 0+nr)}<\widehat{(pr+q\cdot 0,\ p\cdot 0+qr)}$

$\Leftrightarrow \widehat{(mr,nr)}\overset{?}{<}\widehat{(pr,qr)}$

$\Leftrightarrow mr+qr<nr+pr$

$\Leftrightarrow (m+q)r<(n+p)r$ true

But we know that $m+q<n+p$ and $r\in\mathbb{N}^*$

So $(m+q)r<(n+p)r$

ex. 146 (

a) Prove that the definition of the relation „<" on $\mathbb{Q}$ does not depend on the choice of representatives.

Def $\boxed{\frac{a}{b}<\frac{c}{d} \Leftrightarrow (ad-bc)bd<0}$, where $a,b,c,d\in\mathbb{Z}$, $b,d\neq 0$

We have to prove that if $\frac{a}{b}=\frac{a'}{b'}$ i.e. $ab'=a'b$

and $\frac{c}{d}=\frac{c'}{d'}$ i.e. $cd'=c'd$

then we stil have $(a'd'-b'c')b'd'<0$

Remark which simplifies the calculations

We have $\frac{a}{b} = \frac{-a}{-b}$ and $((-a)d-(-b)c)(-b)d<0$

$\Leftrightarrow (ad-bc)bd<0$

Similar for $\frac{c}{d} = \frac{-c}{-d}$

So we don't lose anything if we assume that $b, b', d, d' > 0$

So our hypotheses

$ad<bc$, $ab'=a'b$, $cd'=c'd$

And we must prove that $\underline{a'd'<b'c'}$

We start with $ad<bc \mid \cdot b'$

$\Rightarrow ab'd < bb'c$

$\Rightarrow a'\cancel{b}d < \cancel{b}b'c \mid :b>0$ (simplify with b)

$\Rightarrow a'd<b'c \mid \cdot d'>0$

$\Rightarrow a'dd'<b'cd'$

$\Rightarrow a'\cancel{d}d'<b'c'\cancel{d} \mid :d>0$

$\Rightarrow \underline{a'd'<b'c'}$

Rezultate Duminică : 16:30 – grupa 811
16:45 – grupa 812

Test 2 Duminică 21.01.18 ora 9 sala 6/II