

## Series of real numbers - part 1

### Geometric series

We recall that:

$$\lim_{n \rightarrow \infty} a^n = \begin{cases} +\infty & : a > 1 \\ 1 & : a = 1 \\ 0 & : a \in (-1, 1) \\ \cancel{\exists} & : a \leq -1 \end{cases}$$

Each series of the type

$$\sum_{n \geq m} q^{n-1},$$

for  $q \in \mathbb{R}$  is a **geometric series**. Then:

$$\sum_{n=1}^{\infty} q^{n-1} = \begin{cases} 0 & : q = 0 \\ \frac{1}{1-q} & : q \in (-1, 1) \setminus \{0\} \\ +\infty & : q \geq 1 \end{cases}$$

In the case when  $q \leq -1$ , the geometric series does not posses a sum.

**Proof:**

We study the general term for the sequence of partial sums, given  $n \geq 1$ .

$$s_n = q^0 + q^1 + \dots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$

**Case 1:**  $q = 0$ . In this case  $s_n = 0, \forall n \geq 1$ , therefore its limit 0 as well, thus the sum of the series is 0.

**Case 2:**  $q \in (-1, 1) \setminus \{0\}$ . Then:  $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1 - q^n}{1 - q} = \frac{1 - 0}{1 - q} = \frac{1}{1 - q}$ .

**Case 3:**  $q > 1$ . Then:  $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1 - q^n}{1 - q} = \frac{1 - \infty}{1 - q} = \frac{-\infty}{1 - q} = \infty$  because  $1 - q < 0$ .

**Case 4:**  $q \leq -1$ . We notice that

$$\lim_{n \rightarrow \infty} q^n$$

does not exist, therefore there does not exists the limit  $\lim_{n \rightarrow \infty} s_n$  as well. Thus, in this case the geometric series does not have a sum.

## Telescopic series

If the general term of degree  $n$  of the series of real numbers  $\sum_{n \geq m} x_n$  is defined as the difference of two successive terms of a sequence of real numbers  $(a_n)_{n \geq m}$ , i.e.

$$x_n = a_n - a_{n+1}, \forall n \geq m,$$

it is called a **telescopic series**. If the sequence  $(a_n)_{n \geq m}$  has the limit

$$l = \lim_{n \rightarrow \infty} a_n,$$

then the series  $\sum_{n \geq m} x_n$  has a sum, and it:

$$\sum_{n=m}^{\infty} x_n = a_m - l.$$

**Proof:** We write the general term of degree  $n$  of the sequence of partial sums of the series:

$$\begin{aligned} s_n &= x_m + x_{m+1} + \dots + x_n = a_m - a_{m+1} + a_{m+1} - a_{m+2} + \dots + a_{n-1} - a_n + a_n - a_{n+1} = \\ &= a_m - a_{n+1}. \end{aligned}$$

In conclusion

$$s_n = a_m - a_{n+1}, \forall n \geq m.$$

We notice that  $a_m$  is a constant, thus, due to the fact that the sequence  $(a_n)_{n \geq m}$  has a limit, the sequence  $(s_n)_{n \geq m}$  has a limit as well, and it is:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (a_m - a_{n+1}) = a_m - \lim_{n \rightarrow \infty} a_{n+1} = a_m - l.$$

Since the limit of the sequence of partial sums is the sum of the series, we reach the conclusion that:

$$\sum_{n=m}^{\infty} x_n = a_m - l.$$

## Operations with convergent series

1. Let  $\sum_{n \geq 1} x_n$  and  $\sum_{n \geq 1} y_n$  be two series of real numbers which have the sums:

$$\sum_{n=1}^{\infty} x_n = x, \quad \sum_{n=1}^{\infty} y_n = y.$$

and let  $a, b \in \mathbb{R}$ . If  $ax + by$  is not an undetermined case, then the series

$$\sum_{n \geq 1} ax_n + by_n$$

has a sum and it is:

$$\sum_{n=1}^{\infty} (ax_n + by_n) = ax + by.$$

**2.** In the case of the sum of a series it is very important to notice the first term of the summation. Thus, if the series  $\sum x_n$  has a sum, and if  $p \in \mathbb{N}$ , then

$$\sum_{n=p}^{\infty} x_n = \sum_{n=1}^{\infty} x_n - (x_1 + x_2 + \dots + x_{p-1}).$$

## Exercises

**Exercise 1:** Compute the sums for the following geometric series (if they exists):

$$a) \sum_{n \geq 3} \frac{3}{5^n}, \quad b) \sum_{n \geq 4} \frac{2^{n-3} + (-3)^{n+3}}{5^n}, \quad c) \sum_{n \geq 5} e^n, \quad d) \sum_{n \geq 2} \left(-\frac{1}{\pi}\right)^n \quad e) \sum_{n \geq 3} (-3)^n.$$

**Exercise 2:** Compute the sums of the following telescopic series:

$$a) \sum_{n \geq 1} \frac{1}{4n^2 - 1}, \quad b) \sum_{n \geq 1} \frac{1}{\sqrt{n} + \sqrt{n+1}}, \quad c) \sum_{n \geq 5} \frac{1}{n(n+1)(n+2)}$$

$$d) \sum_{n \geq 1} \ln\left(1 + \frac{1}{n}\right), \quad e) \sum_{n \geq 2} \frac{\ln\left(1 + \frac{1}{n}\right)}{\ln(n^{\ln(n+1)})}.$$

**Exercise3:** Compute the limit of the sequences of real numbers having the general terms for each  $n \in \mathbb{N}$ :

$$a) \quad x_n = \frac{a^n - a^{-n}}{a^n + a^{-n}}, \quad a \neq 0$$

$$b) \quad y_n = \frac{a^n + b^n}{a^{n+1} + b^{n+1}}, \quad a \neq -b$$

$$c) \quad z_n = \frac{1 + a + \dots + a^n}{1 + b + \dots + b^n}, \quad a, b > 0.$$