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CURS 1

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Mathematical logic

Organization

http://math.ubbcluj.ro/~marcus
section „Teaching" → lecture notes (in Romanian)

exam: → no exam
→ 2 tests — week 7 (Nov. 17) saturday; week 14 (Jan. 19) saturday } 1h/each

(+bonus points at the seminar)

attendance: min 75% of the seminars (9 seminars = 18 hours)

prerequisites: sets, functions etc.

Introduction

→ we formally analyse the mathematical proofs
→ we apply mathematical methods to logic
- NO "philosophical logic"
- not computation logic

includes:
→ proof theory
→ set theory
→ calculability (recursion) theory
→ model theory

Ch I - Propositional logic

$\boxed{\text{"Naively"}}$ → a proposition (sentence) is a statement about which we know that it is true or false.

→ we can also form composite sentences using words like "and", "or", ...

! Mathematically, this is not satisfactory.
We introduce a formal language.

Def: the language of propositional logic consists of:

1) symbols – parantheses
– connectives

$\vee$ disjunction or

$\wedge$ conjunction and

$\neg$ negation not

$\longrightarrow$ implication if ... then

$\longleftrightarrow$ if and only if

2) atoms (atomic formulas)

$p, q, r, \ldots, p_1, p_2, \ldots$

• propositional formulas are defined recursively:

1) atomic formulas are formulas
2) if A, B are formulas then $(\neg A)$, $(A \vee B)$, $(A \wedge B)$ $(A \rightarrow B)$, $(A \leftrightarrow B)$ are also formulas

3) no other sequences of symbols is a formula.

ex: 1) $((p \to q) \leftrightarrow (\neg(p \vee (\neg r))))$ is a formula.

2) $((\neg p) \vee q \wedge) \longrightarrow r$ is not a formula.

- **Interpretation of formulas**

→ we assign truth values to every formula:
assume we have a function

$$v : \mathcal{A} \longrightarrow \{0,1\}$$

$\mathcal{A}$: set of atoms
0: false, 1: true (truth values)

then we give a truth value to any formula, by induction (recursion), using the following definitions:

p	$\neg p$
0	1
1	0

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

alternatively, we may write:

$\wedge$	0	1
0	0	1
1	1	0

premise

conclusion

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Remark in practice, we may omit some parantheses by giving a priority order for the connectives:

1) $\neg$

2) $\vee, \wedge$

3) $\rightarrow, \leftrightarrow$

ex: $(p \rightarrow q) \leftrightarrow \neg(p \vee \neg r)$ (is the same as Ex. 1 above)

Relations between formulas

Def: a) $A \underset{\text{implies}}{\Rightarrow} B$ (B is consequence of A)

if $v(A \rightarrow B) = 1$

b) $A \underset{\text{equivalent}}{\Leftrightarrow} B$

if $(v(A \leftrightarrow B) = 1)$

Def: a) the formula A is a tautology

if $v(A) = 1$ for any interpretation.

b) A is a contradiction for any interpretation

if $v(A) = 0$.

c) A is satisfiable if it's not a contradiction.

The decision problem

→ given a formula A, find whether A is a tautology, contradiction or it is satisfiable

Methods for solving the decision pb.

1) by using truth tables
2) by using normal forms.

Def: a) we say that a formula A has disjunctive normal form (DNF) if A is a disjunction of elementary conjunctions:

$A = A_1 \vee \ldots \vee A_m$, where each A_i is a conjunction of atoms or negations of atoms.

e.g. $A_i = p \wedge \neg q \wedge r$ is an elementary conjunction

b) the formula A has conjunctive normal form (CNF) if it is a conjunction of elementary disjunctions:

$A = A_1 \wedge \ldots \wedge A_m$

where each A_i is an elem. disj.

e.g. $A_i = \neg p \vee \neg q \vee r$ is an elem. disjunction

Theorem: Every formula A is equivalent to a formula in DNF or in CNF, which can be found by using certain fundamental tautologies

(see: Thm 1.2.11, p.8)

1) commut., assoc., idempotence $\left\{\begin{array}{l} p \vee p = p \\ p \wedge p = p \end{array}\right.$

2) distributivity, absorbtion $\left\{\begin{array}{l} (p \vee q) \wedge p \Leftrightarrow p \\ (p \wedge q) \vee p \Leftrightarrow p \end{array}\right.$

3) law of double negation $\neg(\neg A) \Leftrightarrow A$

4) law of noncontradiction $A \wedge \neg A \Leftrightarrow 0$

5) law of excluded middle $A \vee \neg A \Leftrightarrow 1$

6) DeMorgan laws $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$

$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$

7) law of implication $A \to B \Leftrightarrow \neg A \vee B$

8) law of equivalence $A \leftrightarrow B \Leftrightarrow (A \to B) \wedge (B \to A)$

Remark DNFs and CNFs are useful, because:

- $A_1 \vee \ldots \vee A_n$ is true $\Leftrightarrow$ at least one of A_i is true.
- $A_1 \wedge \ldots \wedge A_n$ is true $\Leftrightarrow$ all A_is are true.

3. Formal deduction $\to$ we start with some formulas which are called axioms, we deduce other formulas using rules of inference, such as

Modus Ponens (MP)

$$\frac{A, \quad A \to B}{B}$$

This notation means that:

we have $A \wedge (A \to B) \Rightarrow B$

 that is, the formula $C = A \wedge (A \to B) \to B$ is tautolog ex: let us prove that (MP) is a valid inference rule

1) by truth tables:

A	B	A → B	A ∧ (A → B)	C
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

2) by using normal forms:

$$C = A \wedge (A \to B) \to B \iff (A \wedge (\neg A \vee B)) \vee B \iff$$

$$\overset{\text{DeMorgan}}{\iff} \neg A \vee \neg(\neg A \vee B) \vee B \iff \neg A \vee (\neg\neg A \wedge \neg B) \vee B \iff$$

$$\iff \neg A \vee B \vee (A \wedge \neg B) \overset{\text{distr.}}{\iff} \underline{(\neg A \vee B \vee A) \wedge (\neg A \vee B \vee \neg B)} \iff$$

$$\iff (1 \vee B) \wedge (\neg A \vee 1) \iff 1 \wedge 1 \iff 1$$

this is a CNF

Homework:

ex. 1.-13