CHAPTER 1

Affine spaces

Contents

1.1 Group actions

For a group G and a set X, a *left action of G on X* is a map

$$\circleddash : G \times X \to X, \quad (g,x) \mapsto g \circleddash x$$

such that

1. for the identity element $e \in G$ we have $e \circleddash x = x$ $\forall x \in X$ and
2. $\forall a, b \in G$ and $\forall x \in X$: $(ab) \circleddash x = a \circleddash (b \circleddash x)$.

We write $G \circlearrowright X$ and say that G *acts on X on the left*. With such an action X is called a *left G-set*. Similarly, a *right action of G on X* is a map

$$\oslash : X \times G \to X, \quad (x,g) \mapsto x \oslash g$$

such that

1. for the identity element $e \in G$ we have $x \olessthan e = x \; \forall x \in X$ and
2. $\forall a, b \in G$ and $\forall x \in X$: $x \olessthan (ba) = (x \olessthan b) \olessthan a$.

We write $G \circlearrowleft X$ and say that *G acts on X on the right*. With such an action X is called a *right G-set*.

Remark 1.1. Notice that the way in which an element acts is reverted by its inverse

$$g^{-1} \ogreaterthan (g \ogreaterthan x) = x \quad (x \olessthan g) \olessthan g^{-1} = x$$

and that, for an abelian group G, left and right actions are equivalent notions.

Examples 1.2. Group actions are fundamental, here are some examples

1. For any set X the set of bijections $\mathrm{Bij}(X,X)$ from X to X is a group with the composition law and a left action on X is given by
$$(\phi, x) \mapsto \phi(x)$$
If X is finite then $\mathrm{Bij}(X,X) = \mathrm{Sym}_n$ for $n = |X|$, the symmetric group on n letters.
2. Any group G acts on itself ($X = G$) both on the left and on the right via the multiplication map
$$G \times G \to G, \quad (a,b) \mapsto ab.$$
This action can be restricted to any subgroup $H \leq G$. A right action is
$$G \times H \to G, \quad (g,h) \mapsto gh.$$
3. In particular, if G is a vector space V with its additive structure $(V,+)$, we denote the action $V \circlearrowright V$ by
$$(v,w) \mapsto T_v(w) = v + w.$$
We call $T_v(w)$ the *translation* of w by v.
4. A group G acts on itself also by conjugation
$$G \times G \to G, \quad (a,b) \mapsto aba^{-1}$$
For an element $a \in G$, the map $G \to G$, $g \mapsto aga^{-1}$ is commonly denoted by $\mathrm{Ad}(a)$.
5. A left action $G \circlearrowright X$ induces a right action of G on the set of functions $\mathcal{F}(X,\mathbb{R})$
$$G \times \mathcal{F}(X,\mathbb{R}) \to \mathcal{F}(X,\mathbb{R}), \quad (f,g) \mapsto f_{|g}$$
where $f_{|g}(x) = f(g \ogreaterthan x) \; \forall x \in X$.
6. Isometries of the Euclidean plane $\mathbb{E}^2$ and of the Euclidean space $\mathbb{E}^3$. We will encounter them latter in the context of *affine transformations*.

Remark 1.3. In order to shorten notation we will use a dot to replace the symbols $\ogreaterthan$ and $\olessthan$

$$g.x := g \ogreaterthan x, \quad x.g := x \olessthan g.$$

1.2 Orbits and stabilizers

Given a G-space X the *orbit of* $x \in X$ is

$$G.x = \Big\{g.x : g \in G\Big\}$$

and the *stabilizer of* x is

$$G_x = \Big\{g \in G : g.x = x\Big\}.$$

More general, for a subset $Y \subseteq X$, the stabilizer of Y is

$$G_Y = \Big\{g \in G : g.y \in Y \text{ for all } y \in Y\Big\}.$$

Example 1.4. Consider an equilateral triangle ABC. The group of isometries of this triangle is the dihedral group $G = \mathrm{Dih}_3$. It contains an element of order 3, which rotates the triangle permuting the vertices. It also contains 3 reflections r_A, r_B and r_C, corresponding to the 3 midians of the triangle. Moreover, $\mathrm{Dih}_3 \cong \mathrm{Sym}_3$ so G has order 6.

The elements of G act not just on the points A, B and C but on the Euclidean plane $\mathbb{E}^2$ where this triangle is choosen. If we take *any* point $x \in \mathbb{E}$, the orbit $G.x$ is

- a set of 6 points if x does not lie on any median of $\triangle ABC$,
- a set of 3 points, describing an equilateral triangle, if x lies on one of the medians or
- the center of $\triangle ABC$ (x lies on all medians).

The stabilizers can also be recognized:

$$G_A = \{1, r_A\}, \quad G_B = \{1, r_B\}, \quad G_C = \{1, r_C\}.$$

In general a point x is stable under r_A if and only if it lies on the median passing through A.

So in this action $G \circlearrowright \mathbb{E}^2$ the order of the stabilizer of a point $x \in \mathbb{E}^2$ is

- 1 if x does not lie on any median of $\triangle ABC$,
- 2 if x lies on one of the medians,
- $6 = |G|$ if x is the center of $\triangle ABC$.

1.3 Morphisms of G-sets

For a group G and two G-sets X and Y, a map $\phi : X \to Y$ is a *morphism of G-sets* if

$$\phi(g.x) = g.\phi(x) \quad \forall g \in G, x \in X.$$

Furthermore, ϕ is an *isomorphism of G-sets* if it is bijective and its inverse is a morhism of G-sets.

Remark 1.5. We will discuss morhprims later on, in the context of *affine morhisms* and *affine transformations*.

1.4 Principal homogeneous G-spaces

A G-set X is a *principal homogeneous G-space* if

1. the action is transitive: $X = G.x$, $\forall x \in G$ and
2. the stabilizers are trivial: $G_x = \{e\}$, $\forall x \in G$.

Remark 1.6. Notice that, for a transitive action, if one stabilizer is trivial then all stabilizers are trivial.

Examples 1.7. We have already seen homogeneous G-spaces

1. If we restrict the action $G = \mathrm{Dih}_3 \circlearrowright \mathbb{E}^2$ in Exemple 1.4 to an orbit of 6 elements we obtain a finite homogeneous G-space.
2. Consider the group of rotations $G = \mathrm{Rot}$ acting on $\mathbb{E}^2$ by fixing a point O. Then, any circle centered at O is a homogeneous G-space with the obvious action.
3. If W is a vector space and V a subspace, then $G = V$ acts on W via the additive structure. The orbits $V.w = w + V$ are homogeneous G-spaces.

Proposition 1.8. *If X is a principal homogeneous G-space, then, for any $P \in X$, the map*

$$T_{\square}(P) : G \to X, \quad \textit{given by} \quad T_g(P) := T_{\square}(P)(g) = g.P$$

is bijective. In other words, for any $Q \in X$ there is a unique $g \in G$ such that $g.P = P$.

Moreover, $T_{\square}(P)$ is an isomorphism of G-sets when G is considered with multiplication on the left.

Proof. □

Proposition 1.9. *If X is a principal homogeneous G-space, the map*

$$T_{\square} : G \to \mathrm{Bij}(X), \quad \textit{given by} \quad T_g := T_{\square}(g) = (P \mapsto g.P)$$

is an injective group homomorphism. The image $T_{\square}(G) =: T_G$ is called the group of translations of X by G.

Proof. □

1.5 Affine spaces

Let V be a vector space over a field k acting on itself with the additive structure $(V, +)$. This gives $X = V$ the structure of a principal homogeneous V-space called *affine space*.

We refer to V as the *underlying vector space of the affine space X* and denote it by $D(X)$. It is also called the *direction* of X. If V is given, we write $\mathbb{A}(V)$ for X.

The *dimension of X* is that of V

$$\dim X = \dim V$$

an affine space of dimension 0 is a *point* and one of dimension 1 is a *line* and if $\dim X = 2$ it is a *plane*. By convention, we accept $\emptyset$ as an affine space.

Examples 1.10. The notion of *affine space* is designed to treat a vector space by *forgetting about the origin*

1. The prototypical example is that of a vector space acting on itself (as in Examples 1.2).

2. The advantage of using this notion comes from the notion of *affine subspaces*:

Definition. Let X be an affine space. An orbit of a point $x \in X$ under a subspace V of $D(X)$ is called *affine subspace*.

With this notion we cover Euclidean lines and planes in $\mathbb{E}^3$.

Proposition 1.11. *An affine subspace of an affine space is an affine space.*

Definition. Two affine subspaces $Y, Z \subseteq X$ are said to be *parallel* if $D(Y) \subseteq D(Z)$ or $D(Z) \subseteq D(Y)$.

Most of the affine spaces that we encounter are described as follows

Proposition 1.12. *Let V and W be two vector spaces over a field k and let $X = \mathbb{A}(V)$ and $Y = \mathbb{A}(W)$. If $f : V \to W$ is a linear map, then*

1. *for any affine subspace N of Y, $f^{-1}(N)$ is an affine subspace of X,*

2. *if $f^{-1}(N) \neq \emptyset$ then $D(f^{-1}(N)) = f^{-1}(D(N))$*

In particular

1. *all level sets $f^{-1}(w)$ are affine subspace ($w \in W$) and*

2. *if $w \in \mathrm{im}(f)$ then $D(f^{-1}(w)) = \ker(f)$.*

Proof. □

Remark 1.13. One can interpret the last part of the above proposition, by viewing $f^{-1}(w)$ as the solution set S to the non-homogeneous system $f(x) = w$. Any element in S is of the form $v + s_0$ where v is a particular solution to $f(x) = w$ and s_0 is a solution to the homogeneous system $f(x) = 0$, i.e. S is the orbit of v under the vector subspace $\ker(f) \leq V$ so it is an affine space.

1.6 Exercises

Exercise 1. Prove that for an abelian group the notion of left and right actions are equivalent.

Exercise 2. Check if the actions in Examples 1.2 are right or left actions.

Exercise 3. Generalize the discussion of Example 1.4 to a regular n-gon.

Exercise 4. Prove Proposition 1.11.

Exercise 5. In the affine space $X = \mathbb{R}^4$ consider the plane $\alpha = \langle (1,1,1,1), (0,1,0,1) \rangle + (2,4,1,2)$ and the line $\beta = \langle (1,1,-1,1) \rangle + (2,3,-1,1)$. Determine $\alpha \cap \beta$.

Exercise 6. In $\mathbb{R}^4$ consider the affine subspaces

$$\begin{aligned}\alpha &= (2,1,2,1)\\ \beta &= \langle(1,1,1,1)\rangle + (1,3,0,0)\\ \gamma &= \langle(2,1,3,-1),(1,0,2,-2)\rangle + (1,0,1,0)\\ \delta &= \langle(1,0,0,0),(0,1,0,0),(0,0,1,0)\rangle\end{aligned}$$

Which of the following is true?

1. $\alpha \in \beta$
2. $\alpha \in \gamma$
3. $\alpha \in \delta$
4. $\beta \parallel \gamma$
5. $\beta \parallel \delta$
6. $\gamma \parallel \delta$
7. $\beta \subseteq \gamma$
8. $\gamma \subseteq \delta$

Exercise 7. Let V be a vector space of dimension at least 5, $a,b,c \in V$ three distinct points and $\pi = \langle v_1, v_2\rangle + a$ a plane. Determine an affine subvariety of dimension 4 which contains a, b, c and π.

Exercise 8. For a finite field k, consider the affine space $X = k^n$ for some $n \in \mathbb{N}$. Determine

a) the number of points in an affine subspace of X and

b) the number of lines passing through a given point of X.

Exercise 9. Prove Proposition 1.12.

Exercise 10. Which of the following are affine subvarieties?

1. $A = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 - 2 = 0\}$
2. $B = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1 + x_2, 2x_2 + x_3, x_3 - 2x_1) \in A\}$
3. $C = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 0\}$
4. $D = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1^4 + x_2 - 2x_3 + x_4 = 0\}$

Exercise 11. For $n \in \mathbb{N}$ let $a_1,\ldots,a_n$, $b_1,\ldots,b_n \in \mathbb{R}$. Let I be an interval in $\mathbb{R}$ and $g : I \to \mathbb{R} \in \mathcal{C}^\infty(\mathbb{R})$. Which of the following are affine subspaces?

1. $A = \{f \in \mathcal{C}^\infty(\mathbb{R}) : a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + g = 0\}$
2. $B = \{f \in \mathcal{C}^\infty(\mathbb{R}) : a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' \in A\}$

3. $C = \{f \in \mathcal{C}^\infty(\mathbb{R}) : f^3 - 5f^2 + 6f = 0\}$

Exercise 12. Show that the hyperplane $h = a + \langle v_1, \ldots, v_{n-1}\rangle$ of $\mathbb{R}^n$ is described by the equation

$$\begin{vmatrix} x_1 - a_1 & \ldots & x_n - a_n \\ v_{1,1} & \ldots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n-1,1} & \ldots & v_{n-1,n} \end{vmatrix} = 0$$

where $v_i = (v_{i,1}, \ldots, v_{i,n})$ and $a = (a_1, \ldots, a_n)$.