CHAPTER 8

Affine morphisms in dimension 2

Contents

8.1 Euclidean space

Euclidean spaces are modeled on real vector spaces togheter with a distance. More precisely we have an affine space $X = \mathbb{A}(\mathbb{R}^n)$ togheter with a metric d, i.e.

Definition. A *metric (or distance)* on X is a function

$$d : X \times X \to \mathbb{R}$$

such that for all $P, Q, R \in X$

1. $d(P,Q) \geq 0$
2. $d(P,Q) = 0 \Leftrightarrow P = Q$
3. $d(P,Q) = d(Q,P)$
4. $d(P,Q) \leq d(P,R) + d(R,Q)$

Such a function can be defined in terms of the *scalar product* on $D(X) = \mathbb{R}^n$. Recall, this is

$$\langle \_, \_ \rangle : D(X) \times D(X) \to \mathbb{R}, \quad \text{given by} \quad \langle u, v \rangle = \sum_{i=1}^{n} u_i v_i.$$

where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ are with respect to the standard basis of $\mathbb{R}^n$. Then

$$d(P,Q) = \|Q - P\| \quad \text{where} \quad \|Q - P\| = \|\overrightarrow{PQ}\| = \sqrt{\langle \overrightarrow{PQ}, \overrightarrow{PQ} \rangle} \text{ is the norm of } \overrightarrow{PQ}.$$

Remark 8.1. The geometric approach is to assume the existence of a distance and describe the salar product in terms of it. The algebraic approach goes the other way around - while this is quicker to define, it hides the reasons why the definition is the meaningful thing to consider.

Notice also that, since we defined $\langle \_, \_ \rangle$ in terms of a standard basis of $\mathbb{R}^n$ we always assume that an identification $D(X) \cong \mathbb{R}^n$ (as vector spaces) is chosen - there are, of course, several such indentifications.

Definition. We denote by $\mathbb{E}^n$ the n-dimensional real affine space $\mathbb{A}(\mathbb{R}^n)$ endowed with the above distance d.

8.2 Isometries

Definition. An isometry of $\mathbb{E}^n$ is a map $\varphi : \mathbb{E}^n \to \mathbb{E}^n$ such that

$$d(\varphi(P), \varphi(Q)) = d(P,Q) \quad \text{for all} \quad P, Q \in \mathbb{E}^n.$$

The set of all such maps is a group denoted by $\mathrm{Isom}(\mathbb{R}^n)$. As with any map on points, we have an induced map on vectors

$$\mathrm{lin}\,\phi : D(\mathbb{E}^n) \to D(\mathbb{E}^n) \quad \text{given by} \quad (\mathrm{lin}\,\phi)(\overrightarrow{PQ}) = \overrightarrow{\varphi(P)\varphi(Q)}.$$

Remark 8.2. We already have an operator ‘lin’ for affine maps. Right now, the operator introduced above is just a map on vectors. We show next that it is linear and that isometries are affine transformations, i.e. that the operator ‘lin’ defined above is the operator ‘lin’ defined for affine maps.

Proposition 8.3. *If $\varphi \in \mathrm{Isom}(\mathbb{R}^n)$, then $\varphi_0 = \mathrm{lin}\,\varphi$ preserves the vector norm and the scalar product:*

$$\forall v, w \in \mathbb{R}^n \quad \|\varphi_0(v)\| = \|v\|, \quad \langle \varphi_0(v), \varphi_0(w) \rangle = \langle v, w \rangle.$$

Proof. □

Proposition 8.4. *If $\varphi \in \mathrm{Isom}(\mathbb{R}^n)$, then $\varphi_0 = \mathrm{lin}\,\varphi$ is an invertible linear map, i.e. $\varphi_0 \in \mathrm{GL}(\mathbb{R}^n)$.*

Proof. □

Theorem 8.5. *Isometries are affine transformations, i.e.* $\mathrm{Isom}(\mathbb{R}^n) \subseteq \mathrm{AGL}(\mathbb{R}^n)$.

Proof. □

Remark 8.6. Note that the definition of the scalar product implies that the standard basis vectors $e_1,\ldots,e_n$ form an orthonormal basis. For a linear map $f:\mathbb{R}^n\cong D(X)\to D(X)\cong\mathbb{R}^n$, we denote by

$$[f]\quad \text{the matrix of } f \text{ with respect to the standard basis of } X=\mathbb{R}^n.$$

Proposition 8.7. *Let $\varphi:\mathbb{E}^n\to\mathbb{E}^n$ be an affine morphism. The following are equivalent*

1. *φ is an isometry*
2. $[\operatorname{lin}\varphi]^t=[\operatorname{lin}(\varphi)]^{-1}$.

Proof. □

Remark 8.8. Notice that, with Proposition **??**, we have

$$[\operatorname{lin}(\varphi)]^{-1}=[\operatorname{lin}(\varphi)^{-1}]=[\operatorname{lin}(\varphi^{-1})].$$

Definition. An $n\times n$ matrix A such that $A^tA=I_n$ is called *orthogonal matrix*. The set of all such matrices is denoted by $\mathrm{O}(n)$.

Proposition 8.9. *For any $A\in\mathrm{O}(n)$, $\det A\in\{\pm1\}$. $\mathrm{O}(n)$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.*

Proof. □

Definition. Let $\mathrm{SO}(n)=\{A\in\mathrm{O}(n):\det A=1\}$. It is a subgroup of $O(n)$. An affine morphism $\varphi\in\mathrm{Aut}_{\mathrm{aff}}(\mathbb{E}^n)$ with $[\operatorname{lin}\varphi]\in\mathrm{SO}(n)$ is called *dispacement of* $\mathbb{E}^n$. In other words, a displacement of $\mathbb{E}^3$ is an isometry which doesn't change the orientation of $\mathbb{E}^3$.

8.3 Affine transformations in dimension 2

Theorem 8.10 (Chasles)**.** *A displacement of the plane is either a rotation or a translation.*

We make use of the following

Lemma 8.11. *Let A be an 2×2 matrix. Then*

$$A\in\mathrm{SO}(2)\quad\Leftrightarrow\quad A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\quad \textit{for some } \theta\in[0,2\pi).$$

Proof. □

Proof of Theorem 8.10. □

Remark 8.12. Given a non-zero angle of rotation θ and a translation vector v, how does one determine the center of rotation for the displacement

$$\varphi:P\mapsto\mathrm{Rot}_\theta P+v?$$

Solution. □

Remark 8.13. Chasles theorem describes direct isometries. A similar statement can be obtained for indirect isometries: these are either reflections or gliding reflections.

8.4 A factorization of affine transformations in dimension 2

Proposition 8.14. *The following holds*

1. *Any* $\varphi \in \mathrm{AGL}(\mathbb{E}^2)$ *multiplies the oriented area of a geometric figure by* $\det[\mathrm{lin}\,\varphi]$.

2. *Any* $\varphi \in \mathrm{AGL}(\mathbb{E}^3)$ *multiplies the oriented volume of a geometric figure by* $\det[\mathrm{lin}\,\varphi]$.

In particular, φ *preserves the orientation if and only if* $\det[\mathrm{lin}\,\varphi] > 0$.

Proof. □

Definition. For $\varphi \in \mathrm{AGL}(\mathbb{E}^n)$, the matrix

$$\mathbf{G}_\varphi = [\mathrm{lin}\,\varphi]^t \cdot [\mathrm{lin}\,\varphi]$$

is called *the Gram matrix of* φ *(with respect to the standard basis of* $\mathbb{E}^n = \mathbb{R}^n$*)*.

The square roots of the eigenvalues of $\mathbf{G}_f$ are called *the principal dilation coefficients of* φ. The Gram matrix can be thought of as a measure of how far an affine transpormation is from being an isometry.

Analytically, for vectors $v = \sum v_i e_i$ and $w = \sum w_i e_i$ we have

$$\begin{aligned}\langle \varphi(v), \varphi(w) \rangle &= (\mathrm{lin}\,\varphi v)^t (\mathrm{lin}\,\varphi w)\\ &= v^t \mathbf{G}_\varphi w\\ &= \textstyle\sum_{i,j} g_{ij} v_i w_j.\end{aligned}$$

The definitions and previous results then imply the following statement.

Proposition 8.15. *Let* $\varphi \in \mathrm{AGL}(\mathbb{E}^n)$. *The following are equivalent*

1. φ *is an isometry*

2. *The principal dilation coefficients of* φ *are* 1

3. *The Gram matrix of* φ *is the identity matrix*

4. $[\mathrm{lin}\,\varphi]^t = [\mathrm{lin}\,\varphi]^{-1}$

Definition. Let $\varphi \in \mathrm{AGL}(\mathbb{E}^n)$ and $v \in D(\mathbb{E}^n)$. The *dilation of* φ *in the direction of a vector* v is

$$\delta(v) := \frac{\|(\mathrm{lin}\,\varphi)v\|}{\|v\|}.$$

Lemma 8.16. *The eigenvalues of the Gram matrix of an affine transformation are positive. In particular the principal dilation coefficients are positive real numbers.*

Proof. □

Theorem 8.17. *Any affine transformation φ of $\mathbb{E}^2$ can be written in the form*

$$\varphi(P) = (R_2 D R_1)P + \mathbf{v}$$

with respect to the standard basis of $\mathbb{R}^2 \cong \mathbb{E}^2$. Here $\mathbf{v}$ is a translation vector, R_1 and R_2 are linear rotations and D a diagonal transformation whose coefficients are up to sign the principal dilation coefficients of φ.

For the proof of the theorem we need to following lemma. Note that, since $A^t A$ is a symmetric matrix, this lemma is a particular case of the linear algebra statement, which says that any symmetric matrix is diagonalizable in an orthonormal basis.

Lemma 8.18. *For any matrix $A \in \mathrm{GL}(\mathbb{R}^2)$, there is a right oriented basis of eigenvectors for the Gram matrix $\mathbf{G}_A$.*

Proof. □

Poof of Theorem 8.17. □

Corollary 8.19. *Let $\lambda_1 \leq \lambda_2$ be the principal dilation coefficients of $\varphi \in \mathrm{AGL}(\mathbb{E}^2)$. Then, for any vector $v \neq 0$, we have*

$$\lambda_1 \leq \delta(v) \leq \lambda_2$$

Proof. □

8.5 Exercises

Exercise 1. Show that an isometry is bijective.

Exercise 2. Show that

$$\|v_1 + \cdots + v_n\|^2 = \sum_{i=1}^{n} \|v_1\|^2 + \sum_{i<j}^{n} 2\langle v_i, v_j\rangle.$$

Exercise 3. Deduce the formula for $\sin(\alpha + \beta)$ by multiplying rotation matrices.

Definition. If $Y, Z \subseteq \mathbb{E}^n$ are orthogonal affine subspaces, the correponding projection and reflection $\mathrm{Pr}_{Y,Z}$ and $\mathrm{Ref}_{Y,Z}$ are denoted by Pr_Y and Ref_Y and are called *orthogonal projection* and *orthogonal reflection* respectively.

Exercise 4. Show that the orthogonal reflection $\mathrm{Ref}_\pi(x)$ in the plane $\pi : \langle n, x\rangle = p$ is given by

$$\mathrm{Ref}_\pi(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2} n$.

Exercise 5. Use the formula in the previous exercise to calculate $\mathrm{Ref}_\pi^2 = \mathrm{Ref}_\pi \circ \mathrm{Ref}_\pi$.

Exercise 6. Give the algebraic expressions for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4.$$

Exercise 7. Give the algebraic expression of the following displacements of $\mathbb{E}^2$:

1. A translation of one unit in the positive direction of the x-axis followed by a rotation arround the origin of angle $\frac{\pi}{3}$.
2. A rotation of angle $\frac{\pi}{3}$ around the point $(1,1)$.

Exercise 8. Give the planar rotation with 45° having the same center as the rotation

$$x \mapsto \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Exercise 9. A displacement maps $(0,1)$ and $(1,1)$ to $\left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$ and $\left(\frac{2-\sqrt{3}}{2}, \frac{1}{2}\right)$ respectively. Give the algebraic representation (with respect to the standard basis) of this displacement. Find its center if it has one.

Exercise 10. Let δ_1 and δ_2 be the principal dilation coefficients of an affine transformation $\varphi \in \mathrm{AGL}(\mathbb{R}^2)$. Show that φ multiplies the area of a geometric figure by $\delta_1\delta_2$.

Exercise 11. Calculate the principal dilation coefficients of the affine transformation

$$\varphi\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \end{bmatrix}.$$

Exercise 12. Show that if $A, B \in \mathrm{SO}(3)$, then $A \cdot B \in \mathrm{SO}(3)$ and $A^{-1} \in \mathrm{SO}(3)$. In other words, show that $\mathrm{SO}(3)$ is a group.

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