CHAPTER 5

Convex sets (continued)

Contents

5.1 Half-spaces

We restrict here to real affine spaces, i.e. to $\mathbb{R}^n$ with its affine structure. The equation of a hyperplane is

$$\varphi(x_1,\ldots,x_n)+b=a_1x_1+\cdots+a_nx_n+b=0 \quad\Leftrightarrow\quad a_1x_1+\cdots+a_nx_n=b$$

where $a_1,\ldots,a_n,b\in\mathbb{R}$ and where φ is linear map. Using the odering with which $\mathbb{R}$ is endowed we may define

$$H^+_{\varphi,b}=\{x\in\mathbb{R}^n:\varphi\geq b\}\quad\text{and}\quad H^-_{\varphi,b}=\{x\in\mathbb{R}^n:\varphi\leq b\}$$

the *closed half-spaces determined by φ and b.*

Remark 5.1. Notice that

1. If we choose the inequalities to be strict we obtain the corresponding *open* half-spaces.

2. $H^+_{\varphi,b} = H^-_{-\varphi,-b}$ and $H^-_{\varphi,b} = H^+_{-\varphi,-b}$, so, if we label these sets with $+$ and $-$, then this sign depends on the choice of φ and b. When there is no confusion we write

$$H^+ \quad \text{and} \quad H^-$$

3. It is easy to see that halfspaces are convex sets.

5.2 Polyhedra

Definition. A *polyhedron* is the intersection

$$\bigcap_{H\in\mathcal{H}} H$$

of a finite family $\mathcal{H}$ of half-spaces. Further

1. If the intersection is bounded we call the polyhedron a *polytope*.
2. If all halfplanes in H are described by hyperplanes passing through the origin $\mathbf{0} \in \mathbb{R}^n$, we say that the polyhedron is a *polyhedral cone*.

Examples 5.2. The following are polyhedrons

1. The Platonic solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron).
2. The convex hulls of a finite numer of points.
3. In particular, the set of points delimited by regular polygons.
4. Voronoi cells.

Remark 5.3. The algebraic version of an intersection of finitely many half-spaces is as follows. We may assume that the half-space $H_i \in \mathcal{H}$ is described by

$$a_{i1}x_1 + \cdots + a_{in}x_n + b_i \leq 0 \quad (i = \overline{1,r},\ r = |\mathcal{H}|).$$

If the inequality is $\geq$ we may multiply by -1. The system of inequalities can be written in the form

$$Ax \leq b \tag{5.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{r\times n}$ with $r = |\mathcal{H}|$ and $b \in \mathbb{R}^n$.

Remark 5.4. There is a second way of defining cones. For a set of points $P_1,\ldots,P_r \in \mathbb{R}^n$, the corresponding *finitely generated cone* is

$$\mathcal{C} := \operatorname{cone}(P_1,\ldots,P_2) := \left\{ \operatorname{Bar}(\mathbf{0},P_1,\ldots,P_2; 1-\mu_1-\cdots-\mu_n, \mu_1,\ldots,\mu_n) : \mu_1,\ldots,\mu_n \geq 0 \right\}$$

Notice that this is the set of points

$$\mathcal{C} = \left\{ P = \mathbf{0} + \sum_{i=1}^{n} \mu_i (P_i - \mathbf{0}) : \mu_1, \dots \mu_n \geq 0 \right\}.$$

Now $P_i - \mathbf{0} \in D(\mathbb{R}^n) = \mathbb{R}^n$, so, with respect to the standard basis $P_i - \mathbf{0} = \overrightarrow{\mathbf{0}P_i} = (b_{1i}, \dots, b_{ni})$. Algebraically

$$\mathcal{C} = \left\{ B\mu : \mu \geq 0 \right\}$$

where $B = (b_{ij}) \in \mathbb{R}^{n \times r}$ and $\mu = (\mu_1, \dots, \mu_r) \geq \mathbf{0}$ means $\mu_i \geq 0$ for all i.

A theorem of Weyl (see Theorem 5.6) shows that finitely generated cones are polyhedral cones. To prove it we need the following result.

5.3 Cones

Theorem 5.5 (Fourier-Motzkin Elimination)**.** *Let*

$$Ax \leq b \tag{5.2}$$

be a system of linear inequalities with $n \geq 1$ variables and r inequalities. Then there is a system

$$A'x' \leq b' \tag{5.3}$$

with $n-1$ variables and at most $\max(r, \frac{r^2}{2^2})$ inequalities, such that s' is a solution of (5.3) if and only if there is $s_0 \in \mathbb{R}$ such that $s = (s', s_0)$ is a solution to (5.2).

Example. Consider the system of inequalities

$$\begin{array}{rcrcr} -x & + & y & \leq & 2 \\ x & + & 2y & \leq & 4 \\ -2x & - & y & \leq & 1 \\ x & - & 2y & \leq & 2 \\ x & & & \leq & 2 \end{array} \tag{5.4}$$

Draw the figure to illustrate the solution set. If we fix x, what are the conditions which guarantee that (x, y) is a solution to (5.4)? We rewrite the system and impose conditions on y. From the first two conditions we have

$$\begin{array}{rcrcr} y & \leq & 2 & + & x \\ y & \leq & 2 & - & \frac{1}{2}x \end{array}$$

and from the third and fourth we get

$$\begin{array}{rcrcr} -2x & - & 1 & \leq & y \\ \frac{1}{2}x & - & 1 & \leq & y \end{array}$$

The last inequality does not impose any condition on y. Hence (5.4) has a solution for y if and only if

$$\max\left\{-2x - 1, \frac{1}{2}x - 1\right\} \leq y \leq \min\left\{2 + x, 2 - \frac{1}{2}x\right\}$$

i.e. if and only if

$$\begin{array}{rcrcrcr} -2x & - & 1 & \leq & 2 & + & x \\ \frac{1}{2}x & - & 1 & \leq & 2 & + & x \\ -2x & - & 1 & \leq & 2 & - & \frac{1}{2}x \\ \frac{1}{2}x & - & 1 & \leq & 2 & - & \frac{1}{2}x \end{array}$$

Rewriting this and adding the constraint from the fifth equation in (5.4), we obtain

$$\begin{array}{rcl} -x & \leq & 1 \\ -x & \leq & 2 \\ -x & \leq & 6 \\ x & \leq & 3 \\ x & \leq & 2 \end{array}$$

So, this system of inequalities has a solution if and only if (5.4) has a solution, but we have one variable less. We can make one more step and eliminate x as well

$$\max\{-1,-2,-6\} \leq x \leq \min\{2,3\}.$$

This tells us that if $-1 \leq x \leq 2$ the system (5.4) has a solution. The range obtained is in fact the porjection $(x,y) \mapsto x$ of the solution set for (5.4). □

Theorem 5.6 (Weyl's Theorem). *A non-empty finitely generated cone is a polyhedral cone.*

Proof. □

Remark 5.7. Since the proof is constructive we can use the elimination method on the system (**??**) to obtain the system of linear inequalities defining the given cone.

We may use this result of Weyl to prove a statement which is of importance in mathematical optimization.

Theorem 5.8 (Farkas' Lemma). *For any $P_1,\dots,P_r,Q \in \mathbb{R}^n$, exactly one of the following holds*

1. $Q \in \operatorname{cone}(P_1,\dots,P_r)$ *or*
2. *there is a vector half-space H of $\mathbb{R}^n$ such that*

$$P_1,\dots,P_r \in H \quad \textit{and} \quad Q \notin H.$$

Proof. □

5.4 Convex hulls in dimension 2

There are several algorithms for determining the convex hull of a set of points in $\mathbb{R}^2$. We discuss two of them. In doing so we view $\mathbb{R}^2$ as the Euclidean plane $\mathbb{E}^2$. So, in addition to the affine structure we also consider distances and angles.

5.4.1 Andrew's monotone chain algorithm

Data: List of Points $P = (P_1, \dots, P_n)$
Result: Determines the ordered vertices of conv(P)
Sort the points by x-coordinate, when equal by y-coordinate;
$L \leftarrow \emptyset$;
for *i from* 1 *to* n **do**
  $m \leftarrow |L|$;
  while $m > 2$ *and* $(\overrightarrow{L_{m-1}L_m}, \overrightarrow{L_mP_i})$ *left-oriented* **do**
    Remove L_m from L and decrease m;
  Append P_i to L;
$U \leftarrow \emptyset$;
for *i from* n *to* 1 **do**
  $m \leftarrow |U|$;
  while $m > 2$ *and* $(\overrightarrow{U_{m-1}U_m}, \overrightarrow{U_mP_i})$ *left-oriented* **do**
    Remove U_m from U and decrease m;
  Append P_i to U;
Remove the last point from L and the last point from U;
return L concatenated with U;

5.4.2 Graham's scan

Data: List of Points $P = (P_1, \dots, P_n)$
Result: Determines the ordered vertices $H = (H_1, H_2, \dots)$ of conv(P)
$n \leftarrow |\mathcal{P}|$;
$P_1 \leftarrow$ point with smalles y value;
Sort P increasingly by $\measuredangle(\overrightarrow{P_1P_i}, Ox)$;
$H \leftarrow (P_1, P_2)$;
for *i in* $\{2, \dots, n\}$ **do**
  $m \leftarrow |H|$;
  while $(\overrightarrow{H_{m-1}P_i}, \overrightarrow{H_{m-1}H_m})$ *left-oriented* **do**
    Remove H_m from H and decrease m;
  Append P_i to H;

5.5 Voronoi cells

Given a set of points $P_1, \dots, P_n$ in the Euclidean space $\mathbb{E}$. The *Voronoi cell* corresponding to P_i is the set of points closer to P_i than to the other points

$$\mathcal{C}_i = \left\{ Q \in \mathbb{E}^n : d(Q, P_i) = \min_j d(Q, P_j) \right\}.$$

Since for two given points P_i and P_j, the equation

$$d(Q,P_i) = d(Q,P_j)$$

defines a hyperplane H_{ij}. It is easy to see that a Voronoi cell is an intersection of half-spaces and in particular that it is convex.

5.6 Exercises

Exercise 1. Check that the algebebraic description of a finitely generated cone in Remark 5.4 is correct.

Exercise 2. Consider the system of inequalities

$$\begin{array}{rcrcr} -2x & - & 3y & \leq & -12 \\ 2x & - & 3y & \leq & -12 \\ -x & - & y & \leq & 0 \\ & & y & \leq & 1 \\ x & & & \leq & 2 \end{array}$$

Use Fourier-Motzkin Elimination to eliminate the variables.

Exercise 3. In $\mathbb{E}^2$ consider the points $(2,2)$, $(7,1)$, $(6,3)$, $(4,3)$, $(3,0)$ $(3,4)$ and $(5,2)$. Use the two algorithms discussed to determine their convex hull.

Exercise 4. In two algorithms for convex hulls discussed above use angles and the notion of orientation. Is it possible to restate them solely in terms of half-spaces?

Exercise 5. What are the Voronoi cells of the vertices of an equilateral triangel. Generalize this to any regular polygon.

Exercise 6. Describe the Voronoi cells of the trapezoid with vertices $(-1,2)$, $(3,2)$, $(5,0)$, $(-2,0)$.

Exercise 7. What are the Voronoi cells of $\{(n,m) : n,m \in \mathbb{N}\} \subseteq \mathbb{R}^2$?

Exercise 8. Consider $M = \{(2n,2m) : n,m \in \mathbb{N}\} \subseteq \mathbb{R}^2$. What are the vertices of the Voronoi cells of $M \cup ((1,1)+M)$?

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