CHAPTER 6

Affine morphism

Contents

6.1 Morhism of affine spaces

Definition. Let X and Y be two affine spaces. A map $\varphi : X \to Y$ is a *morphism of affine spaces* if it preserves the barycenters:

$$\forall P_1, \dots, P_n \in X \quad \forall \mu_1, \dots \mu_n \in k \quad \text{with} \quad \sum_{i=1}^{n} \mu_i = 1$$

we have

$$\varphi(\mathrm{Bar}(P_1, \dots, P_n; \mu_1, \dots \mu_n)) = \mathrm{Bar}(\varphi(P_1), \dots, \varphi(P_n); \mu_1, \dots \mu_n)).$$

Examples 6.1. We will encounter many more examples in the following sections.

1. Projections on hyperplans along a given direction. Eg. $X = \mathbb{R}^2 \to \mathbb{R} = Y$, $(x, y) \mapsto x$ is affine.

2. Any linear map between vector spaces is an affine morphism of the corresponding affine spaces. In fact we will prove that affine morphism are up to translations linear.

Remark 6.2. The composition of two affine morphisms is an affine morphism.

Definition. We denote the set of affine morphisms $X \to Y$ by $\mathrm{Hom}_{\mathrm{aff}}(X, Y)$.

- An *affine endomorphism* is an element of $\mathrm{End}_{\mathrm{aff}}(X) := \mathrm{Hom}_{\mathrm{aff}}(X, X)$
- An *affine isomorphism* is a bijective element in $\mathrm{Hom}_{\mathrm{aff}}(X, Y)$ with inverse in $\mathrm{Hom}_{\mathrm{aff}}(Y, X)$. We denote the set of these maps by $\mathrm{Iso}_{\mathrm{aff}}(X, Y)$.
- An *affine automorphism* is an element of $\mathrm{Aut}_{\mathrm{aff}}(X) := \mathrm{Iso}_{\mathrm{aff}}(X, X)$.

Remark 6.3. In view of the previous remark, $\mathrm{Aut}_{\mathrm{aff}}(X)$ is a group called *group of affine transformations*. Its elements are called *affine transformations*. It is sometimes denoted by

$$\mathrm{AGL}(X) := \mathrm{Aut}_{\mathrm{aff}}(X).$$

6.2 Structure of affine morphisms

Proposition 6.4. *Let X and Y be two affine spaces. A map $\varphi : X \to Y$ is affine if and only if for any $P_0 \in X$ the map*

$$\varphi_0 : D(X) \to D(Y) \quad \text{given by} \quad \varphi_0(v) = \varphi(P_0 + v) - P_0$$

is linear.

Remark 6.5. If $\varphi : X \to Y$ is affine, the map φ_0 in the previous proposition does not depend on P_0. It is called *the linear part of* φ and we denote it by

$$\mathrm{lin}(\varphi) := \varphi_0.$$

Proof of Proposition 5.4. □

Corollary 6.6. *Let V be a k-vector space and $\varphi \in \mathrm{End}_{\mathrm{aff}}(V)$. Then φ decomposes as*

$$\varphi = t_{\varphi(\mathbf{0})} \circ \mathrm{lin}(\varphi).$$

Proof. □

Proposition 6.7. *Let X, Y, Z be three affine spaces and $\varphi \in \mathrm{Hom}_{\mathrm{aff}}(X, Y)$, $\psi \in \mathrm{Hom}_{\mathrm{aff}}(Y, Z)$. Then $\psi \circ \varphi \in \mathrm{Hom}_{\mathrm{aff}}(X, Z)$ and*

$$\mathrm{lin}(\psi \circ \varphi) = \mathrm{lin}(\psi) \circ \mathrm{lin}(\varphi).$$

If in addition φ is bijective, then the map φ^{-1} is affine and

$$\mathrm{lin}(\varphi^{-1}) = \mathrm{lin}(\varphi)^{-1}.$$

Proof. □

Corollary 6.8. *The map 'linear part'*

$$\mathrm{lin} : \mathrm{AGL}(X) \to \mathrm{GL}(D(X))$$

is a homomorphism of groups with kernel the translations $T(D(X))$.

Proof. □

6.3 Affine subspaces associated to affine morphisms

Corollary 6.9. *Let $\varphi : X \to Y$ be an affine morphism. Then*

1. *Its image $Z = \mathrm{im}(\varphi)$ is an affine subspace of Y under the action of $V = \mathrm{im}(\mathrm{lin}\,\varphi)$.*
2. *For all $Q \in Y$, the preimage $\varphi^{-1}(Q)$ is either empty or an affine subspace of X under the action of $\ker(\mathrm{lin}\,\varphi)$.*
3. *We have*
$$\dim X = \dim \mathrm{im}(\varphi) + \dim \ker(\mathrm{lin}\,\varphi)$$
4. *In general the image and the preimage of an affine space under an affine morphism is an affine subspace (possibly $\emptyset$).*

Proof. □

Remark 6.10. See Remark **??** in Chapter 1.

Corollary 6.11. *The following holds*

1. *An affine morphism $\varphi : X \to Y$ is injective if and only if $\mathrm{lin}\,\varphi$ is (i.e. $\ker \mathrm{lin}\,\varphi = \{0\}$).*
2. *An affine morphism $\varphi : X \to Y$ is surjective if and only if $\mathrm{lin}\,\varphi$ is.*
3. *An affine morphism $\varphi : X \to Y$ is surjective if and only if $\dim \mathrm{im}\,\varphi = \dim Y$.*
4. *An affine morphism $\varphi : X \to Y$ is an isomorphism if and only if $\mathrm{lin}\,\varphi$ is bijetive.*

Proof. □

6.4 Exercises

Exercise 1. Let $\varphi, \varphi' : X \to Y$ be two affine maps with the same linear part $\mathrm{lin}\,\varphi = \mathrm{lin}\,\varphi'$. Show that φ and φ' are equal if and only if they are equal in one point.

Exercise 2. Let $f : X \to X$ be an affine transformation. Show that

1. if f fixes A and B, then it fixes the line AB;
2. the image of the midpoint of a segment is midpoint of the image of the segment;
3. the image of a parallelogram is a prallelogram
4. the image of a plane is a plane
5. the image of two parallel lines are two parallel lines.

Exercise 3. Let $f : X \to X$ be an affine transformation.

1. What is the differnece between a figure fixed by f and and one that is invariant under f?
2. Show that if f fixes three non-collinear points of a plane it fixes the plane.
3. Show that if $\dim X = 3$ and f fixes four non-coplanar points then is the identity on X.

Exercise 4. Une *affinité* φ is a transformation of the plane X which fixes a line $L_1 \subseteq X$ and leaves invariant a second line $L_2 \nparallel L_1$. Show that φ is uniquely determined by the image of a point $A \in L_2$.

Exercise 5. Consider the affine transformation of $\mathbb{R}^2$ given by

$$f(x) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

1. Does the application have a fixed point?
2. If yes, what is the fixed point?
3. Compute the algebraic form of $f^2 = f \circ f$.
4. Compute the algebraic form of f^{-1}.

Bibliography

[1] D. Andrica, *Geometrie*, Cluj-Napoca, 2017.

[2] P.A. Blaga, *Geometrie şi grafică pe calculator - note de curs*, Cluj-Napoca, 2016.

[3] M. Craioveanu, I.D. Albu, *Geometrie afină şi euclidiană*, Timişoara, 1982.

[4] GH. Galbură, F. Radó, *Geometrie*, Bucureşti, 1979.

[5] P. Michele, *Géométrie - notes de cours*, Lausanne, 2016.

[6] A. Paffenholz, *Polyhedral Geometry and Linear Optimization*, Darmstadt, 2013.

[7] C.S. Pintea, *Geometrie afină - note de curs*, Cluj-Napoca, 2017.

[8] I.P. Popescu, *Geometrie afină si euclidiană*, Timişoara, 1984.

[9] F. Radó, B. Orbán, V. Groze, A. Vasiu, *Culegere de probleme de geometrie*, Cluj-Napoca, 1979.

[10] M. Troyanov, *Cours de géométrie*, Lausanne, 2011.