Subject: ANALIZĂ - $\boxed{\text{CURS 1}}$

Date: 08 | 03 | 2018

Differential and Integral Calculus in $\mathbb{R}^n$

Part I - Differentiability of functions of several variables
- Topology in $\mathbb{R}^n$
- Differentiability of functions of several variables

$\longrightarrow$ Midterm exam (6 May 2018, 8:00)
50% $\rightarrow$ 3 problems + 1 theory
Mandatory

Part II - Integrability of functions of several variables
- Multiple integrals
- Functions of bounded variation
- Line integrals — integrale curbilinii
- Surface integrals

$\longrightarrow$ Written exam

$\boxed{\text{Chapter 1.}}$ Topology in $\mathbb{R}^n$

$\boxed{1.}$ The Euclidean space $\mathbb{R}^n$

1.1. Definition (the vector space $\mathbb{R}^n$)

Let $n \in \mathbb{N} := \{1, 2, 3, \dots\}$

(We define) Set $\mathbb{R}^n := \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{R}, \forall j = 1, \dots, n\}$

On the set $\mathbb{R}^n$ we consider the internal law $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$,

$$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto x+y \in \mathbb{R}^n$$

as well as an external law $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$,

$$\forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n \longmapsto \alpha \cdot x \in \mathbb{R}^n$$

defined as follows:

Let $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, $y := (y_1, \dots, y_n) \in \mathbb{R}^n$, and let $\alpha \in \mathbb{R}$. We define

$$x + y := (x_1 + y_1, \dots, x_n + y_n), \qquad \alpha \cdot x := (\alpha x_1, \dots, \alpha x_n)$$

It is known from the linear algebra course that $\mathbb{R}^n$ endowed (înzestrată) with this two laws is a vector space over the field of real numbers.

The origin of this vector space is $O_n := (0, \dots, 0)$

The symmetric of some $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is $-x := (-x_1, \dots, -x_n)$

Convention: We write αx instead of $\alpha \cdot x$

The elements of $\mathbb{R}^n$ can be considered from three points of view (depending of the problem which we have)

- n-tuples: $(x_1, \dots, x_n)$
  $\updownarrow$
- points: $M(x_1, \dots, x_n)$
  $\updownarrow$
- vectors: $\overrightarrow{OM}$

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1.2. Definition (the standard / canonical basis in $\mathbb{R}^n$). Set

$e_1 := (1, 0, 0, .., 0)$
$e_2 := (0, 1, 0, .., 0)$
$e_n := (0, 0, 0, .., 1)$

Then $\{e_1, e_2, ..., e_n\}$ is an algebric basis in the vector space $\mathbb{R}^n$, called the standard or the canonical basis in $\mathbb{R}^n$.

In $\mathbb{R}^2$: $e_1 = (1, 0) = \vec{i}$
$e_2 = (0, 1) = \vec{j}$

In $\mathbb{R}^3$: $e_1 = (1, 0, 0) = \vec{i}$
$e_2 = (0, 1, 0) = \vec{j}$
$e_3 = (0, 0, 1) = \vec{k}$

For every $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$
we have $x = x_1 e_1 + x_2 e_2 + ... + x_n e_n$

1.3. Definition (the scalar / dot / inner product in $\mathbb{R}^n$)

Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be arbitrary elements of $\mathbb{R}^n$.

The scalar product of the vectors x and y is the real number defined by:

$$\boxed{(1)\quad \langle x, y \rangle := x_1 y_1 + ... + x_n y_n}$$

Other notation $x \cdot y$

It is easy to verify that the scalar product has the following properties:

1° $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in \mathbb{R}^n$

2° $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}, \ \forall x, y \in \mathbb{R}^n$

3° $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in \mathbb{R}^n$

4° $\langle x, x \rangle > 0 \quad \forall x \in \mathbb{R}^n, \ x \neq 0_n$

ASTEA

The notion of a scalar product can be defined axiomatically as follows:
Let X be a real vector space.
A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called a scalar product on X if it satisfies the following set of axioms:

(SP_1) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in X$

(SP_2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}$, $\forall x, y \in X$

ASTEA satisfac

(SP_3) $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$

(SP_4) $\langle x, x \rangle > 0 \quad \forall x \in X$ but $x \neq 0_X$ (the origin of $X \to$ zero)

The ordered pair $(X, \langle \cdot, \cdot \rangle)$ ~~consisting~~ is called a PREHILBERTIAN space or a scalar product space.

From the properties 1° → 4° ⇒ the ordered pair $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ where the scalar product $\langle \cdot, \cdot \rangle$ is defined by (1) is a scalar product space called the EUCLIDIAN SPACE $\mathbb{R}^n$

1.4. Theorem (the Cauchy–Bunyakovsky–Schwarz inequality).

For all $x, y \in \mathbb{R}^m$ it holds

$$\boxed{(*) \quad |\langle x, y\rangle| \le \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle}}$$

Proof: At the seminar.

$(*) \iff |x_1 y_1 + \dots + x_m y_m| \le \sqrt{x_1^2 + \dots + x_m^2}\ \sqrt{y_1^2 + \dots + y_m^2} \quad \Big|^2$

$\iff (x_1 y_1 + \dots + x_m y_m)^2 \le (x_1^2 + \dots + x_m^2)(y_1^2 + \dots + y_m^2)$ (this must be known by us from high school)

1.5. Definition (the Euclidean norm in $\mathbb{R}^m$).

Let $x := (x_1, \dots, x_m) \in \mathbb{R}^m$

The Euclidean norm of x is defined by

$$\boxed{(2) \quad \|x\| := \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + \dots + x_m^2}}$$

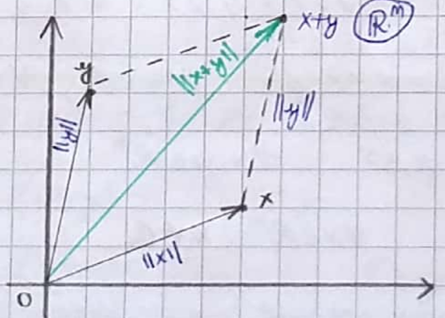
It is easy to verify that the Euclidean norm has the following properties:

1° $\|x\| = 0 \iff x = 0_m$

2° $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{R},\ \forall x \in \mathbb{R}^m$

3° $\|x+y\| \le \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^m$

↳ the triangle inequality



Also, the notion of norm can be determined axiomatically.

Let X be a real vector space.

A function $\|\cdot\| : X \to [0, \infty)$ is said to be a norm on X if it satisfies the following set of axioms:

(N_1) $\|x\| = 0 \iff x = 0_X$

(N_2) $\|\alpha x\| = |\alpha|\, \|x\| \quad \forall \alpha \in \mathbb{R},\ \forall x \in X$.

(N_3) $\|x+y\| \le \|x\| + \|y\| \quad \forall x, y \in X$

The ordered pair $(X, \|\cdot\|)$ is called a real normed space

From $1° \div 3° \Rightarrow (\mathbb{R}^m, \|\cdot\|)$, where $\|\cdot\|$ is the Euclidean norm defined by (2) is a real normed space.

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Examples

Besides the Euclidean norm on $\mathbb{R}^m$ one can introduce also other norms

a) For instance, let $p \in [1, \infty)$, and we define

This satisfies properties from $(N_1) \to (N_3)$ $\leftarrow$ $\|x\|_p := (|x_1|^p + \dots + |x_m|^p)^{\frac{1}{p}}$ $\quad \forall\, x = (x_1, \dots, x_m) \in \mathbb{R}^m$

It was proved at the seminar that $\|\cdot\|_p$ is a norm on $\mathbb{R}^m$, called the p-norm in $\mathbb{R}^m$.

The 2-norm ≡ the Euclidean norm

1-norm ≡ the Minkowski norm.

$$\|x\|_1 = |x_1| + \dots + |x_m|$$

b) Define

$$\|x\|_\infty := \max\{|x_1|, \dots, |x_m|\}, \quad \forall\, x = (x_1, \dots, x_m) \in \mathbb{R}^m$$

$\|\cdot\|_\infty$ is a norm on $\mathbb{R}^m$, called the TCHEBYSEV in $\mathbb{R}^m$.

! **Remark:** The C-B-S inequality can be written as: (another form of CBS ineq)

$$\boxed{|\langle x, y\rangle| \le \|x\| \cdot \|y\|} \quad \forall\, x, y \in \mathbb{R}^m$$

1.6. Definition (the Euclidean distance in $\mathbb{R}^m$)

Let $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$

The Euclidean distance between x and y is defined by:

formula 3 $\leftarrow$
$$\boxed{(3) \quad d(x, y) := \|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_m - y_m)^2}}$$

It can be verified that the Euclidean distance has the following properties:

1° $d(x,y) = 0 \iff x = y$

2° $d(x,y) = d(y,x) \quad \forall\, x, y \in \mathbb{R}^m$

3° $d(x,z) \le d(x,y) + d(y,z) \quad \forall\, x, y, z \in \mathbb{R}^m$

↳ the triangle inequality

Also, the notion of a distance can be defined axiomatically.

Let X be an arbitrary nonempty set.

A function $d : X \times X \to [0, \infty)$ is called a distance or a metric on X if it satisfies the following set of axioms:

(M_1) $d(x,y) = 0 \iff x = y$

(M_2) $d(x,y) = d(y,x) \quad \forall\, x, y \in X$

(M_3) $d(x,z) \le d(x,y) + d(y,z) \quad \forall\, x, y, z \in X$

The ordered pair (X, d) is called a metric space.

From 1° ÷ 3° $\Rightarrow$ $(\mathbb{R}^m, d)$, where d is the Euclidean distance, defined by (3), is a metric space

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Examples

ⓐ. Every norm on a real vector space generates a distance by means of the formula:

$$d(x, y) := \|x - y\|$$

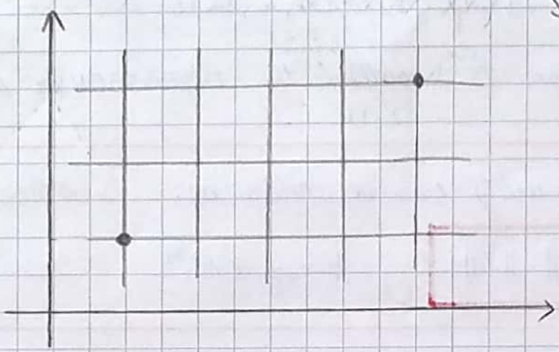
$\|\cdot\|_p$ generates the p-distance in $\mathbb{R}^m$

$$d_p(x, y) = \|x - y\|_p = \left(|x_1 - y_1|^p + \dots + |x_m - y_m|^p\right)^{\frac{1}{p}} \quad \forall \left.\begin{array}{l} x = (x_1, \dots, x_m) \\ y = (y_1, \dots, y_m) \end{array}\right| \in \mathbb{R}^m$$

$d_2 \equiv$ the Euclidean distance

$d_1 \equiv$ the Minkowski distance / the taxicab distance (distanța taximetristului)

cea mai rentabilă în America, pt. că străzile sunt ca în graficul din stânga.



ⓑ $\|\cdot\|_\infty$ generates the Tchebysev distance

$$d_m(x, y) = \|x - y\|_\infty = \max_{1 \le j \le m} |x_j - y_j|$$

2. The topological structure of $\mathbb{R}^m$

2.1. Definition (balls).

Let $a \in \mathbb{R}^m$, and let $r > 0$. Put

$$B(a, r) := \{x \in \mathbb{R}^m \mid d(x, a) < r\} = \{x \in X \mid \|x - a\| < r\}$$

↳ the open ball of radius r about a. (and center a)

$$\overline{B}(a, r) := \{x \in \mathbb{R}^m \mid d(x, a) \le r\} = \{x \in \mathbb{R}^m \mid \|x - a\| \le r\}$$

↳ the closed ball of radius r about a. (and center a)

Example

Let $\overline{B}_2(0_m, 1) = \{x \in \mathbb{R}^m \mid \|x\|_2 \le 1\}$

Euclidean

$\overline{B}_1(0_m, 1) = \{x \in \mathbb{R}^m \mid \|x\|_1 \le 1\}$

Minkowski

$\overline{B}_\infty(0_m, 1) = \{x \in \mathbb{R}^m \mid \|x\|_\infty \le 1\}$

TCHEBYSEV

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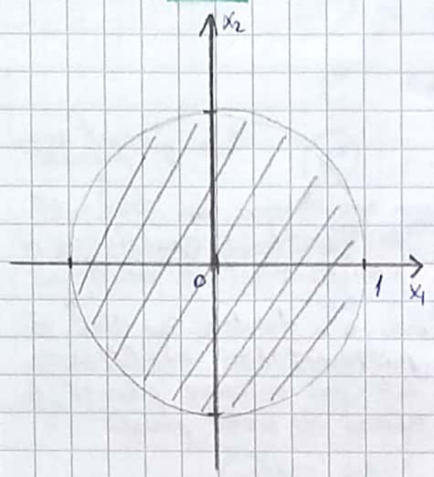
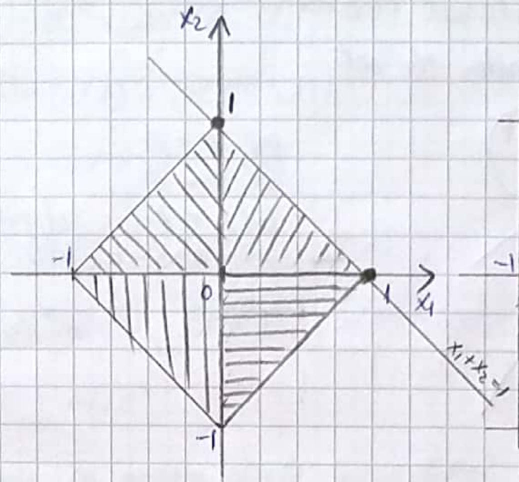
$\boxed{m=1}$ $\overline{B_2}(0,1)=\{x\in\mathbb{R} \mid |x|\leq 1\}=[-1,1]$

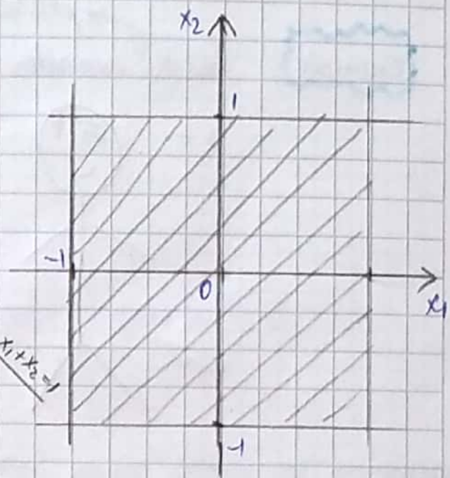
$\overline{B_1}(0,1)=[-1,1]$

$\overline{B_\infty}(0,1)=[-1,1]$

$\boxed{m=2}$







$x_1^2+x_2^2\leq 1$

$|x_1|+|x_2|\leq 1$

I. $x_1, x_2\geq 0$: $x_1+x_2\leq 1$

$\max\{|x_1|, |x_2|\}\leq 1 \Leftrightarrow$

$\Leftrightarrow |x_1|\leq 1$ and $|x_2|\leq 1$

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2. 2. Definition (neighbourhoods)

Let $x\in\mathbb{R}^m$, and let $V\subseteq\mathbb{R}^m$
The set V is called a neighbourhood of x if

$\exists\, r>0$ s.t. $B(x,r)\subseteq V$

We denote by $\mathcal{V}(x)$ = the family of all neighbourhoods of x

2. 3. Definition

Let $A\subseteq\mathbb{R}^m$, and let $x\in\mathbb{R}^m$. Then x is said to be:

- an interior point of A if $A\in\mathcal{V}(x)$ i.e. $\exists\, r>0$ s.t. $B(x,r)\subseteq A$
- an adherent (cluster point) of A if $\forall\, V\in\mathcal{V}(x)$: $V\cap A\neq\emptyset$
- a boundary point of A if $\forall\, V\in\mathcal{V}(x)$: $V\cap A\neq\emptyset$ and $V\cap(\mathbb{R}^m\setminus A)\neq\emptyset$
- a limit point of A if $\forall\, V\in\mathcal{V}(x)$: $V\cap A\setminus\{x\}\neq\emptyset$
- an isolated point of A if $\exists\, V\in\mathcal{V}(x)$: $V\cap A=\{x\}$
- an exterior point of A if $\mathbb{R}^m\setminus A\in\mathcal{V}(x)$, i.e. $\exists\, r>0$ s.t. $B(x,r)\subseteq\mathbb{R}^m\setminus A$

The set consisting of all interior points of A is called the interior of A and it's denoted by $\operatorname{int} A$, $\mathring{A}$.

The set of all exterior points of A is called the exterior of A and it's denoted by $\operatorname{ext} A$.

The set of all adherent points of A is called the closure of A and it's denoted by $cl\,A$, $\overline{A}$.

The set of all boundery points of A is called the boundery of A and it's denoted by $bd\,A$, ∂A.

The set of all limit points of A is called the derivative of A and it's denoted by A'.

Example In $\mathbb{R}^2$ consider the set → planul tablei

$A =$ 

$int\, A = \emptyset$

$cl\, A = A$

$bd\, A = A$

$A' = A \setbminus \{a, b, c\}$

$a, b, c =$

• în jurul unui punct

bd: oricum am desena un disc, el intersectează mulțimea și complementara mulțimii

A': oricum am desena un disc, el intersectează mulțimea într-un oricare punct în afară de acel punct

2.4. Definition (open sets and closed sets)

$A \subseteq \mathbb{R}^n$. A set A is said to be an open set if $A \in \mathcal{V}(x)$ $\forall x \in A$, (nând meu)

i.e. $\forall x \in A$ $\exists r > 0$ s.t. $B(x, r) \subseteq A$

The set A is called closed if the set $\mathbb{R}^n \setminus A$ is open (it's complement is open).

2.5. **Example**

Every open ball in $\mathbb{R}^n$ is an open set

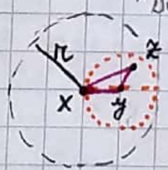
Every closed ball in $\mathbb{R}^n$ is a closed set.

Proof:

Let $x \in \mathbb{R}^n$, and let $r > 0$

? $B(x, r)$ is open $\Leftrightarrow$ $\forall y \in B(x, r)$ $\exists r' > 0$ s.t. $B(y, r') \subseteq B(x, r)$

*Desenul e doar intuitiv, nu facem dem. în funcție de el.



Let $y \in B(x, r) \Rightarrow d(y, x) < r$

Let $r' := r - d(y, x) > 0$

We prove that $B(y, r') \subseteq B(x, r)$. ✓

Let $z \in B(y, r')$.

$$d(z, x) \leq \underbrace{d(z, y)}_{< r'} + d(y, x) < r' + d(y, x) \underset{\text{by def of } r'}{=} r \Rightarrow$$

$\Rightarrow d(z, x) < r \Rightarrow z \in B(x, r)$

$\overline{B}(x, r)$ is closed → homework

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2.6. Theorem Given $A \subseteq \mathbb{R}^n$, the following relations hold:

1° $\text{int}\, A \subseteq A \subseteq \text{cl}\, A$

2° $\text{ext}\, A = \text{int}\,(\mathbb{R}^n \setminus A)$

3° $\text{cl}\, A = \mathbb{R}^n \setminus \text{int}\,(\mathbb{R}^n \setminus A)$

4° $\text{bd}\, A = (\text{cl}\, A) \cap (\text{cl}\,(\mathbb{R}^n \setminus A))$

5° $\text{cl}\, A = (\text{int}\, A) \cup (\text{bd}\, A)$

6° $\mathbb{R}^n = (\text{int}\, A) \cup (\text{bd}\, A) \cup (\text{ext}\, A)$

7° $(\text{int}\, A) \cap (\text{bd}\, A) = (\text{ext}\, A) \cap (\text{bd}\, A) = (\text{int}\, A) \cap (\text{ext}\, A) = \emptyset$

10° $\text{cl}\, A = A \cup A'$

! Those which are in green circle, prof. GRAD must prove at seminar. !

2.7. Proposition For every $A \subseteq \mathbb{R}^n$, the set $\text{int}\, A$ is open, while the set $\text{cl}\, A$ is closed.

Proof ? $\text{int}\, A$ is open $\Leftrightarrow \forall x \in \text{int}\, A \quad \exists r > 0$ s.t. $B(x,r) \subseteq \text{int}\, A$

Let $x \in \text{int}\, A \Rightarrow \exists r > 0$ s.t. $B(x,r) \subseteq A$. We prove that $B(x,r) \subseteq \text{int}\, A$

Let $y \in B(x,r)$.
We have already proved that $B(x,r)$ is open. $\Big\} \Rightarrow \exists r' > 0$ s.t. $B(y,r') \subseteq B(x,r)$
But $B(x,r) \subseteq A$

$\Rightarrow B(y,r') \subseteq A \Rightarrow y \in \text{int}\, A$

Hence $B(x,r) \subseteq \text{int}\, A$

? $\text{cl}\, A$ is closed $\Leftrightarrow \mathbb{R}^n \setminus \text{cl}\, A$ is open

But $\mathbb{R}^n \setminus \text{cl}\, A = \underbrace{\text{int}\,(\mathbb{R}^n \setminus A)}_{\text{open}}$

2.8. Theorem (characterization of open sets)

$A \subseteq \mathbb{R}^n$ is open $\Leftrightarrow A = \text{int}\, A$

Proof $\boxed{\Leftarrow}$ If $A = \text{int}\, A$
$\boxed{P2.7.} \Rightarrow \text{int}\, A$ is open $\Big\} \Rightarrow A$ is open

$\boxed{\Rightarrow}$ Assume that A is open

? $A = \text{int}\, A \Leftrightarrow A \subseteq \text{int}\, A$ (the other inclusion holds for every set A)

Let $x \in A \xrightarrow{A \text{ is open}} \exists r > 0$ s.t. $B(x,r) \subseteq A \Rightarrow x \in \text{int}\, A$

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2.9. Theorem (characterization of closed sets)

$$bd A \subseteq A \overset{②}{\Longleftrightarrow} A \subseteq \mathbb{R}^m \text{ is closed} \overset{①}{\Longleftrightarrow} A = cl A$$

Proof ① $\Leftarrow$ If $A = cl A$; P2.7 $\Rightarrow cl A$ is closed $\Big\} \Rightarrow A$ is closed

① $\Rightarrow$ Assume that A is closed $\Rightarrow \mathbb{R}^m \setminus A$ is open $\overset{T2.8}{\Longrightarrow} int(\mathbb{R}^m \setminus A) = \mathbb{R}^m \setminus A$; But $cl A = \mathbb{R}^m \setminus int(\mathbb{R}^m \setminus A)$ $\Big\} \Rightarrow$

$\Rightarrow cl A = A$.

② $\Rightarrow$ Assume $bd A \subseteq A$; We know $int A \subseteq A$ $\Big\} \Rightarrow \underbrace{(int A) \cup (bd A)}_{cl A} \subseteq A \Rightarrow cl A \subseteq A \Rightarrow cl A = A \overset{1}{\Rightarrow}$

$\Rightarrow A$ is closed

② $\Leftarrow$ Assume A is closed $\overset{1}{\Rightarrow} A = cl A$; But $cl A = (int A) \cup (bd A)$ $\Big\} \Rightarrow bd A \subseteq A$.

3. Sequences in $\mathbb{R}^m$

3.1. Definition (convergent sequences in $\mathbb{R}^m$)

Every function $f : \mathbb{N} \to \mathbb{R}^m$ is called a sequence in $\mathbb{R}^m$

$\boxed{\text{if } f(k) = x_k \quad \forall k \in \mathbb{N}.}$

Then the seq. will be denoted by $(x_k)_{k\in\mathbb{N}}$, or $(x_k)_{k\geq 1}$, or (x_k).

Let (x_k) be a seq. in $\mathbb{R}^m \Rightarrow x_k \in \mathbb{R}^m, \forall k \in \mathbb{N}$.

$x_k = (x_{k1}, x_{k2}, \dots, x_{km})$

Let (x_k) be a seq. in $\mathbb{R}^m$, and let $x \in \mathbb{R}^m$.

The seq. x_k is said to be convergent to x (or x is said to be a limit of the seq. (x_k)) if:

index or rank

$$\boxed{\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0 : \|x_k - x\| < \varepsilon} \iff \big|\|x_k - x\| - 0\big| < \varepsilon$$

For 3.2. proof

It is imediately seen that a seq. in $\mathbb{R}^m$ has at most one limit.

(de la rf. rândului) If this limit exists, then the seq. is called convergent.

Notation: $\lim_{k\to\infty} x_k = x$, $(x_k) \to x$.

3.2. Theorem Let (x_k) be a seq. in $\mathbb{R}^m$, and let $x \in \mathbb{R}^m$. Then

$$\boxed{\lim_{k\to\infty} x_k = x \iff \lim_{k\to\infty} \|x_k - x\| = 0.}$$

Proof: Obvious

3.3. Theorem Let (x_k) and (y_k) be convergent sequence in $\mathbb{R}^m$, let (α_k) be a convergent sequence of real numbers, let $x := \lim\limits_{k\to\infty} x_k$, let $y := \lim\limits_{k\to\infty} y_k$, and let $\alpha := \lim\limits_{k\to\infty} \alpha_k$. Then:

$$\boxed{\lim_{k\to\infty} (x_k + y_k) = x + y, \quad \lim_{k\to\infty} \alpha_k x_k = \alpha x}$$

Proof: Obvious.

Ex. $x_k = \left(\frac{k}{k+1}, \frac{k^2}{k^3+1}\right)$ seq. in $\mathbb{R}^2$

$x_k = \left(\left(1+\frac{1}{k}\right)^k, k \sin\frac{1}{k}, \frac{1}{k}\right)$ seq. in $\mathbb{R}^3$

3.4. Theorem. Let (x_k) be a seq. in $\mathbb{R}^m$, $x_k = (x_{k1}, x_{k2}, \dots, x_{km})$ $(k \in \mathbb{N})$

and let $\overline{x} := (\overline{x_1}, \overline{x_2}, \dots, \overline{x_m}) \in \mathbb{R}$. Then:

$$\boxed{\lim_{k\to\infty} x_k = \overline{x} \iff \lim_{k\to\infty} x_{kj} = \overline{x_j}, \ \forall j = \overline{1,m}}$$

Proof: $\boxed{\Rightarrow}$ Assume that $(x_k) \longrightarrow \overline{x} \Rightarrow \lim\limits_{k\to\infty} \|x_k - \overline{x}\| = 0.$

Fix an arbitrary $j \in \{1, \dots, m\}$. Then we have:

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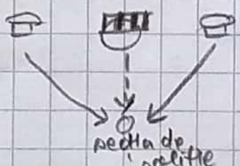


$$\|x_k - \overline{x}\| = \sqrt{(x_{k1} - \overline{x}_1)^2 + \dots + (x_{km} - \overline{x}_m)^2} \geq |x_{kj} - \overline{x}_j| \geq 0$$

$\xrightarrow{k\to\infty} 0$

$\Rightarrow \lim\limits_{k\to\infty} |x_{kj} - \overline{x}_j| = 0 \Rightarrow \lim\limits_{k\to\infty} x_{kj} = \overline{x}_j$

$\boxed{\Leftarrow}$ Assume that $\lim\limits_{k\to\infty} x_{kj} = \overline{x}_j \ \forall j = \overline{1,m}$. We have:

$$0 \leq \|x_k - \overline{x}\| = \sqrt{(x_{k1} - \overline{x}_1)^2 + \dots + (x_{km} - \overline{x}_m)^2} \leq \underbrace{|x_{k1} - \overline{x}_1| + \dots + |x_{km} - \overline{x}_m|}$$

$\xrightarrow{k\to\infty} 0$

Subject: Date: ....../....../......

Ex. $\lim_{k\to\infty}\left(\left(1+\frac{1}{k}\right)^k, k \sin\frac{1}{k}, \frac{1}{k}\right) = (e, 1, 0)$

$k\sin\frac{1}{k} = \dfrac{\sin\frac{1}{k}}{\frac{1}{k}}$

ca un șir să fie convergent:

$\rightarrow$ 3.5.

$\rightarrow$

3.5. Definition (Cauchy sequences)

A sequence (x_k) in $\mathbb{R}^m$ is called a Cauchy seq. if:

rank / index k_0

$$\boxed{\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k, l \geq k_0 : \|x_k - x_l\| < \varepsilon}$$

3.6. Theorem. Let (x_k) be a seq. in $\mathbb{R}^m$, $x_k = (x_{k1}, x_{k2}, \dots, x_{km})$. Then:

$$\boxed{(x_k) \text{ is a Cauchy seq.} \Leftrightarrow (x_{kj})_{k\geq 1} \text{ is Cauchy}, \forall j = \overline{1,m}}$$

Proof: Is analogous to the proof of $\boxed{T\ 3.4.}$ $\rightarrow$ de la convergență

3.7. Theorem (A.L. Cauchy)

A seq. of points in $\mathbb{R}^m$ is convergent $\Leftrightarrow$ it is a Cauchy seq.

Proof: Let (x_k) be a seq. in $\mathbb{R}^m$, $x_k =$

(x_k) converges $\overset{\boxed{T3.4.}}{\Longleftrightarrow}$ $(x_{kj})_{k\geq 1}$ converges $\forall j = \overline{1,m}$

$\overset{\text{Cauchy Th. in } \mathbb{R}}{\Longleftrightarrow}$ $(x_{kj})_{k\geq 1}$ is a Cauchy seq. $\forall j = \overline{1,m}$

$\overset{\boxed{T3.6}}{\Longleftrightarrow}$ (x_k) is a Cauchy seq. in $\mathbb{R}^m$

3.8. Theorem (characterization of cluster points by using sequences).

Let $A \subseteq \mathbb{R}^m$, and let $x \in \mathbb{R}^m$. Then:

$x \in \operatorname{cl} A \Leftrightarrow \exists (x_k)$ seq. in A s.t. $(x_k) \to x$

Proof $\boxed{\Rightarrow}$ Assume that $x \in \operatorname{cl} A \Rightarrow \forall V \in \mathcal{V}(x) : V \cap A \neq \emptyset$.
In particular, for $V := B(x, \frac{1}{k}) \Rightarrow \forall k \geq 1 : B(x, \frac{1}{k}) \cap A \neq \emptyset$

For $\forall k \geq 1 \quad \exists x_k \in B(x, \frac{1}{k}) \cap A \Rightarrow \begin{cases} x_k \in A & \forall k \geq 1 \\ \|x_k - x\| < \frac{1}{k} \Rightarrow \lim_{k\to\infty} x_k = x \end{cases}$

$\boxed{\Leftarrow}$ Assume that $\exists (x_k)$ seq. in A s.t. $(x_k) \rightarrow x$

Let V be a neighbourhood of $x \Rightarrow \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq V$.

Since $(x_k) \rightarrow x \Rightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0 : \|x_k - x\| < \varepsilon$

$$\left.\begin{array}{l} \Rightarrow \forall k \geq k_0 : \quad x_k \in B(x, \varepsilon) \subseteq V \\ \qquad\qquad\quad\; x_k \in A \end{array}\right\} \Rightarrow \forall k \geq k_m : \; x_k \in V \cap A$$

$\Downarrow$

$V \cap A \neq \emptyset$

Hence $x \in \operatorname{cl} A$.

3.9. Corollary (characterization of limit points by using sequences)

Let $A \subseteq \mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Then:

$$x \in A' \iff \exists (x_k) \text{ seq. in } A \setminus \{x\} \text{ s.t. } (x_k) \rightarrow x.$$

$\forall$ pct. aderent este și pct. de acumulare, dar invers NU!

Proof: Follows by $\boxed{T\ 3.8}$, taking into account that $x \in A' \iff x \in \operatorname{cl}(A \setminus \{x\})$

3.10. Corollary (characterization of closed sets by using sequences).

A set $A \subseteq \mathbb{R}^n$ is closed $\iff$ the limit of every convergence sequence of ~~points in~~ points in A belongs also to A.

Proof: $\boxed{\Rightarrow}$ Assume A is closed. Let (x_k) be an arbitrary convergent sequence of points in A, and let $x := \lim_{k \to \infty} x_k$.

$$\left.\begin{array}{l} \text{By } \boxed{T\ 3.8} \Rightarrow x = \operatorname{cl} A \\ \text{But } \operatorname{cl} A = A \text{ because } A \text{ is closed} \end{array}\right\} \Rightarrow x \in A.$$

$\boxed{\Leftarrow}$ Assume that the limit of every convergent sequence of points in A belongs also in A.

? A is closed $\iff A = \operatorname{cl} A \iff \operatorname{cl} A \subseteq A$.

Let $x \in \operatorname{cl} A \xRightarrow{T3.8} \exists (x_k)$ seq. in A s.t. $(x_k) \rightarrow x \xrightarrow{\text{our hypothesis}} x \in A$

$\boxed{4.}$ Compact sets in $\mathbb{R}^m$

4.1. Definition (compact sets)

Let $A \subseteq \mathbb{R}^m$.
A family $\underbrace{(A_i)_{i\in I}}_{\text{of subsets of } \mathbb{R}^m}$ is called a covering of A if: $\boxed{A \subseteq \bigcup_{i \in I} A_i.}$

If all the sets A_i are open $(i \in I)$ then the covering $(A_i)_{i\in I}$ is called open.
The set A is called compact if every covering of it, one can extract a finite subcovering.

$$\boxed{\begin{array}{l} A \text{ is compact} \Leftrightarrow \forall\, (A_i)_{i\in I} \text{ open covering of } A \\ \qquad \exists\, J \subseteq I,\ J = \text{finite s.t. } A \subseteq \bigcup_{j\in J} A_j \end{array}}$$

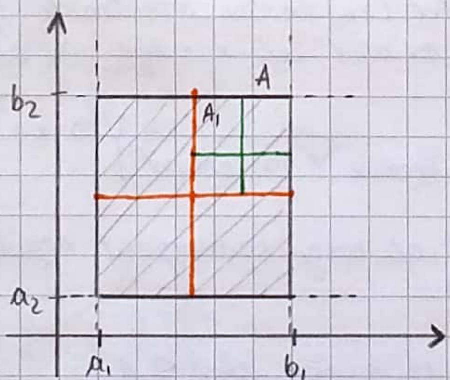
4.2. $\boxed{\text{Example}}$ A set $A \subseteq \mathbb{R}^m$ is called a closed hypercube if it has the form:

$$A = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_m, b_m]$$

where $b_1 - a_1 = b_2 - a_2 = \dots = b_m - a_m = l$.

Every closed hypercube in $\mathbb{R}^m$ is a compact set.

Proof: $\boxed{m=2}$ $A = [a_1, b_1] \times [a_2, b_2]$, $b_1 - a_1 = b_2 - a_2 = l$



Suppose that A IS NOT COMPACT.
$\Rightarrow$ there exists an open covering $(G_i)_{i\in I}$ of A from which one cannot extract a finite subcovering

- We split A in 4 equal squares. At least one of this squares cannot be covered by a finite number of sets (G_i)

Let us denote by $A_1 = [a_{11}, b_{11}] \times [a_{21}, b_{21}]$

where $a_1 \le a_{11} < b_{11} \le b_1$, $a_2 \le a_{21} < b_{21} \le b_2$

$$b_{11} - a_{11} = b_{21} - a_{21} = \frac{l}{2}$$

- Next, we split A_1 in 4 equal squares. At least one of them cannot be covered by a finite number of sets (G_i), denoted by A_2.

$$A_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}]$$
$$a_{11} \le a_{12} < b_{12} \le b_{11}$$
$$a_{21} \le a_{22} < b_{22} \le b_{21} \qquad b_{12} - a_{12} = b_{22} - a_{22} = \frac{l}{2^2}$$

Continuing this argument by induction $\Rightarrow$ ... - next course -

Continuing this argument by induction, it follows that there exists a sequence of squares $(A_k)_{k\geq1}$, $A_k = [a_{1k}, b_{1k}] \times [a_{2k}, b_{2k}]$ with the following properties:

- each A_k is contained in A
- $A_{k+1} \subseteq A_k \Leftrightarrow a_{1k} \leq a_{1,k+1} < b_{1,k+1} \leq b_{1k}$
  $a_{2k} \leq a_{2,k+1} < b_{2,k+1} \leq b_{2k}$ $\forall k \geq 1$
- $b_{1k} - a_{1k} = b_{2k} - a_{2k} = \frac{l}{2^k}$

$\Rightarrow ([a_{1k}, b_{1k}])_{k\geq1}$ and $([a_{2k}, b_{2k}])_{k\geq1}$ are decreasing sequences of closed intervals, having length $= \frac{l}{2^k} \xrightarrow{k\to\infty} 0$.

$\xRightarrow{\text{sem I}} \bigcap_{k=1}^{\infty} [a_{1k}, b_{1k}] = \{x_1^*\}$ and $\bigcap_{k=1}^{\infty} [a_{2k}, b_{2k}] = \{x_2^*\}$

Put $x^* := (x_1^*, x_2^*)$. Then, $x^* \in A_k$, $\forall k \in \mathbb{N}^* \Rightarrow x^* \in A \subseteq \bigcup_{i\in I} G_i \Rightarrow \exists i^* \in I$ s.t. $x^* \in G_{i^*}$, G_{i^*} is open $\Big\} \Rightarrow$

$\Rightarrow \exists r > 0$ s.t. $B(x^*, r) \subseteq G_{i^*}$

Take $k \in \mathbb{N}$ sufficiently large s.t.

? } ⟶ cred că mai trebuie scris ceva aici → completez mâine

Let $x = (x_1, x_2) \in A_k$

$$\|x - x^*\| = \sqrt{(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2} \leq \sqrt{\frac{l^2}{2^{2k}} + \frac{l^2}{2^{2k}}} = \sqrt{2} \cdot \frac{l}{2^k} < r$$

$x, x^* \in A_k = [a_{1k}, b_{1k}] \times [a_{2k}, b_{2k}]$

$\Rightarrow x_1, x_1^* \in [a_{1k}, b_{1k}] \Rightarrow |x_1 - x_1^*| \leq b_{1k} - a_{1k} = \frac{l}{2^k}$

$\Rightarrow (x_1 - x_1^*)^2 \leq \frac{l^2}{2^{2k}}$

Analogously $(x_2 - x_2^*)^2 \leq \frac{l^2}{2^{2k}}$

$\Rightarrow \forall x \in A_k: \|x - x^*\| < r \Rightarrow A_k \subseteq B(x^*, r) \subseteq G_{i^*}$

$\Rightarrow A_k$ can be covered by one set in the family $(G_i)_{i\in I}$, namely G_{i^*}

→ contradiction

⇒ proof complete

4.3. Definition (bounded sets)

A set $A \subseteq \mathbb{R}^n$ is called bounded if it is contained in certain closed ball $\overline{B}(a, r) \Leftrightarrow$

$$A \subseteq \mathbb{R}^n \text{ bounded} \Leftrightarrow \exists a \in \mathbb{R}^n, \exists r > 0 : A \subseteq \overline{B}(a, r)$$

4.4. Theorem (Characterization of compact sets)

Let $A \subseteq \mathbb{R}^n$. Then, the following statements are equivalent:

1° A is compact.

2° Each infinite subset of A has a limit point belonging to A.

3° A is sequentially compact (i.e. every sequence of points in A has a subsequence converges to an element of A)

4° A is closed and bounded

Proof: $\boxed{1° \Rightarrow 2°}$ Assume that A is compact. Let A_0 be an arbitrary infinite subset of A. Suppose that A_0 does not have a limit point in A.

$\Rightarrow \forall x \in A,\ x \notin A_0' \Rightarrow \forall x \in A \quad \exists r_x > 0$ s.t. $B(x, r_x) \cap A_0 \setminus \{x\} = \emptyset$

$$\Updownarrow$$

$$B(x, r_x) \cap A_0 \subseteq \{x\}$$

$\Rightarrow \forall x \in A \ \ \exists r_x > 0$ s.t. $B(x, r_x) \cap A_0 \subseteq \{x\}$

The family of open balls $(B(x, r_x))_{x \in A}$ is an open covering of A.

$\xRightarrow{A \text{ is compact}}$ this open covering possesses a finite subcovering

$\Rightarrow \exists x_1, \dots, x_k \in A$ s.t. $A \subseteq B(x_1, r_{x_1}) \cup \dots \cup B(x_k, r_{x_k})$

$\Rightarrow A = A_0 \cap A \subseteq A_0 \cap (B(x_1, r_{x_1}) \cup \dots \cup B(x_k, r_{x_k}))$

$$= \underbrace{(A_0 \cap B(x_1, r_{x_1}))}_{\subseteq \{x_1\}} \cup \dots \cup \underbrace{(A_0 \cap B(x_k, r_{x_k}))}_{\subseteq \{x_k\}}$$

$\Rightarrow A_0 \subseteq \{x_1, \dots, x_m\}$ $\Rightarrow\Leftarrow$ A_0 is infinite — this means contradiction

$\boxed{2° \Rightarrow 3°}$ Let (x_k) be an arbitrary sequence of points in A, and let us define $A_0 := \{x_k \mid k = 1, 2, 3, \dots\}$.

Case I. A_0 is finite $\Rightarrow$ there is a term of the sequence occurring infinitely many times $\Rightarrow$ (x_k) has a constant subsequence, which is obviously convergent.

Case II. A_0 is infinite $\xRightarrow{\text{hyp } 2°}$ $\exists x \in A$ s.t. $x \in A_0'$

$\Rightarrow \forall V \in \mathcal{V}(x): \ V \cap A_0 \setminus \{x\} \neq \emptyset$ $\xRightarrow{\text{Prove it!}}$

$\Rightarrow \forall V \in \mathcal{V}(x): \ V \cap A_0$ is infinite

Take $V = B(x, 1) \Rightarrow B(x,1) \cap A_0$ is infinite $\Rightarrow \exists x_{k_1} \in A_0 \cap B(x, 1) \Rightarrow \|x_{k_1} - x\| < 1$

Take $V = B(x, \frac{1}{2}) \Rightarrow B(x, \frac{1}{2}) \cap A_0$ is infinite $\Rightarrow \exists k_2 > k_1$ s.t. $x_{k_2} \in A_0 \cap B(x, \frac{1}{2})$

$$\Downarrow$$

$$\|x_{k_2} - x\| < \tfrac{1}{2}$$

$\vdots$ (inductively)

$\Rightarrow \exists (k_j)$ sequence of natural strictly increasing s.t.

$\|x_{k_j} - x\| < \frac{1}{j} \quad \forall j \geq 1 \Rightarrow \lim_{j \to \infty} \|x_{k_j} - x\| = 0 \Rightarrow \lim_{j \to \infty} x_{k_j} = x$

$\boxed{3° \Rightarrow 4°}$ Assume that A is sequentially compact.

- Suppose that A is not closed $\xRightarrow[\text{using seqs}]{\text{charact. of closed sets}}$ there exists a convergent sequence (x_k) of points in A, whose limit does not belong to A.

Let $x := \lim_{k \to \infty} x_k$.

Since (x_k) is a sequence in A $\xRightarrow{A \text{ seq. compact}}$ $\exists (x_{k_j})_{j \geq 1}$ subseq. of (x_k) and $\exists x' \in A$ s.t. $(x_{k_j}) \xrightarrow{j \to \infty} x'$

But $(x_{k_j}) \xrightarrow{j \to \infty} x$

$\Downarrow$ uniqueness of the limit

$x = x'$

$\Downarrow$ $x' \in A$

$x \in A$ $\Rightarrow\Leftarrow$

Hence A must be closed.

• Suppose that A is not bounded $\Rightarrow A \not\subseteq \bar{B}(0_m, k),\ k\in\mathbb{N} \Rightarrow$

$\Rightarrow \exists\, x_k \in A$ s.t. $x_k \notin \bar{B}(0_m,k) \Leftrightarrow \|x_k\| > k$

$\Rightarrow k_j$ sir cresc. de nr. nat.

Then (x_k) is a seq. in $A \xrightarrow[\text{compact}]{A \text{ is seq.}} \exists\,(x_{k_j})_{j\ge1}$ subseq. of (x_k) and $\exists\, x\in A$ s.t. $(x_{k_j}) \xrightarrow{j\to\infty} x$

$\Rightarrow \lim\limits_{j\to\infty}\|x_{k_j}-x\|=0$

But $\|x_{k_j}-x\| \ge \underbrace{\|x_{k_j}\|}_{>k_j} - \|x\| > k_j - \|x\| \xrightarrow{j\to\infty}\infty$ $\Rightarrow\Leftarrow$

Hence A must be bounded.

$\boxed{4° \Rightarrow 1°}$ Assume that A is closed and bounded. Let (G_i) be an arbitrary open covering of A.

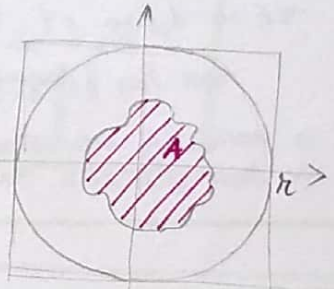
A is bounded $\Rightarrow \exists\, r>0$ s.t. $A\subseteq \bar{B}(0_m,r)$

But $\bar{B}(0_m,r)\subseteq[-r,r]^m$

A is closed $\Rightarrow \mathbb{R}^m\setminus A$ is open

$\Rightarrow (G_i)_{i\in I}\cup\{\mathbb{R}^m\setminus A\}$ is an open covering of $[-r,r]^m$

We proved that $[-r,r]^m$ is compact $\Bigg\} \Rightarrow$



$\Rightarrow \exists\, J\subseteq I$, J - finite s.t. $[-r,r]^m \subseteq \left(\bigcup\limits_{i\in J} G_i\right)\cup(\mathbb{R}^m\setminus A)$

$A\subseteq[-r,r]^m$ $\Bigg\} \Rightarrow$

$\Rightarrow A\subseteq\left(\bigcup\limits_{i\in J}G_i\right)\cup(\mathbb{R}^m\setminus A)$

But $A\cap(\mathbb{R}^m\setminus A)=\emptyset$ $\Bigg\} \Rightarrow A\subseteq\bigcup\limits_{i\in J}G_i$

Hence A is compact

5. Limits of vector function

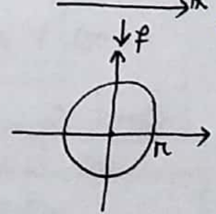
5.1. Definition

• A function $f:A\to\mathbb{R}^m$, where $A\subseteq\mathbb{R}$ is called a vector function of a real variable

Ex: $f:\mathbb{R}\to\mathbb{R}^2,\ f(t)=(r\cos t,\ r\sin t)$ cerc

$f:\mathbb{R}\to\mathbb{R}^3,\ f(t)=(r\cos\omega t,\ r\sin\omega t,\ vt)$ elipsă

(ω) omega



• A function $f:A\to\mathbb{R}$, where $A\subseteq\mathbb{R}^m$ is called real or scalar function of vector value.

Ex. $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=e^{-x^2-y^2}$ surface

• A function $f:A\to\mathbb{R}^m$, where $A\subseteq\mathbb{R}^n$ is called a vector function of a vector variable

Ex $f:\mathbb{R}^3\to\mathbb{R}^2$

$f(x,y,z)=\left(\underbrace{\sin(x+y+z)}_{f_1(x,y,z)},\ \underbrace{x^2-y^2+2z^2}_{f_2(x,y,z)}\right)$

Let $A \subseteq \mathbb{R}^n$, and let $f: A \to \mathbb{R}^m$. We define $f_1, \dots, f_m : A \to \mathbb{R}$ as follows: take $x \in A \Rightarrow f(x) \in \mathbb{R}^m \Rightarrow f(x)$ is of the form $f(x) = (y_1, \dots, y_m)$. Set $f_1(x) := y_1, \dots, f_m(x) := y_m$.

The functions $f_1, \dots, f_m$ defined in this way are called the scalar components of the vector function f.

$$\boxed{\forall x \in A : f(x) = f_1(x), \dots, f_m(x)}$$

We write $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$ each time we want to emphasize the scalar components of f.

5.2. Definition (the limit of a vector function at a point)
Let $A \subseteq \mathbb{R}^n$, let $a \in A'$, let $f: A \to \mathbb{R}^m$, and let $b \in \mathbb{R}^m$.
Then f is said to have limit b at the point a if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x \in A \setminus \{a\} \text{ with } \|x - a\| < \delta$$

$$\text{one has } \|f(x) - b\| < \varepsilon.$$

It is immediately seen that f can have at most one limit at the point a.
If b is the limit of f at a, then we write: $\lim_{x \to a} f(x) = b$.

*pt. limite de funcții vectoriale nu există regula lui L'Hospital

5.3. Theorem (characterization of the limit of a vector function by means of sequences, Heine's theorem)
Let $A \subseteq \mathbb{R}^n$, $a \in A'$, $f: A \to \mathbb{R}^m$, $b \in \mathbb{R}^m$.
Then $\lim_{x \to a} f(x) = b \iff \forall (x_k)$ seq. in $A \setminus \{a\}$ with $\lim_{k \to \infty} x_k = a$ one has $\lim_{k \to \infty} f(x_k) = b$.

Proof: $\boxed{\Rightarrow}$ Assume that $\lim_{x \to a} f(x) = b$. Let (x_k) be an arbitrary sequence in $A \setminus \{a\}$ with $\lim_{k \to \infty} x_k = a$.

Let $\varepsilon > 0 \Rightarrow \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $\|x - a\| < \delta : \|f(x) - b\| < \varepsilon$,
Since $(x_k) \to a \Rightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0 : \|x_k - a\| < \delta \Rightarrow$
$\Rightarrow \forall k \geq k_0$ we have $x_k \in A \setminus \{a\}$ and $\|x_k - a\| < \delta$ $\Rightarrow$

$\Rightarrow \forall k \geq k_0 : \|f(x_k) - b\| < \varepsilon$

$\boxed{\Leftarrow}$ Suppose, by reduction at absurdum, that f does not have the limit b at the point $a \Rightarrow$

$\Rightarrow \exists \varepsilon > 0$ s.t. $\forall \delta > 0 \ \exists x \in A \setminus \{a\}$ with $\|x - a\| < \delta$ s.t. $\|f(x) - b\| \geq \varepsilon$
In particular, taking $\delta = \frac{1}{k} \Rightarrow \forall k \in \mathbb{N} \ \exists x_k \in A \setminus \{a\}$ with $\|x_k - a\| < \frac{1}{k}$ s.t. $\|f(x_k) - b\| \geq \varepsilon$

$\Rightarrow (x_k)$ is a seq. in $A \setminus \{a\}$, with $(x_k) \to a$ but $f(x_k) \not\to b$ because $\|f(x_k) - b\| \geq \varepsilon, \forall k \in \mathbb{N}$ $\Rightarrow\Leftarrow$ hypothesis

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!Remark: Heine's theorem is usefull when we want to prove that a function does not have a limit in a point.

ex.: Prove that the limit $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Solution: $f: \mathbb{R}^2\setminus\{(0,0)\} \longrightarrow \mathbb{R},\ f(x,y) = \frac{xy}{x^2+y^2}$

$$\left.\begin{aligned} &\text{Let } x_k = \left(\tfrac{1}{k}, \tfrac{1}{k}\right) \xrightarrow{k\to\infty} (0,0);\ f(x_k) = f\left(\tfrac{1}{k}, \tfrac{1}{k}\right) = \tfrac{1}{2} \xrightarrow{k\to\infty} \tfrac{1}{2} \\ &\phantom{\text{Let }} y_k = \left(\tfrac{1}{k}, \tfrac{2}{k}\right) \xrightarrow{k\to\infty} (0,0);\ f(y_k) = f\left(\tfrac{1}{k}, \tfrac{2}{k}\right) = \tfrac{2}{5} \xrightarrow{k\to\infty} \tfrac{2}{5} \end{aligned}\right\} \Rightarrow$$

$\xRightarrow[\text{theorem}]{\text{Heine's}} \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

5.4. Theorem

Let $A \subseteq \mathbb{R}^n$, $a \in A'$, $f: A \to \mathbb{R}^m$, and let $b \in \mathbb{R}^m$.

Then $\boxed{\lim\limits_{x\to a} f(x) = b \iff \lim\limits_{x\to a} \|f(x) - b\| = 0}$

Proof: Obvious

5.5. Theorem

Let $A \subseteq \mathbb{R}^n$, $a \in A'$, $f = (f_1, \dots, f_m): A \longrightarrow \mathbb{R}^m$, $b = (b_1, \dots, b_m) \in \mathbb{R}^m$

Then $\lim\limits_{x\to a} f(x) = b \iff \lim\limits_{x\to a} f_j(x) = b_j \quad \forall j \in \{1, \dots, m\}$

Proof: Apply Heine's Theorem + $\boxed{T\,3.4}$

6. Continuity of vector function

6.1. Definition

Let $A \subseteq \mathbb{R}^n$, let $f: A \to \mathbb{R}^m$, and let $a \in A$.

The function f is said to be continuous at a if:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \forall x \in A \text{ with } \|x - a\| < \delta$$

$$\text{one has } \|f(x) - f(a)\| < \varepsilon$$

!Remark: If a is an isolated point of A, then f is continuous at a.

6.2. Theorem (Characterization of continuity using sequences)

Let $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$, $a \in A$. Then:

f is continuous at $a \iff \forall (x_k)$ seq. in A, with $\lim\limits_{k\to\infty} x_k = a$, we have $\lim\limits_{k\to\infty} f(x_k) = f(a)$

Proof: is similar to the proof of Heine's Theorem

→ temă de casă facultativă

6.3. Theorem (continuity vs limit)

Let $A \subseteq \mathbb{R}^m$, $f: A \to \mathbb{R}^m$, $a \in A \cap A'$.

Then f is continuous at $a \iff \exists \lim_{x \to a} f(x) = f(a)$

Proof: homework

6.4. Theorem

Let $A \subseteq \mathbb{R}^n$, $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$, $a \in A$.

Then f is continuous at $a \iff f_i$ is continuous at a, $\forall i = \overline{1,m}$

Proof: homework

6.5. Theorem

If A is a compact subset of $\mathbb{R}^n$, and $f: A \to \mathbb{R}^m$ is continuous on A, then $f(A)$ is compact in $\mathbb{R}^m$.

Proof: ? $f(A)$ is compact $\xLeftrightarrow[\text{of compact set}]{\text{Charact.}}$ $f(A)$ is seq. compact

Let (y_k) be an arbitrary seq. in $f(A)$

$\Rightarrow \forall k \in \mathbb{N}$ $\exists x_k \in A$ s.t. $y_k = f(x_k)$

$\left.\begin{array}{l} \Rightarrow (x_k) \text{ is a seq. of points in } A \\ A \text{ compact} \end{array}\right\} \Rightarrow \exists (x_{k_j})_{j \geq 1}$ subseq. of (x_k)

and $\exists x \in A$ s.t. $(x_{k_j}) \xrightarrow{j \to \infty} x$

Since f is continuous at $a \Rightarrow \lim_{j \to \infty} f(x_{k_j}) = f(x)$

$\Rightarrow \lim_{j \to \infty} y_{k_j} = f(x)$, i.e. $(y_{k_j})_{j \geq 1}$ is subseq. of (y_k) converging to $f(x) \subseteq f(A)$

6.6. Theorem (K. Weierstrass)

If $A \subseteq \mathbb{R}^m$ is compact and $f: A \to \mathbb{R}$ is continuous on A, then f is bounded and attains its bounds.

Proof: By $\boxed{T6.5}$ $\Rightarrow f(A)$ is compact $\Rightarrow$

$\Rightarrow f(A)$ is bounded $\Rightarrow f$ is bounded

and

closed $\Rightarrow \inf f(A) \in f(A)$ and $\sup f(A) \in f(A) \Rightarrow$

$\Rightarrow f$ attains its bounds.

ANALIZA

Ch. II: Differentiability of functions of several numbers

COURSE 4
ANALIZA
23.03.2018

1. The normed space of linear mappings

1.1. Definition

A function $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is called a linear mapping if:

$$\forall \alpha, \beta \in \mathbb{R}, \forall x, y \in \mathbb{R}^m : \varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$$

$$L(\mathbb{R}^m, \mathbb{R}^m) := \{\varphi : \mathbb{R}^m \to \mathbb{R}^m \mid \varphi \text{ is linear mapping}\}$$

If $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m) \Rightarrow \varphi(0_m) = 0_m$

$\varphi(-x) = -\varphi(x) \quad \forall x \in \mathbb{R}^m$

$\forall k \in \mathbb{N}, \forall \alpha_1, \dots, \alpha_k \in \mathbb{R}, \forall x_1, \dots, x_k \in \mathbb{R}^m$:

$\varphi(\alpha_1 x_1 + \dots + \alpha_k x_k) = \alpha_1 \varphi(x_1) + \dots + \alpha_k \varphi(x_k)$

1.2. Theorem. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$. Then:

$\varphi \in L(\mathbb{R}^m, \mathbb{R}^m) \Leftrightarrow \exists v_1, \dots, v_m \in \mathbb{R}^m$ s.t. $\forall x = (x_1, \dots, x_m) \in \mathbb{R}^m$:

$$\varphi(x) = x_1 v_1 + \dots + x_m v_m$$

Proof: $\boxed{\Rightarrow}$ Assume $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$. Let $v_1 := \varphi(e_1), \dots, v_m := \varphi(e_m)$.

Then $\forall x = (x_1, \dots, x_m) \in \mathbb{R}^m$: $x = x_1 e_1 + \dots + x_m e_m \Rightarrow$

$\Rightarrow \varphi(x) = \varphi(x_1 e_1 + \dots + x_m e_m) = x_1 \varphi(e_1) + \dots + x_m \varphi(e_m) = x_1 v_1 + \dots + x_m e_m$.

$\boxed{\Leftarrow}$ Let $\alpha, \beta \in \mathbb{R}$, and let $x, y \in \mathbb{R}^m$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$

$\Rightarrow \alpha x + \beta y = (\alpha x_1 + \beta y_1, \dots, \alpha x_m + \beta y_m)$

$\Rightarrow \varphi(\alpha x + \beta y) = (\alpha x_1 + \beta y_1) v_1 + \dots + (\alpha x_m + \beta y_m) v_m$

$= \alpha(x_1 v_1 + \dots + x_m v_m) + \beta(y_1 v_1 + \dots + y_m v_m) = \alpha \varphi(x) + \beta \varphi(y)$

$\Rightarrow \varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$

1.3. Corollary. Let $\varphi : \mathbb{R}^m \to \mathbb{R}$. Then:

$$\varphi \in L(\mathbb{R}^m, \mathbb{R}) \Leftrightarrow \exists v \in \mathbb{R}^m \text{ s.t. } \forall x \in \mathbb{R}^m : \varphi(x) = \langle v, x \rangle$$

1.4. Definition (matrix of a linear mapping).

Let $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$.

Let $v_1 := \varphi(e_1) \Rightarrow v_1 \in \mathbb{R}^m \Rightarrow v_1$ is of the form $v_1 = (v_{11}, \dots, v_{1m})$

$v_2 := \varphi(e_2) \Rightarrow v_2 \in \mathbb{R}^m \Rightarrow v_2$ is of the form $v_2 = (v_{21}, \dots, v_{2m})$

$\vdots$

$v_m := \varphi(e_m) \Rightarrow v_m \in \mathbb{R}^m \Rightarrow v_m$ is of the form $v_m = (v_{m1}, \dots, v_{mm})$

Let

$$[\varphi] := \begin{pmatrix} v_{11} & v_{21} \dots & v_{m1} \\ v_{21} & v_{22} \dots & v_{m2} \\ \vdots & \vdots & \vdots \\ v_{1m} & v_{2m} \dots & v_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}).$$

$$\forall x = (x_1, \dots, x_m) : \varphi(x) = [\varphi] \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

Convention: in matrix relations an arbitrary ~~vector~~ vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m \xleftrightarrow[\text{with}]{\text{is identified}} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathcal{M}_{m,1}(\mathbb{R})$

$\forall x \in \mathbb{R}^m : \varphi(x) = [\varphi] \cdot x$

$\forall x, y \in \mathbb{R}^m : \langle x, y \rangle = x_1 y_1 + \dots + x_m y_m = \underbrace{(x_1 \ x_2 \ \dots \ x_m)}_{\text{row matrix}} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$

$\boxed{\langle x, y \rangle = x^t \cdot y}$ $\qquad \|x^2\| = \langle x, x \rangle = x^t \cdot x$ → x transposed

1.5. Theorem

If $\varphi, \psi \in L(\mathbb{R}^n, \mathbb{R}^m)$, and $a, b \in \mathbb{R}$
then $a\varphi + b\psi \in L(\mathbb{R}^n, \mathbb{R}^m)$, and $[a\varphi + b\psi] = a[\varphi] + b[\psi]$

! Remark: $\boxed{T1.5.} \Rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space w.r.to the usual operations, and the function

$$\forall \varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \mapsto [\varphi] \in \mathcal{M}_{m,n}(\mathbb{R})$$

is an isomorphism between the vector spaces $L(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathcal{M}_{m,n}(\mathbb{R})$.

1.6. Theorem. If $\varphi \in (\mathbb{R}^n, \mathbb{R}^m)$ and $\psi \in L(\mathbb{R}^m, \mathbb{R}^p)$, then $\psi \circ \varphi \in L(\mathbb{R}^n, \mathbb{R}^p)$ and $[\psi \circ \varphi] = [\psi] \cdot [\varphi]$.

1.7. Theorem Let $\varphi = (\varphi_1, \dots, \varphi_m) : \mathbb{R}^n \to \mathbb{R}^m$. Then:

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \Leftrightarrow \varphi_i \in L(\mathbb{R}^n, \mathbb{R}),\ \forall i = \overline{1, m}$$

1.8. Theorem. Let $\varphi \in L(\mathbb{R}^n, \mathbb{R}^n)$. Then the following assertions are equivalent:

1° φ is bijective
2° φ is injective
3° φ is surjective
4° $[\varphi]$ is invertible $\Leftrightarrow \det[\varphi] \neq 0$

1.9. Theorem. If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^n)$ is bijective, then $\varphi^{-1} \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $[\varphi^{-1}] = [\varphi]^{-1}$

1.10. Theorem Every linear mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function.

Def. Let $A \subseteq \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is said to be a Lipschitz function if $\exists\, \alpha \geq 0$ s.t. $\forall x, y \in A$: $\|f(x) - f(y)\| \leq \alpha \|x - y\|$.
If $\alpha \in [0,1)$, then f is said to be a contraction.

Proof: Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n \Rightarrow x = x_1 e_1 + \dots + x_n e_n \xRightarrow{\varphi \text{ is linear}}$

$$\Rightarrow \|\varphi(x)\| = \|x_1 \varphi(e_1) + \dots + x_n \varphi(e_n)\| \leq$$

$$\leq \|x_1 \varphi(e_1)\| + \dots + \|x_n \varphi(e_n)\|$$

$$= \underbrace{|x_1|}_{\leq \|x\|} \cdot \|\varphi(e_1)\| + \dots + \underbrace{|x_n|}_{\leq \|x\|} \cdot \|\varphi(e_n)\|$$

$$\leq \|x\| \underbrace{\left(\|\varphi(e_1)\| + \dots + \|\varphi(e_n)\| \right)}_{:= \alpha}$$

$\Rightarrow \alpha \geq 0$ and
$\|\varphi(x)\| \leq \alpha \|x\|$, $\forall x \in \mathbb{R}^n$
$\Rightarrow \forall x, y \in \mathbb{R}^n$ we have:
$\|\varphi(x) - \varphi(y)\| = \|\varphi(x - y)\| \leq \alpha \|x - y\|$
$\Rightarrow \varphi$ is a Lipschitz function.

! Remark: being a Lipschitz function, every linear mapping is continuous.

1.11. **Definition** (norm of a linear mapping)

Let $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m) \Rightarrow \varphi$ is continuous on $\mathbb{R}^m$.

Let $S^{m-1} := \{x \in \mathbb{R}^m \mid \|x\| = 1\} = \{(x_1, \dots, x_m) \mid x_1^2 + \dots + x_m^2 = 1\}$

↳ the unit sphere in $\mathbb{R}^m$

It is evideately seen that S^{m-1} is closed and bounded, hence it is a compact set. Then, by Weierstrass theorem $\Rightarrow$ the continuous function $\forall x \in S^{m-1} \mapsto \|\varphi(x)\| \in \mathbb{R}$ is bounded on S^{m-1} and attains its bounds $\Rightarrow$ we may define

$$\|\varphi\| := \max_{x \in S^{m-1}} \|\varphi(x)\| = \max_{x_1^2 + \dots + x_m^2 = 1} \|\varphi(x_1, \dots, x_m)\|$$

↳ the norm of φ

1.12. **Theorem**. Let $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$ and $\psi \in L(\mathbb{R}^m, \mathbb{R}^p)$. Then:

1° $\forall x \in \mathbb{R}: \|\varphi(x)\| \le \|\varphi\| \cdot \|x\|$ — prop. used

2° $\|\psi \circ \varphi\| \le \|\psi\| \cdot \|\varphi\|$

versor = vector de lungime 1 $\Rightarrow$ aparține sferei unitate S^{m-1}

Proof: **1°** Let $x \in \mathbb{R}^m$. If $x = 0_m \Rightarrow$ 1° holds with equality

If $x \ne 0_m \Rightarrow \frac{1}{\|x\|} x \in S^{m-1} \xRightarrow{\text{def of } \|\varphi\|}$

$$\Rightarrow \|\varphi\| \ge \left\| \varphi\left(\frac{1}{\|x\|} x\right) \right\| = \frac{1}{\|x\|} \cdot \|\varphi(x)\| \Rightarrow \|\varphi\| \cdot \|x\| \ge \|\varphi(x)\|$$

2° Take $x_0 \in S^{m-1}$ s.t. $\|\psi \circ \varphi\| = \|(\psi \circ \varphi)(x_0)\| \Rightarrow$

$$\Rightarrow \|\psi \circ \varphi\| = \|\psi(\varphi(x_0))\| \underset{1°}{\le} \|\psi\| \cdot \underbrace{\|\varphi(x_0)\|}_{\le \|\varphi\|} \le \|\psi\| \cdot \|\varphi\|$$

1.13. **Theorem**. The function $\|\cdot\| : L(\mathbb{R}^m, \mathbb{R}^m) \to [0, \infty)$

$$\|\varphi\| := \max_{x \in S^{m-1}} \|\varphi(x)\| \quad \forall \varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$$

is a norm on the real vector space $L(\mathbb{R}^m, \mathbb{R}^m)$

Proof: We have to verify the axioms of the norm.

(N_1) $\|\varphi\| = 0 \Leftrightarrow \varphi = \theta$, where $\theta : \mathbb{R}^m \to \mathbb{R}^m$, $\theta(x) = 0_m$, $\forall x \in \mathbb{R}^m$

$\boxed{\Rightarrow}$ $\|\varphi\| = 0 \Rightarrow \|\varphi(x)\| \le \|\varphi\| \cdot \|x\| = 0 \Rightarrow \|\varphi(x)\| = 0 \Rightarrow \varphi(x) = 0_m \ \forall x \in \mathbb{R}^m \Rightarrow \varphi = \theta$

$\boxed{\Leftarrow}$ If $\varphi = \theta \Rightarrow \|\varphi\| = 0$

(N_2) $\forall \alpha \in \mathbb{R}, \forall \varphi \in L(\mathbb{R}^m, \mathbb{R}^m): \|\alpha\varphi\| = |\alpha| \cdot \|\varphi\|$

Let $\alpha \in \mathbb{R}$, and let $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$. Take $x_0 \in S^{m-1}$ s.t. $\|\varphi\| = \|\varphi(x_0)\|$.

Then: • $\forall x \in S^{m-1}: \|(\alpha\varphi)(x)\| = \|\alpha\varphi(x)\| = |\alpha| \cdot \overbrace{\|\varphi(x)\|}^{\le \|\varphi\|} \le |\alpha| \cdot \|\varphi\|$

• $\|(\alpha\varphi)(x_0)\| = \|\alpha\varphi(x_0)\| = |\alpha| \cdot \|\varphi(x_0)\| = |\alpha| \cdot \|\varphi\|$ $\Big\} \Rightarrow$

$$\Rightarrow |\alpha| \cdot \|\varphi\| = \max_{x \in S^{m-1}} \|(\alpha\varphi)(x)\| = \|\alpha\varphi\|$$

(N_3) $\forall \varphi, \psi \in L(\mathbb{R}^m, \mathbb{R}^m): \|\varphi + \psi\| \le \|\varphi\| + \|\psi\|$

Let $\varphi, \psi \in L(\mathbb{R}^m, \mathbb{R}^m)$. Choose $x_0 \in S^{m-1}$ s.t. $\|\varphi + \psi\| = \|(\varphi + \psi)(x_0)\|$

$$\Rightarrow \|\varphi + \psi\| = \|\varphi(x_0) + \psi(x_0)\| \le \underbrace{\|\varphi(x_0)\|}_{\le \|\varphi\|} + \underbrace{\|\psi(x_0)\|}_{\le \|\psi\|} \le \|\varphi\| + \|\psi\|$$

! Remark: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A \cap A'$ (a from A and a is an antiderivative point of A (punct de acumulare))

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2. The derivative of a vector function of real variable

2.1. Definition. Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}^m$, and let $a \in A \cap A'$.

If $\exists\, b \in \mathbb{R}^m$ (vector) s.t. $\lim_{x \to a} \frac{1}{x-a}(f(x) - f(a)) = b$

then f is said to be differentiable at A.
The vector b is called the derivative of x at a, denoted by $f'(a)$.

2.2. Theorem. Let $A \subseteq \mathbb{R}$, $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$, and let $a \in A \cap A'$. Then:

1° If f is differentiable at a $\Rightarrow f_1, \dots, f_m$ are diff. at a and

(1) $f'(a) = (f_1'(a), \dots, f_m'(a))$

2° If $f_1, \dots, f_m$ are diff. at a $\Rightarrow$ f is diff. at a and (1) holds

! Remark: The mean value theorem (teorema de medie) does not hold for vector functions.

Counter example: $f: [0, 2\pi] \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$

Suppose that $\exists\, c \in (0, 2\pi)$ s.t. $f(2\pi) - f(0) = 2\pi \cdot f'(c)$

But $f(2\pi) = (1, 0)$
$f(0) = (1, 0)$ $\Rightarrow f(2\pi) - f(0) = (0, 0)$
$f'(x) = (-\sin x, \cos x)$

$\Rightarrow (0,0) = 2\pi(-\sin c, \cos c)$
$\Downarrow$
$\sin c = 0$ and $\cos c = 0$ $\cancel{}$

! Recapitulare:
→ tabel derivate
→ tabel integrale
Pt. examen.

- $f = (f_1, \dots, f_m): [a, b] \to \mathbb{R}^m$

If $\exists\, c \in (a, b)$ s.t. $f(b) - f(a) = (b-a) f'(c) \Rightarrow \begin{cases} f_1(b) - f_1(a) = (b-a) f_1'(c) \\ \vdots \\ f_m(b) - f_m(a) = (b-a) f_m'(c) \end{cases} \Rightarrow$

$\Rightarrow$ If we apply tha Lagranges' mean value theorem to the functions $f_1, \dots, f_m$, then we get the same intermediate point c

2.3. Theorem (the mean value theorem for vector functions of a real variable)

Let $f: [a, b] \to \mathbb{R}^m$ be a function continuous on $[a, b]$ and differentiable on (a, b). Then $\exists\, c \in (a, b)$ s.t.

$$\|f(b) - f(a)\| \le (b-a) \cdot \|f'(c)\|$$

Proof: If $f(a) = f(b) \Rightarrow c$ can be chosen arbitrarily in (a, b)

If $f(a) \ne f(b)$, then we define

$$v := \frac{1}{\|f(b) - f(a)\|}[f(b) - f(a)] \in \mathbb{R}^m, \quad v = (v_1, \dots, v_m)$$

$$g: [a, b] \to \mathbb{R}, \quad g(x) = \langle v, f(x) \rangle = v_1 f_1(x) + \dots + v_m f_m(x),$$

where $f_1, \dots, f_m$ are the scalar components of f.

Since f is continuous on $[a, b] \Rightarrow f_1, \dots, f_m$ are continuous on $[a, b]$
f is diff. on $(a, b) \Rightarrow f_1, \dots, f_m$ are diff. on (a, b) $\Rightarrow$

$\Rightarrow$ g is continuous on $[a, b]$ and diff. on (a, b)

By applying the Lagranges' mean value theorem to $g \Rightarrow$

$\Rightarrow \exists\, c \in (a, b)$ s.t. $g(b) - g(a) = (b - a) g'(c)$ (1)

But $g(b)-g(a)=\langle v, f(b)\rangle - \langle v, f(a)\rangle = \langle v, f(b)-f(a)\rangle =$

$$=\left\langle \frac{1}{\|f(b)-f(a)\|}\|f(b)-f(a)\|, \ f(b)-f(a)\right\rangle$$

$$=\frac{1}{\|f(b)-f(a)\|}\langle f(b)-f(a), f(b)-f(a)\rangle =$$

$$=\frac{1}{\|f(b)-f(a)\|}\cdot\|f(b)-f(a)\|^2=\|f(b)-f(a)\| \quad (2)$$

$$g'(x)=v_1 f_1'(x)+\dots+v_m f_m'(x)=\langle v, f'(x)\rangle \Rightarrow g'(c)=\langle v, f'(c)\rangle \quad (3)$$

By (1) (2) (3) $\Rightarrow \|f(b)-f(a)\|=(b-a)\langle v, f'(c)\rangle \le (b-a)\cdot|\langle v, f'(c)\rangle| \overset{C-B-S}{\le}$

$$\le (b-a)\underbrace{\|v\|}_{1}\cdot\|f'(c)\|$$

Cauchy Buniakowski Stoltz

$$\boxed{|\langle x, y\rangle| \le \|x\|\cdot\|y\|}$$

$\Rightarrow \|f(b)-f(a)\| \le (b-a)\,\|f'(c)\|$

! Remark: $A\subseteq\mathbb{R}$, $f: A\to\mathbb{R}$, $a\in A\cap A'$

f is diff. at $a \iff \exists \lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}=:c\in\mathbb{R}$

$\iff \exists\, c\in\mathbb{R}$ s.t. $\lim\limits_{x\to a}\frac{f(x)-f(a)-c(x-a)}{x-a}=0$

$\overset{!}{\iff} \exists\, \varphi\in L(\mathbb{R},\mathbb{R}): \lim\limits_{x\to a}\frac{f(x)-f(a)-\varphi(x-a)}{x-a}=0$

3. Fréchet differentiability of vector functions of vector variable.

3.1. Lemma Let $A\subseteq\mathbb{R}^n$, $a\in \operatorname{int} A$, $f: A\to\mathbb{R}^m$, $\varphi_1, \varphi_2\in L(\mathbb{R}^n,\mathbb{R}^m)$.

If $\lim\limits_{x\to a}\frac{1}{\|x-a\|}\left[f(x)-f(a)-\varphi_i(x-a)\right]=0_m$ for $i=1,2$,

then $\varphi_1=\varphi_2$.

Without proof.

3.2. Definition Let $A\subseteq\mathbb{R}^n$, let $a\in\operatorname{int} A$, and let

The function f is called Fréchet differentiable at a if

$\exists\,\varphi\in L(\mathbb{R}^n,\mathbb{R}^m)$ s.t.

$$\lim_{x\to a}\frac{1}{\|x-a\|}\left[f(x)-f(a)-\varphi(x-a)\right]=0_m$$

If f is Fréchet diff. at a, then, from Lemma 3.1. $\Rightarrow$ there exists a unique linear mapping satisfying the above inequality. This unique linear mapping is called the Fréchet differential of f at a, denoted by $df(a)$.

!! Remark: $f: A\to\mathbb{R}^m$, $a\in\operatorname{int} A$ differentiable at a

$\Rightarrow df(a)\in L(\mathbb{R}^n,\mathbb{R}^m)$

$\forall\, x\in\mathbb{R}^n \Rightarrow df(a)(x)\in\mathbb{R}^m$

We have

(*) $$\boxed{\lim_{x\to a}\frac{1}{\|x-a\|}\left[f(x)-f(a)-df(a)(x-a)\right]=0_m}$$

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3.3. Proposition. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$. Then:

1° If f is Fréchet differentiable at $a \Rightarrow \exists\, \omega : A \to \mathbb{R}^m$ s.t.

(1) $\lim_{x \to a} \omega(x) = 0_m$

and

(2) $\forall x \in A : f(x) = f(a) + df(a)(x-a) + \|x-a\|\omega(x)$

2° If $\exists\, \omega : A \to \mathbb{R}^m$ and $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

(3) $\lim_{x \to a} \omega(x) = 0_m$

and

(4) $\forall x \in A : f(x) = f(a) + \varphi(x-a) + \|x-a\|\omega(x)$

then f is Fréchet differentiable at a and $df(a) = \varphi$.

Proof: 1° f Fréchet differentiable at $a \Rightarrow$ (*) holds. Define $\omega : A \to \mathbb{R}^m$ by

$$\omega(x) := \frac{1}{\|x-a\|}\left[f(x) - f(a) - df(a)(x-a)\right] \text{ if } x \in A \setminus \{a\}$$

$$\omega(a) := 0_m$$

$\Rightarrow \omega$ satisfies (1) and (2)

2° By (4) $\Rightarrow \lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - \varphi(x-a)\right] = \lim_{x \to a} \omega(x) \overset{(3)}{=} 0_m$

$\xRightarrow{def}$ f is Fréchet diff. at a and $df(a) = \varphi$

COURSE 5

3.4. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$ be a Fréchet diff. function at a. Then f is continuous at a.

Proof: ? f continuous at $a \Leftrightarrow \exists \lim_{x \to a} f(x) = f(a)$

f is Fréchet diff. at $a \xRightarrow{P\,3.3} \exists\, \omega : A \to \mathbb{R}^m$ s.t.

$\lim_{x \to a} \omega(a) = 0_m$

$\forall x \in A : f(x) = f(a) + df(a)(x-a) + \|x-a\|\omega(x) \Rightarrow$

$$\Rightarrow 0 \le \|f(x) - f(a)\| = \|df(a)(x-a) + \|x-a\|\omega(x)\|$$
$$\le \|df(a)(x-a)\| + \|x-a\| \cdot \|\omega(x)\|$$
$$\le \|df(a)\| \cdot \|x-a\| + \|x-a\| \cdot \|\omega(x)\| \xrightarrow{x \to a} 0$$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

3.5. Theorem Let $A \subseteq \mathbb{R}$, let $a \in \operatorname{int} A$, and let $f : A \to \mathbb{R}^m$.

Then: 1° If f is diff. at $a \Rightarrow f$ is Fréchet diff. at a and

(1) $df(a)(x) = x \cdot f'(a) \quad \forall x \in \mathbb{R}$

2° If f is Fréchet diff. at $a \Rightarrow f$ is diff. at a and (1) holds.

Without proof.

3.6. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \text{int}\, A$, $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$.

Then: **1°** If f is Fréchet diff. at $a \Rightarrow f_1, \dots, f_m$ are Fréchet diff. at a, and

(2) $df(a) = (df_1(a)$

4. Directional derivatives

4.1. Definition. (directional derivative)

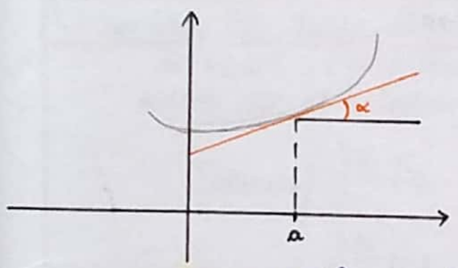
Let $A \subseteq \mathbb{R}^n$, $a \in \text{int}\, A$, $f: A \to \mathbb{R}^m$, $v \in \mathbb{R}^n$

If the limit $\boxed{\lim\limits_{t \to 0} \frac{1}{t}\left[f(a+tv) - f(a)\right] = b \in \mathbb{R}^m}$ exists

then f is said to be diff. at a in the direction v.

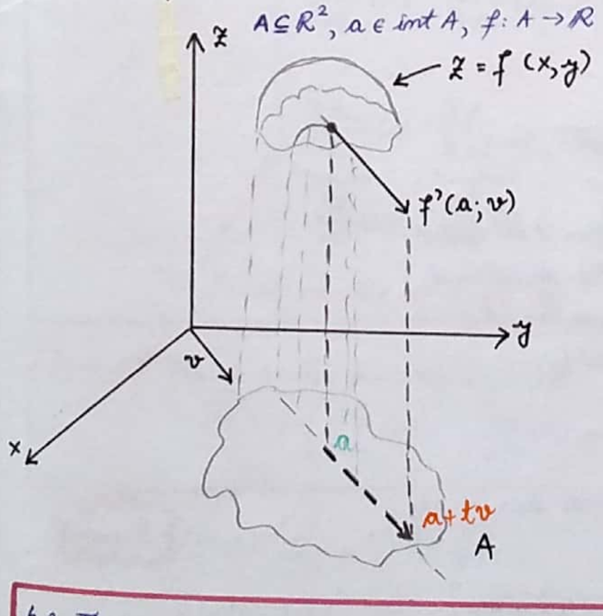
The vector $b \in \mathbb{R}^m$ is called the directional derivative of f at the point a in the direction v.

Denote it: $\boxed{f'(a; v)}$



$f'(a) = tg\,\alpha$

$A \subseteq \mathbb{R}^2$, $a \in \text{int}\, A$, $f: A \to \mathbb{R}$



4.2. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \text{int}\, A$, $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$, $v \in \mathbb{R}^n$.

Then: **1°** If f is diff at a in the direction $v \Rightarrow$

$\Rightarrow f_1, \dots, f_m$ are diff. at a in the direction v and

(1) $\boxed{f'(a; v) = (f_1'(a; v), \dots, f_m'(a; v))}$

2° If $f_1, \dots, f_m$ are diff. at a in the direction $v \Rightarrow$

$\Rightarrow f$ is diff. at a in the direction v and (1) holds

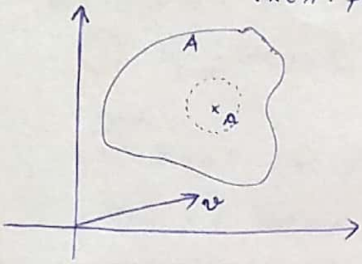
4.3. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \text{int } A$, $f: A \to \mathbb{R}^m$. If f is Fréchet diff. at a then f is diff. at a in every direction and

$$\boxed{\forall v \in \mathbb{R}^n : f'(a; v) = df(a)(v)}$$

Proof: Let $v \in \mathbb{R}^n$. Since f is Fréchet diff. at a $\xRightarrow{P3.3}$

$\Rightarrow \exists \omega : A \to \mathbb{R}^m$ s.t. $\lim\limits_{x \to a} \omega(x) = 0_m$ (2)

$\forall x \in A : f(x) = f(a) + df(a)(x-a) + \|x-a\| \omega(x)$ (3)



$a \in \text{int } A \Rightarrow \exists r > 0$ s.t. $B(a,r) \subseteq A$

$\|(a+tv) - a\| = \|tv\| = |t| \cdot \|v\| < r$ for $|t|$ suff. small

$\Rightarrow \exists \delta > 0$ s.t. $\forall t \in [-\delta, \delta]$:

$a + tv \in B(a,r) \subseteq A$

Replacing x by $a+tv$ in (3) $\Rightarrow$

$\Rightarrow \forall t \in [-\delta, \delta]$:

$$f(a+tv) = f(a) + df(a)(tv) + \|tv\| \omega(a+tv)$$

$$= f(a) + t\,df(a)(v) + |t| \cdot \|v\| \omega(a+tv)$$

$$\Rightarrow \frac{1}{t}[f(a+tv) - f(a)] = df(a)(v) + \underbrace{\frac{|t|}{t}}_{=\pm 1} \|v\| \underbrace{\omega(a+tv)}_{\xrightarrow{t \to 0}\, 0_m}, \forall t \in [-\delta, \delta] \setminus \{0\}$$

$$\Rightarrow \exists \lim_{t \to 0} \frac{1}{t}[f(a+tv) - f(a)] = df(a)(v) \Rightarrow$$

$\Rightarrow f$ is diff. at a in the direction v and $f'(a; v) = df(a)(v)$

5. PARTIAL DERIVATIVES

5.1. Definition. Let $A \subseteq \mathbb{R}^n$, let $a \in \text{int} A$, let $f: A \to \mathbb{R}^m$, and let $\{e_1, \dots, e_n\}$ be the cannonical /

If f is diff. at a in the direction e_j then f is called partially diff. at a w.r.t. the variable x_j. The directional derivative $f'(a; e_j)$ is called the partial derivative of f at w.r.t. the variable x_j and will be denoted by:

$$\frac{\partial f}{\partial x_j}(a), \ f'_{x_j}(a), \ \partial_j f(a)$$

If f is partially diff. w.r.t. all variables then we say that f is partially diff.

$$\frac{\partial f}{\partial x_j}(a) = f'(a; e_j) = \lim_{t \to 0} \frac{1}{t}[f(a + te_j) - f(a)]$$

$$= \lim_{t \to 0} \frac{1}{t}[f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)]$$

$x_j = a_j + t$

$$= \lim_{x_j \to a_j} \frac{1}{x_j - a_j}[f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n)]$$

Example $f:(0,+\infty)\times(0,+\infty)\to\mathbb{R}^2,\ f(x,y)=\left(\frac{1}{xy},\ \arctan\frac{x}{y}\right)$

$$\frac{\partial f}{\partial x}(x,y)=\left(-\frac{1}{x^2y},\ \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}\right)=\left(-\frac{1}{x^2y},\ \frac{y}{x^2+y^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y)=\left(-\frac{1}{xy^2},\ \frac{1}{1+\frac{x^2}{y^2}}\cdot x\cdot\left(-\frac{1}{y^2}\right)\right)=\left(-\frac{1}{xy^2},\ -\frac{x}{x^2+y^2}\right)$$

5.2. Theorem Let $A\subseteq\mathbb{R}^m$, $a\in \text{int}A$, $f=(f_1,\dots,f_m):A\to\mathbb{R}^m$ and $j\in\{1,\dots,n\}$. Then:

1° If f is partially diff. at a w.r.t. x_j $\Rightarrow$
$\Rightarrow f_1,\dots,f_m$ are partially diff. at a w.r.t. x_j and

(1) $$\frac{\partial f}{\partial x_j}(a)=\left(\frac{\partial f_1}{\partial x_j}(a),\dots,\frac{\partial f_m}{\partial x_j}(a)\right)$$

2° If $f_1,\dots,f_m$ are partially diff. at a w.r.t. x_j $\Rightarrow$ f is partially diff. at a w.r.t. x_j and (1) holds

5.3. Definition (the Jacobi matrix of a function)
Let $A\subseteq\mathbb{R}^m$, $a\in\text{int}A$, $f=(f_1,\dots,f_m):A\to\mathbb{R}^m$ which is partially diff. at a. Define:

$$J(f)(a)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \dots & \frac{\partial f_1}{\partial x_m}(a)\\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \dots & \frac{\partial f_2}{\partial x_m}(a)\\ \dots & \dots & \dots & \dots\\ \underbrace{\frac{\partial f_m}{\partial x_1}(a)}_{\frac{\partial f}{\partial x_1}(a)} & \underbrace{\frac{\partial f_m}{\partial x_2}(a)}_{\frac{\partial f}{\partial x_2}(a)} & \dots & \underbrace{\frac{\partial f_m}{\partial x_m}(a)}_{\frac{\partial f}{\partial x_m}(a)}\end{pmatrix}$$

The Jacob matrix of f at a

! Remark: For a scalar function $f:A\to\mathbb{R}$
$$J(f)(a)=\left(\frac{\partial f}{\partial x_1}(a)\ \ \frac{\partial f}{\partial x_2}(a)\ \dots\ \frac{\partial f}{\partial x_m}(a)\right)\in \mathcal{M}\ ??$$

Example $f(x,y)=\left(\frac{1}{xy},\ \arctan\frac{x}{y}\right)$

$$J(f)(x,y)=\begin{pmatrix}-\frac{1}{x^2y} & -\frac{1}{xy^2}\\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2}\end{pmatrix}$$

Define $$\boxed{\nabla f(a):=\left(\frac{\partial f}{\partial x_1}(a),\frac{\partial f}{\partial x_2}(a),\dots,\frac{\partial f}{\partial x_m}(a)\right)\in\mathbb{R}^m}$$

↳ nabla f at a — the gradient of f at a

$$\boxed{J(f)(a)=\nabla f(a)^t}$$

5.4. Theorem Let $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}^m$ be a Fréchet diff. function at a. Then:

1° f is partially diff. at a and $[df(a)] = J(f)(a)$

2° $\forall\, h = (h_1, \dots, h_m) \in \mathbb{R}^m$ we have

$$df(a)(h) = h_1 \frac{\partial f}{\partial x_1}(a) + \dots + h_m \frac{\partial f}{\partial x_m}(a)$$

Proof 1° Also $\frac{\partial f}{\partial x_j}(a) = f'(a; e_j) \overset{\boxed{T4.2}}{=} df(a)(e_j)$, $\forall j = \overline{1,m}$ (1)

The fact that f is partially diff. at a follows by theorem 4.2.

$\frac{\partial f}{\partial x_j}(a) = $ column j in $J(f)(a)$ (2)

$df(a)(e_j) = $ column j in $[df(a)]$ (3)

By 1, 2, 3 $\Rightarrow [df(a)] = J(f)(a)$

2° $\forall\, h = (h_1, \dots, h_m) \in \mathbb{R}^m$, $h = h_1 e_1 + \dots + h_m e_m$

$df(a)(h) = df(a)(h_1 e_1 + \dots + h_m e_m) = h_1\, df(a)(e_1) + \dots + h_m\, df(a)(e_m)$

$\overset{(1)}{=} h_1 \frac{\partial f}{\partial x_1}(a) + \dots + h_m \frac{\partial f}{\partial x_m}(a)$

! Remark: f Fréchet diff. $\Rightarrow f$ diff. at a in all directions $\Rightarrow$

$\Rightarrow f$ partially diff. at a

$\Downarrow$

f is cont. at a.

5.5. Theorem Let $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$ be a function satisfying the following conditions:

(i) $\exists\, r > 0$ s.t. $B(a,r) \subseteq A$ and f is partially diff. on $B(a,r)$

(ii) the function $\forall x \in B(a,r) \longmapsto \frac{\partial f}{\partial x_j}(x) \in \mathbb{R}$ is continuous at a, $\forall j = 1, \dots, m$

Then f is Fréchet diff. at a

Proof: Let $T: \mathbb{R}^m \to \mathbb{R}$, $T(h) = h_1 \frac{\partial f}{\partial x_1}(a) + \dots + h_m \frac{\partial f}{\partial x_m}(a)$, $\forall\, h = (h_1, \dots, h_m) \in \mathbb{R}^m$

Obviously, $T \in L(\mathbb{R}^m, \mathbb{R})$

? f is Fréchet diff. at a and $df(a) = T$ $\overset{def}{\Longleftrightarrow} \lim\limits_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - T(x-a)\right] = 0 \Leftrightarrow$

$\Leftrightarrow \forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $\|x-a\| < \delta$ we have $\left|\frac{1}{\|x-a\|}\left[f(x) - f(a) - T(x-a)\right] - 0\right| \leq \varepsilon \Leftrightarrow$

$\Leftrightarrow \forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $\forall x \in A$ with $\|x-a\| < \delta$:

$|f(x) - f(a) - T(x-a)| \leq \varepsilon \|x-a\|$

Let $\varepsilon' > 0$

Since $\frac{\partial f}{\partial x_j}$ is cont. at $a \Rightarrow \exists\, \delta_j > 0$ s.t. $x \in B(a,r)$ with

$\|x-a\| < \delta_j$: $\left|\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(a)\right| < \varepsilon' = \frac{\varepsilon}{m}$

Set $\delta := \min\{\delta_1, \delta_2, \dots, \delta_m, r\} > 0$

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Let $x \in A$ with $\|x-a\|<\delta$

$\boxed{m=2}$ $f(x)-f(a)=f(x_1,x_2)-f(a_1,a_2)$

$$=f(x_1,x_2)-f(a_1,x_2)+f(a_1,x_1)-f(a_1,a_2)$$

By the Lagrange mean value theorem applied to $f(\cdot,x_2) \Rightarrow$

$\Rightarrow \exists\, c_1$ between a_1 and x_1 s.t.

$$f(x_1,x_2)-f(a_1,x_2)=(x_1-a_1)\cdot\frac{\partial f}{\partial x_1}(c_1,x_2)$$

By the Lagrange mean value theorem applied to $f(a_1,\cdot) \Rightarrow$

$\Rightarrow \exists\, c_2$ between a_2 and x_2 s.t.

$$f(a_1,x_2)-f(a_1,a_2)=(x_2-a_2)\frac{\partial f}{\partial x_2}(a_1,c_2)$$

$$\Rightarrow |f(x)-f(a)-f(x-a)|=\left|(x_1-a_1)\frac{\partial f}{\partial x_1}(c_1,x_2)+(x_2-a_2)\frac{\partial f}{\partial x_2}(a_1,c_2)-(x_1-a_1)\frac{\partial f}{\partial x_1}(a_1,a_2)-\right.$$

$$\left.-(x_2-a_2)\frac{\partial f}{\partial x_2}(a_1,a_2)\right|\le$$

$$\le \underbrace{|x_1-a_1|}_{\le\|x-a\|}\underbrace{\left|\frac{\partial f}{\partial x_1}(c_1,x_2)-\frac{\partial f}{\partial x_1}(a_1,a_2)\right|}_{<\varepsilon' \text{ because } \|(c_1,x_2)-(a_1,a_2)\|<\delta_1}+\underbrace{|x_2-a_2|}_{\le\|x-a\|}\underbrace{\left|\frac{\partial f}{\partial x_2}(a_1,c_2)-\frac{\partial f}{\partial x_2}(a_1,a_2)\right|}_{<\varepsilon' \text{ because } \|(a_1,c_2)-(a_1,a_2)\|<\delta_2}\le$$

Prove it!

$$\le 2\varepsilon'\|x-a\| = \varepsilon''\|x-a\|$$

5.6. Theorem Let $A\subseteq\mathbb{R}^m$, $a\in \operatorname{int} A$, $f:A\to\mathbb{R}^m$ a function satisfying the following conditions:

(i) $\exists\, r>0$ s.t. $B(a,r)\subseteq A$ and f is partially diff. on $B(a,r)$

(ii) the function $\forall x\in B(a,r)\longmapsto \frac{\partial f}{\partial x_j}(x)\in\mathbb{R}^m$ is continuous at a $\forall j=1,\dots,m$

Then f is Fréchet diff. at a.

Proof: Apply the previous theorem to deduce that $f_1,\dots,f_m$ (the scalar comp. of f) are Fréchet diff. at a and after that apply and after that apply theorem 3.6.

5.7. Corollary Let $A\subseteq\mathbb{R}^m$ be an open set and let $f:A\to\mathbb{R}^m$ be partially diff. on A s.t. all the partial derivatives $\frac{\partial f}{\partial x_j}$ are continuous on A.

Then f is Fréchet diff. on A.

6. The Chain Rule

! Remark: $\boxed{(f\circ g)'(x) = f'(g(x))\cdot g'(x)}$

6.1. Theorem Let $A\subseteq \mathbb{R}^n$, $a\in \operatorname{int} A$, $B\subseteq \mathbb{R}^m$, $g: A\to B$ s.t. g is Fréchet diff. at a and $g(a)\in \operatorname{int} B$, $f: B\to \mathbb{R}^p$ s.t. f is Fréchet diff. at $g(a)$. Then $f\circ g: A\to \mathbb{R}^p$ is Fréchet diff. at a, and

$$\boxed{\begin{aligned} d(f\circ g)(a) &= df(g(a))\circ dg(a)\\ J(f\circ g)(a) &= J(f)(g(a))\cdot J(g)(a)\end{aligned}}$$

Proof:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f\circ g"'] & B \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^n \arrow[r, "dg(a)"] \arrow[dr, dashed, "d(f\circ g)(a)"'] & \mathbb{R}^m \arrow[d, "\psi"] \\ & \mathbb{R}^p \end{tikzcd}$$

Set $b := g(a)$
$\varphi := dg(a)$
$\psi := df(b)$

g is Fréchet diff. at a and $dg(a)=\varphi \Rightarrow \exists\, \rho: A\to \mathbb{R}^m$ s.t.

(1) $\lim\limits_{x\to a} \rho(x) = 0_m$

(2) $\forall x\in A: g(x) = g(a) + \varphi(x-a) + \|x-a\|\cdot \rho(x)$

f is Fréchet diff. at $b=g(a)$ and $df(b)=\psi \Rightarrow \exists\, \nabla: B\to \mathbb{R}^p$ s.t.

(3) $\lim\limits_{u\to b} \nabla(u) = 0_p$

(4) $\forall u\in B: f(u) = f(b) + \psi(u-b) + \|u-b\|\,\nabla(u)$

Without loss of generality we may assume that $\rho(a)=0_m$, $\nabla(b)=0_p$ i.e. ρ is continuous at a, while ∇ is continuous at b.

Replacing u in (4) by $g(x) \Rightarrow$

$$\forall x\in A: f(g(x)) = f(b) + \psi(g(x)-b) + \|g(x)-b\|\,\nabla(g(x))$$
$$= f(g(a)) + \psi(\underbrace{g(x)-g(a)}_{\varphi(x-a)+\|x-a\|\,\rho(x)}) + \|g(x)-g(a)\|\,\nabla(g(x))$$

$$\Rightarrow \forall x\in A: (f\circ g)(x) = (f\circ g)(a) + \psi(\varphi(x-a)) + \|x-a\|\,\psi(\rho(x)) + \|g(x)-g(a)\|\,\nabla(g(x))$$

$$(f\circ g)(x) = (f\circ g)(a) + (\psi\circ\varphi)(x-a) + \|x-a\|\,\psi(\rho(x)) + \|g(x)-g(a)\|\cdot\nabla(g(x))$$

$$\Rightarrow \forall x\in A: (f\circ g)(x) = (f\circ g)(a) + (\psi\circ\varphi)(x-a) + \|x-a\|\,\omega(x)$$

where $\omega: A\to \mathbb{R}^p$ is defined by

$$\omega(x) := \varphi(\rho(x)) + \frac{\|g(x)-g(a)\|}{\|x-a\|}\,\nabla(g(x)) \text{ for } x\neq a$$

$$\omega(a) := 0_p$$

To finish the proof we need only to show that the limit $\lim\limits_{x\to a}\omega(x) \overset{?}{=} 0_p$

10.04.2018. $?\ \lim\limits_{x\to a}\omega(x) = 0_p$

$$\forall x\in A\setminus\{a\}: \|\omega(x)\| = \left\|\varphi(\rho(x)) + \frac{\|g(x)-g(a)\|}{\|x-a\|}\cdot\nabla(g(x))\right\|$$
$$\leq \|\varphi(\rho(x))\| + \frac{\|g(x)-g(a)\|}{\|x-a\|}\cdot\|\nabla(g(x))\| \quad (*)$$

By (2) $\Rightarrow \|g(x)-g(a)\| = \|\varphi(x-a) + \|x-a\|\,\rho(x)\|$
$$\leq \|\varphi(x-a)\| + \|x-a\|\cdot\|\rho(x)\|$$
$$\leq \|\varphi\|\cdot\|x-a\| + \|x-a\|\cdot\|\rho(x)\| \quad (**)$$

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By (*) and (**) $\Rightarrow \|\omega(x)\| \le \|f(\rho(x))\| + \dfrac{\|f\|\cdot\cancel{\|x-a\|} + \cancel{\|x-a\|}\cdot\|\rho(x)\|}{\cancel{\|x-a\|}} \cdot \|\nabla(g(x))\|$

$$\le \|f\|\cdot\underbrace{\|\rho(x)\|}_{\xrightarrow{x\to a} 0} + \Big(\|f\| + \underbrace{\|\rho(x)\|}_{\xrightarrow{x\to a} 0}\Big)\cdot \underbrace{\|\nabla(g(x))\|}_{\xrightarrow{x\to a} 0}$$

g Fréchet diff. at $a \Rightarrow g$ is continuous at a
∇ is continuous at $b = g(a)$ $\Big\} \Rightarrow$

$\Rightarrow \nabla\circ g$ is continuous at a

$\Rightarrow \lim\limits_{x\to a} (\nabla\circ g)(x) = (\nabla\circ g)(a) = \nabla(g(a)) = \nabla(b) = 0_p$

$\Rightarrow \lim\limits_{x\to a} \|\nabla(g(x))\| = 0$

$\Rightarrow \lim\limits_{x\to a} \|\omega(x)\| = 0 \Rightarrow \lim\limits_{x\to a} \omega(x) = 0_p$

COURSE 6 — week 7

7. Second order partial derivatives

7.1. Definition Let $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$, and let $i, j \in \{1, \dots, m\}$. Assume that $\exists\, V \in \mathcal{V}(a)$ s.t. $V \subseteq A$ s.t.:

(i) f is partially diff. w.r. to x_i on V

(ii) the function

$$\forall\, x \in V \longmapsto \frac{\partial f}{\partial x_i}(x) \in \mathbb{R} \quad (*)$$

is partially diff. w.r. to x_j at a

In this case f is said to be twice partially diff. w.r. to the variables (x_i, x_j) at a. The partial derivative w.r. to x_j of the function (*) at the point a is called the second order partial derivative ? ? ? ? ? ? ?
and it will be denoted by:

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) \cdot f''_{x_i x_j}(a)$$ → be careful at the order of the notations

When $i = j$ $\quad \dfrac{\partial^2 f}{\partial x_i^2}(a), f''_{x_i^2}(a)$

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(a)$$

$$f''_{x_i x_j}(a) = \left(f'_{x_i}\right)'_{x_j}(a)$$

A function of m variables can have m^2 second order partial derivatives of the second order. If all these derivatives exists, then we reconstruct all the matrix:

$$\boxed{H(f)(a)} = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(a) & \dfrac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_m \partial x_1}(a) \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \dfrac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_m \partial x_2}(a) \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_m}(a) & \dfrac{\partial^2 f}{\partial x_2 \partial x_m}(a) & \cdots & \dfrac{\partial^2 f}{\partial x_m^2}(a) \end{pmatrix}$$

↓
the HESSIAN matrix of f at a

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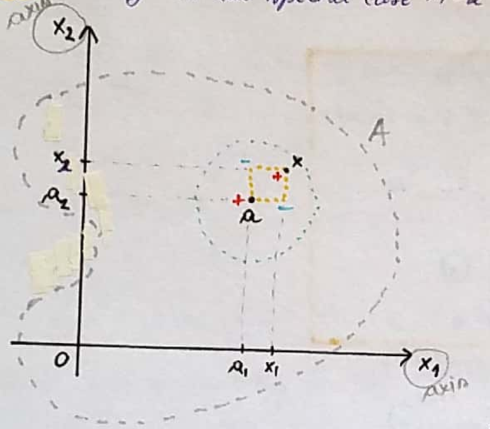
! Remark: In general, it is possible that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) \neq \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$$

7.2. Theorem (the Schwarz test) Let $A \subseteq \mathbb{R}^m$ be an open set, let $f: A \to \mathbb{R}$, and let $i, j \in \{1, \dots, m\}$, $i \neq j$. Suppose that f is twice partially diff. on A both w.r.to (x_i, x_j) and w.r.to (x_j, x_i). If $\frac{\partial^2 f}{\partial x_j \partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are continuous at some point $a \in A$, then

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$$

Proof: Only in the special case $m = 2 \Rightarrow i = 1, j = 2$



Let $\varepsilon > 0$. Since $\frac{\partial^2 f}{\partial x_2 \partial x_1}$ and $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ are continuous at $a \Rightarrow$

$\Rightarrow \exists \delta > 0$ s.t. $B(a, \delta) \subseteq A$ and $\forall x \in B(a, \delta)$:

$$\left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) - \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) \right| < \varepsilon \text{ and } \left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right| < \varepsilon$$

Select a point $x = (x_1, x_2) \in B(a, \delta)$ s.t. $x_1 > a_1$ and $x_2 > a_2$. Set

$$\alpha := f(x_1, x_2) + f(a_1, a_2) - f(x_1, a_2) - f(a_1, x_2).$$

Note that $\alpha = g(x_1) - g(a_1)$, where

$g: [a_1, x_1] \to \mathbb{R}$, $g(t) := f(t, x_2) - f(t, a_2)$

By the Lagrange MV Theorem $\Rightarrow \exists c_1 \in (a_1, x_1)$ s.t. $g(x_1) - g(a_1) = (x_1 - a_1) g'(c_1)$

But $g'(t) = \frac{\partial f}{\partial x_1}(t, x_2) - \frac{\partial f}{\partial x_1}(t, a_2)$ $\Rightarrow$

$$\Rightarrow \alpha = (x_1 - a_1)\left[\frac{\partial f}{\partial x_1}(c_1, x_2) - \frac{\partial f}{\partial x_1}(c_1, a_2)\right] \quad (1)$$

Let $h: [a_2, x_2] \to \mathbb{R}$, $h(t) := \frac{\partial f}{\partial x_1}(c_1, t)$

By the Lagrange MV Theorem $\Rightarrow \exists c_2 \in (a_2, x_2)$ s.t.

$h(x_2) - h(a_2) = (x_2 - a_2) h'(c_2)$

But $h(x_2) - h(a_2) = \frac{\partial f}{\partial x_1}(c_1, x_2) - \frac{\partial f}{\partial x_1}(c_1, a_2)$ $\Rightarrow$

$h'(t) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(c_1, t)$

$$\Rightarrow \frac{\partial f}{\partial x_1}(c_1, x_2) - \frac{\partial f}{\partial x_1}(c_1, a_2) = (x_2 - a_2) \cdot \frac{\partial^2 f}{\partial x_2 \partial x_1}(c_1, c_2) \quad (2)$$

By (1), (2) $\Rightarrow$ $\boxed{\alpha = (x_1 - a_1)(x_2 - a_2) \frac{\partial^2 f}{\partial x_2 \partial x_1}(c_1, c_2)}$ (3)

Let $c := (c_1, c_2)$

$\|c - a\| = \sqrt{(c_1 - a_1)^2 + (c_2 - a_2)^2} < \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$

$< (x_1 - a_1)^2 < (x_2 - a_2)^2$

$= \|x - a\| < \delta \Rightarrow c \in B(a, \delta)$

Analogously, by applying the Lagrange MVTheorem first for the function

$$g_1 : [a_2, x_2] \to \mathbb{R},\ g_1(t) = f(x_1, t) - f(a_1, t)$$

and after that for the function

$$h_1 : [a_1, x_1] \to \mathbb{R},\ h_1(t) = \frac{\partial f}{\partial x_2}(t, d_2)$$

it follows that there exists a point d

$\exists\ d = (d_1, d_2) \in B(a, \delta)$ s.t.

$$\boxed{\alpha = (x_1 - a_1)(x_2 - a_2) \cdot \frac{\partial^2 f}{\partial x_1 \partial x_2}(d_1, d_2)} \quad (4)$$

By (3), (4) $\Rightarrow \dfrac{\partial^2 f}{\partial x_2 \partial x_1}(c) = \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(d)$

$$\Rightarrow \left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right| =$$

$$= \left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) + \frac{\partial^2 f}{\partial x_1 \partial x_2}(d) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right|$$

$$\leq \underbrace{\left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_2 \partial x_1}(c) \right|}_{< \varepsilon} + \underbrace{\left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(d) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right|}_{< \varepsilon} < 2\varepsilon$$

$$\Rightarrow \left| \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) - \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) \right| < 2\varepsilon,\ \forall \varepsilon > 0 \quad \Big| \varepsilon \searrow 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)$$

8. The second order Fréchet differential

! Remark

$A \subseteq \mathbb{R}$ open interval
$f : A \to \mathbb{R}$ is differentiable on A
$f' : A \to \mathbb{R}$

$\Bigg\{$ $A \subseteq \mathbb{R}^m$ open set
$f : A \to \mathbb{R}$ is Fréchet diff. on A
$df : A \to L(\mathbb{R}^m, \mathbb{R})$

8.1. Definition Let $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}$.
Suppose that $\exists\ V \in \mathcal{V}(a)$ s.t. $V \subseteq A$ and:

(i) f is Fréchet diff. on V
(ii) $\forall\ i \in \{1, \dots, m\}$ the function
$\forall x \in V \longmapsto \dfrac{\partial f}{\partial x_i}(x) \in \mathbb{R}$
is Fréchet diff. at a.

Then is said to be twice Fréchet differentiable at some point, then
It can be proved that if is twice
it ? ?

8.2. Definition Let $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$ a twice Fréchet diff. function at a. We define $d^2 f(a): \mathbb{R}^m \to \mathbb{R}$

$$\boxed{d^2 f(a)(h) := \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 f}{\partial x_j \partial x_i}(a)\, h_i h_j} \quad \forall\, h = (h_1, \dots, h_m) \in \mathbb{R}^m$$

$\to$ the second Fréchet diff. of f at the point a

! Remark: Let $M := (a_{ij}) \in M_m(\mathbb{R})$, and let $\Phi: \mathbb{R}^m \to \mathbb{R}$,

$$\boxed{\Phi(h) := \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, h_i h_j} \quad \forall\, h = (h_1, \dots, h_m) \in \mathbb{R}^m$$

$\to$ the quadratic form generated by the matrix M

$\Rightarrow$ $d^2 f(a)$ is a quadratic form, namely the quadratic form generated by the Hessian matrix $H(f)(a)$

! Remark: $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$ is Fréchet diff. at $a \Rightarrow$

$$\Rightarrow \lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - df(a)(x-a)\right] = 0$$

8.3. Theorem If $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$ is twice Fréchet diff. at $a \Rightarrow$

$$\Rightarrow \lim_{x \to a} \frac{1}{\|x-a\|^2}\left[f(x) - f(a) - df(a)(x-a) - \frac{1}{2} d^2 f(a)(x-a)\right] = 0$$

Without proof.

! Remark: $\displaystyle \lim_{x \to a} \frac{f(x) - T_m f(x; a)}{(x-a)^m} = 0$

9. Necessary and sufficient optimality conditions

9.1. Definition Let $A \subseteq \mathbb{R}^m$, $a \in A$, $f: A \to \mathbb{R}$. If $\exists\, V \in \mathcal{V}(a)$ s.t. $x \in V \cap A$ one has

(*) $f(a) \le f(x)$ (respectively $f(a) \ge f(x)$)

the a is called a local minimum (respectively local maximum) for small f if inequality (*) hols for all $x \in$?? then m is said to be a global minimum (respectively global maximum) for f.

The local minima and local maxima is called the local extrema (global minima and global maxima is called the global extrema.

9.2. Theorem (P. Fermat) Let $A \subseteq \mathbb{R}^m$, let $f: A \to \mathbb{R}$, and let a be a point satisfying the following conditions:

(i) a is a local extremum for f

(ii) $a \in \operatorname{int} A$

(iii) f is Fréchet diff. at a

Then $\nabla f(a) = 0_m \Leftrightarrow \dfrac{\partial f}{\partial x_j}(a) = 0$, $\forall\, j = \overline{1, m}$

Proof: Fix $j \in \{1, \dots, m\}$. Assume that a is a local minimum for f.

Since $a \in \text{int} A \Rightarrow \exists\, \delta > 0$ s.t. the hypercube

$$[a_1 - \delta, a_1 + \delta] \times [a_2 - \delta, a_2 + \delta] \times \dots \times [a_m - \delta, a_m + \delta] \subseteq A$$

Let $g : [a_j - \delta, a_j + \delta] \to \mathbb{R}$, $g(t) := f(a_1, \dots, a_{j-1}, t, a_{j+1}, \dots, a_m)$

$\Rightarrow a_j$ is a local minimum for $g \xRightarrow{\text{Fermat Theorem}}$ $\left.\begin{array}{l} g'(a_j) = 0 \\ \text{But } g'(a_j) = \dfrac{\partial f}{\partial x_j}(a) \end{array}\right\} \Rightarrow \dfrac{\partial f}{\partial x_j}(a) = 0$

! Remark: Let $A \subseteq \mathbb{R}^m$, $a \in \text{int} A$, $f : A \to \mathbb{R}$

- If f is Fréchet diff. at a and $\nabla f(a) = 0_m$, then a is called a CRITICAL/STATIONARY point for f. By the Fermat th. $\Rightarrow$ each local extremum of f from int A must be a CRITICAL point. Unfortunately, not each critical point is a local extremum.

Counterexample for T.9.2.

$f(x, y) = x^2 - y^2$

$\nabla f(x, y) = (2x, -2y)$

$\nabla f(0,0) = (0,0) \Rightarrow (0,0)$ is a critical point for f

$f(x,0) = x^2 > 0 = f(0,0) \quad \forall x \in \mathbb{R} \setminus \{0\} \Rightarrow (0,0)$ is not a local maximum for f

$f(0,y) = -y^2 < 0 = f(0,0) \quad \forall y \in \mathbb{R} \setminus \{0\} \Rightarrow (0,0)$ is not a local minimum for f.

The surface of equation $z = f(x, y) = x^2 - y^2$ is an hyperbolic paraboloid which in the neighborhood of the origin looks like a saddle. Due to this fact, the critical points of a function which are not local extrema are called SADDLE points.

9.3. Definition Let $M = (a_{ij}) \in M_m(\mathbb{R})$, and let $\Phi : \mathbb{R}^m \to \mathbb{R}$,

$$\boxed{\Phi(h_1, \dots, h_m) = \sum_{i,j=1}^{m} a_{ij} \cdot h_i h_j}$$ be the quadratic form generated by M.

The quadratic form Φ is called:

- positive semidefinite if $\Phi(h) \geq 0 \quad \forall h \in \mathbb{R}^m$
- positive definite if $\Phi(h) > 0 \quad \forall h \in \mathbb{R}^m \setminus \{0_m\}$
- negative semidefinite if $\Phi(h) \leq 0 \quad \forall h \in \mathbb{R}^m$
- negative definite if $\Phi(h) < 0 \quad \forall h \in \mathbb{R}^m \setminus \{0_m\}$
- indefinite if $\exists\, a, b \in \mathbb{R}^m$ s.t. $\Phi(a) < 0 < \Phi(b)$

Examples

$\Phi(h_1, h_2) = 2h_1^2 + 5h_2^2 \to$ positive definite

$\Phi(h_1, h_2) = -2h_1^2 - 5h_2^2 \to$ negative definite

$\Phi(h_1, h_2) = 2h_1^2 - 5h_2^2 \to$ indefinite

$\Phi(h_1, h_2) = 2h_1^2 \to$ positive semidefinite

9.4. Theorem (characterization of positive definite quadratic form)

Let $\Phi : \mathbb{R}^m \to \mathbb{R}$ be a quadratic form. Then

Φ is positive definite $\Leftrightarrow \exists\, \alpha > 0$ s.t. $\forall\, h \in \mathbb{R}^m : \Phi(h) \geq \alpha \cdot \|h\|^2$ (*)

Proof: $\boxed{\Leftarrow}$ Obvious

$\boxed{\Rightarrow}$ Φ being a quadratic form $\Rightarrow \Phi$ is continuous on $\mathbb{R}^m$

$S^{m-1} = \{ h \in \mathbb{R}^m \mid \|h\| = 1 \}$ is a compact set $\Bigg\} \xrightarrow{\text{T. Weierstrass}}$

$\Rightarrow \Phi$ is bounded and attains its bounds on S^{m-1}

Partial exam:
- unul din cele de pe listă la teorie (poate fi de dem. pt. un caz particular sau notat diferit sau formulat diferit) = 3pct. / 10
- 3 probleme

Set $\alpha := \min\limits_{h \in S^{m-1}} \Phi(h) \xrightarrow{\Phi \text{ is positive definite}} \exists\, h_0 \in S^{m-1}$ s.t. $\alpha = \Phi(h_0) > 0$

If $h = 0_m \Rightarrow$ (*) holds with equality

If $h \neq 0_m \Rightarrow \frac{1}{\|h\|} h \in S^{m-1} \Rightarrow$

$\Rightarrow \alpha \leq \Phi\left(\frac{1}{\|h\|} \cdot h\right) = \frac{1}{\|h\|^2} \Phi(h) \Big/ \cdot \|h\|^2 \Rightarrow \Phi(h) \geq \alpha \|h\|^2$, hence (*) holds

But $\Phi(th) = t^2 \Phi(h)$, $\forall\, t \in \mathbb{R}$, $\forall\, h \in \mathbb{R}^m$

9.5. Theorem (necessary and sufficient optimality conditions)

Let $A \subseteq \mathbb{R}^m$, let $a \in \operatorname{int} A$, and let $f : A \to \mathbb{R}$ be a twice Fréchet differentiable function at a. Then the following assertions are true:

1° If a is a local minimum (resp. local maximum) for f, then $\nabla f(a) = 0_m$ and $d^2 f(a)$ is a positive semidefinite (resp. negative semidefinite) quadratic form.

2° If $\nabla f(a) = 0_m$ and $d^2 f(a)$ is a positive definite (resp. negative definite) quadratic form, then a is

Proof: poza

27.04.2018.

$\boxed{\text{COURSE 8}}$

examen partial 6 mai, ora 8^{00}
sala Augustin Maior

– continue for proof of Th 7.5 –

? $d^2f(a)$ is positive semidefinite $\Leftrightarrow \forall h \in \mathbb{R}^m : d^2f(a)(h) \geq 0$

Let $h \in \mathbb{R}^m$ and let $\varepsilon > 0$.

- Since a is a local minimum for $f \Rightarrow \exists V \in \mathcal{V}(a)$ s.t.

$$\forall x \in V \cap A : f(a) \leq f(x) \quad (1)$$

- We know that (see Theorem 8.3)

$$\lim_{x \to a} \frac{1}{\|x-a\|^2} \left[f(x) - f(a) - \underbrace{df(a)(x-a)}_{=\langle \underbrace{\nabla f(a)}_{=0_m}, x-a \rangle = 0} - \frac{1}{2} d^2f(a)(x-a) \right] = 0$$

$$\Rightarrow \lim_{x \to a} \frac{1}{\|x-a\|^2} \left[f(x) - f(a) - \frac{1}{2} d^2f(a)(x-a) \right] = 0$$

$\Rightarrow \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}, \|x-a\| < \delta$:

$$\left| \frac{f(x) - f(a) - \frac{1}{2} d^2f(a)(x-a)}{\|x-a\|^2} - 0 \right| < \varepsilon \quad \Big| \cdot \|x-a\|^2$$

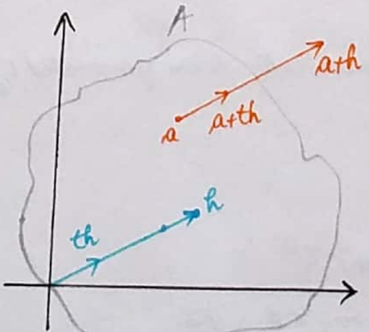
$\Rightarrow \forall x \in A$ with $\|x-a\| < \delta$:

$$|f(x) - f(a) - \frac{1}{2} d^2f(a)(x-a)| \leq \varepsilon \|x-a\|^2$$

$\Rightarrow \forall x \in A \cap B(a,\delta)$: $f(x) - f(a) - \frac{1}{2} d^2f(a)(x-a) \leq \varepsilon \|x-a\|^2$

$\Rightarrow \forall x \in A \cap B(a,\delta)$: $f(x) - f(a) \leq \frac{1}{2} d^2f(a)(x-a) + \varepsilon \|x-a\|^2 \quad (2)$

By (1) and (2) $\Rightarrow \forall x \in \underbrace{A \cap V \cap B(a,\delta)}_{\in \mathcal{V}(a)}$: $0 \leq \frac{1}{2} d^2f(a)(x-a) + \varepsilon \|x-a\|^2 \quad (3)$



$\Rightarrow \exists t > 0$ sufficiently small s.t. $a + th \in A \cap V \cap B(a,\delta)$

Replacing x by $a + th$ in (3) $\Rightarrow$

$$0 \leq \frac{1}{2} d^2f(a)(th) + \varepsilon \|th\|^2$$

$$0 \leq \frac{t^2}{2} d^2f(a)(h) + \varepsilon t^2 \|h\|^2 \quad \Big| \cdot \frac{1}{t^2} \Rightarrow$$

$$\Rightarrow 0 \leq \frac{1}{2} d^2f(a)(h) + \varepsilon \|h\|^2, \forall \varepsilon > 0$$

$$\boxed{\Phi(th) = t^2 \Phi(h)}$$

Letting $\varepsilon \searrow 0 \Rightarrow d^2f(a)(h) \geq 0$.

$2°$ Suppose, for instance, that $\nabla f(a) = 0_m$ and that $d^2 f(a)$ is a positive definite quadratic form $\Rightarrow$

$\Rightarrow \exists \alpha > 0$ s.t. $\forall h \in \mathbb{R}^m: d^2f(a)(h) \geq \alpha \|h\|^2$ ⊛

? a is a local minimum for f

We know that

$$\lim_{x \to a} \frac{1}{\|x-a\|^2}\left[f(x) - f(a) - \underbrace{df(a)(x-a)}_{=\langle \nabla f(a), x-a \rangle = 0} - \frac{1}{2} d^2f(a)(x-a)\right] = 0$$

$$\lim_{x \to a} \frac{1}{\|x-a\|^2}\left[f(x) - f(a) - \frac{1}{2} d^2f(a)(x-a)\right] = 0$$

Take $\varepsilon := \alpha/2$ $\Rightarrow \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $\|x-a\| < \delta$

$$\left|\frac{f(x) - f(a) - \frac{1}{2}d^2f(a)(x-a)}{\|x-a\|^2} - 0\right| < \varepsilon \quad \Big| \cdot \|x-a\|^2$$

$\Rightarrow \forall x \in A \cap B(a, \delta): \left|f(x) - f(a) - \frac{1}{2} d^2f(a)(x-a)\right| \leq \varepsilon \|x-a\|^2$

$\Rightarrow \forall x \in A \cap B(a, \delta): \quad -\varepsilon\|x-a\|^2 \leq f(x) - f(a) - \frac{1}{2}d^2f(a)(x-a)$

$\Rightarrow \forall x \in A \cap B(a, \delta): f(x) - f(a) \geq \frac{1}{2} d^2f(a)(x-a) - \varepsilon \|x-a\|^2 \overset{⊛}{\geq}$

$\geq \frac{\alpha}{2}\|x-a\|^2 - \varepsilon \|x-a\|^2 = 0$

$\Rightarrow \forall x \in A \cap B(a, \delta): f(x) - f(a) \geq 0 \Leftrightarrow f(a) \leq f(x)$

$\Rightarrow x$ is a local minimum for f

8.5. Remarks

a. The Sylvester Test: let $M := (a_{ij}) \in M_m(\mathbb{R})$ be a symmetric matrix and let $\Phi: \mathbb{R}^m \to \mathbb{R}$,

$$\Phi(h_1, \dots, h_m) = \sum_{i,j=1}^{m} a_{ij} h_i h_j \text{ be the quadratic form generated by } M.$$

Let $\Delta_k := \det(a_{ij})_{1 \leq i,j \leq k}$, $k = \overline{1,m}$

Then the following assertions are true:

$1°$ Φ is positive definite $\Leftrightarrow \Delta_k > 0, \forall k = \overline{1,m}$

$2°$ Φ is negative definite $\Leftrightarrow -\Phi$ is positive definite

$\Updownarrow$ $\quad\quad$ $\downarrow$ is generated by $M' = -M$

$(-1)^k \Delta_k > 0, \forall k = \overline{1,m}$ $\quad\quad$ $\Delta'_k = \det(-a_{ij})_{1 \leq i,j \leq k} = (-1)^k \Delta_k$

$3°$ Φ is positive semidefinite $\Rightarrow \Delta_k \geq 0, \forall k = \overline{1,m}$

$4°$ Φ is negative semidefinite $\Rightarrow (-1)^k \Delta_k \geq 0, \forall k = \overline{1,m}$

(b) If $d^2f(a)$ is indefinite $\Rightarrow$ a is a saddle point

Example: Determine the critical points of $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2+y^2+z^2-2xyz$ and specify their nature.

Solution: The critical points of f are solutions to the system

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z)=0 \\ \frac{\partial f}{\partial y}(x,y,z)=0 \\ \frac{\partial f}{\partial z}(x,y,z)=0 \end{cases} \Leftrightarrow \begin{cases} 2x-2yz=0 \\ 2y-2xz=0 \\ 2z-2xy=0 \end{cases} \Leftrightarrow \begin{cases} x=yz \\ y=xz \\ z=xy \end{cases}$$

$x = xy^2 \Rightarrow x(1-y^2)=0 \Rightarrow x=0$ or $y=\pm 1$

- $x=0 \Rightarrow y=0,\ z=0 \Rightarrow \boxed{(0,0,0)}$
- $y=1 \Rightarrow \begin{cases} x=z \\ 1=x^2 \Rightarrow x=\pm 1 \end{cases} \Rightarrow \boxed{(1,1,1)}, \boxed{(-1,1,-1)}$
- $y=-1 \Rightarrow \begin{cases} x=-z \\ -1=-x^2 \Rightarrow x=\pm 1 \end{cases} \Rightarrow \boxed{(1,-1,-1)}, \boxed{(-1,-1,1)}$

$\to$ critical points of f

Step II. $\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2 = \frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{\partial^2 f}{\partial z^2}(x,y,z)$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = \frac{\partial^2 f}{\partial x \partial y}(x,y,z) = -2z$$

Analogously:

$$\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = -2y$$

$$\frac{\partial^2 f}{\partial z \partial y}(x,y,z) = \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = -2x$$

$$H(f)(x,y,z) = \begin{pmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{pmatrix}$$

Step III. 1. $H(f)(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\Delta_1 = 2 > 0$

$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$\Delta_3 = 8 > 0$

$\Rightarrow (0,0,0)$ is a local minimum for f.

2. $H(f)(1,1,1) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$

$\Delta_1 = 2 \geq 0$

$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0 \geq 0$

$\Delta_3 = \begin{vmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{vmatrix} = 8-8-8-8-8-8 = -32 \leq 0$

$\Rightarrow (1,1,1)$ is a saddle point

Method II: $d^2f(1,1,1)(h_1,h_2,h_3) = \frac{\partial^2 f}{\partial x^2}(1,1,1)h_1^2 + \frac{\partial^2 f}{\partial y^2}(1,1,1)h_2^2 + \frac{\partial^2 f}{\partial z^2}(1,1,1)h_3^2$

$+ 2\frac{\partial^2 f}{\partial y \partial x}(1,1,1)h_1h_2 + 2\frac{\partial^2 f}{\partial y \partial z}(1,1,1)h_2h_3 + 2\frac{\partial^2 f}{\partial z \partial x}(1,1,1)h_1h_3$

$d^2 f(1,1,1)(h_1, h_2, h_3) = 2h_1^2 + 2h_2^2 + 2h_2^2 - 4h_1h_2 - 4h_2h_3 - 4h_1h_3$

$d^2 f(1,1,1)(1,0,0) = 2 > 0$

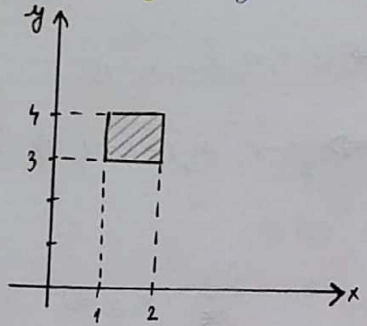
$d^2 f(1,1,1)(1,1,1) = -6 < 0$

$\Rightarrow d^2 f(1,1,1)$ is an indefinite quadratic form $\Rightarrow$

$\Rightarrow (1,1,1)$ is a saddle point.

CHAPTER 3. MULTIPLE INTEGRALS

$$\int_1^2 \int_3^4 \frac{1}{(x+y)^2}\, dx\, dy = \int_1^2 \left(\int_3^4 \frac{1}{(x+y)^2}\, dy \right) dx$$



$$= \int_3^4 \left(\int_1^2 \frac{1}{(x+y)^2}\, dx \right) dy$$

$$I = \int_1^2 \left(-\frac{1}{x+y} \Big|_{y=3}^{y=4} \right) dx = \int_1^2 \left(-\frac{1}{x+4} + \frac{1}{x+3} \right) dx$$

$$= -\ln(x+4)\Big|_1^2 + \ln(x+3)\Big|_1^2 = -\ln 6 + \ln 5 + \ln 5 - \ln 4$$

$$\boxed{I = \ln \frac{25}{24}}$$

1. The Riemann integral over generalized rectangles in $\mathbb{R}^n$

1.1 Definition (generalized rectangles) $T \subseteq \mathbb{R}^m$

A set $T \subseteq \mathbb{R}^m$ is called a generalized rectangle if it has the form: $T = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_m, b_m]$

where $a_1 < b_1,\ a_2 < b_2, \dots, a_m < b_m$ are real numbers.

We define

$m(T) := (b_1 - a_1)(b_2 - a_2)\dots(b_m - a_m)$

↳ the volume (the measure) of T

$m=1$ $T = [a_1, b_1]$ $m(T) = b_1 - a_1$ the length of T

$m=2$ $T = [a_1, b_1] \times [a_2, b_2]$ $m(T) = (b_1 - a_1)(b_2 - a_2)$ the area of T

$m=3$ $T = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ $m(T) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$ the volume of T

1.2 Definition (partitions of a generalized rectangle)

Let $[a,b]$ be a compact interval, and let $\Delta = (x_0, x_1, \dots, x_k)$ be a partition of $[a,b] \Rightarrow x_0 = a < x_1 < \dots < x_k = b$.

We write $\Delta = \{[x_0, x_1], [x_1, x_2], \dots, [x_{k-1}, x_k]\}$

Letting $X_j = [x_{j-1}, x_j] \Rightarrow \Delta = \{X_1, X_2, \dots, X_k\}$

Let $T = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_m, b_m]$ be a generalized rectangle in $\mathbb{R}^m$

let $\Delta_1 = \{X_1^1, X_2^1, \dots, X_{k_1}^1\} \in \text{Part}[a_1, b_1]$

$\Delta_2 = \{X_1^2, X_2^2, \dots, X_{k_2}^2\} \in \text{Part}[a_2, b_2]$

$\dots$

$\Delta_m = \{X_1^m, X_2^m, \dots, X_{k_m}^m\} \in \text{Part}[a_m, b_m]$

Set $\Delta_1 \times \Delta_2 \times \ldots \times \Delta_m := \{ X^1_{j_1} \times X^2_{j_2} \times \ldots \times X^m_{j_m} \mid j_1 = \overline{1, k_1}, j_2 = \overline{1, k_2}, \ldots, j_m = \overline{1, k_m} \}$

↳ the product of the partitions $\Delta_1, \Delta_2, \ldots, \Delta_m$

A set π is called a partition of the generalized rectangle T if it is of the form $\pi = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_m$, where $\Delta_1 \in \text{Part}[a_1, b_1], \ldots, \Delta_m \in \text{Part}[a_m, b_m]$

Define

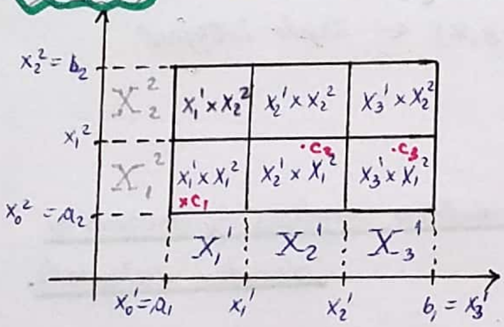
$$\|\pi\| := \max\{\|\Delta_1\|, \|\Delta_2\|, \ldots, \|\Delta_m\|\}$$

↳ the norm (mesh) of the partition π.

In some cases we will denote the partition of T by enumerating its elements, namely: $\pi = \{T_1, T_2, \ldots, T_p\}$ where each T_i is of the form $X^1_{j^1} \times \ldots \times X^m_{j^m}$

$$p = k_1 \cdot k_2 \cdot \ldots \cdot k_m$$

Example $T = [a_1, b_1] \times [a_2, b_2]$



$\pi = \{ X^1_1 \times X^2_1, X^1_2 \times X^2_1, X^1_3 \times X^2_1, X^1_1 \times X^2_2, X^1_2 \times X^2_2, X^1_3 \times X^2_2 \}$

$\pi = \{ T_1, T_2, T_3, T_4, T_5, T_6 \}$

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Given a generalized rectangle T, we denote by $\text{Part}(T)$ = the set of all partitions of T

1.3. Definition (intermediate points associated with a partition)
Let $T \subseteq \mathbb{R}^m$ be a generalized rectangle, and let $\pi = \{T_1, T_2, \ldots, T_p\} \in \text{Part}(T)$.

A set of points $\xi = \{c_1, c_2, \ldots c_p\}$ is called a system of intermediate points associated with the partition π if $c_1 \in T_1, c_2 \in T_2, \ldots, c_p \in T_p$

Let $IP(\pi)$ = the set of all systems of intermediate points associated with π.

1.4. Definition (Riemann sums) Let $T \subseteq \mathbb{R}^m$ be a generalized rectangle, let $\pi = \{T_1, T_2, \ldots, T_p\}$ be a partition of T and let $\xi = \{c_1, c_2, \ldots, c_p\} \in IP(\pi)$

Define

$$\sigma(f, \pi\, \xi) = \sum_{j=1}^{p} f(c_j) \cdot m(T_j)$$

↳ the Riemann sum associated with f, π, ξ

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1.5. Definition. Let $T \subseteq \mathbb{R}^m$ be a generalized rectangle, and let $f: T \to \mathbb{R}$

The function f is said to be Riemann integrable over T if there exists a real number I with the following property:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall \pi \in Part(T)$ with $\|\pi\| < \delta$

and $\forall \xi \in IP(\pi)$: $|\sigma(f, \pi, \xi) - I| < \varepsilon$

It is imideately seen that if f is Riemann integrable over T, then the number I with the above property is unique. It is called the Riemann integrable of f over T and it is denoted by $\int_T f(x)\,dx$, $\int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} f(x_1, x_2, \dots, x_m)\,dx_1\,dx_2 \dots dx_m$

x is a vector variable NOT a real variable

$m=2$ $\quad \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dx\,dy \leftarrow$ double integral

$m=3$ $\quad \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z) \leftarrow$ triple integral

2. Computation of multiple integrals by means of iterated integrals.

2.1. Definition (iterated integrals)

Let $S \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^n$ be generalized rectangles

$\Rightarrow S \times T$ is a generalized rectangle in $\mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$

Let $f: S \times T \to \mathbb{R}$

$\forall x \in S$ let $f_x: T \to \mathbb{R}$, $f_x(y) = f(x,y) \leftarrow$ the section of f through x

$f_x = f(x, \cdot)$

$\forall y \in T$ let $f_y: S \to \mathbb{R}$, $f_y(x) = f_y(x) = f(x,y) \leftarrow$ the section of f through y

$f_y = f(\cdot, y)$

The function f is said to be partially integrable over $S \times T$ if: $\forall x \in S$ the section f_x is Riemann integrable over T

cartezian product

$\forall y \in T$ the section f_y is Riemann integrable over S

In this case we may define the functions

$F_1: S \to \mathbb{R}$, $F_1(x) := \int_T f_x(y)\,dy = \int_T f(x,y)\,dy$

$F_2: T \to \mathbb{R}$, $F_2(y) := \int_S f_y(x)\,dx = \int_S f(x,y)\,dx$.

Suppose moreover that:

F_1 is Riemann integrable over S, F_2 is Riemann integrable over T

$\Rightarrow \int_S F_1(x)\,dx = \int_S \left(\int_T f(x,y)\,dy \right) dx \leftarrow$ the iterated integral of f in the order (y, x)

$$\int_T F_2(y)\,dy = \int_T\left(\int_S f(x,y)\,dx\right)dy \leftarrow \text{the iterated integral of } f \text{ in the order } (x,y)$$

$\boxed{\text{COURSE 8}}$ - week 9

04.05.2018.

$\boxed{\text{2.2 Theorem (G. Fubini)}}$ Let $S \subseteq \mathbb{R}^m$, $T \subseteq \mathbb{R}^n$ be generalized rectangles, and let $f : S\times T \to \mathbb{R}$ be a function which is both Riemann integrable and partially integrable over $S\times T$. Then $F_1 \in R(S)$, $F_2 \in R(T)$ and

$$\int_{S\times T} f(x,y)\,dx\,dy = \int_S F_1(x)\,dx = \int_T F_2(y)\,dy.$$

Proof: Let $I := \int_{S\times T} f(x,y)\,dx\,dy$. We prove that $F_1 \in R(S)$ and $\int_S F_1(x)\,dx = I$

(the fact that $F_2 \in R(T)$ and $\int_T F_2(y)\,dy = I$ can be proved analogously)

? $F_1 \in R(S)$ and $\int_S F_1(x)\,dx = I \iff \forall \varepsilon>0\ \exists \delta>0$ s.t. $\forall \pi_1 \in Part(S)$ with $\|\pi_1\|<\delta$, $\forall \eta \in IP(\pi_1)$:

$$|\underbrace{\sigma}_{\text{SIGMA}}(F_1,\pi_1,\underbrace{\eta}_{\text{TETA}}) - I| < \varepsilon.$$

Let $\varepsilon>0$. Since $f \in R(S\times T)$ and $\int_{S\times T} f(x,y)\,dx\,dy = I \Rightarrow$

$\Rightarrow \exists \delta>0$ s.t. $\forall \pi \in Part(S\times T)$ with $\|\pi\|<\delta$ and

and $\forall \xi \in IP(\pi)$: $\quad |\sigma(f,\pi,\underbrace{\xi}_{\text{csi}}) - I| < \varepsilon$

intermediate POINTS

Let $\pi_1 = \{S_1,\dots,S_p\} \in Part(S)$ with $\|\pi_1\|<\delta$ and let $\eta = \{a_1,\dots,a_p\} \in IP(\pi_1)$

$$\sigma(F_1,\pi_1,\eta) = \sum_{i=1}^{p} F_1(a_i)\,m(S_i)$$

$F_1(a_i) = \int_T f_{a_i}(y)\,dy \Rightarrow \forall i \in \{1,\dots,p\}\ \exists \delta_i>0$ s.t.

$\forall \pi_2 \in Part(T)$ with $\|\pi_2\|<\delta_i$

$\|\pi\|$ = the mesh of π

$\forall \underbrace{\zeta}_{\text{ZETA}} \in IP(\pi_2)$: $|\sigma(f_{a_i},\pi_2,\zeta) - F_1(a_i)| < \frac{\varepsilon}{2m(S)}$

Choose $\pi_2 \in Part(T)$ s.t. $\|\pi_2\| < \min\{\delta,\delta_1,\delta_2,\dots,\delta_p\}$, $\pi_2 = \{T_1,\dots,T_q\}$,

choose $\zeta = \{b_1,\dots,b_q\} \in IP(\pi_2)$, and after that define

$$\pi := \{S_i\times T_j \mid i=\overline{1,p},\ j=\overline{1,q}\},\quad \xi = \{(a_i,b_j) \mid i=\overline{1,p},\ j=\overline{1,q}\}$$

$\Rightarrow \pi \in Part(S\times T)$, $\xi \in IP(\pi)$, $\|\pi\|<\delta$

$$|\sigma(F_1,\pi_1,\eta) - I| = |\sigma(F_1,\pi_1,\eta) - \sigma(f,\pi,\xi) + \sigma(f,\pi,\xi) - I|$$
$$\le |\sigma(F_1,\pi_1,\eta) - \sigma(f,\pi,\xi)| + \underbrace{|\sigma(f,\pi,\xi) - I|}_{<\varepsilon/2}$$

$$< \frac{\varepsilon}{2} + \left| \sum_{i=1}^{p} F_1(a_i)\, m(S_i) - \sum_{i=1}^{p} \sum_{j=1}^{\ell} \underbrace{f(a_i,b_j)}_{\overset{\shortparallel}{f_{a_i}(b_j)}}\, \underbrace{m(S_i \times T_j)}_{\overset{\shortparallel}{m(S_i)\cdot m(T_j)}} \right|$$

$$\Rightarrow \left| \sigma(F_1,\pi_1,\eta) - I \right| < \frac{\varepsilon}{2} + \left| \sum_{i=1}^{p} \left[F_1(a_i) - \sum_{j=1}^{\ell} f_{a_i}(b_j)\, m(T_j) \right] m(S_i) \right|$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{p} \left| F_1(a_i) - \underbrace{\sum_{j=1}^{\ell} f_{a_i}(b_j)\, m(T_j)}_{\sigma(f_{a_i},\pi_2,\zeta)} \right| m(S_i)$$

$$\Rightarrow \left| \sigma(F_1,\pi_1,\eta) - I \right| < \frac{\varepsilon}{2} + \sum_{i=1}^{p} \underbrace{\left| F_1(a_i) - \sigma(f_{a_i},\pi_2,\zeta) \right|}_{<\varepsilon'} m(S_i)$$

$$\Rightarrow \left| \sigma(F_1,\pi_1,\eta) - I \right| < \frac{\varepsilon}{2} + \varepsilon' \underbrace{\sum_{i=1}^{p} m(S_i)}_{m(S)} = \frac{\varepsilon}{2} + \varepsilon' m(S) = \varepsilon$$

! Remark: The equality in the Fubini theorem (2.2.) can be written equivalently as

$$\underbrace{\int_{S\times T} f(x,y)\,dx\,dy}_{m+n \text{ variables}} = \underbrace{\int_S \Big(\underbrace{\int_T f(x,y)\,dy}_{n \text{ variables}} \Big) dx}_{m \text{ variables}} = \underbrace{\int_T \Big(\underbrace{\int_S f(x,y)\,dx}_{m \text{ variables}} \Big) dy}_{n \text{ variables}}$$

integralele multiple
→ pt. calculul momentelor de inerție (energia cinetică de rotație

3. The Riemann integral over bounded sets in $\mathbb{R}^m$

Let $A \subseteq \mathbb{R}^m$, and let $f : A \to \mathbb{R}$ be a given function.

Define $\overline{f} : \mathbb{R}^m \to \mathbb{R}$, $\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$

→ the null extension of f.

3.1. Definition. Let $A \subseteq \mathbb{R}^m$ be a bounded set, and $f : A \to \mathbb{R}$.

It can be proved that if $\exists\, T \subseteq \mathbb{R}^m$ generalized rectangle s.t. $A \subseteq T$ and $\overline{f}\big|_T \in R(T)$ (Riemann integrable over T), then $\forall\, S \subseteq \mathbb{R}^m$ generalized rectangle s.t. $A \subseteq S$ we have

$$\overline{f}\big|_S \in R(S) \text{ and } \int_T \overline{f}(x)\,dx = \int_S \overline{f}(x)\,dx$$

Due to this remark, we may give the following definition:

f is said to be Riemann integrable over A if $\exists\, T \subseteq \mathbb{R}^m$ generalized rectangle s.t. $A \subseteq T$ and $\bar{f}|_T \in \mathcal{R}(T)$. In this case, the real number $\int_T \bar{f}(x)\,dx$ is called the Riemann integral of f over A and it is denoted by $\underbrace{\int\cdots\int_A}_{m} f(x_1,\dots,x_m)\,dx_1\dots dx_m$

$m=2 \quad \iint_A f(x,y)\,dx\,dy$

$m=3 \quad \iiint_A f(x,y,z)\,dx\,dy\,dz$

3.2. Definition Let $A \subseteq \mathbb{R}^m$ be a bounded set.

If the constant function equal to 1 on A is Riemann integrable over A, then the set A is said to be JORDAN MEASURABLE. In this case, we define

$$m(A) := \underbrace{\int\cdots\int_A}_{m} dx_1\dots dx_m$$

→ the JORDAN measure of A

$m=2 \quad m(A) = \iint_A dx\,dy$ ← area of A

$m=3 \quad m(A) = \iiint_A dx\,dy\,dz$ ← volume of A

! Remark: It should be noted that not every bounded set is JORDAN measurable. For instance, it can be proved that $A := [0,1] \cap \mathbb{Q}$ is not JORDAN measurable.

CURSUL 10 . 8°°
AVRAM IANCU ET. 3
MARȚI, 8 mai

4. Computation of Riemann integrals over bounded sets by means of iterated integrals

4.1. Definition (sections and projections of a set)

Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$

$\forall\, x \in \mathbb{R}^m$ we define $A_x := \{ y \in \mathbb{R}^n \mid (x,y) \in A \}$

→ section of A through x

$p_1(A) := \{ x \in \mathbb{R}^m \mid A_x \neq \emptyset \}$

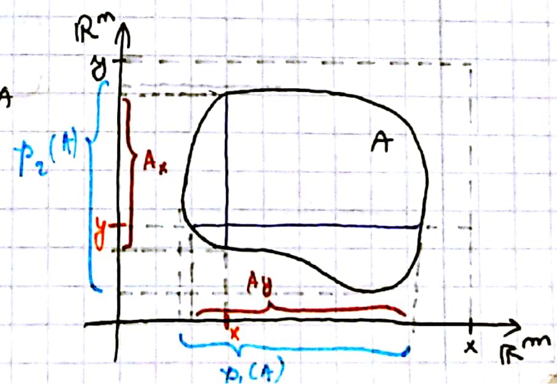
→ the first projection of A

$\forall\, y \in \mathbb{R}^n$ we define $A_y := \{ x \in \mathbb{R}^m \mid (x,y) \in A \}$

→ section of A through y

$p_2(A) := \{ y \in \mathbb{R}^n \mid A_y \neq \emptyset \}$

→ the second projection of A



4.2. Definition (iterated integrals). Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$, $f: A \to \mathbb{R}$. Then f is said to be partially integrable over A_x if

- $\forall x \in p_1(A)$ f_x is Riemann integrable over A_x

$$f_x : A_x \to \mathbb{R}, \quad f_x(y) := f(x,y)$$

- $\forall y \in p_2(A)$ f_y is Riemann integrable over A_y

$$f_y : A_y \to \mathbb{R}, \quad f_y(x) := f(x,y)$$

In this case we can define the functions:

$$F_1 : p_1(A) \to \mathbb{R}, \quad F_1(x) := \int_{A_x} f_x(y)\,dy = \int_{A_x} f(x,y)\,dy$$

$$F_2 : p_2(A) \to \mathbb{R}, \quad F_2(y) = \int_{A_y} f_y(x)\,dx = \int_{A_y} f(x,y)\,dx$$

If F_1 is Riemann integrable over $p_1(A)$, then its integral

$$\int_{p_1(A)} F_1(x)\,dx = \int_{p_1(A)} \left(\int_{A_x} f(x,y)\,dy \right) dx$$

is called iterated integral of f over A in the order (y,x)

If F_2 is Riemann integrable over $p_2(A)$, then its integral

$$\int_{p_2(A)} F_2(y)\,dy = \int_{p_2(A)} \left(\int_{A_y} f(x,y)\,dx \right) dy$$

is called iterated integral of f over A in the order (x,y)

4.3 Theorem (G. Fubini) Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be a bounded set and let $f: A \to \mathbb{R}$ s.t. f is both Riemann and partially integrable over A. Then F_1 is Riemann integrable over $p_1(A)$, F_2 is Riemann integrable over $p_2(A)$ and

$$\int_A f(x,y)\,dx\,dy = \int_{p_1(A)} F_1(x)\,dx = \int_{p_2(A)} F_2(y)\,dy.$$

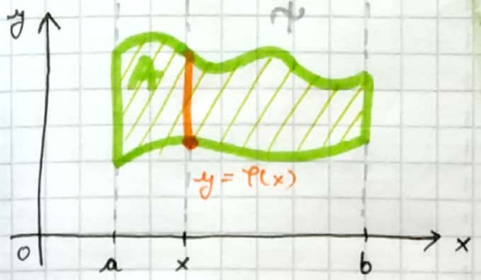
Remark!

$$\underbrace{\int_A f(x,y)\,dx\,dy}_{m+n \text{ vars.}} = \underbrace{\int_{p_1(A)} \Bigg(\underbrace{\int_{A_x} f(x,y)\,dy}_{n \text{ vars.}} \Bigg) dx}_{m \text{ vars.}} =$$

$$= \underbrace{\int_{p_2(A)} \Bigg(\underbrace{\int_{A_y} f(x,y)\,dx}_{m \text{ vars.}} \Bigg) dy}_{n \text{ vars.}}$$

4.4. Definition (simple sets w.r.t. an axis in $\mathbb{R}^2$)
$A \subseteq \mathbb{R}^2$ is said to be simple w.r.t. Oy if $\exists\, a, b \in \mathbb{R}$, $a < b$ and $\exists\, \varphi, \psi : [a,b] \to \mathbb{R}$ φ, ψ continuous, $\varphi \leq \psi$ s.t.

$$A = \{(x,y) \mid a \leq x \leq b,\ \varphi(x) \leq y \leq \psi(x)\}$$

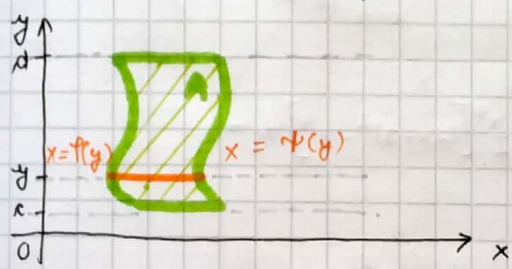


$p_1(A) = [a,b]$

$A_x = [\varphi(x), \psi(x)]\ \forall x \in [a,b]$

$A \subseteq \mathbb{R}^2$ is said to be simple w.r.t. Ox if $\exists\, c, d \in \mathbb{R}$, $c < d$ and $\exists\, \varphi, \psi : [c,d] \to \mathbb{R}$ φ, ψ continuous, $\varphi \leq \psi$ s.t.

$$A = \{(x,y) \mid c \leq y \leq d,\ \varphi(y) \leq x \leq \psi(y)\}$$



4.5. Corollary. Let $A \subseteq \mathbb{R}^2$ be simple w.r.t. Oy (respectively Ox), and let $f : A \to \mathbb{R}$ be a continuous function. Then

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right) dx$$

$$\left(\text{resp. } \iint_A f(x,y)\,dx\,dy = \int_c^d \left(\int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx \right) dy \right)$$

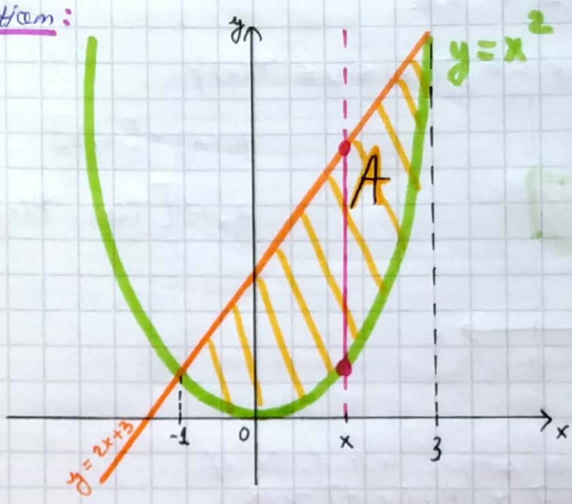
In particular, for $f = 1$,

$$m(A) = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} dy \right) dx = \int_a^b [\psi(x) - \varphi(x)]\,dx$$

$$\left(\text{resp. } m(A) = \int_c^d [\psi(y) - \varphi(y)]\,dy \right)$$

Example Compute $I=\iint_A x^2y\,dxdy$, where $A\subseteq\mathbb{R}^2$ is the set bounded by the parabola $y=x^2$ and by the line $y=2x+3$

Solution:



$$\begin{cases} y=x^2 \\ y=2x+3 \end{cases} \qquad x^2-2x-3=0 \qquad I=\int_{x=-1}^{x=3}\left(\int_{y=x^2}^{y=2x+3} x^2y\,dy\right)dx$$

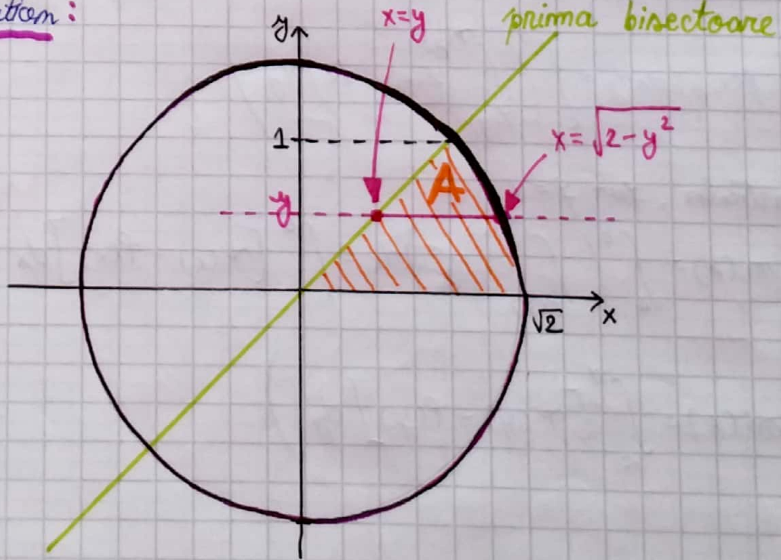
$$x_1=-1 \quad y_1=1$$
$$x_2=3 \quad y_2=9$$

$$I=\int_{-1}^{3}\frac{x^2y^2}{2}\Bigg|_{y=x^2}^{y=2x+3}dx=\int_{-1}^{3}\frac{x^2}{2}\left[(2x+3)^2-x^4\right]dx=$$

$$=\frac{1}{2}\int_{-1}^{3}x^2(4x^2+12x+9)\,dx-\frac{1}{2}\int_{-1}^{3}x^6\,dx=\dots$$

Example Evaluate $I=\iint_A\frac{x}{y^2+1}\,dx\,dy$, where A is the set defined by $A=\{(x,y)\mid x\geq y\geq 0,\ x^2+y^2\leq 2\}$

Solution:



$$\begin{cases} y=x \\ x^2+y^2=2 \end{cases} \Rightarrow 2y^2=2 \Rightarrow y=\pm 1$$

$y=1$

$$I=\int_{y=0}^{y=1}\left(\int_{x=y}^{x=\sqrt{2-y^2}}\frac{x}{y^2+1}\,dx\right)dy=\int_0^1\frac{1}{y^2+1}\cdot\frac{x^2}{2}\Bigg|_{x=y}^{x=\sqrt{2-y^2}}dy=$$

$$= \int_0^1 \frac{2-y^2-y^2}{2(y^2+1)}\,dy = \int_0^1 \frac{1-y^2}{1+y^2}\,dy = \int_0^1 \frac{-1-y^2+2}{1+y^2}\,dy =$$

$$= \int_0^1 \left(-1+\frac{2}{y^2+1}\right)dy = -y\Big|_0^1 + 2\cdot \arctan y\Big|_0^1$$

$$\boxed{I = -1+\frac{\pi}{2}}$$

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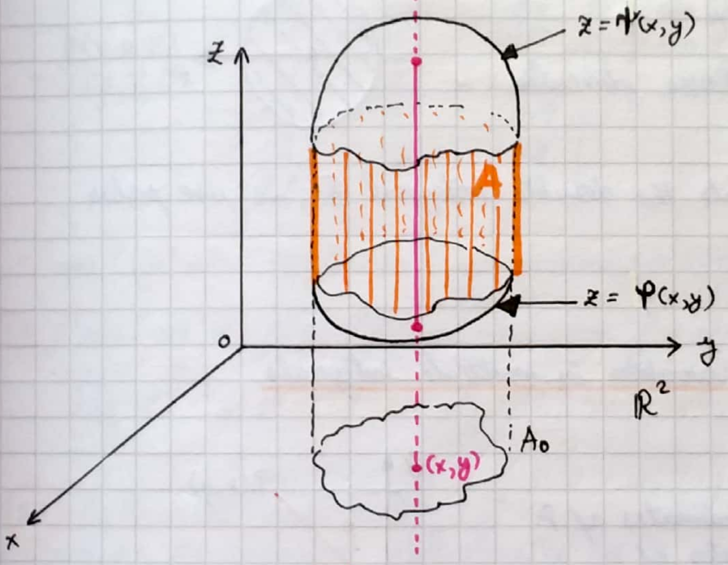
4.6 Definition (simple sets wrt an axis in $\mathbb{R}^3$)
$A \subseteq \mathbb{R}^3$ is said to be simple wrt Oz if
$\exists\, A_0 \subseteq \mathbb{R}^2$ measurable JORDAN
$\exists\, \varphi, \psi : A_0 \to \mathbb{R}$ s.t. $\varphi \le \psi$ and

$$A = \{(x,y,z) \mid (x,y) \in A_0,\ \varphi(x,y) \le z \le \psi(x,y)\}$$



$pr_1(A) = A_0$

$A_{(x,y)} = [\varphi(x,y), \psi(x,y)]$

4.7 Corollary. Let $A \subseteq \mathbb{R}^3$ be a simple set wrt Oz, and let $f: A \to \mathbb{R}$ be a continuous bounded function. Then

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{A_0}\left(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\right)dx\,dy.$$

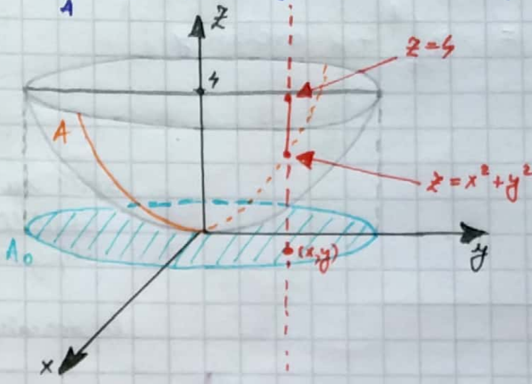
If $f=1$, $m(A) = \iint_{A_0} [\psi(x,y) - \varphi(x,y)]\,dx\,dy$

Example

Evaluate $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$, where $A = \{(x,y,z) \mid x^2+y^2 \le z \le 4\}$

Solution:

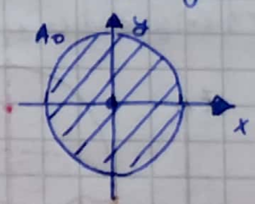
PARABOLOID ELLIPTIC



$z = x^2+y^2$

$$I = \iint_{A_0}\left(\int_{z=x^2+y^2}^{z=4}(x^2+y^2)\,dz\right)dx\,dy = \iint_{A_0}(x^2+y^2)\,z\Big|_{z=x^2+y^2}^{z=4}$$

$$= \iint_{A_0}(x^2+y^2)(4-x^2-y^2)\,dx\,dy$$

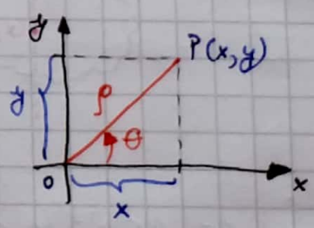


$A_0: \begin{cases} z=4 \\ x^2+y^2=z \end{cases}$ Raza discului = 2

In order to evaluate the double integral ⊛, we use polar coordinates (see below).

5. Change of variables in multiple integrals.

5.1 Polar coordinates



x, y = the Cartesian coordinates of P.

ρ, θ = the Polar coordinates of P.

$$\iint_A f(x,y)\,dx\,dy = \iint_B f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

$\cos\theta = \frac{x}{\rho}$; $\sin\theta = \frac{y}{\rho}$ $\Rightarrow \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

$$\iint_A f(x,y)\,dx\,dy = \iint_B f(\rho\cos\theta, \rho\sin\theta)\,\frac{D(x,y)}{D(\rho,\theta)}\,d\rho\,d\theta$$

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho$$

1, 2, 3 replaced by 1, 2, 3 respectively

$$\left[\iint_{\underbrace{A}_{1}} f\underbrace{(x,y)}_{2}\,\underbrace{dx\,dy}_{3} = \iint_{\underbrace{B}_{1}} f\underbrace{(\rho\cos\theta, \rho\sin\theta)}_{2}\cdot\underbrace{\rho\,d\rho\,d\theta}_{3}\right.$$

$$I = \iint_{A_0}(x^2+y^2)(4-x^2-y^2)\,dx\,dy\ ;\ \begin{matrix}\rho \in [0,2] \\ \theta \in [0,2\pi]\end{matrix} \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

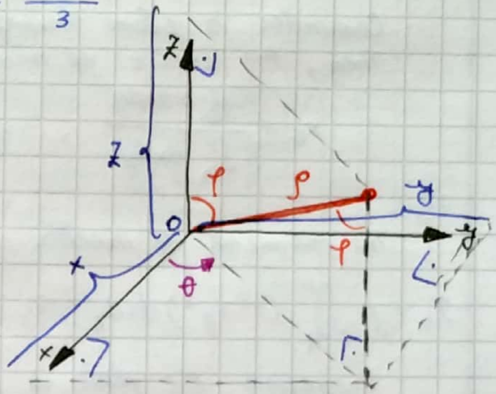
The set „B" is $[0,2]\times[0,2]$.

$$I=\iint_B(\rho^2\cos^2\theta+\rho^2\sin^2\theta)(4-\rho^2\cos^2\theta-\rho^2\sin^2\theta)\rho\,d\rho\,d\theta=$$

$$=\int_0^2\int_0^{2\pi}\rho^3(4-\rho^2)\,d\rho\,d\theta=\int_0^2(4\rho^3-\rho^5)\,d\rho\cdot\int_0^{2\pi}d\theta=$$

$$=2\pi\left(\rho^4-\frac{\rho^6}{6}\right)\Big|_0^2=2\pi\left(16-\frac{64}{6}\right)=\frac{32\pi}{3}$$

5.2 Spherical coordinates



x,y,z = the cartesian coordinates of P.
ρ,θ,φ = the spherical coordinates of P.

$$\left.\begin{array}{l}x=OP'\cdot\cos\theta\\ OP'=OP\cdot\sin\varphi=\rho\sin\varphi\end{array}\right\}\Rightarrow$$

$$\Rightarrow\boxed{\begin{array}{l}x=\rho\sin\varphi\cos\theta\\ y=\rho\sin\varphi\sin\theta\\ z=\rho\cos\varphi\end{array}}$$

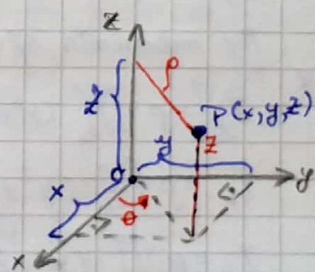
$$\frac{D(x,y,z)}{D(\rho,\varphi,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial\rho}&\frac{\partial x}{\partial\varphi}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\rho}&\frac{\partial y}{\partial\varphi}&\frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial\rho}&\frac{\partial z}{\partial\varphi}&\frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\sin\varphi\cos\theta&\rho\cos\varphi\cos\theta&-\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta&\rho\cos\varphi\sin\theta&\rho\sin\varphi\cos\theta\\ \cos\varphi&-\rho\sin\varphi&0\end{vmatrix}=$$

$$=\cos\varphi(\rho^2\sin\varphi\cos\varphi\cos^2\theta+\rho^2\sin\varphi\cos\varphi\sin^2\theta)+\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta+\rho\sin^2\theta\sin^2\varphi)=$$

$$=\rho^2\sin\varphi\cos^2\varphi+\rho^2\sin^3\varphi=\rho^2\sin\varphi(\cos^2\varphi+\sin^2\varphi)=\rho^2\sin\varphi$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iiint_B f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

5.3 Cylindrical coordinates



x,y,z = the cartesian coordinates of P.
ρ,θ,z = the cylindrical coordinates of P.

$$\boxed{\begin{array}{l}x=\rho\cos\theta\\ y=\rho\sin\theta\\ z=z\end{array}}$$

$$\frac{D(x,y,z)}{D(\rho,\theta,z)}=\begin{vmatrix}\frac{\partial x}{\partial\rho}&\frac{\partial x}{\partial\theta}&\frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial\rho}&\frac{\partial y}{\partial\theta}&\frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial\rho}&\frac{\partial z}{\partial\theta}&\frac{\partial z}{\partial z}\end{vmatrix}=\begin{vmatrix}\cos\theta&-\rho\sin\theta&0\\ \sin\theta&\rho\cos\theta&0\\ 0&0&1\end{vmatrix}=\rho$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iiint_B f(\rho\cos\theta,\rho\sin\theta,z)\,\rho\,d\rho\,d\theta\,dz$$

6. Applications of multiple integrals in physics (moments of inertia)

6.1 Centres of mass and moments of inertia for homogeneous plane plates.

Consider a plane having the shape of a set $A \subseteq \mathbb{R}^2$.
Then, the center of mass of this plate has the coordinates

$$x_G = \frac{\iint_A x\,dx\,dy}{\iint_A dx\,dy} \quad ; \quad y_G = \frac{\iint_A y\,dx\,dy}{\iint_A dx\,dy}$$

The moment of inertia of the plate w.r.t O (the origin) is

$$I_0 = \rho \iint_A (x^2+y^2)\,dx\,dy .$$

Supposing the plate is homogeneous and its density is equal to ρ.

6.2 Centres of mass and moments of inertia for homogeneous bodies.

Consider a homogeneous body having the shape of a set $A \subseteq \mathbb{R}^3$.
Then, its center of mass has the coordinates

$$x_G = \frac{\iiint_A x\,dx\,dy\,dz}{\iiint_A dx\,dy\,dz} \quad ; \quad y_G = \frac{\iiint_A y\,dx\,dy\,dz}{\iiint_A dx\,dy\,dz} \quad ;$$

$$z_G = \frac{\iiint_A z\,dx\,dy\,dz}{\iiint_A dx\,dy\,dz}$$

The moments of inertia of the body w.r.t. the three axis are:

$$I_z = \rho \iiint_A (x^2+y^2)\,dx\,dy\,dz$$

$$I_x = \rho \iiint_A (y^2+z^2)\,dx\,dy\,dz$$, where ρ denotes the density of the body.

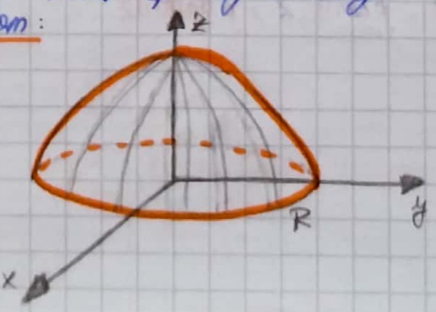
$$I_y = \rho \iiint_A (x^2+z^2)\,dx\,dy\,dz$$

$$I_0 = \rho \iiint_A (x^2+y^2+z^2)\,dx\,dy\,dz$$ ← the moment of inertia w.r.t. O

Application

Find the centre of mass of an homogeneous semisphere. Find also the moment of inertia of the semisphere w.r.t. its axis of symmetry.

Solution:



Due to symmetry, we have $x_G = y_G = 0$.

$$\iiint_A z\,dx\,dy\,dz = \ldots$$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\rho \in [0, R], \quad \varphi \in \left[0, \frac{\pi}{2}\right], \quad \theta \in [0, 2\pi]$$

$$\iiint_A z\,dx\,dy\,dz = \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho \cos\varphi \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^R \rho^3 d\rho \cdot \int_0^{\frac{\pi}{2}} \sin\varphi \cdot \cos\varphi \, d\varphi \cdot \int_0^{2\pi} d\theta = \frac{R^4}{4} \cdot \frac{\sin^2\varphi}{2}\Big|_0^{\frac{\pi}{2}} \cdot 2\pi = \frac{\pi R^4}{4}.$$

$$\iiint_A dx\,dy\,dz = \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^R \rho^2 d\rho \cdot \underbrace{\int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi}_{-\cos\varphi\big|_0^{\pi/2} = 1} \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi} = \frac{2\pi R^3}{3}.$$

$$z_G = \frac{\frac{\pi R^4}{4}}{\frac{2\pi R^3}{3}} = \frac{\pi R^4}{4} \cdot \frac{3}{2\pi R^3} = \frac{3R}{8}$$

$G\left(0, 0, \frac{3R}{8}\right)$ (centrul de greutate)

(moment of inertia : (suppose the density is equal to $\boxed{1}$)

$$I_z = \boxed{1} \cdot \iiint_A (x^2 + y^2)\,dx\,dy\,dz = \boxed{1} \cdot \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho^2 \sin^2\varphi \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^R \rho^4 d\rho \cdot \int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi \cdot \int_0^{2\pi} d\theta = 2\pi \frac{R^5}{5} \cdot \int_0^{\frac{\pi}{2}} (1 - \cos^2\varphi) \sin\varphi \, d\varphi =$$

$$= \frac{2\pi R^5}{5} \left(-\cos\varphi + \frac{\cos^3\varphi}{3}\right)\Big|_0^{\frac{\pi}{2}} = \frac{2\pi R^5}{5} \cdot \frac{2}{3} = \frac{4\pi R^5}{15}.$$

Chapter 4.

FUNCTIONS OF BOUNDED VARIATION.

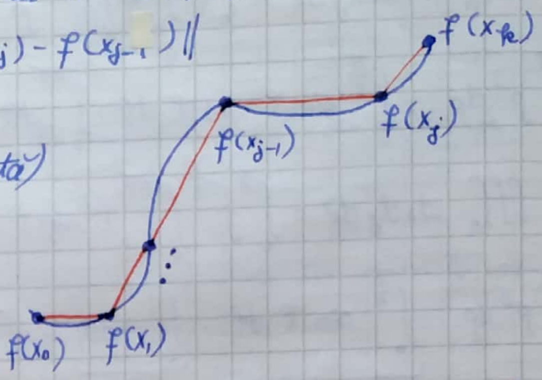
1. Definitions and notations.

1.1. Definition

Let $f:[a,b]\to\mathbb{R}^m$ and let $\Delta := (x_0, x_1, \dots, x_k) \in \text{Part}[a,b]$

Set $V(f,\Delta) := \sum_{j=1}^{k} \| f(x_j) - f(x_{j-1}) \|$

(În plan, cam așa arată)



$\Rightarrow V(f,\Delta) =$ length of the polygonal line joining the points $f(x_0), f(x_1), \dots, f(x_k)$

Define $\overset{b}{\underset{a}{V}}(f) := \sup_{\Delta\in \text{Part}[a,b]} V(f,\Delta)$ ← total variation of f on $[a,b]$

$\overset{b}{\underset{a}{V}}(f) \in [0,\infty]$

If $\overset{b}{\underset{a}{V}}(f) < \infty$ (if the total variation is finite), then the function f is said to be of bounded variation.

$BV([a,b],\mathbb{R}^m) = \{ f:[a,b]\to\mathbb{R}^m \mid \overset{b}{\underset{a}{V}}(f) < \infty \}$

When $m=1 \Rightarrow BV[a,b] = BV([a,b],\mathbb{R})$

1.2. Examples

a) Every monotone function $f:[a,b]\to\mathbb{R}$ is of bounded variation, and $\overset{b}{\underset{a}{V}}(f) = |f(b)-f(a)|$.

Indeed, for every $\Delta = (x_0, x_1, \dots, x_k) \in \text{Part}[a,b]$

$$V(f,\Delta) = \sum_{j=1}^{k} \underbrace{|f(x_j) - f(x_{j-1})|}_{\pm(f(x_j)-f(x_{j-1}))} = \pm \sum_{j=1}^{k} (f(x_j) - f(x_{j-1})) =$$

$= \pm(f(x_k) - f(x_0)) = \pm(f(b) - f(a))$.

$\Rightarrow \forall \Delta \in \text{Part}[a,b]: V(f,\Delta) = |f(b)-f(a)| \Rightarrow \overset{b}{\underset{a}{V}}(f) = |f(b)-f(a)|$

b) Suppose that $f:[a,b]\to\mathbb{R}^m$ is a Lipschitz function $\Rightarrow$

$\Rightarrow \exists\, \alpha>0$ s.t. $\|f(x)-f(x')\| \le \alpha |x-x'|$, $\forall x,x'\in[a,b]$.

Then, f is of bounded variation and $\overset{b}{\underset{a}{V}}(f) \le \alpha(b-a)$

Indeed, for each $\Delta = (x_0, x_1, \dots, x_k) \in \text{Part}[a,b]$ we have:

$$V(f,\Delta) = \sum_{j=1}^{k} \underbrace{\| f(x_j) - f(x_{j-1}) \|}_{\leq \alpha (x_j - x_{j-1})} \leq \alpha \sum_{j=1}^{k} (x_j - x_{j-1}) = \alpha \cdot (x_k - x_0) = \alpha \cdot (b-a)$$

$$\Rightarrow V(f,\Delta) \leq \alpha (b-a), \ \forall \Delta \in \text{Part}[a,b] \Rightarrow \bigvee_a^b (f) \leq \alpha (b-a)$$

2. Properties of the total variation

2.1. Theorem (monotonicity of variation w.r.t. partition)

Let $f:[a,b] \to \mathbb{R}^m$, and let $\Delta, \Delta' \in \text{Part}[a,b]$ with $\Delta \leq \Delta'$.
Then, $V(f,\Delta) \leq V(f,\Delta')$

Proof: It suffices to consider one of the case when Δ' is obtained from Δ, by adding exactly one point.

$\Delta = (x_0, x_1, \dots, x_k)$

$\Delta' = (x_0, x_1, \dots, x_{i-1}, \dots, x_k)$

$$V(f,\Delta') = \sum_{j=1}^{i-1} \| f(x_j) - f(x_{j-1}) \| + \overbrace{\| f(x') - f(x) \|}^{①} +$$

$$+ \overbrace{\| f(x_i) - f(x') \|}^{②} + \sum_{j=i+1}^{k} \| f(x_j) - f(x_{j-1}) \|$$

$$① + ② \geq \| f(x_i) - f(x_{i-1}) \|$$

$$\Rightarrow V(f,\Delta') \geq \sum_{j=1}^{k} \| f(x_j) - f(x_{j-1}) \| = V(f,\Delta)$$

2.2. Theorem

If $f, g : [a,b] \to \mathbb{R}^m$ are functions of bounded variation SCHAUM SERIES and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is of bounded variation, and $\bigvee_a^b (\alpha f + \beta g) \leq |\alpha| \cdot \bigvee_a^b (f) + |\beta| \cdot \bigvee_a^b (g)$.

Without proof.

2.3. Theorem (additivity of the total variation w.r.t. the interval)

Let $f:[a,b] \to \mathbb{R}^m$ and let $c \in (a,b)$.

Then $\bigvee_a^b (f) = \bigvee_a^c (f) + \bigvee_c^b (f)$.

Proof: We first prove that ① $\bigvee_a^b (f) \leq \bigvee_a^c (f) + \bigvee_c^b (f)$

Let $\Delta \in \text{Part}[a,b]$ be arbitrarily chosen, let $\bar{\Delta} := \Delta \cup (a,c,b)$

$\Rightarrow \Delta \leq \bar{\Delta}$ and $\exists \Delta_1 \in \text{Part}[a,c]$, $\exists \Delta_2 \in \text{Part}[c,b]$ s.t. $\bar{\Delta} = \Delta_1 \cup \Delta_2$.

$$\Rightarrow V(f,\Delta) \overset{\text{Th } 2.1}{\leq} V(f,\bar{\Delta}) = \underbrace{V(f,\Delta_1)}_{\leq \bigvee_a^c (f)} + \underbrace{V(f,\Delta_2)}_{\leq \bigvee_c^b (f)} \leq \bigvee_a^c (f) + \bigvee_c^b (f) \Rightarrow$$

$$\Rightarrow \forall \Delta \in \text{Part}[a,b]: V(f,\Delta) \leq \bigvee_a^c (f) + \bigvee_c^b (f) \Big|_{\Delta \in \text{Part}[a,b]}^{\sup}$$

$\Rightarrow$ ① holds

Now, we prove the opposite inequality:

② $\bigvee_a^c(f) + \bigvee_c^b(f) \le \bigvee_a^b(f)$

$\forall \Delta_1 \in Part[a,b]$ and $\forall \Delta_2 \in Part[c,b]$ we have:

$\underbrace{V(f,\Delta_1)+V(f,\Delta_2)} = V(\underbrace{f,\Delta_1 \vee \Delta_2}_{\in Part[a,b]}) \le \underbrace{\bigvee_a^b(f)} \Big|_{\Delta_1 \in Part[a,c]}^{sup}$

$\bigvee_a^c(f) + V(f,\Delta_2) \le \bigvee_a^b(f) \Big|_{\Delta_2 \in Part[c,b]}^{sup}$

$\bigvee_a^c(f) + \bigvee_c^b(f) \le \bigvee_a^b(f)$

By ① and ② $\Rightarrow \bigvee_a^b(f) = \bigvee_a^c(f) + \bigvee_c^b(f)$

2.4 Theorem (the Jordan decomposition theorem)

If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then $\exists f_1, f_2 :[a,b]\to\mathbb{R}$ nondecreasing s.t. $f = f_1 \cdot f_2$

Proof Let $f_1:[a,b]\to\mathbb{R}$, $f_1(x) := \bigvee_a^x(f)$.

Let $x_1, x_2 \in [a,b]$ s.t. $x_1 < x_2$. By T2.3 $\Rightarrow$

$\Rightarrow \underbrace{\bigvee_a^{x_2}(f)}_{f_1(x_2)} = \underbrace{\bigvee_a^{x_1}(f)}_{f_1(x_1)} + \bigvee_{x_1}^{x_2}(f) \Rightarrow f_1(x_2) - f_1(x_1) = \bigvee_{x_1}^{x_2}(f) \ge 0 \Rightarrow$

$\Rightarrow f_1(x_1) \le f_1(x_2) \Rightarrow f_1$ is nondecreasing

Let $f_2:[a,b]\to\mathbb{R}$, $f_2(x) = f_1(x) - f(x) = \bigvee_a^x(f) - f(x)$

Then $f = f_1 - f_2$.

Is f_2 nondecreasing? We have to prove it is.

Let $x_1, x_2 \in [a,b]$ s.t. $x_1 < x_2$.

$f_2(x_2) - f_2(x_1) = \left[\bigvee_a^{x_2}(f) - f(x_2)\right] - \left[\bigvee_a^{x_1}(f) - f(x_1)\right] =$

$= \bigvee_{x_1}^{x_2}(f) - \left[f(x_2) - f(x_1)\right] \ge 0$ which is true, because

$\bigvee_{x_1}^{x_2}(f) \ge |f(x_2) - f(x_1)| \ge f(x_2) - f(x_1)$

$\hookrightarrow (x_1, x_2) \in Part[x_1, x_2]$.

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2.5 **Theorem**

Let $f:[a,b]\to \mathbb{R}^m$ be a function of the class C^1_f on $[a,b]$.

Then:

1° f is of bounded variation

2° The function $v_f:[a,b]\to\mathbb{R}$, defined by $v_f(x):=\overset{x}{\underset{a}{V}}(f)$ is differentiable on $[a,b]$ and $v_f'(x)=\|f'(x)\|,\ \forall x\in[a,b]$.

3° One has $\boxed{\overset{b}{\underset{a}{V}}(f)=\int_a^b \|f'(x)\|\,dx}$

Proof: 1° Since f is a C^1 function $\Rightarrow$ f is differentiable and $f':[a,b]\to\mathbb{R}^m$ is continuous $\xrightarrow{W.Th}$ $\exists\,\alpha>0$ s.t. $\|f'(x)\|\le\alpha$, $\forall x\in[a,b]$. Let $x,x'\in[a,b]$, $x<x'$. By the M.V. Theorem, for a vector function of real variable $\Rightarrow$ $\exists$ $c\in(x,x')$ s.t.

$$\|f(x')-f(x)\|\le \underbrace{\|f'(c)\|}_{\le\alpha}(x'-x)\le\alpha(x'-x)$$

$\Rightarrow$ f is a Lipschitz function $\Rightarrow$ f is of bounded variation.

2° ? v_f is diff. on $[a,b]$, $v_f'(x)=\|f'(x)\|$, $\forall x\in[a,b]$.

$\Leftrightarrow$ (i) v_f is right diff. on $[a,b)$ and $(v_f)_r'(x)=\|f'(x)\|$ $\forall x\in[a,b]$.

(ii) v_f is left diff. on $(a,b]$ and $(v_f)_l'(x)=\|f'(x)\|$ $\forall x\in(a,b]$

We only prove (i). Let us take $x_0\in[a,b)$? v_f is right diff. at x_0 and $(v_f)_r'(x_0)=\|f(x_0)\|$

$$\Leftrightarrow\ \exists \lim_{x\searrow x_0}\frac{v_f(x)-v_f(x_0)}{x-x_0}=\|f'(x_0)\|\quad (*)$$

Let $x\in(x_0,b)$ arbitrarily chosen.

By the Weierstrass theorem $\Rightarrow$ $\exists\ c_x\in[x_0,x]$ s.t. $\|f'(c_x)\|=$

$$\|f'(c_x)\|=\max_{y\in[x_0,x]}\|f'(y)\|.$$

Let $\Delta=(x_0,x_1,\dots,x_k)\in \text{Part}[x_0,x]$:

$$V(f,\Delta)=\sum_{j=1}^{k}\|f(x_j)-f(x_{j-1})\|$$

By the MV Theorem for vector functions of a real variable

$\Rightarrow$ $\forall j\in\{1,\dots,k\}$, $\exists$ $c_j\in(x_{j-1},x_j)$ s.t.

$$\|f(x_j)-f(x_{j-1})\|\le\underbrace{\|f'(c_j)\|}_{\le\|f'(c_x)\|}(x_j-x_{j-1})\le\|f'(c_x)\|(x_j-x_{j-1})$$

$$\Rightarrow V(f,\Delta)\le\|f'(c_x)\|\underbrace{\sum_{j=1}^{k}(x_j-x_{j-1})}_{=x-x_0}=\|f'(c_x)\|(x-x_0)$$

In conclusion: $V(f,\Delta)\le\|f'(c_x)\|(x-x_0)$, $\forall\,\Delta\in\text{Part}[x_0,x]$ $\Rightarrow$

$\xrightarrow{\sup_{\Delta\in\text{Part}[x_0,x]}}$ $\sup_{\Delta\in\text{Part}[a,b]}V(f,\Delta)=\overset{x}{\underset{x_0}{V}}(f)\le\|f'(c_x)\|(x-x_0)$ $\Rightarrow$ $v_f(x)-v_f(x_0)\le\|f'(c_x)\|(x-x_0)\Big|\cdot\frac{1}{x-x_0}$

But $\overset{x}{\underset{x_0}{V}}(f)=\overset{x}{\underset{a}{V}}(f)-\overset{x_0}{\underset{a}{V}}(f)=v_f(x)-v_f(x_0)$

$$\frac{v_f(x)-v_f(x_0)}{x-x_0} \le \|f'(c_x)\| \quad ①$$

On the other hand, we have:

$$v_f(x)-v_f(x_0) = \bigvee_{x_0}^{x}(f) \underset{(x_0,x)\,\in\, \text{Part}[x_0,x]}{\ge} \|f(x)-f(x_0)\| \;\Big|\cdot \frac{1}{x-x_0}$$

$$\Rightarrow \frac{v_f(x)-v_f(x_0)}{x-x_0} \ge \frac{1}{x-x_0}\|f(x)-f(x_0)\| = \left\|\frac{1}{x-x_0}[f(x)-f(x_0)]\right\| \quad ②$$

$$①\text{ and }② \Rightarrow \left\|\frac{1}{x-x_0}[f(x)-f(x_0)]\right\| \le \frac{v_f(x)-v_f(x_0)}{x-x_0} \le \|f'(c_x)\|$$

sandwich theorem

$x \searrow x_0 \Rightarrow c_x \to x_0$

$$\|f'(x_0)\|$$

It follows that ⊛ holds.

3° We have: $\bigvee_a^b (f) = v_f(b) \overset{v_f(a)=0}{=} v_f(b) - v_f(a) = v_f(x)\Big|_a^b =$

$$= \int_a^b v_f'(x)\,dx = \int_a^b \|f'(x)\|\,dx.$$

Proof now complete.

Chapter 5.

LINE INTEGRALS.

1 Parametrized paths and curves

1.1. Definition

A continuous function $\gamma : [a,b] \to \mathbb{R}^m$ is called a parametrized path in $\mathbb{R}^m$.

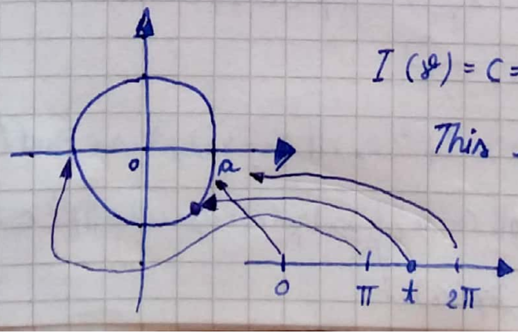
The set

$$\boxed{I(\gamma) := \{\gamma(t) \mid t \in [a,b]\} \subseteq \mathbb{R}^m}$$

↳ the image / geometrical support of the parametrized path γ.

1.2 Examples

a) $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(t) = (a\cos t, a\sin t)$ is a parametr path in $\mathbb{R}^2$



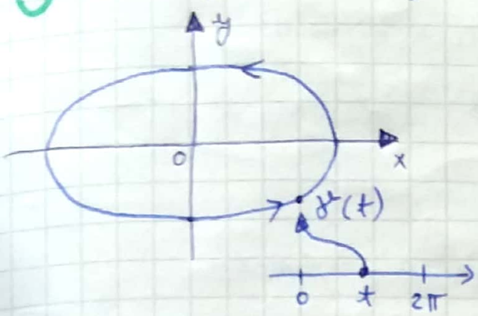
$$I(\gamma) = C = \{(x,y) \mid x^2+y^2 = a^2\}$$

This is a circle.

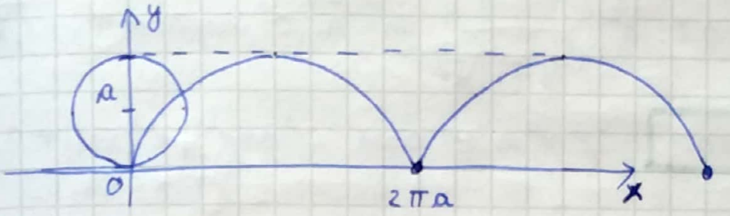
(b) $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(t) = (a\cos t, b\sin t)$, $a, b > 0$.

$I(\gamma) = E = \left\{(x,y) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$

This is an ellipse.

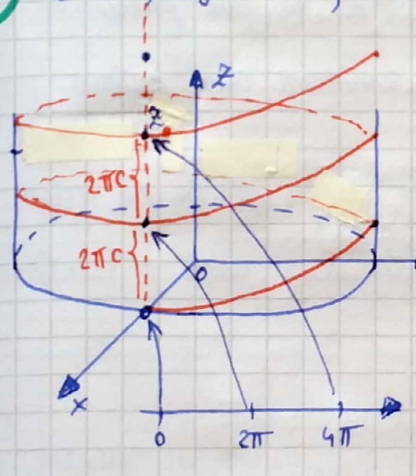


(c) The cycloide (cicloida ; ro.)



$\gamma(t) = \left(a(t - \sin t), a(1 - \cos t)\right)$, $t \in [0, 2\pi]$.

(d) $\gamma : [0, 4\pi] \to \mathbb{R}^3$, $\gamma(t) = (a\cos t, a\sin t, ct)$



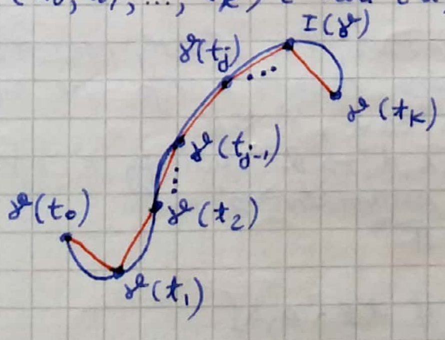
1.2 Definition (arclength of a parametrized path)

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a parametrized path.

Define $\boxed{\ell(\gamma) := \bigvee_a^b (\gamma) = \sup_{\Delta \in \text{Part}[a,b]} V(\gamma, \Delta)}$

→ the arclength of γ

$\Delta = (t_0, t_1, \dots, t_k) \in \text{Part}[a,b]$



$\boxed{V(\gamma, \Delta) = \sum_{j=1}^{k} \| \gamma(t_j) - \gamma(t_{j-1}) \|}$

→ the total length of the polygonal line joining the points $\gamma(t_0), \gamma(t_1), \dots, \gamma(t_k)$

In general, $\ell(\gamma) \in [0, \infty]$.

If $\ell(\gamma) < \infty$, then the parametrized path γ is said to be RECTIFIABLE.

1.3 Definition (special parametrized paths)

A parametrized path $\gamma : [a,b] \to \mathbb{R}$ is called:

- **closed** if $\gamma(a) = \gamma(b)$ (starting & ending points coincide)
- **simple** if $\gamma|_{(a,b)}$ is injective
- **of the class C^1** if γ is a C^1 function
- **piercewise C^1** if $\exists \Delta = (a_0, a_1, \dots, a_k) \in \text{Part}[a,b]$ s.t. $\gamma|_{[a_{j-1}, a_j]}$ is a C^1 function $\forall j = \overline{1,k}$
- **smooth** if γ is of class C^1 and $\|\gamma'(t)\| \neq 0$, $\forall t \in [a,b]$

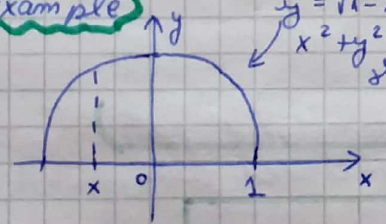
1.4 Definition

Let $\gamma : [a,b] \to \mathbb{R}^n$, $\rho : [c,d] \to \mathbb{R}^n$ be two paramtr. paths.

We say that γ and ρ have common end points if $\gamma(a) = \rho(a)$ and $\gamma(b) = \rho(b)$.

The paramtr. paths γ and ρ are said to be equiv. if $\exists$ a homomorphism $\varphi : [a,b] \to [c,d]$ (homomorphism: ρ is bijective, continuous, with ρ^{-1} continuous) s.t. $\gamma = \varphi \circ \varphi$.

Example



$y = \sqrt{1-x^2}$

$x^2 + y^2 = 1$

$\gamma : [0,\pi] \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$

$\rho : [-1,1] \to \mathbb{R}^2$, $\rho(x) = (x, \sqrt{1-x^2})$

γ and ρ are equivalent via $\varphi : [0,\pi] \to [-1,1]$, $\varphi(t) = \cos t$

Notation:

We write $\gamma \sim \rho$ to express that γ and ρ are equivalent

If $\gamma \sim \rho \Rightarrow \exists \varphi : [a,b] \to [c,d]$ homomorphism s.t. $\gamma = \rho \circ \varphi$.

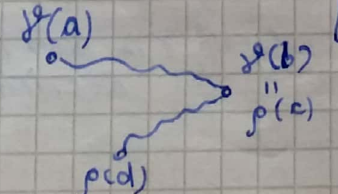
Being injective and continuous, φ is strictly monotonic.

If φ is strictly increasing, then γ and ρ are said to be directly equivalent. In this case, $\gamma \overset{\sim}{\sim} \rho$.

If $\gamma : [a,b] \to \mathbb{R}^n$, $\rho : [c,d] \to \mathbb{R}^n$ satisfy $\gamma(b) = \rho(c)$ then, we can define a new parametrized path

$\gamma \vee \rho : [a, b+d-c] \to \mathbb{R}^n$

$$(\gamma \vee \rho)(t) = \begin{cases} \gamma(t), & \text{if } t \in [a,b] \\ \rho(t-b+c), & \text{if } t \in [b, b+d-c] \end{cases}$$



1.5 Definition (curves and oriented curves)

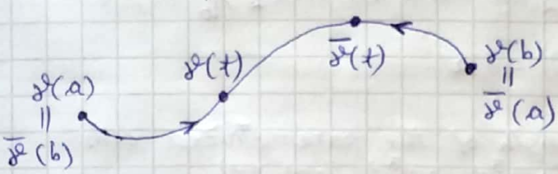
Let $\mathcal{D}^m$ = the set of all parametrized paths in $\mathbb{R}^m$.
It can be proved that the binary relations „$\sim$" and „$\approx$" are, in fact, equivalence relations on $\mathcal{D}^m$.

An equivalence class on $\mathcal{D}^m$ w.r.t. the first relation, „$\sim$" is called curve (in the sense of Jordan's) in $\mathbb{R}^m$.
An equivalence class on $\mathcal{D}^m$ w.r.t. the second relation „$\approx$" is called oriented curve (in the sense of Jordan's) in $\mathbb{R}^m$.

1.6 Definition

If $\gamma : [a,b] \to \mathbb{R}^m$ is a paramtr. path, then

$\overline{\gamma} : [a,b] \to \mathbb{R}^m$, $\overline{\gamma}(t) = \gamma(a+b-t)$ is also a paramtr. path, called the opposite of γ.



$\gamma \sim \overline{\gamma}$, but $\gamma \not\approx \overline{\gamma}$

1.7 Proposition

Let $\gamma : [a,b] \to \mathbb{R}^m$ and $\rho : [c,d] \to \mathbb{R}^m$ be equiv. paramtr. paths.

Then:

1°. if γ is closed $\Rightarrow$ ρ is closed
2°. if γ is simple $\Rightarrow$ ρ is simple
3°. if γ is rectifiable $\Rightarrow$ ρ is rectifiable, and $\ell(\gamma) = \ell(\rho)$
4°. One has $I(\gamma) = I(\rho)$

Without proof

1.8 Definition

Let Γ be a curve in $\mathbb{R}^m$. The curve Γ is said to be:

- closed, if $\exists$ a closed parametrized path $\gamma \in \Gamma$
- simple, if $\exists$ a simple parametrized path $\gamma \in \Gamma$
- rectifiable, if $\exists$ rectifiable parametrized path $\gamma \in \Gamma$
  $\ell(\Gamma) = \ell(\gamma)$

The image of the curve Γ is defined as $I(\Gamma) = I(\gamma)$, where $\gamma \in \Gamma$

1.9 Theorem.

If $\gamma : [a,b] \to \mathbb{R}^m$ is a piercewise C^1 parametrized path, then γ is rectifiable and

$$\boxed{\ell(\gamma) = \int_a^b \|\gamma'(t)\| \, dt}$$

Example

1. Determine the arclength of a branch of the cycloide.
↳ buclă

Solution:

$\gamma(t) = (a(t - \sin t), a(1 - \cos t)), t \in [0, 2\pi]$

$\ell(\gamma) = \int_0^{2\pi} \|\gamma'(t)\| dt$

$\gamma'(t) = (a(1 - \cos t), a \sin t)$

$\|\gamma'(t)\|^2 = a^2(1-\cos t)^2 + a^2 \sin^2 t$

$= a^2(1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1})$

$= 2a^2\underbrace{(1-\cos t)}_{2\sin^2\frac{t}{2}} = 4a^2 \sin^2\frac{t}{2}$

$\|\gamma'(t)\| = 2a \sin\frac{t}{2}$

$\ell(\gamma) = \int_0^{2\pi} 2a \sin\frac{t}{2}\, dt = 2a\left(2 - \cos\frac{t}{2}\right)\Big|_0^{2\pi} =$

$= 2a(2-(2)) = 8a$

2. Determine the arclength of the parametrized path $\gamma : [0, 2\pi] \to R^3$, $\gamma(t) = (\cos^3 t, \sin^3 t, \cos 2t)$

Solution: $\ell(\gamma) = \int_0^{2\pi} \|\gamma'(t)\| dt$

$\gamma'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t, -2\sin 2t)$

$\|\gamma'(t)\|^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 4\sin^2 2t$

$= 9\sin^2 t \cos^2 t + 4 \cdot 4\sin^2 t \cos^2 t =$

$= 25 \sin^2 t \cdot \cos^2 t$

$\|\gamma'(t)\| = 5|\sin t \cos t|$

$\ell(\gamma) = \int_0^{2\pi} 5|\sin t \cos t| dt$

$f(t) = |\sin t \cos t|$ is periodic with $T = \frac{\pi}{2}$

$f\left(t + \frac{\pi}{2}\right) = \left|\sin\left(t+\frac{\pi}{2}\right)\cos\left(t + \frac{\pi}{2}\right)\right| = |\cos t \cdot (-\sin t)| = f(t)$

$\ell(\gamma) = 4\int_0^{\frac{\pi}{2}} 5 \sin t \cos t\, dt = 20 \sin^2\frac{t}{2}\Big|_0^{\frac{\pi}{2}} = 10$

2. Integration of scalar functions along a parametrized path (line integrals of the first kind)

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a piecewise C^1 parametrized path, and let $f : I(\gamma) \to \mathbb{R}$ be a continuous function. Then, the integral of f along γ is defined by:

$$\boxed{\int_\gamma f(x)\, ds = \int_a^b f(\gamma(t)) \cdot \|\gamma'(t)\|\, dt}$$

Example

Evaluate $I = \int_\gamma y^2 ds$, where $\gamma : [0, 2\pi] \to \mathbb{R}^2$,

$\gamma(t) = (\underbrace{a(t - \sin t)}_{x(t)}, \underbrace{a(1-\cos t)}_{y(t)}), a > 0$

Solution: $I = \int_0^{2\pi} (a(1-\cos t))^2 \cdot \underbrace{\|\gamma'(t)\|}_{2a\sin\frac{t}{2}} dt =$

$= \int_0^{2\pi} a^2(1-\cos^2 t)^2 \cdot a \cdot \sin\frac{t}{2} = 2a^3 \int_0^{2\pi} 4\sin^4\frac{t}{2} \cdot \sin\frac{t}{2}\, dt =$

$= 2a^3 \int_1^{-1} 4(1-y^2)^2 = 16a^3\left(y - \frac{2y^3}{3} + \frac{y^5}{5}\right)\Big|_{-1}^{1} = 16a^3\left(2-\frac{4}{3}+\frac{2}{5}\right) = \ldots$

$\cos\frac{t}{2} = y$

$-\sin\frac{t}{2}\cdot\frac{1}{2}dt = dy$

$\sin\frac{t}{2}\, dt = -2dy$

Physical meaning of the line integral of the first kind:

Consider a material string (fir, ro.) having the form $I(\gamma)$, where $\gamma: [a,b] \to \mathbb{R}^3$ is a parametrized path.

The string is not homogeneous, but its linear density (g/cm) $f(x,y,z)$ is known at each point $(x,y,z) \in I(\gamma)$.

Then, the mass of the string is

$$\boxed{m = \int_\gamma f(x,y,z)\, ds}$$

COURSE 11 - week 12.

3. Integration of vector fields along a parametrized path (line integrals of the second kind)

3.1. Definition

Let $A \subseteq \mathbb{R}^n$. A vector field on A is any continuous function $\vec{F}: A \to \mathbb{R}^n$.

If, in addition, A is open, then $\vec{F}$ is called a gradient field if $\exists\, U: A \to \mathbb{R}$ s.t. $\nabla U = \vec{F}$.

In this case, U is called the potential of the vector field $\vec{F}$.

$$\vec{F} = (F_1, \ldots, F_n)$$

$$\nabla U = \vec{F} \iff \frac{\partial U}{\partial x_1} = F, \ldots, \frac{\partial U}{\partial x_n} = F_n$$

3.2. Definition

Let $A \subseteq \mathbb{R}^n$, let $\vec{F}: A \to \mathbb{R}^n$ be a vector field on A, let $\gamma: [a,b] \to A$ be a piecewise C_1 parametrized path.

Then, the integral of F vector along γ is defined as the real number

$\sum_{i=1}^{n} \int_a^b (F_i \circ \gamma)(t) \cdot \gamma_i'(t)\, dt$ and it will be denoted by $\int_\gamma \vec{F} \cdot d\vec{s}$

$$\boxed{\int_\gamma \vec{F} \cdot d\vec{s} = \sum_{i=1}^{n} \int_a^b F_i(\gamma(t)) \cdot \gamma_i'(t)\, dt}$$

Here, $\vec{F} = (F_1, \ldots, F_n)$; $\gamma = (\gamma_1, \ldots, \gamma_n)$

The differential form notation:

$$\int_\gamma \vec{F}\cdot d\vec{s} = \int_\gamma F_1\, dx_1 + \dots + F_n\, dx_n$$

Examples

1 Let $a>0$, let $\gamma : [0,2\pi] \to \mathbb{R}^2$,

$$\gamma(t) = (a(t-\sin t),\ a(1-\cos t)),$$

and let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\vec{F}(x,y) = (2a-y)\vec{i} + x\vec{j}$$

Evaluate $\int_\gamma \vec{F}\cdot d\vec{s}$

Solution: $\displaystyle\int_\gamma \vec{F}\cdot d\vec{s} = \int_0^{2\pi} (2a - a(1-\cos t))\cdot(a(t-\sin t))'\,dt +$

$$+\int_0^{2\pi} a(t-\sin t)\cdot(a(1-\cos t))'\,dt =$$

$$= \int_0^{2\pi} (a+a\cos t)\cdot a(1-\cos t)\,dt + \int_0^{2\pi} a(t-\sin t)\cdot a\sin t\,dt =$$

$$= a^2\int_0^{2\pi} \overbrace{(1-\cos^2 t)}^{\sin^2 t}\,dt + a^2\int_0^{2\pi} (t\sin t - \sin^2 t)\,dt =$$

$$= a^2\int_0^{2\pi} t\sin t\,dt = a^2\int_0^{2\pi} t(-\cos t)'\,dt =$$

$$= a^2 t(-\cos t)\Big|_0^{2\pi} + a^2\underbrace{\int_0^{2\pi}\cos t\,dt}_{-\sin t\big|_0^{2\pi}=0} = \boxed{-2\pi a^2}$$

2 Evaluate $I = \int_\gamma y\,dx - x\,dy + (x^2+y^2+z^2)\,dz$,

where: $\gamma : [0,\pi] \to \mathbb{R}^3$, $\gamma(t) = (\sin t - t\cos t,\ \cos t + t\sin t,\ 1+t)$

Solution:

$$I = \int_0^{\pi} (\cos t + t\sin t)\cdot(\sin t - t\cos t)'\,dt - \int_0^{\pi} (\sin t - t\cos t)(\cos t + t\sin t)'\,dt +$$

$$+\int_0^{\pi} (\sin^2 t - 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t + 2t\sin t\cos t + t^2\sin^2 t + 1 + 2t + t^2)$$

$$\cdot (1+t)'\,dt.$$

$$I = \int_0^{\pi} (\cos t + t\sin t)(\cos t - \cos t + t\sin t)\,dt -$$

$$-\int_0^{\pi} (\sin t - t\cos t)(-\sin t + \sin t + t\cos t)\,dt + \int_0^{\pi} (2 + 2t^2 + 2t)\,dt$$

$$I = \int_0^{\pi} (3t^2 + 2t + 2)\,dt = (t^3 + t^2 + 2t)\Big|_0^{\pi} = \boxed{\pi^3 + \pi^2 + 2\pi}$$

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3.3 The physical meaning of line integrals of the second kind

Consider a vector field $\vec{F} = (P, Q, R) : \mathbb{R}^3 \to \mathbb{R}^3$.
$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ and consider the parametrized path $\gamma : [a,b] \to \mathbb{R}^3$, $\gamma(t) = (x(t), y(t), z(t))$.

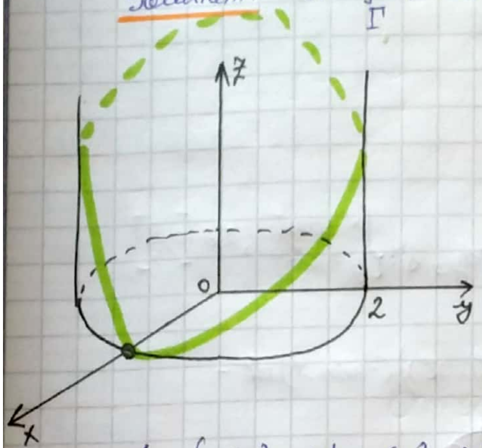
Then, $\displaystyle\int_\gamma \vec{F} \cdot d\vec{s} = \int_\gamma P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$,

represents the work done by the vector field when a particle is moving on a trajectory having the shape $I(\gamma)$.

Example

Determine the work done by the vector field $\vec{F}(x,y,z) = (x^2+y)\vec{i} + (y^2+z)\vec{j} + (z^2+x)\vec{k}$ along the oriented curve Γ, having the image $I(\Gamma) = \{(x,y,z) \mid x^2+y^2=4,\ x+z=2\}$.

Solution: $\displaystyle W = \int_\Gamma \vec{F} \cdot d\vec{s} = \int_\gamma \vec{F} \cdot d\vec{s}$, where $\gamma \in \Gamma$



A parametrization of γ can be obtained by using cylinder coordinates

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases} \qquad x^2+y^2=4 \Leftrightarrow \rho^2 = 4 \Rightarrow \rho = 2.$$

$$\gamma : \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \\ z = 2 - x = 2 - \cos\theta \end{cases}, \quad \theta \in [0, 2\pi]$$

$$W = \int_\gamma (x^2+y)\,dx + (y^2+z)\,dy + (z^2+x)\,dz =$$

$$= \int_0^{2\pi} (4\cos^2\theta + 2\sin\theta)(-2\sin\theta)\,d\theta + \int_0^{2\pi} (4\sin^2\theta + 2 - 2\cos\theta) \cdot 2\cos\theta\,d\theta +$$

$$+ \int_0^{2\pi} (4 - 8\cos\theta + 4\cos^2\theta + 2\cos\theta) \cdot 2\sin\theta\,d\theta =$$

$$= \cancel{-8\int_0^{2\pi} \cos^2\theta \sin\theta\,d\theta} - 4\int_0^{2\pi} \sin^2\theta\,d\theta + \overbrace{4\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta}^{= \frac{\sin^3\theta}{3}\Big|_0^{2\pi} = 0} +$$

$$+ \underbrace{4\int_0^{2\pi} \cos\theta\,d\theta}_{\sin\theta|_0^{2\pi} = 0} - 4\int_0^{2\pi} \cos^2\theta\,d\theta + \underbrace{8\int_0^{2\pi} \sin\theta\,d\theta}_{= -\cos\theta|_0^{2\pi} = 0} - 12\int_0^{2\pi} \sin\theta\cos\theta\,d\theta +$$

$$+ \cancel{8\int_0^{2\pi} \cos^2\theta\sin\theta\,d\theta}$$

$$\Rightarrow W = -4\int_0^{2\pi} d\theta = \boxed{-8\pi}$$

3.4 Integration of gradient fields - the Liebniz-Newton formula

Let $A \subseteq \mathbb{R}^m$ be an open set, let $\vec{F} : A \to \mathbb{R}^m$, $\vec{F} = (F_1, \dots, F_m)$ be a gradient field on A and let $U : A \to \mathbb{R}$ be a potential of $\vec{F}$ (i.e. $\nabla U = \vec{F} \Leftrightarrow \frac{\partial U}{\partial x_1} = F_1, \dots, \frac{\partial U}{\partial x_m} = F_m$).

Then, for every piecewise C_1 parametrized path $\gamma : [a,b] \to A$, one has:

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_\gamma F_1\,dx_1 + \dots + F_m x_m = \boxed{U(\gamma(b)) - U(\gamma(a)) \overset{not.}{=} U\Big|_{\gamma(a)}^{\gamma(b)}}$$

Proof: We have $\int_\gamma \vec{F}\cdot d\vec{r} = \sum_{i=1}^{m} \int_a^b F_i(\gamma(t))\cdot \gamma_i'(t)\,dt =$

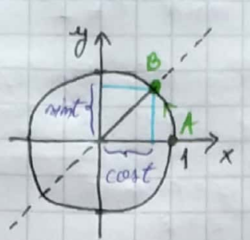
$$= \sum_{i=1}^{m}\int_a^b \frac{\partial U}{\partial x_i}(\gamma(t))\cdot \gamma_i'(t)\,dt = \int_a^b \underbrace{\sum_{i=1}^{m}\left(\frac{\partial U}{\partial x_i}\circ\gamma\right)(t)\cdot \frac{d\gamma_i}{dt}(t)}_{=\frac{d}{dt}(U\circ\gamma)(t)}\,dt =$$

$$= \int_a^b (U\circ\gamma)'(t)\,dt = (U\circ\gamma)\Big|_a^b = U(\gamma(b)) - U(\gamma(a))$$

Example

Evaluate $I = \int_\Gamma (2x\cos(xy) - x^2 y\sin(xy))dx - x^3\sin(xy)\,dy$, where Γ is the oriented curve whose image is the circle about the origin with radius 1 lying between points $A(1,0)$ and $B\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$ oriented counterclockwise.

Solution:



$$I = \int_\gamma (2x\cos(xy) - x^2 y\sin(xy))dx - x^3\sin(xy)\,dy,\quad \gamma: \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ t\in\left[0,\frac{\pi}{4}\right]$$

$$I = \int_0^{\frac{\pi}{4}} (2\cos t\cos(\sin t\cos t) - \cos^2 t\sin t\,\sin(\sin t\cos t))(-\sin t)\,dt -$$

$$- \int_0^{\frac{\pi}{4}} \cos^3 t\cdot\sin(\sin t\cos t)\cdot\cos t\,dt.$$

"HORROR integral"
(actual quotation of Mr. Trif)

? Is the vector field $\vec{F}(x,y) = (2x\cos(x,y) - x^2 y\sin(xy))\vec{i} - x^3\sin(xy)\vec{j}$ a gradient field?

? $\exists\, U:\mathbb{R}^2\to\mathbb{R}$ s.t. $\begin{cases} \frac{\partial U}{\partial x}(x,y) = 2x\cos(xy) - x^2 y\sin(xy) \\ \frac{\partial U}{\partial y}(x,y) = -x^3\sin(xy) \end{cases}$

By using the guessing method $\Rightarrow U(x,y) = x^2\cos(xy)$
By the Liebniz-Newton formula

$$I = U\Big|_A^B = U(B) - U(A) = U\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right) - U(1,0)$$

$$\boxed{I = \frac{1}{2}\cos\frac{1}{2} - 1}$$

Second method (not guessing, but actual systematic method)

$$\frac{\partial U}{\partial y}(x,y) = -x^3\sin(xy) \Rightarrow U(x,y) = \int -x^3\sin(xy)\,dy =$$

$$= x^2\int -\sin(xy)\cdot x\,dy = x^2\cos(xy) + \varphi(x) \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial U}{\partial x}(x,y) = 2x\cos(x,y) - x^2 y\sin(xy) + \varphi'(x) \\ &\text{But } \frac{\partial U}{\partial x}(x,y) = 2x\cos(xy) - x^2 y\sin(xy) \end{aligned}\right\} \Rightarrow \varphi'(x) = 0 \Rightarrow \varphi(x) = c\ (\text{const.})$$

$$\boxed{U(x,y) = x^2\cos(xy) + c}\quad (\text{metoda sistematica})$$

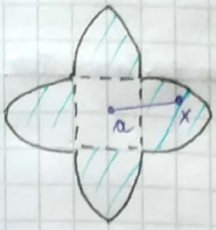
TEORIE PENTRU ANALIZĂ 2, PARTEA A DOUA:

1. Fubini's theorem for generalized rectangles
2. Additivity of the total variation w.r.t. the interval
3. The formula of the total variation for C_1 functions (theorem 2.5)
4. Grin's formula (mai târziu sau în ultimul curs).

3.5. Definition

A set $A \subseteq \mathbb{R}^n$ is said to be star-shaped (stelată) w.r.t. some of its points a, if $\forall x \in A, \forall t \in [0,1]: (1-t)a + tx \in A$

If A is convex $\Rightarrow$ A is starshaped w.r.t. every point $a \in A$.



star-shaped but non-convex set

(stelată) $\Leftrightarrow$ $\exists$ un punct din figură de unde vedem orice punct al mulțimii)

3.6. How to recognize a gradient field (H. Poincare's LEMMA)

Let $A \subseteq \mathbb{R}^m$ be an open set starshaped set w.r.t. some point $a \in A$ and let $\vec{F} = (F_1, \dots, F_m): A \to \mathbb{R}^m$ be a vector field of the class C^1 on A.

Then $\vec{F}$ is a gradient field $\Leftrightarrow$ $\boxed{\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}}$, $\forall i, j \in \{1, \dots, m\}, i \neq j$

Example

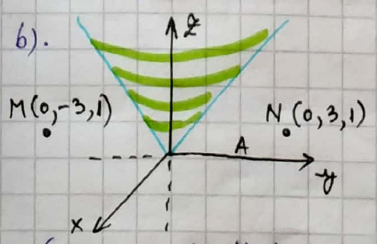
Consider the set $A := \{(x,y,z) \mid z < \sqrt{x^2+y^2}\}$ and the vector field on A defined by

$$\vec{F}(x,y,z) = \underbrace{\frac{x}{x^2+y^2+z^2}}_{P(x,y,z)}\vec{i} + \underbrace{\left(\frac{y}{x^2+y^2+z^2} + y^2(z^3+1)\right)}_{Q(x,y,z)}\vec{j} + \underbrace{\left(\frac{z}{x^2+y^2+z^2} + z^2(y^3+1)\right)}_{R(x,y,z)}\vec{k}$$

a) Establish whether A is open / starshaped / convex.

b) Give an example of a C^1 parametrized path γ, joining the points $M(0,-3,1)$ and $N(0,3,1)$, whose image is included in A.

c) Find $\int_\gamma \vec{F} \cdot d\vec{s}$ where γ is a parametrized path as in b).



a) $z = \sqrt{x^2+y^2}$

$x^2+y^2-z^2=0$

A is open

A is star-shaped w.r.t. each point $(0,0,a), a<0$

A isn't convex, because $M \in A$, $N \in A$ but $[MN] \not\subseteq A$

(convexă $\Leftrightarrow$ $\forall$ două puncte, segmentul ce le unește e inclus complet în A).

b) $\gamma: \begin{cases} x = 3\cos t \cdot 3\sin t \\ y = 3\sin t \cdot 3\cos t \\ z = 1 \end{cases} \quad t \in [\pi, 2\pi]$

c) $I = \int_M^N \vec{F} \cdot d\vec{s}$

by Poincare's lemma

? $\vec{F}$ is a gradient field $\Leftrightarrow$ $\begin{cases} \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} \\ \dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} \end{cases}$

$$\frac{\partial P}{\partial z}(x,y,z) = x\left(-\frac{1}{(x^2+y^2+z^2)^2}\cdot 2z\right) = -\frac{2xz}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial R}{\partial x}(x,y,z) = \frac{-2xz}{(x^2+y^2+z^2)^2}$$

$\Rightarrow \vec{F}$ is a gradient field $\Rightarrow \exists\, U : A \to \mathbb{R}$ s.t.

$$\nabla U = \vec{F} \Rightarrow \begin{cases} \frac{\partial U}{\partial x} = P \\ \frac{\partial U}{\partial y} = Q \\ \frac{\partial U}{\partial z} = R \end{cases}$$

Cum se determ. sistematic potențialul, aflăm în ep. următor.

COURSE 12 - week 14 08.06.2018.

- continuing from COURSE 11:

- $\frac{\partial U}{\partial x}(x,y,z) = P(x,y,z) = \frac{x}{x^2+y^2+z^2}$

$\Rightarrow U(x,y,z) = \frac{1}{2}\int \frac{2x}{x^2+y^2+z^2}\,dx = \frac{1}{2}\ln(x^2+y^2+y^2) + \varphi(y,z)$

$\Rightarrow \frac{\partial U}{\partial y}(x,y,z) = \frac{y}{x^2+y^2+z^2} + \frac{\partial \varphi}{\partial y}(y,z)$

- But $\frac{\partial U}{\partial y}(x,y,z) = Q(x,y,z) = \frac{y}{x^2+y^2+z^2} + y^2(z^3+1)$ $\Rightarrow$

$\Rightarrow \frac{\partial \varphi}{\partial y}(y,z) = y^2(z^3+1)$

$\Rightarrow \varphi(y,z) = \int y^2(z^3+1)\,dy = \frac{y^3}{3}(z^2+1) + \psi(z)$

constantă

$= +c =$ ceva care nu depinde de y

$\Rightarrow U(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2) + \frac{y^3}{3}(z^3+1) + \psi(z)$

$\Rightarrow \frac{\partial U}{\partial z}(x,y,z) = \frac{z}{x^2+y^2+z^2} + y^3z^2 + \psi'(z)$

- But $\frac{\partial U}{\partial z}(x,y,z) = R(x,y,z) = \frac{z}{x^2+y^2+z^2} + z^2(y^3+1$ $\Rightarrow$

$\Rightarrow \psi'(z) = z^2 \Rightarrow \psi(z) = \frac{z^3}{3} + c$

$\Rightarrow U(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2) + \frac{y^3}{3}(z^3+1) + \frac{z^3}{3} + c$

$$\int_M^N \vec{F}\cdot d\vec{s} = U(N) - U(M) = U(0,3,1) - U(0,-3,1$$

$$= \left(\frac{1}{2}\ln 10 + 18 + \frac{1}{3} + c\right) - \left(\frac{1}{2}\ln 10 - 18 + \frac{1}{3} + c\right) = \boxed{36}$$

* După ce dem. că un câmp de vectori e un câmp de gradiente, o să îi putem ~~afla~~ calcula potențialul. (Task of the above problem)

~~4. Green's formula~~

4. Green's formula

4.1. Definition. Let $D \subseteq \mathbb{R}^2$ be a simple set w.r. to Oy

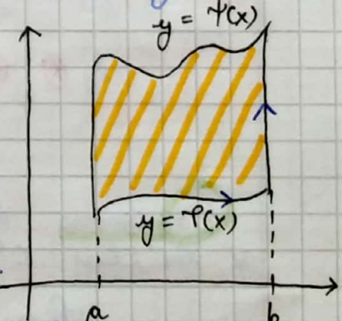
$\Rightarrow \exists\ a,b \in \mathbb{R},\ a<b$

$\exists\ \varphi,\psi : [a,b] \to \mathbb{R}$ continuous

with $\varphi(x) \le \psi(x),\ \forall x \in [a,b]$

~~s.t.~~

s.t. $D = \{(x,y) \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$



Consider the following parametrized paths:

$\gamma_1 : [a,b] \to \mathbb{R}^2,\ \gamma_1(t) = (t, \varphi(t))$

$\gamma_2 : [0,1] \to \mathbb{R}^2,\ \gamma_2(t) = (b, (1-t)\varphi(b) + t\psi(b))$

$\gamma_3 : [a,b] \to \mathbb{R}^2,\ \gamma_3(t) = (t, \psi(t))$

$\gamma_4 : [0,1] \to \mathbb{R}^2,\ \gamma_4(t) = (a, (1-t)\psi(a) + t\varphi(a))$

The parametrized path, defined by

$$\boxed{\partial\Delta := \gamma_1 \vee \gamma_2 \vee \overline{\gamma_3} \vee \gamma_4}$$

is called the boundary of Δ oriented in the positive sense.
Analogously it can be defined the boundary oriented in the positive sense for a simple set w.r.to Ox

4.2 Lemma. Let $A \subseteq \mathbb{R}^2$ be an open set, let $P: A \to \mathbb{R}$ be a function of the class C^1, let $\Delta \subseteq A$ be a simple set w.r.to Oy.
Then:

$$\circlearrowleft\int_{\partial\Delta} P(x,y)\,dx = -\iint_{\Delta} \frac{\partial P}{\partial y}(x,y)\,dx\,dy$$

means that the boundary $\partial\Delta$ is oriented in the positive sense

Proof:

$$\circlearrowleft\int_{\partial\Delta} P(x,y)\,dx = \int_{\gamma_1} P(x,y)dx + \int_{\gamma_2} P(x,y)dx - \int_{\gamma_3} P(x,y)\,dx + \int_{\gamma_4} P(x,y)\,dx =$$

$$= \int_a^b P(t,\varphi(t))\,(t)'dt + \int_0^1 P(b,(1-t)\varphi(b)+t\psi(b))\,\underbrace{(b)'}_{=0}dt$$

$$- \int_a^b P(t,\psi(t))\,(t)'dt + \int_0^1 P(a,(1-t)\psi(a)+t\varphi(a))\underbrace{(a)'}_{=0}dt$$

$$= \int_a^b P(t,\varphi(t))dt - \int_a^b P(t,\psi(t))\,dt \quad (*)$$

Fubini's theorem

$$-\iint_{\Delta} \frac{\partial P}{\partial y}(x,y)\,dx\,dy \overset{\downarrow}{=}$$

$$= -\int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \frac{\partial P}{\partial y}(x,y)\,dy\right) dx = -\int_a^b P(x,y)\Big|_{y=\varphi(x)}^{y=\psi(x)} dx$$

$$= -\int_a^b \left[P(x,\psi(x)) - P(x,\varphi(x))\right] dx =$$

$$= \int_a^b P(x,\varphi(x))\,dx - \int_a^b P(x,\psi(x))\,dx \quad (**)$$

By $(*)$, $(**)$ $\Rightarrow \circlearrowleft\int_{\partial\Delta} P(x,y)dx = \iint_{\Delta} \frac{\partial P}{\partial y}(x,y)\,dxdy.$

4.3 Lemma. Let $A \subseteq \mathbb{R}^2$ be an open set, let $Q: A \to \mathbb{R}$ be a function of the class C^1, and let $\Delta \subseteq A$ be a simple set w.r.to Ox. Then:

$$\circlearrowleft\int_{\partial\Delta} Q(x,y)\,dy = \iint_{\Delta} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

4.4. Theorem (Green's ~~theor.~~ formula)

~~Let A⊆~~

Let $A \subseteq \mathbb{R}^2$ be an open set, let $P, Q : A \to \mathbb{R}$ be functions of the class C^1, and let $D \subseteq A$ be a simple set w.r. to both Ox and Oy. Then:

$$\boxed{\oint_{\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right] dx\,dy}$$

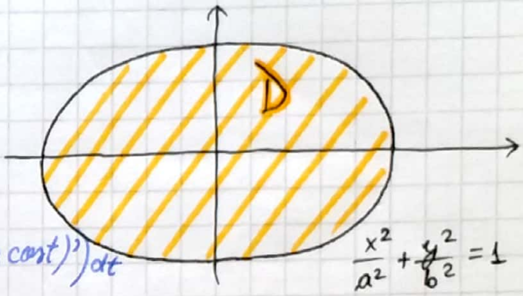
Proof: Follows by Lemma 4.2. + Lemma 4.3.

4.5. Corollary. If $D \subseteq \mathbb{R}^2$ is simple w.r. to both Ox and Oy, then:

$$\boxed{\mathcal{A}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx}$$

Application:

$$\partial D : \begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \quad t \in [0, 2\pi]$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\mathcal{A}(D) = \frac{1}{2}\int_0^{2\pi} \left(a\cos t \cdot (b\sin t)' - b\sin t\,(a\cos t)'\right) dt$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

Pentru examen:

- câte o - integrală: dublă / triplă / curbilinie
- teorie (cele 4 de pe listă)

END OF ANALISE 2.