

LECTURE

6

HIGHER ORDER DERIVATIVES. TAYLOR SERIES AND POWER SERIES

Higher order derivatives

Definition 6.1 Let $A \subseteq \mathbb{R}$, $c \in A \cap A'$ and $f : A \rightarrow \mathbb{R}$. We say that f is twice differentiable at c if $\exists V \in \mathcal{V}(c)$ such that f is differentiable on $A \cap V$ and f' is differentiable at c . If f is twice differentiable at c , then we write

$$f^{(2)}(c) := f''(c) := (f')'(c).$$

In general, for $n \in \mathbb{N}$, ≥ 2 , we say that f is n -times differentiable at c if $\exists V \in \mathcal{V}(c)$ such that f is $(n-1)$ -times differentiable on $A \cap V$ and $f^{(n-1)}$ is differentiable at c . If f is n -times differentiable at c , then we write

$$f^{(n)}(c) := (f^{(n-1)})'(c).$$

If B is a nonempty subset of A , we say that f is n -times differentiable on B if it is n -times differentiable at every point of B . In this case, the function $f^{(n)} : B \rightarrow \mathbb{R}$, $x \in B \mapsto f^{(n)}(x) \in \mathbb{R}$ is called the n^{th} derivative of f on B .

We say that f is infinitely differentiable at c if f is n -times differentiable at c for every $n \in \mathbb{N}$. Notational conventions: $f^{(0)} := f$ and $f^{(1)} := f'$.

Approximation of differentiable functions by Taylor polynomials

Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n -times differentiable at x_0 , we want to find a polynomial function $P : \mathbb{R} \rightarrow \mathbb{R}$, of degree (at most) n , such that

$$\begin{cases} P(x_0) &= f(x_0) \\ P'(x_0) &= f'(x_0) \\ P''(x_0) &= f''(x_0) \\ &\vdots \\ P^{(n)}(x_0) &= f^{(n)}(x_0). \end{cases} \quad (6.1)$$

We are looking for P of the form

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n.$$

By (6.1) we deduce that

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!}, \quad \dots, \quad a_n = \frac{f^{(n)}(x_0)}{n!}.$$

So, there is a unique polynomial P of degree (at most) n satisfying (6.1).

Definition 6.2 Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n -times differentiable at x_0 , the polynomial function $T_n : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$T_n(x) := f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad (6.2)$$

is called the n^{th} Taylor polynomial of f (centered) at x_0 .

Remark 6.3 The n^{th} Taylor polynomial of f at x_0 is also denoted by $T_n(f; x_0)$. However, we simply write $T_n(x)$ instead of $T_n(f; x_0)(x)$ for all $x \in \mathbb{R}$.

Remark 6.4 Since the Taylor polynomial satisfies

$$\begin{cases} T_n(x_0) &= f(x_0) \\ T'_n(x_0) &= f'(x_0) \\ T''_n(x_0) &= f''(x_0) \\ &\vdots \\ T_n^{(n)}(x_0) &= f^{(n)}(x_0), \end{cases} \quad (6.3)$$

it approximates the function f on a neighborhood of x_0 , i.e.,

$$f(x) \simeq f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

In particular, for $n = 1$ we obtain

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0).$$

Definition 6.5 Let $I \subseteq \mathbb{R}$ be an interval, $x_0 \in I$, $f : I \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$. Supposing that f is n -times differentiable at x_0 , the function $R_n : I \rightarrow \mathbb{R}$, defined by

$$R_n(x) := f(x) - T_n(x), \quad \forall x \in I, \quad (6.4)$$

is called the remainder of the approximation of f by T_n around x_0 . Whenever R_n is given explicitly, we get the so-called Taylor formula:

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}_{T_n(x)} + R_n(x), \quad \forall x \in I.$$

Theorem 6.6 (Taylor-Lagrange) Let $f : I \rightarrow \mathbb{R}$ be a function which is $(n+1)$ -times differentiable on I for some $n \in \mathbb{N} \cup \{0\}$. Then, for any distinct points $x, x_0 \in I$ there exists a point $c \in \mathbb{R}$, $\min\{x_0, x\} < c < \max\{x_0, x\}$, such that

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \quad (6.5)$$

In other words, we have $f(x) = T_n(x) + R_n(x)$, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \quad (6.6)$$

Proof. Consider any distinct points $x, x_0 \in I$. Without loss of generality we can assume that $x_0 < x$. By (6.3) and (6.4) we have

$$R_n^{(k)}(x_0) = 0, \quad \forall k \in \{0, 1, \dots, n\}.$$

By Cauchy's Generalized Mean Value Theorem 5.48, applied to the functions

$$x \mapsto R_n(x) \quad \text{and} \quad x \mapsto (x - x_0)^{n+1}$$

on the interval $[x_0, x]$, there exists $c_1 \in (x_0, x)$ such that

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R_n(x) - R_n(x_0)}{(x - x_0)^{n+1} - (x_0 - x_0)^{n+1}} = \frac{R'_n(c_1)}{(n+1)(c_1 - x_0)^n}.$$

Applying now Cauchy's Generalized Mean Value Theorem to the functions

$$x \mapsto R'_n(x) \quad \text{and} \quad x \mapsto (n+1)(x - x_0)^n$$

on the interval $[x_0, c_1]$, we deduce that there is $\exists c_2 \in (x_0, c_1)$ such that

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R'_n(c_1)}{(n+1)(c_1 - x_0)^n} = \frac{R'_n(c_1) - R'_n(x_0)}{(n+1)(c_1 - x_0)^n - (n+1)(x_0 - x_0)^n} = \frac{R''_n(c_2)}{(n+1)n(c_2 - x_0)^{n-1}}.$$

Continuing in this way we find $c_{n+1} \in (x_0, c_n) \subseteq (x_0, c_{n-1}) \subseteq \dots \subseteq (x_0, x)$ such that

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R_n^{(n+1)}(c_{n+1})}{(n+1)!}. \quad (6.7)$$

On the other hand, recalling that T_n is a polynomial of degree at most n , we deduce by (6.4) that $R_n^{(n+1)}(c_{n+1}) = f^{(n+1)}(c_{n+1}) - T_n^{(n+1)}(c_{n+1}) = f^{(n+1)}(c_{n+1})$. Hence, by choosing $c := c_{n+1}$, we infer from (6.7) that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \quad \square$$

Remark 6.7 (6.5) is called the Taylor's formula with the remainder in Lagrange's form (6.6).

Remark 6.8 Assume that, for some $a, b \in I$ with $a < x_0 < b$, there exists $M \in \mathbb{R}$ such that $|f^{(n+1)}(x)| \leq M$ for all $x \in [a, b]$. Then, the error of approximation of $f(x)$ by $T_n(x)$ can be estimated by

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}(x - x_0)^{n+1}, \quad \forall x \in [a, b].$$

Corollary 6.9 (Local optimality conditions) Let $f : I \rightarrow \mathbb{R}$ be a function, defined on an interval $I \subseteq \mathbb{R}$. If f is n -times differentiable ($n \in \mathbb{N}$, $n \geq 2$) at $x^0 \in \text{int}, I$ and

$$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0 \neq f^{(n)}(x_0),$$

then the following assertions hold true:

- 1° If n is even and $f^{(n)}(x_0) > 0$, then x_0 is a local minimum point of f .
- 2° If n is even and $f^{(n)}(x_0) < 0$, then x_0 is a local maximum point of f .
- 3° If n is odd, then x_0 is not a local extremum point of f .

Example 6.10 Let $f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = 24 \cos x + 12x^2 - x^4$ for all $x \in \mathbb{R}$. It is easy to check that

$$f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = f^{(5)}(0) = 0 \neq -24 = f^{(6)}(0),$$

hence $x_0 = 0$ is local maximum point of f .

Taylor series

Definition 6.11 Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series

$$\sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad (6.8)$$

is called the Taylor series of f around x_0 .

If $J \subseteq I$ is a nonempty set such that for every $x \in J$ the series (6.8) converges and its sum is $f(x)$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \quad (6.9)$$

then we say that f can be expanded as a Taylor series around x_0 on J . In this case, (6.9) is called the Taylor expansion of $f(x)$ around x_0 on J .

Remark 6.12 For any $x \in I$, the partial sums of the Taylor series (6.8) are given by

$$\sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = T_n(x), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Thus, the series (6.8) converges if and only if its sum is finite, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n := s(x) := \lim_{n \rightarrow +\infty} T_n(x) \in \mathbb{R}$$

and, according to (6.4), we have

$$\lim_{n \rightarrow +\infty} R_n(x) = f(x) - \lim_{n \rightarrow +\infty} T_n(x) = f(x) - s(x).$$

Therefore, by Definition 6.11, f can be expanded as a Taylor series around x_0 on J if and only if

$$\lim_{n \rightarrow +\infty} R_n(x) = 0, \quad \forall x \in J.$$

Example 6.13 (Taylor expansion of the exponential function around 0) Let $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = e^x.$$

Note that $\forall k \in \mathbb{N}, \forall x \in \mathbb{R}, f^{(k)}(x) = e^x$, so $\forall k \in \mathbb{N}, f^{(k)}(0) = 1$. Let $n \in \mathbb{N}, x \in \mathbb{R}$. Then there exists c between 0 and x such that

$$e^x = 1 + \frac{1}{1!}x + \dots + \frac{1}{n!}x^n + R_n(x),$$

where $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$.

Since $0 \leq |R_n(x)| \leq e^c \frac{|x|^{n+1}}{(n+1)!}$ and $\lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0$, it follows that $\lim_{n \rightarrow \infty} R_n(x) = 0$.

Therefore, f can be expanded as a Taylor series around 0 on \mathbb{R} :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}.$$

Example 6.14 (Taylor expansion of sine function around 0) The function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \sin x,$$

can be expanded as a Taylor series around 0 on \mathbb{R} :

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, \quad \forall x \in \mathbb{R}.$$

Example 6.15 (Taylor expansion of cosine function around 0) The function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \cos x,$$

can be expanded as a Taylor series around 0 on \mathbb{R} :

$$\cos x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!}, \quad \forall x \in \mathbb{R}.$$

Power series

Definition 6.16 Let $(a_n)_{n \geq 0}$ be a sequence of real numbers and let $c \in \mathbb{R}$. A series of type

$$\sum_{n \geq 0} a_n (x - c)^n, \quad \text{where } x \in \mathbb{R}, \quad (6.10)$$

is called power series centered at x with coefficients a_n . The set

$$C := \{x \in \mathbb{R} \mid \text{the series (6.10) converges}\}$$

is called the convergence set of the power series.

Theorem 6.17 (Abel) There exists $R \in [0, +\infty) \cup \{+\infty\}$ such that the power series (6.10) converges absolutely whenever $0 \leq |x - c| < R$ and diverges whenever $|x - c| > R$.

Definition 6.18 The unique R satisfying the conditions of Abel's Theorem 6.17 is called the radius of convergence of the power series.

Theorem 6.19 (Cauchy-Hadamard) If the limit

$$L := \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \in [0, \infty) \cup \{+\infty\}$$

exists, then the power series (6.10) has the radius of convergence $R = \begin{cases} 1/L & \text{if } 0 < L < +\infty \\ 0 & \text{if } L = +\infty \\ +\infty & \text{if } L = 0. \end{cases}$

Corollary 6.20 If the limit

$$L := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, \infty) \cup \{+\infty\}$$

exists, then the power series (6.10) has the radius of convergence $R = \begin{cases} 1/L & \text{if } 0 < L < +\infty \\ 0 & \text{if } L = +\infty \\ +\infty & \text{if } L = 0. \end{cases}$

Example 6.21 1) For the geometric series $\sum_{n \geq 1} (x - c)^n$, centered at any number $c \in \mathbb{R}$, we have

$R = 1$ and $C = (c - 1, c + 1)$.

2) For $\sum_{n \geq 1} \frac{1}{n} x^n$ we have $R = 1$ and $C = [-1, 1)$.

3) For $\sum_{n \geq 1} \frac{(-1)^n}{n} x^n$ we have $R = 1$ and $C = (-1, 1]$.

4) For $\sum_{n \geq 1} \frac{1}{n!} (x - c)^n$, centered at any $c \in \mathbb{R}$, we have $R = +\infty$ and $C = \mathbb{R}$.

5) For $\sum_{n \geq 1} n! (x + 1)^n$ we have $R = 0$ and $C = \{-1\}$.