CHAPTER 3

Affine hulls

Contents

3.1 Affine hulls

Definition. For a subset M of the affine space X, the *affine hull* of M is

$$\operatorname{aff}(M) = \bigcap \{Y : \text{ affine subspace of } X \text{ containing } M\}.$$

Proposition 3.1. *For $M, N \subseteq X$ we have*

1. $M \subseteq \operatorname{aff}(M)$,
2. $\operatorname{aff}(M)$ *is an affine space,*
3. *if Y is an affine space containing M then* $\operatorname{aff}(M) \subseteq Y$,
4. *if $M \subseteq N$ then* $\operatorname{aff}(M) \subseteq \operatorname{aff}(N)$,
5. $\operatorname{aff}(M) = M$ *if and only if M is an affine space,*
6. $\operatorname{aff}(\operatorname{aff}(M)) = \operatorname{aff}(M)$.

Remark 3.2. In Exercise **??**, we defined the operator ℓ. For any $M \subseteq X$ we have

$$\ell(M) \subseteq \ell^2(M) \subseteq \cdots \subseteq \operatorname{aff}(M).$$

Notice that $\ell = \operatorname{aff}$ if and only if $\dim X \leq 1$.

If $P_0, \ldots, P_n$ is an affine basis. Then the points describe de corners of a simplex. At the m-th iteration, $\ell^m(P_0, \ldots, P_n)$ contains all affine subspaces in which the m-dimensional facets of the simplex lie.

Corollary 3.3. *For any $M \subseteq X$ we have*

$$\operatorname{aff}(X) = \left\{ \operatorname{Bar}(P_1, \ldots, P_m; \mu_1, \ldots, \mu_m) : \forall m \in \mathbb{N}, \forall P_1, \ldots, P_m \in X \text{ and } \forall \mu_1, \ldots, \mu_m \in k \text{ with } \sum_{i=1}^{n} \mu_i = 1 \right\}.$$

In fact, if $\dim X = n$, then

$$\operatorname{aff}(X) = \left\{ \operatorname{Bar}(P_0, \ldots, P_m; \mu_0, \ldots, \mu_m) : \forall m \leq n, \forall P_0, \ldots, P_n \in X \text{ and } \forall \mu_0, \ldots, \mu_n \in k \text{ with } \sum_{i=1}^{n} \mu_i = 1 \right\}.$$

Proposition 3.4. *Let Y and Z be two affine subspaces of X. For any $y \in Y$ and $z \in Z$*

$$\operatorname{aff}(Y \cup Z) = y + D(Y) + D(Z) + \langle z - y \rangle.$$

Proof. □

Proposition 3.5. *Let Y and Z be two affine subspaces of X. For any $y \in Y$ and $z \in Z$*

$$A \cap B \neq \emptyset \quad \Leftrightarrow \quad \langle z - y \rangle \subseteq D(Y) + D(Z).$$

Proof. □

Corollary 3.6. *Let Y and Z be two affine subspaces of X. If they have a common point P, then*

$$\operatorname{aff}(Y \cup Z) = P + D(Y) + D(Z) \quad \text{and} \quad Y \cap Z = P + D(Y) \cap D(Z)$$

3.2 Dimension

The dimension of an affine space X is $\dim D(X)$. In particular, it equals the maximal number n such that there exist points $P_0, \ldots, P_n \subseteq X$ in general position.

From Propositions 3.4 and 3.5 we obtain

Theorem 3.7. *Let Y and Z be two finite dimensional affine subspaces of X.*

1. *If $Y \cap Z \neq \emptyset$, then*
$$\dim \operatorname{aff}(Y \cup Z) = \dim(Y) + \dim(Z) - \dim(Y \cap Z).$$

2. *If $Y \cap Z = \emptyset$, then*

$$\dim \operatorname{aff}(Y \cup Z) = \dim(D(Y) + D(Z)) + 1.$$

Definition. We say that two affine subspaces $Y, Z \subseteq X$ are in *general position* if $\dim \operatorname{aff}(Y \cup Z)$ is as big as possible, or equivalently, if $\dim(Y \cap Z)$ is as small as possible

$$\dim(Y \cap Z) = \dim(Y) + \dim(Z) - \dim \operatorname{aff}(Y \cup Z).$$

If a subset $M \subseteq X$ is a finite union of affine spaces

$$M = \bigcup_{i=1}^{n} Y_i \quad \text{then} \quad \dim M = \max_i \dim Y_i.$$

3.3 Parallelism

Two affine subspaces Y and Z of X are parallel if $D(Y) \subseteq D(Z)$ or $D(Z) \subseteq D(Y)$.

Proposition 3.8. *Let Y be an affine subspace of X and H a hyperplane of X. Then*

$$Y \cap H = \emptyset \quad \Rightarrow \quad Y \parallel H.$$

Proof. □

Corollary 3.9. *Let L be a line in X intersecting the hyperplane H of X in a point. If L' is any line parallel to L, then L' intersects H, i.e. $L' \cap H \neq \emptyset$.*

Proof. □

3.4 The lattice of affine subspaces

Theorem 3.10. *The set of affine subspaces of X is a complete lattice with*

$$\inf_{Y \in \mathcal{Y}} = \bigcap_{y \in \mathcal{Y}} Y \quad \textit{and} \quad \sup_{Y \in \mathcal{Y}} = \operatorname{aff}\left(\bigcup_{y \in \mathcal{Y}} Y \right)$$

for any family $\mathcal{Y}$ of affine subspaces.

3.5 Cartesian coordinates

It should be clear at this point, how Cartesian coordinates are to be obtained for an affine subspace Y of $\mathbb{R}^n$. We fix any point $O \in Y$ which we call *origin*, so

$$Y = O + D(Y).$$

We then fix a basis $(v_1,\ldots,v_m)$ of the vector space $D(Y)$ and call

$$O+\langle v_i\rangle_{\mathbb{R}}$$

the *axis* of the Cartesian system for Y. Further

$$Y = O+\langle v_1,\ldots,v_m\rangle = O+\{t_1v_1+\cdots+t_mv_m : t_1,\ldots,t_m\in\mathbb{R}\}$$

so, for any $P\in Y$ we have unique scalars t_i such that

$$P = O+t_1v_1+\cdots+t_mv_m \quad\Leftrightarrow\quad \begin{cases} p_1 = o_1+t_1v_{11}+\cdots+t_mv_{m1}\\ \ldots\\ p_m = o_m+t_1v_{1n}+\cdots+t_mv_{mn}\end{cases}$$

Where $P=(p_1,\ldots,p_n), O=(o_1,\ldots,o_n)\in\mathbb{R}^n$ and $v_i=(v_{i1},\ldots,v_{in})\in D(\mathbb{R}^n)=\mathbb{R}^n$.

3.6 Exercises

Exercise 1. Let Y and Z be affine subspaces of an affine subspace over k. If $|k|>2$ and $Y\cap Z\neq\emptyset$ show that

$$\mathrm{aff}(Y\cap Z)=\{ty+(1-t)z : t\in k,\, y\in Y,\, z\in Z\}.$$

Why are the two conditions needed?

Exercise 2. For the set $\mathcal{S}$ of vertices of a cube in $\mathbb{R}^3$ describe

$$\ell(\mathcal{S}),\quad \ell^2(\mathcal{S})\quad\text{and}\quad \ell^3(\mathcal{S}).$$

In general, what is $\ell^n(\mathcal{S})$ for a subset $\mathcal{S}\subset\mathbb{R}^n$? Show that if $\ell^m(\mathcal{S})\neq\mathrm{aff}(\mathcal{S})$, then

$$\dim\ell^{m+1}(\mathcal{S})=\dim\ell^m(\mathcal{S})+1.$$

Exercise 3. Let Y be an affine subspace of an affine space X. For $P\in X\setminus Y$, is the set

$$\bigcup_{Q\in Y}\mathrm{aff}\{Q,P\}$$

affine?

Exercise 4. Let Y be a d-dimensional affine subspace of X. Show that for any $P\in X$ there exists a unique d-dimensional affine subspace Z of X such that

$$Z\parallel Y.$$

Exercise 5. Show that in a 4-dimensional affine space any two hyperplanes which intersect non-trivially have a plane in common.

Exercise 6. Determine all relative positions of two planes α and β in a 4-dimensional affine space. Give $\dim\mathrm{aff}(\alpha\cup\beta)$ in each case.

Exercise 7. Let Y and Z be two affine subspaces of dimension d. Show that $Y \parallel Z$ if and only if Y and Z lie in a $d+1$ dimensional affine subspace.

Exercise 8. In an affine n-dimensional space X let H be a hyperplane and Y a d-dimensional affine subspace. Show that exactly one of the following holds

1. $\dim(H \cap Y) = d-1$,
2. $H \parallel Y$.

Exercise 9. Consider the following affine subspaces of $\mathbb{R}^4$

$$Y : \begin{cases} x_1 + x_3 - 2 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 &= 0 \end{cases}$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + x_3 - 3x_4 &= -1 \\ x_1 - x_2 + 3x_4 &= 3 \end{cases}$$

1. Determine the dimensions of Y and Z.
2. What are the parametric equations of the two affine subspaces?
3. Show that $Y \parallel Z$.

Exercise 10. In $\mathbb{R}^n$ ($n \geq 2$) consider the line

$$L = P + \langle (v_1, \ldots, v_n) \rangle$$

and the hyperplane

$$H : \alpha_1 x_1 + \ldots \alpha_n x_n + \beta = 0.$$

Show that $L \parallel H$ if and only if

$$\alpha_1 v_1 + \ldots \alpha_n v_n = 0.$$

Exercise 11. Show that for any subsets M, N of an affine space X, we have

$$\mathrm{aff}(\mathrm{aff}(M) \cup \mathrm{aff}(N)) = \mathrm{aff}(M \cup N).$$

Exercise 12. In $\mathbb{R}^5$ consider the vectors

$$\begin{aligned} a &= (1,0,0,2,0) \\ b &= (0,2,0,0,1) \\ c &= (1,2,0,0,0) \\ d &= (0,0,0,2,1) \end{aligned}$$

and the affine spaces

$$A = a + \langle b, c \rangle \quad \text{and} \quad B = c + \langle b, d \rangle$$

Determine

$$A \cap B \quad \text{and} \quad \mathrm{aff}(A \cup B).$$

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