Calculus on $\mathbb{R}$ Anca Grad, Ph.D., Lect. Proff.
Homework 4 **Week 4-5, 2017**

Sequences of Real Numbers

Mandatory Excercises

Exercise 1: Study the monotonicity, boundedness and convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, having the general term:

$$a)\quad x_n = \frac{2^n + 3^n}{5^n},\quad b)\quad x_n = \frac{(-1)^n}{n},\quad c)\quad x_n = \frac{2^n}{n!},\quad d)\quad x_n = \frac{n}{n^2+1}.$$

Exercise2: Using the characterising theorem with ε prove that

$$a)\lim_{n\to\infty}\frac{n}{n^2+1} = 0 \qquad b)\lim_{n\to\infty}\frac{n^2}{-2n+4} = -\infty.$$

Exercise 3: Compute the limit of the sequences of real numbers having the following general terms:

$$a)\quad \frac{5^n+1}{7^n+1},\qquad b)\quad \frac{4^n+(-2)^n}{4^{n-1}+2},\qquad c)\quad \left(\sin\frac{\pi}{10}\right)^n,\qquad d)\sqrt{9n^2+2n+1}-3n,$$

$$e)\quad \left(5+\frac{1-2n^3}{3n^4+2}\right)^2,\qquad f)\quad \sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1},\qquad g)\quad \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}},$$

$$h)\quad \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right).$$

Exercise 4: Let $t\in\mathbb{R}$.

a) Prove that there exists an increasing sequence of rational numbers converging to t.

b) Prove that there exists a decreasing sequence of irrational numbers converging to t.

Exercise 5: Let $a > 0$ and let $x_0 \in \mathbb{R}$ be such that $0 < x_0 < \frac{1}{a}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, defined recursively by:

$$x_{n+1} = 2x_n - ax_n^2, \forall n \in \mathbb{N}.$$

Study the convergence of the sequence by following the next steps:

a) Prove by induction that $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$.

b) Prove by induction that $0 < x_n, \forall n \in \mathbb{N}$.

c) By using a) and b) prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

d) Compute the limit of the sequence.

Elective Exercises

Elective 1: Study the nature (monotonicity and boundedness) of the sequence having the general term

$$x_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{n^2}\right),\ n \geq 2.$$

In case of convergence compute the limit.

Elective 2: Calculaţi limitele următoareleor şirui de numere reale având ca termen general:

$$a)\ \frac{3^n}{4^n}, \quad b)\ \frac{2^n + (-2)^n}{3^n}, \quad c)\ \frac{5 - n^3}{n^2 + 1}, \quad d)\ \left(2 + \frac{4^n + (-5)^n}{7^n + 1}\right)^{2n^3 - n^2},$$

$$e)\ \frac{1 + 2 + \ldots + n}{n^2}, \quad f)\ \left(\frac{n^3 + 4n + 1}{2n^3 + 5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}, \quad g)\ (\cos(-2013))^n,$$

$$h)\ \left(\frac{n^5 + 3n + 1}{2n^5 - n^4 + 3}\right)^{\frac{3n-n^4}{n^3+1}}.$$

Elective 3: Compute the limit for each of the following sequences:

$$a)\ \left(1 + \frac{1}{-n^3 + 3n}\right)^{n^2 - n^3}, \quad b)\ (3n^2 + 5)ln\left(1 + \frac{1}{n^2}\right),$$

$$c)\frac{n^n}{1^1+2^2+...+n^n}$$

$$d)\frac{x_1+2x_2+...+nx_n}{n^2},$$

where $(x_n)_{n\in\mathbb{N}}$ is a converging sequence having the limit $x\in\mathbb{R}$.