CHAPTER 2

Coordinates

Contents

2.1 Chasles' notation

An affine space X is a vector space where we forget the origin. In particular it doesn't make sense to add points in X. However, we can make sense out of subtracting points. Namely, for two points $P, Q \in X$ there is a unique vector $v \in D(X)$ such that $Q = P + v$ (see Proposition **??**). We denote this vector by

$$Q - P \quad \text{or, traditionally, by} \quad \overrightarrow{PQ}$$

further, with this notation, we have

$$P + \overrightarrow{PQ} = P + (Q - P) = Q.$$

Proposition 2.1 (Chasles' relation)**.** *For $A, B, C \in X$ we have*

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}, \quad \textit{in particular} \quad \overrightarrow{AB} = -\overrightarrow{BA}.$$

Proof. □

The notation adopted here can be extended:

Lemma 2.2. *For $n \geq 1$ consider points $P_1, \dots, P_n \in X$ and scalars $a_1, \dots, a_n \in k$ such that*

$$\sum_{i=1}^{n} a_i = 0.$$

For any point $P_0 \in X$, the vector

$$\sum_{i=1}^{n} a_i (P_i - P_0) = \sum_{i=1}^{n} a_i \overrightarrow{P_0 P_i}$$

does not depend on P_0 and we denote it by

$$\sum_{i=1}^{n} a_i P_i.$$

Proof. □

Remark 2.3. If in the above proposition, we take $n = 2$, $P_1 = P$, $P_2 = Q$, $\mu_1 = -1$ and $\mu_2 = 1$, we obtain $Q - P = \overrightarrow{PQ}$.

2.2 Barycentric coordinates

Proposition 2.4. *For $n \geq 1$ consider points $P_1, \dots, P_n \in X$ and scalars $\mu_1, \dots, \mu_n \in k$ such that*

$$\sum_{i=1}^{n} \mu_i = 1.$$

For any point $P_0 \in X$, the point

$$P_0 + \Big(\sum_{i=1}^{n} \mu_i P_i - P_0 \Big) \tag{2.1}$$

does not depend on P_0 and we denote it with

$$\sum_{i=1}^{n} \mu_i P_i. \tag{2.2}$$

Proof. □

Definition. We call the point (2.1) in the above proposition the *barycenter of the points $P_1, \dots, P_n$ with respect to the weights $\mu_1, \dots, \mu_n \in k$* and we denote it by

$$\mathrm{Bar}(P_1, \dots, P_n; \mu_1, \dots, \mu_n) = \sum_{i=1}^{n} \mu_i P_i.$$

If all weights are equal we use the shorter notation

$$\mathrm{Bar}(P_1, \dots, P_n) = \sum_{i=1}^{n} \frac{1}{n} P_i.$$

Example 2.5. The barycenter $\mathrm{Bar}(P_1,\ldots,P_n)$ is the centroid of the points. In particular, if $P_1P_2P_3$ is a triangle, then $\mathrm{Bar}(P_1,P_2,P_3)$ is the intersection of the medians. Similarly, if $P_1P_2P_3P_4$ is a tetrahedron, then $\mathrm{Bar}(P_1,P_2,P_3,P_4)$ is the intersection of the medians of the tetrahedron (recall, these are the lines connecting a vertex with the centroid of the opposite face).

More generally one can think of $\mathrm{Bar}(P_1,\ldots,P_n;\mu_1,\ldots,\mu_n)$ as being the center of mass of the system of points $P_1,\ldots,P_n$ if the point P_i has mass μ_i.

Definition. In an n-dimensional affine space X, the points $P_0,\ldots,P_m$ $(m\leq n)$ are said to be *in general position* if

$$\overrightarrow{P_0P_1},\ldots,\overrightarrow{P_0P_m}\in D(X) \tag{2.3}$$

are linearly independent. A set of $(n+1)$ points $(P_0,\ldots,P_n)$ in general position is called *affine basis*. The expression (2.2) is called an *affine combination*.

Remark 2.6. It is clear that the condition (2.3) is equivalent to

$$\overrightarrow{P_iP_1},\ldots,\overrightarrow{P_iP_{i-1}},\overrightarrow{P_iP_{i+1}},\ldots,\overrightarrow{P_iP_m}\quad\text{being linearly independent.}$$

Proposition 2.7. *Let $(P_0,\ldots,P_n)$ be an affine basis of X. For any $P\in X$ there exists a unique set of scalar $(\mu_0,\ldots,\mu_n)$, with $\sum_{i=0}^n\mu_i=1$ such that*

$$P=\mathrm{Bar}(P_0,\ldots,P_n;\mu_0,\ldots,\mu_n).$$

Proof. □

Definition. The scalars $(\mu_0,\ldots,\mu_n)$ are *the barycentric coordinates of P with respect to the affine basis* $(P_0,\ldots,P_n)$.

Corollary 2.8. *Let X be an affine space. A subset $Y\subseteq X$ is an affine subspace if and only if*

$$\forall P_1,\ldots,P_m\in Y\ \textit{and}\ \forall\mu_1,\ldots,\mu_m\in k\ \textit{with}\ \sum_{i=1}^n\mu_i=1\ \textit{we have}\ \mathrm{Bar}(P_1,\ldots,P_m;\mu_1,\ldots,\mu_m)\in Y.$$

2.3 Lines

For two points P,Q in the affine space X, we denote by PQ the line

$$PQ:P+\langle\overrightarrow{PQ}\rangle=\Big\{P+t(Q-P):t\in k\Big\}=\Big\{\mathrm{Bar}(P,Q;1-t,t):t\in k\Big\}=\Big\{(1-t)P+tQ:t\in k\Big\}$$

where in the second and last equality we use Chasles' notation.

Proposition 2.9. *Let k be a field of at least three elements and X an affine space X over k. A subset $Y\subseteq X$ is an affine subspace if and only if*

$$\forall P,Q\in Y\ \textit{we have}\ PQ\subseteq Y. \tag{2.4}$$

Proof. □

Example 2.10. For the field $\mathbb{F}_2=\{0,1\}$ of two elements consider the 2-dimensional affine space $X=\mathbb{F}_2^2$. The subset $M=\{(0,0),(0,1),(1,0)\}$ contains the lines passing through any two of its points. However, one checks that it is not an affine subspace of X.

2.4 Cross-ratios

For a point P on the line AB we have a unique $t \in k$ such that

$$P = \mathrm{Bar}(A,B;1-t,t) = (1-t)A + tB.$$

If $P \neq A$ and $P \neq B$, we denote by

$$(A,B|P) = \frac{t}{1-t}$$

the *ratio* of how P divides the segment $[AB] = \{(1-t)A + tB : t \in [0,1]\}$. If another point Q (distinct from A, B and P) is chosen on AB, *the cross-ratio of A, B, P and Q* is

$$(A,B|P,Q) = \frac{(A,B|P)}{(A,B|Q)} = \frac{t}{1-t}\frac{1-s}{s}.$$

Proposition 2.11. *Consider distinct collinear points A, B and $C = \mathrm{Bar}(A,B;1-t,t)$. Then*

1. $(A,B|C) = \mu$
2. $(B,C|A) = -\frac{\mu}{1+\mu}$
3. $(C,A|B) = -\frac{1}{1+\mu}$
4. $(B,A|C) = \frac{1}{\mu}$
5. $(C,B|A) = -\frac{1+\mu}{\mu}$
6. $(A,C|B) = -(1+\mu)$

Proposition 2.12. *Consider distinct collinear points A, B, C and D. If $(A,B|C,D) = \lambda$, the possible values for the cross-ratios of these four points are*

$$\frac{1}{\lambda}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad 1-\frac{1}{\lambda}, \quad -\frac{\lambda}{1-\lambda}.$$

2.5 Exercises

Exercise 1. Two pairs of points (A,B) and (C,D) in an affine space X are called *equipollent* if

$$\overrightarrow{AB} = \overrightarrow{CD}$$

Show that for such equipollent pairs we have

$$\overrightarrow{AC} = \overrightarrow{BD}.$$

Exercise 2. Show that the set M in Example 2.10 is not an affine subspace.

Exercise 3. For an affine space X we define the operator ℓ by

$$\ell(M) = \{tP + (1-t)Q : P, Q \in M\} \subseteq X$$

for any subset $M \subseteq X$.

1. For $X = \mathbb{R}^2$ and $M = \{(1,0),(0,2),(0,0)\}$ describe $\ell(M)$ and $\ell^2(M) = \ell(\ell(M))$.
2. For $X = \mathbb{R}^2$ and $M = \{(1,0,0),(0,1,0),(0,0,1)\}$ describe $\ell(M)$, $\ell^2(M)$ and $\ell^3(M)$.
3. Show that the sequence
$$M \subseteq \ell(M) \subseteq \ell^2(M) \subseteq \dots$$
is stationary if X is finite dimensional.

Exercise 4. Consider a triangle ABC in a real affine space. Let $D \in [AB]$ and $E \in AC$ be such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{3}{4}.$$

Consider D' and E' given by

$$\overrightarrow{EE'} = 3\overrightarrow{BE} \quad \text{and} \quad \overrightarrow{DD'} = 3\overrightarrow{CD}.$$

Show that A, D' and E' are collinear.

Exercise 5. Let X be an affine space. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that

$$\overrightarrow{AC'} = \lambda\overrightarrow{BC'} \quad \text{and} \quad \overrightarrow{AB'} = \mu\overrightarrow{CB'}.$$

The lines BB' and CC' meet in the point M. For $O \in X$ show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda\overrightarrow{OB} - \mu\overrightarrow{OC}}{1-\lambda-\mu}.$$

Exercise 6. Consider the triangle ABC in a real affine space X. Let G be its centroid, H the orthocenter, I the incenter and O the circumcenter. For a point $P \in X$ let

$$\mathbf{r}_A = \overrightarrow{OA}, \quad \mathbf{r}_B = \overrightarrow{OB} \quad \text{and} \quad \mathbf{r}_C = \overrightarrow{OC}.$$

Show that

1. $\overrightarrow{PG} = \frac{\mathbf{r}_A+\mathbf{r}_B+\mathbf{r}_C}{3}$
2. $\overrightarrow{PI} = \frac{a\mathbf{r}_A+b\mathbf{r}_B+c\mathbf{r}_C}{a+b+c}$
3. $\overrightarrow{PH} = \frac{(\tan A)\mathbf{r}_A+(\tan B)\mathbf{r}_B+(\tan C)\mathbf{r}_C}{\tan A+\tan B+\tan C}$
4. $\overrightarrow{PO} = \frac{(\sin 2A)\mathbf{r}_A+(\sin 2B)\mathbf{r}_B+(\sin 2C)\mathbf{r}_C}{\sin 2A+\sin 2B+\sin 2C}$

Exercise 7. Consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $M = AB' \cap A'B$ and $N = AA' \cap BB'$

Exercise 8. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

Exercise 9 (Möbius' Theorem)**.** For two points A, B in an affine space, consider the line AB. Further consider the points P_1, P_2 and P_3 in $AB \setminus \{A, B\}$. Show that

$$(A, B|P_1, P_2)(A, B|P_2, P_3)(A, B|P_3, P_1) = 1.$$

Generalize this to n points.

Exercise 10 (Menelaus' Theorem)**.** Consider a triangle ABC in an affine space and the points $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ (distinct from A, B and C). Show that A_1, B_1, C_1 are collinear if and only if

$$(B, C|A_1)(C, A|B_1)(A, B|C_1) = -1.$$

Exercise 11. For a set $\mathcal{S} = \{A_1, \ldots, A_p\}$ the barycenter

$$\mathrm{Bar}(\mathcal{S}) = \mathrm{Bar}(A_1, \ldots, A_p)$$

is sometimes refered to as the *center of mass*.

1. Let $\mathcal{S}_i = \mathcal{S} \setminus \{A_i\}$. Show that

$$\mathrm{Bar}(S) = \mathrm{Bar}(\mathrm{Bar}(\mathcal{S}_1), \ldots, \mathrm{Bar}(\mathcal{S}_p))$$

2. Let $\mathcal{S}' = \{A_1, \ldots, A_q\}$ and $\mathcal{S}'' = \{A_{q+1}, \ldots, A_p\}$. Show that

$$\mathrm{Bar}(S) = \frac{q}{p}\mathrm{Bar}(S') + \frac{p-q}{p}\mathrm{Bar}(S'')$$

3. Discuss the cases $p = 2, 3, 4$ when the points are in general position.