2.2 $GL_m(\mathbb{R}) = \{A \in \mathcal{M}_m(\mathbb{R}) \mid \det(A) \neq 0\}$

Determinant property

$\det(x \cdot y) = \det x \cdot \det y$

• s.s. $(\mathcal{M}_m(\mathbb{R}), \cdot)$

$\forall A, B \in GL_m(\mathbb{R}) : A \cdot B \in GL_m(\mathbb{R})$?

$\det(A) \neq 0$

$\det(B) \neq 0$

Let $C \in GL_m(\mathbb{R})$ $\overset{?}{\Leftrightarrow}$ $\det(C) \neq 0 \Leftrightarrow \det(A \cdot B) \neq 0 \Leftrightarrow$

$C = A \cdot B$ $\Leftrightarrow \underset{\substack{\neq \\ 0}}{\det A} \cdot \underset{\substack{\neq \\ 0}}{\det B} \neq 0 \Rightarrow \det(C) \neq 0$ True $\Rightarrow$

$\Rightarrow C \in GL_m(\mathbb{R})$

$\Rightarrow A \cdot B \in GL_m(\mathbb{R})$

$\Rightarrow$ stable subset

$(GL_m(\mathbb{R}), \cdot)$ group?

• associativity (inherited from $(\mathcal{M}_m(\mathbb{R}), \cdot)$)

$\forall A, B, C \in GL_m(\mathbb{R})$, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$?

$$\left(\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \\ a_{m1} & \dots & a_{mm} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \\ b_{m1} & \dots & b_{mm} \end{pmatrix}\right) \cdot \begin{pmatrix} c_{11} & \dots & c_{1m} \\ \vdots & & \\ c_{m1} & \dots & c_{mm} \end{pmatrix} = \dots$$

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• identity $\exists E \in GL_m(\mathbb{R}), \forall A \in GL_m(\mathbb{R}) : E \cdot A = A \cdot E = A$?

$(\mathcal{M}_m(\mathbb{R}), \cdot)$ monoid $\Rightarrow$ We choose a candidate: J_m for $GL_m(\mathbb{R})$

We know $A \cdot J_m = J_m \cdot A = A$

$\det(J_m) = 1 \neq 0 \Rightarrow J_m \in GL_m(\mathbb{R})$ $\Big\} \Rightarrow E = J_m$

• inverse $A \cdot A^{-1} = J_m$ / det $\Leftrightarrow$ $\det(A \cdot A^{-1}) = \det(J_m) = 1 \Leftrightarrow$

$\Leftrightarrow \det(A^{-1}) \cdot \det(A) = 1 \Leftrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

$\det(A) \neq 0$ because $A \in GL_m(\mathbb{R})$ $\Bigg\} \Rightarrow \det(A^{-1}) \neq 0$

$\Rightarrow A^{-1} \in GL_m(\mathbb{R})$

So, $(GL_m(\mathbb{R}), \cdot)$ group