

AWC Annual Meeting: Convergence and curvature of phylogenetic Markov chains

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(joint work with Chris Whidden and Erick Matsen)

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Question

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Answer

I don't know

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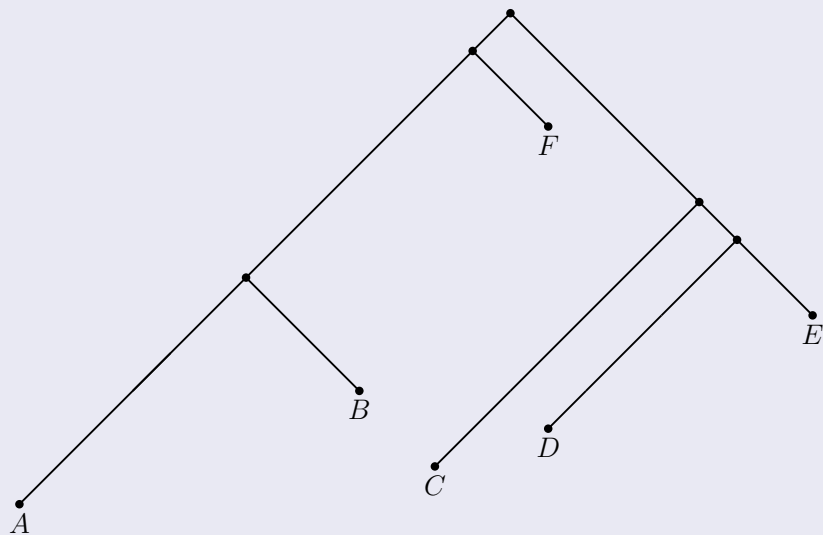
Even more specifically: Why no reliable convergence criterion is known?

Answer

I don't know geometry well.

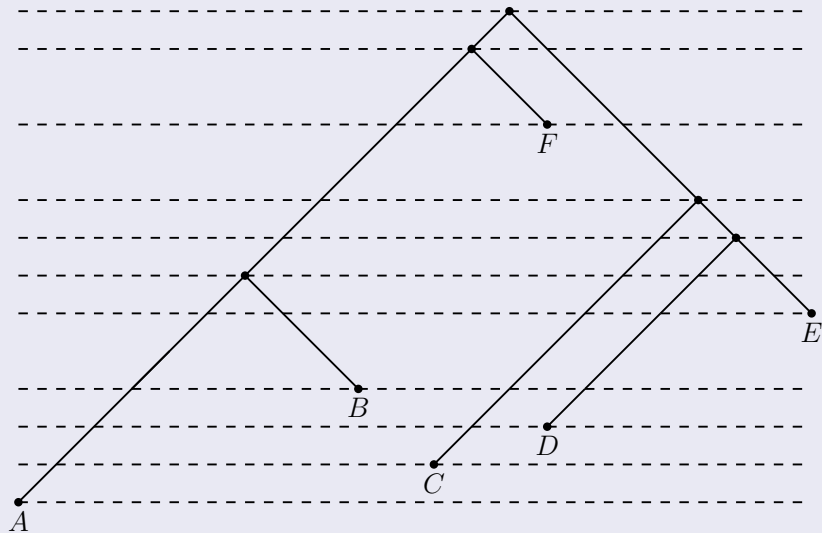
Discrete time-trees

Definition (Ranked tree topology)



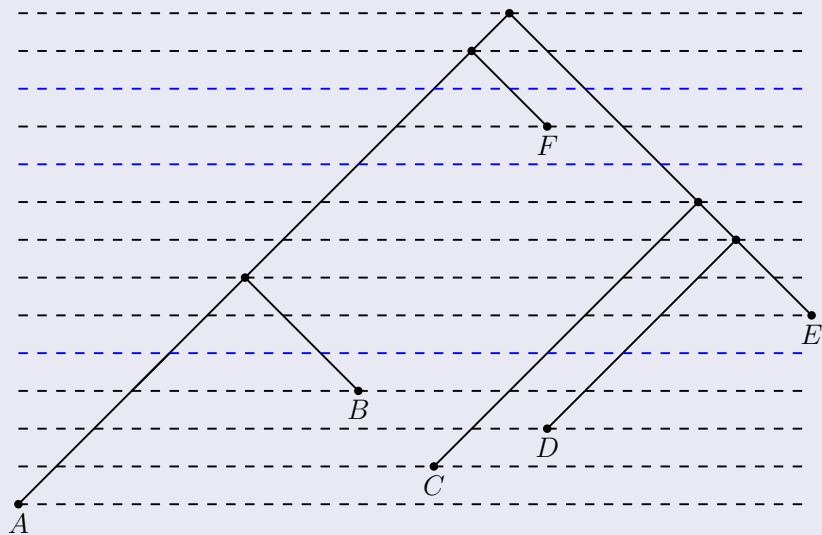
Discrete time-trees

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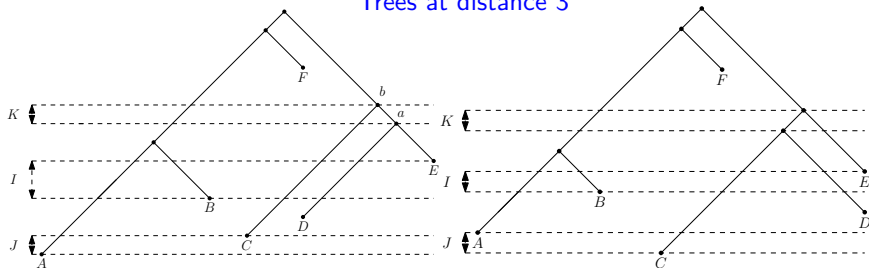
Discrete time-trees

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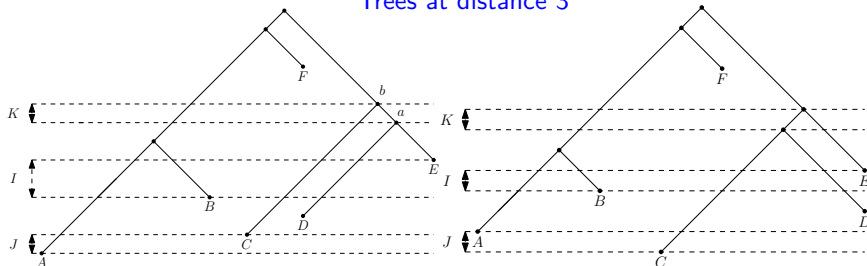
Discrete time-tree space

Trees at distance 3

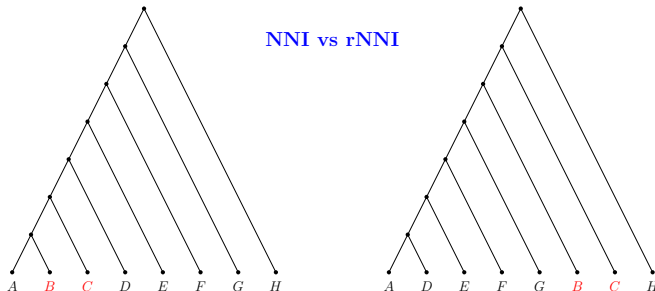


Discrete time-tree space

Trees at distance 3



NNI vs rNNI



Definition (Ollivier [2009])

Let (\mathcal{T}, d) be a metric (tree) space with a random walk

$$m = (m_T)_{T \in \mathcal{T}}.$$

Let $T, R \in \mathcal{T}$ be two distinct points (trees). The Ricci-Ollivier curvature of (\mathcal{T}, d, m) along \overrightarrow{TR} is

$$\kappa_m(T, R) = 1 - \frac{W(m_T, m_R)}{d(T, R)},$$

where $W(\cdot, \cdot)$ is the earth mover's distance.

Negative VS positive

$$\kappa_m(T, R) \leq 0 \iff W(m_T, m_R) \geq d(T, R)$$

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Take-home message

Negative curvature is bad.

Curvature of Markov chains on graphs

Theorem (Ollivier [2009])

If (\mathcal{T}, d) is a geodesic space then curvature is a local property.

Definition

Let (\mathcal{T}, d) be a graph with a Markov chain m . Then the *curvature of the Markov chain m* on the graph \mathcal{T} is the greatest number χ_m such that

$$\chi_m \leq \kappa_m(T, R) \text{ for adjacent } T \text{ and } R.$$

Trivial observation

Under a distance-one random walk, the following is true for any finite metric d and any pair of points T, R :

$$\frac{-2}{d(T, R)} \leq \kappa(T, R) \leq \frac{2}{d(T, R)}.$$

For now, we consider three simplest random walks on various phylogenetic tree spaces.

- Metropolis-Hastings random walk: Choose a tree from the one neighbourhood and accept it with probability

$$\min\left(1, \frac{|N_1(T_{old})|}{|N_1(T_{new})|}\right).$$

- Uniform random walk.
- Uniform p -lazy random walk, where p is the laziness probability.

Theorem (G, Whidden, Matsen [2015])

Let T and R be adjacent trees. Then both the asymptotic curvature of the space with p -lazy uniform random walk and the curvature of the space with uniform random walk are at least

$$\kappa(T, R) \geq \frac{-n^2 + 2n}{3.5n^2 - 15n + 16} \geq -2/5 \quad \text{in rSPR space,}$$

$$\kappa(T, R) \geq -\frac{4}{n-1} \quad \text{in DtT space,}$$

$$\kappa(T, R) \geq -\frac{4}{n-2} \quad \text{in NNI space,}$$

$$\kappa(T, R) \geq -\frac{8}{n-1} \quad \text{in rNNI space.}$$

The bounds are tight.

Theorem (G, Whidden, and Matsen [2015])

Let T and R be adjacent trees. Then the curvature of the following spaces with uniform random walk satisfy

$$\begin{array}{ll}\kappa(T, R) \leq \frac{6n - 17}{3n^2 - 13n + 14} & \text{in rSPR space,} \\ \kappa(T, R) \leq \frac{1}{2(n - 1)} & \text{in DtT space,} \\ \kappa(T, R) \leq \frac{1}{2(n - 2)} & \text{in NNI space, and} \\ \kappa(T, R) \leq \frac{1}{n - 1} & \text{in rNNI space.}\end{array}$$

The bounds are tight.

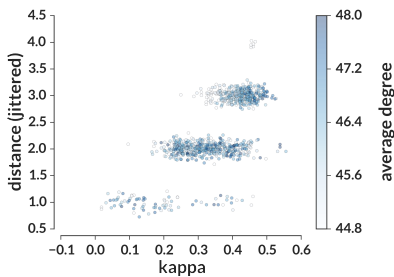
Theorem (G, Whidden, and Matsen [2015])

Let $\{T_n \mid n \in \mathbb{N}\}$ and $\{S_n \mid n \in \mathbb{N}\}$ be two sequences of phylogenetic trees such that $d(T_n, R_n) = 1$ for all n . Then

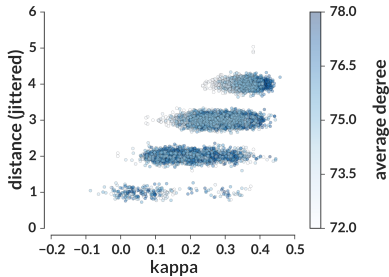
$$\lim_{n \rightarrow \infty} \kappa_n(T_n, S_n) = 0$$

for the uniform random walk on the SPR graph^{}, the NNI graph, the rNNI-graph, and the DtT-graph.*

^{*}For the SPR graph, we have to bound the size of the subtree which is getting moved.



(a) 6 taxa



(b) 7 taxa

Figure: Scatter plot of $\kappa(\text{MH}; T_1, T_2)$ values versus $d_{\text{SPR}}(T_1, T_2)$ for the rSPR graph. Colour displays the average degree of T_1 and T_2 . Distance values randomly perturbed ("jittered") a small amount to avoid superimposed points.

Take-home message

$$1.001^{10000} = 21916.68 \dots$$

BUT

$$0.999^{10000} = 0.000045 \dots$$

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- The curvature of basic random walks is normally positive.
- Although the spaces flatten out when the number of taxa n grows, there always are negatively curved pieces.
- Importantly, the number of those pieces grows with n .

Thank you for your attention!



Yann Ollivier

Ricci curvature of Markov chains on metric spaces

J. Functional Analysis, 256, 3, 810–864, 2009



Alex Gavryushkin and Alexei Drummond

The space of ultrametric phylogenetic trees

arXiv preprint arXiv:1410.3544, 2014



Chris Whidden and Frederick A. Matsen IV

Quantifying MCMC exploration of phylogenetic tree space

Systematic Biology, doi:10.1093/sysbio/syv006, 2015



Chris Whidden and Frederick A. Matsen IV

Ricci-Ollivier curvature of two random walks on rooted phylogenetic subtree-prune-regraft graph

To appear in the proceedings of the *Thirteenth Workshop on Analytic Algorithmics and Combinatorics*, 2015



Alex Gavryushkin, Chris Whidden, and Frederick A. Matsen IV

Random walks over discrete time-trees

To appear on the *arXiv*, 2015



<https://github.com/gavruskin/tTauCurvature>