Macroeconometrics Assignment 2 - Group 5 Report

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Exercise 1

In this section, we (i) list and justify the selection of nine macroeconomic variables used to forecast 'Hours Worked', (ii) Source and transform the data for the ten variables into a data matrix, and (iii) Plot the data matrix and comment with observations.

Selections
To forecast Hours worked (Index), we selected: - Household Savings Ratio (Proportion) - Total Unemployed persons (000s); - CPI Index Change (Percent change from previous period); - Real Unit Labour Costs (Index); - GDP Growth (Chain Volume Measure, Percentage change) - Inventories (Chain Volume Measure, $millions) - Public Capital (Gross fixed capital formation - Chain volume measure, $millions) - Disposable Income per Capita (Real net national disposable income per capita, $) - Terms of Trade (Percentage change from previous period)

All measures are at the national level, seasonally adjusted and measured quarterly for consistency of the analysis.

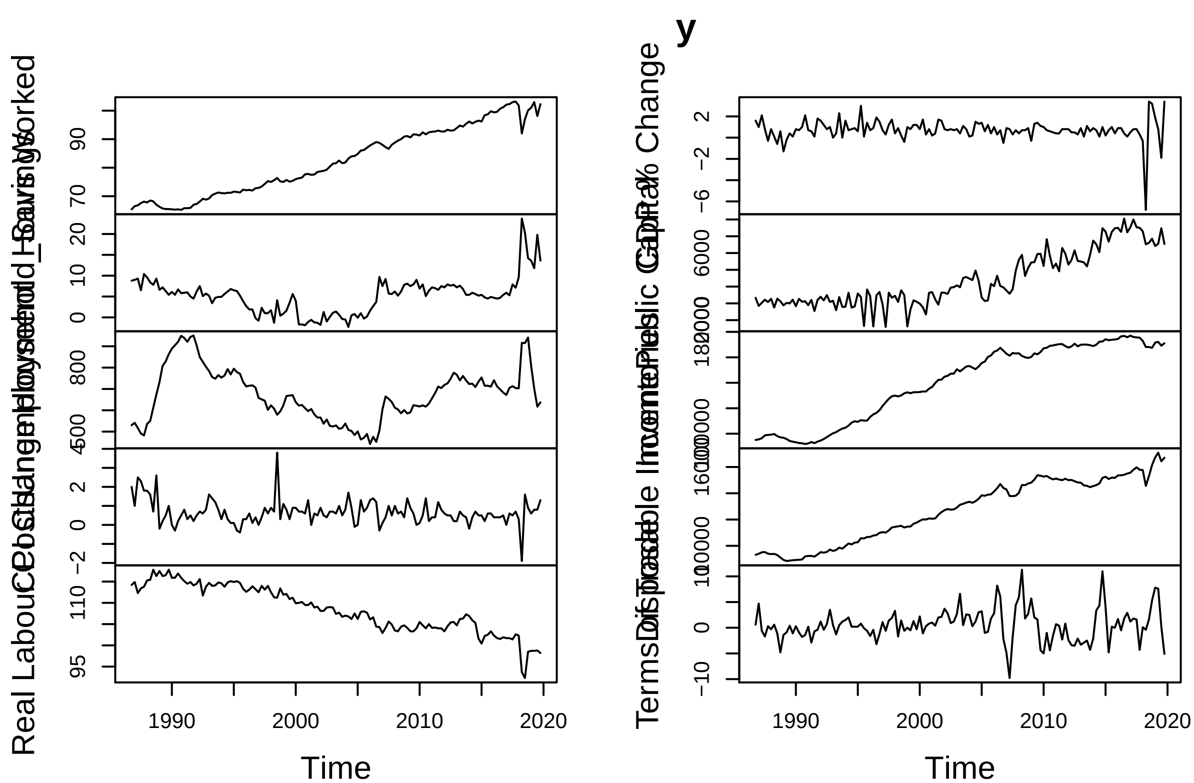
Unemployment and Real Unit Labour Costs relate to Hours worked through obvious labour supply and demand relationships between price of labour (wages) as well as the size of the workforce. Household savings and Disposable Income will relate to Labour Supply and hence Hours Worked as agents supply less labour when they can rely on greater savings/income.

We consider GDP Growth, Terms of Trade, Public Capital and Inventories to have a relationship with Hours Worked, as when the demand for goods or level of investment changes (i.e. capital), this will affect the demand for labour for production, and thus the number of hours supplied by employers. Similarly there will be a relationship between CPI (prices of household goods and services purchased) and labour demand and supply, with employers and employees making decisions based on their expenditures.

```
# INSTALLING REQUIRED PACKAGES
library(readrba)
library(readabs)
library(tidyr)
library(dplyr)
library(ggplot2)
library(lubridate)
library(reticulate)
library(mvtnorm)
library(HDInterval)
library(tinytex)
set.seed(123456)

# DATA COLLECTION AND TRANSFORMING
###########################################################
##VAR1
```

```r
HOURS                   = read_abs(series_id ="A2304428W")
HOURS                   = HOURS %>%slice(-c(1:101))
HOURS                   = HOURS[c(4,6)]

##VAR2
HHSAVINGS               = read_abs(series_id = "A2323382F")
HHSAVINGS               = HHSAVINGS %>%slice(-c(1:101))
HHSAVINGS               = HHSAVINGS[c(4,6)]

##VAR3
UNEMPLOYMENT            = read_abs(series_id = "A2454521V")
UNEMPLOYMENT            = UNEMPLOYMENT %>%slice(-c(1:109))
UNEMPLOYMENT            = UNEMPLOYMENT[c(4,6)]

##VAR4
CPI_CHANGE_Q            = read_abs(series_id = "A2325850V")
CPI_CHANGE_Q            = CPI_CHANGE_Q %>%slice(-c(1:153))
CPI_CHANGE_Q            = CPI_CHANGE_Q[c(4,6)]

##VAR5
REAL_LABOUR_COST        = read_abs(series_id = "A2433071F")
REAL_LABOUR_COST        = REAL_LABOUR_COST %>%slice(-c(1:5))
REAL_LABOUR_COST        = REAL_LABOUR_COST[c(4,6)]

##VAR6
GDP_PCT_CHANGE          = read_abs(series_id = "A2304370T")
GDP_PCT_CHANGE          = GDP_PCT_CHANGE %>%slice(-c(1:101))
GDP_PCT_CHANGE          = GDP_PCT_CHANGE[c(4,6)]

##VAR7
PUBLIC_CAPITAL          = read_abs(series_id = "A2454459T")
PUBLIC_CAPITAL          = PUBLIC_CAPITAL %>%slice(-c(1:109))
PUBLIC_CAPITAL          = PUBLIC_CAPITAL[c(4,6)]

##VAR8
INVENTORIES             = read_abs(series_id = "A3538852F")
INVENTORIES             = INVENTORIES %>%slice(-c(1:5))
INVENTORIES             = INVENTORIES[c(4,6)]

##VAR9
DISP_INC_PC             = read_abs(series_id = "A2304416L")
DISP_INC_PC             = DISP_INC_PC %>% slice(-c(1:101))
DISP_INC_PC             = DISP_INC_PC[c(4,6)]

##VAR10
TOT                     = read_abs(series_id = "A2304400V")
TOT                     = TOT %>% slice(-c(1:101))
TOT                     = TOT[c(4,6)]

# Create Y and X
y_VEC1                  = merge(HOURS, HHSAVINGS, by = 'date')
y_VEC2                  = merge(UNEMPLOYMENT, CPI_CHANGE_Q, by = 'date')
y_VEC3                  = merge(REAL_LABOUR_COST, GDP_PCT_CHANGE, by ='date')
```

```r
y_VEC4              = merge(PUBLIC_CAPITAL, INVENTORIES, by ='date')
y_VEC5              = merge(DISP_INC_PC, TOT, by ='date')

y1                  = merge(y_VEC1, y_VEC2, by = 'date')
y2                  = merge(y_VEC3, y_VEC4, by = 'date')
y3                  = merge(y1, y2, by ='date')

y                   = merge(y3, y_VEC5, by = 'date')
y                   = y %>% slice(-c(1:8))
y                   = ts(y[,c(2:11)], start = c(1986,4), frequency = 4, names=c(
                          "Hours Worked","Household_Savings","Unemployment",
                          "CPI Change","Real Labour Costs",
                          "GDP % Change","Public Capital","Inventories",
                          "Disposable Income","Terms of Trade"))
#Plot macro data and comment on trends
plot(y)
```



Observations
For many variables, we see an upward trend over time which follows Hours Worked (which has steadily increased over time). The trend is less obvious when considering variables which are percentage changes from the previous period. We also see a downward trend in real labour unit costs.

Exercise 2

In this section we document the likelihood function of the model, and the probability density function of A and Sigma which follows joint matrix-variate normal-inverse Wishart distribution, and includes two hyperparameters.

The VAR model in matrix notation is:

$$Y = XA + E$$

$$E|X \sim \mathcal{MN}_{T\times N}(\mathbf{0}_{T\times N}, \Sigma, I_T)$$

Equivalently,

$$Y = XA + E$$

$$Y|X, A, \Sigma \sim \mathcal{MN}_{T\times N}(XA, \Sigma, I_T)$$

The likelihood function is

$$L(A, \Sigma\,|Y, X) = (2\pi)^{-\frac{NT}{2}} \det(\Sigma)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} tr[\Sigma^{-1}(Y - XA)'(Y - XA)]\right\}$$

$$= (2\pi)^{-\frac{NT}{2}} \times \det(\Sigma)^{-\frac{T}{2}} \times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(A - \hat{A})'X'X(A - \hat{A})\right]\right\}$$
$$\times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(Y - X\hat{A})'(Y - X\hat{A})\right]\right\}$$

where $\hat{A} = (X'X)^{-1}X'Y$.

The pdf of the joint matrix-variate normal-inverse Wishart prior distribution for parameter matrices A and Σ is

$$p(A, \Sigma\,|\kappa_A, \kappa_\Sigma) = \mathcal{MNIW}_{K\times N}\left(\underline{A}, \kappa_A\underline{V}, \kappa_\Sigma\underline{S}, \underline{\nu}\right)$$
$$= \left\{2^{\frac{N+(K+\underline{\nu})}{2}} \pi^{\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu} + 1 - n}{2}\right)\right] \det\left(\kappa_A\underline{V}\right)^{\frac{N}{2}} \det\left(\kappa_\Sigma\underline{S}\right)^{-\frac{\underline{\nu}}{2}}\right\}^{-1}$$
$$\times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A - \underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A - \underline{A}\right)\right]\right\}$$
$$\times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

We also specify the parameters determining the prior distribution matrices

The parameters determining the prior distribution matrices are specified as the following:

$\underline{A} = [\mathbf{0}_{N\times 1} \quad I_N \quad \mathbf{0}_{N\times(p-1)N}]'$

Macroeconomic variables are unit-root nonstationary and are well-characterised by a multivariate random walk process.

$\underline{V} = diag([\kappa_\Sigma \quad \kappa_A(\boldsymbol{p}^{-2}\otimes\iota_N')])$, where $\boldsymbol{p} = [1 \quad 2 \quad \ldots \quad p]$ and ι_N is a $N \times 1$ vector of ones

This is to determine the prior shrinkage, where κ_A determines the overall shrinkage level for autoregressive slopes and κ_Σ determines the overall shrinkage for the constant term.

$\kappa_A = 0.02^2$, $\kappa_\Sigma = 100$

These are to express the stylised facts that the data are little informative about the values of the constant term and strongly favour the unit-root hypothesis.

$\underline{S} = diag(\hat{\Sigma})$, where $\hat{\Sigma} = \frac{1}{T}(Y - X\hat{A})'(Y - X\hat{A})$

$\underline{\nu} = N + 1$

This is the degrees of freedom, simply set to $N + 1$ such that $\underline{\nu} > N$.

$N = 10, p = 4.$

Exercise 3 - Joint conditional posterior distribution

The joint conditional posterior distribution of matrices A and Σ given the data matrices Y and X and hyper-parameters κ_A and κ_Σ is

$$p(A,\Sigma\,|Y,X,\kappa_A,\kappa_\Sigma) \propto L(A,\Sigma\,|Y,X)p(A,\Sigma\,|\kappa_A,\kappa_\Sigma)$$

$$p(A,\Sigma\,|Y,X,\kappa_A,\kappa_\Sigma) \propto \det(\Sigma)^{-\frac{T}{2}} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(A-\hat{A})'X'X(A-\hat{A})\right]\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(Y-X\hat{A})'(Y-X\hat{A})\right]\right\} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}\left(A-\hat{A}\right)'X'X\left(A-\hat{A}\right)\right] - \frac{1}{2}tr\left[\Sigma^{-1}\left(Y-X\hat{A}\right)'\left(Y-X\hat{A}\right)\right] - \frac{1}{2}tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2}\left\{tr\left[\Sigma^{-1}\left(A-\hat{A}\right)'X'X\left(A-\hat{A}\right)\right] + tr\left[\Sigma^{-1}\left(Y-X\hat{A}\right)'\left(Y-X\hat{A}\right)\right] + tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\}\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2}tr\left\{\Sigma^{-1}\left[\left(A-\hat{A}\right)'X'X\left(A-\hat{A}\right) + \left(Y-X\hat{A}\right)'\left(Y-X\hat{A}\right) + \left(A-\underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\}\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2}tr\left\{\Sigma^{-1}\left[A'X'XA - A'X'X\hat{A} - \hat{A}'X'XA + \hat{A}'X'X\hat{A} + Y'Y - Y'X\hat{A} - \hat{A}'X'Y + \hat{A}'X'X\hat{A} + A'\left(\kappa_A\underline{V}\right)^{-1}A - A'\left(\kappa_A\underline{V}\right)^{-1}\underline{A} - \underline{A}'\left(\kappa_A\underline{V}\right)^{-1}A + \underline{A}'\left(\kappa_A\underline{V}\right)^{-1}\underline{A}\right]\right\}\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2}tr\left\{\Sigma^{-1}\left[A'X'XA - A'X'Y - ((X'X)^{-1}X'Y)'X'XA + 2((X'X)^{-1}X'Y)'X'X\hat{A} + Y'Y - Y'X(X'X)^{-1}X'Y - ((X'X)^{-1}X'Y)'X'Y + A'\left(\kappa_A\underline{V}\right)^{-1}A - A'\left(\kappa_A\underline{V}\right)^{-1}\underline{A} - \underline{A}'\left(\kappa_A\underline{V}\right)^{-1}A + \underline{A}'\left(\kappa_A\underline{V}\right)^{-1}\underline{A}\right]\right\}\right\} \times \exp\left\{-\frac{1}{2}tr\left[\Sigma^{-1}(\kappa_\Sigma\underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'X'XA - A'X'Y - Y'XA + Y'Y + A'\left(\kappa_A \underline{V}\right)^{-1} A - A'\left(\kappa_A \underline{V}\right)^{-1} \underline{A} - \underline{A}'\left(\kappa_A \underline{V}\right)^{-1} A + \underline{A}'\left(\kappa_A \underline{V}\right)^{-1} \underline{A}\right]\right\}\right\} \times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'X'XA - A'X'Y - Y'XA + Y'Y + A'\underline{V}^{-1}\kappa_A^{-1} A - A'\underline{V}^{-1}\kappa_A^{-1}\underline{A} - \underline{A}'\underline{V}^{-1}\kappa_A^{-1} A + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A}\right]\right\}\right\} \times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'X'XA - A'X'Y - Y'XA + Y'Y + A'\underline{V}^{-1}\kappa_A^{-1} A - A'\underline{V}^{-1}\kappa_A^{-1}\underline{A} - \underline{A}'\underline{V}^{-1}\kappa_A^{-1} A + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} + \kappa_\Sigma \underline{S}\right]\right\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'(X'X + \underline{V}^{-1}\kappa_A^{-1})A - A'(X'Y + \underline{V}^{-1}\kappa_A^{-1}\underline{A}) - (Y'X + \underline{A}'\underline{V}^{-1}\kappa_A^{-1})A + Y'Y + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} + \kappa_\Sigma \underline{S}\right]\right\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'\left(X'X + \underline{V}^{-1}\kappa_A^{-1}\right)A - A'\left(X'Y + \underline{V}^{-1}\kappa_A^{-1}\underline{A}\right) - \left(Y'X + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\right)A + \overline{A}'\overline{V}^{-1}\overline{A} - \overline{A}'\overline{V}^{-1}\overline{A} + Y'Y + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} + \kappa_\Sigma \underline{S}\right]\right\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[\left(A'\left(X'X + \underline{V}^{-1}\kappa_A^{-1}\right)A - A'\left(X'Y + \underline{V}^{-1}\kappa_A^{-1}\underline{A}\right) - \left(Y'X + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\right)A + \overline{A}'\overline{V}^{-1}\overline{A}\right) - \left(\overline{A}'\overline{V}^{-1}\overline{A} - Y'Y - \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} - \kappa_\Sigma \underline{S}\right)\right]\right\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'\left(X'X + \underline{V}^{-1}\kappa_A^{-1}\right)A - A'\left(X'Y + \underline{V}^{-1}\kappa_A^{-1}\underline{A}\right) - \left(Y'X + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\right)A + \overline{A}'\overline{V}^{-1}\overline{A}\right]\right\} + \frac{1}{2} tr\left\{\Sigma^{-1}\left[\overline{A}'\overline{V}^{-1}\overline{A} - Y'Y - \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} - \kappa_\Sigma \underline{S}\right]\right\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{T+\underline{v}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'\left(X'X + \underline{V}^{-1}\kappa_A^{-1}\right)A - A'\left(X'Y + \underline{V}^{-1}\kappa_A^{-1}\underline{A}\right) - \left(Y'X + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\right)A + \overline{A}'\overline{V}^{-1}\overline{A}\right]\right\}\right\} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[Y'Y + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} + \kappa_\Sigma \underline{S} - \overline{A}'\overline{V}^{-1}\overline{A}\right]\right\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{\overline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[A'\overline{V}^{-1}A - A'\overline{V}^{-1}\overline{A} - \overline{A}'\overline{V}^{-1}A + \overline{A}'\overline{V}^{-1}\overline{A}\right]\right\}\right\} \times$$

$$\exp\left\{-\frac{1}{2} tr\{\Sigma^{-1}\overline{S}\}\right\}$$

$$= \det(\Sigma)^{-\left(\frac{\overline{\nu}+N+K+1}{2}\right)} \times \exp\left\{-\frac{1}{2} tr\left\{\Sigma^{-1}(A-\overline{A})'\overline{V}^{-1}(A-\overline{A})\right\}\right\} \times \exp\left\{-\frac{1}{2} tr\{\Sigma^{-1}\overline{S}\}\right\}$$

where

$$\overline{V} = \left(X'X + \underline{V}^{-1}\kappa_A^{-1}\right)^{-1}$$

$$\overline{A} = \overline{V}(X'Y + \underline{V}^{-1}\kappa_A^{-1}\underline{A})$$

$$\overline{S} = Y'Y + \underline{A}'\underline{V}^{-1}\kappa_A^{-1}\underline{A} + \kappa_\Sigma \underline{S} - \overline{A}'\overline{V}^{-1}\overline{A}$$

$$\overline{\nu} = T + \underline{\nu}$$

Therefore,

$$p(A, \Sigma \,|Y, X, \kappa_A, \kappa_\Sigma) \sim \mathcal{MNIW}_{K\times N}(\overline{A}, \overline{V}, \overline{S}, \overline{\nu})$$

Exercise 4

We propose that the conditionally conjugate prior distribution $(\kappa_A, \kappa_\Sigma)$ follow an independent inverse gamma 2 and gamma distribution such that

$$p(\kappa_A, \kappa_\Sigma) = p(\kappa_A)p(\kappa_\Sigma)$$

$$p(\kappa_A) = \mathcal{IG}2(\underline{s}_{\kappa_A}, \underline{\nu}_{\kappa_A}) = \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1}\left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \kappa_A^{-\frac{\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\}$$

$$p(\kappa_\Sigma) = \mathcal{G}amma(\underline{\nu}_{\kappa_\Sigma}, \underline{s}_{\kappa_\Sigma}) = \frac{\kappa_\Sigma^{\underline{\nu}_{\kappa_\Sigma}-1}\exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}}{\underline{s}_{\kappa_\Sigma}^{\underline{\nu}_{\kappa_\Sigma}}\Gamma(\underline{\nu}_{\kappa_\Sigma})} = \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}}\Gamma(\underline{\nu}_{\kappa_\Sigma})^{-1}\kappa_\Sigma^{\underline{\nu}_{\kappa_\Sigma}-1}\exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}$$

Hence the pdf is such that

$$p(\kappa_A, \kappa_\Sigma) = \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1}\left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \kappa_A^{-\frac{\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}}\Gamma(\underline{\nu}_{\kappa_\Sigma})^{-1}\kappa_\Sigma^{\underline{\nu}_{\kappa_\Sigma}-1}\exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}$$

Exercise 5

Since

$$p(\kappa_A, \kappa_\Sigma, A, \Sigma \mid Y, X) \propto L(A, \Sigma \mid Y, X)\, p(\kappa_A, \kappa_\Sigma, A, \Sigma)$$

where $L(A, \Sigma \mid Y, X)$ is not related to κ_A and κ_Σ,

hence

$$p(\kappa_A, \kappa_\Sigma, A, \Sigma \mid Y, X) \propto p(A, \Sigma \mid \kappa_A, \kappa_\Sigma)\, p(\kappa_A)\, p(\kappa_\Sigma)$$

The full conditional posterior distribution for the hyper-parameter κ_A is

$$p(\kappa_A \mid Y, X, A, \Sigma, \kappa_\Sigma) \propto p(A, \Sigma \mid \kappa_A, \kappa_\Sigma) p(\kappa_A)$$

$$p(\kappa_A \mid Y, X, A, \Sigma, \kappa_\Sigma) \propto \left\{ 2^{\frac{N+(K+\underline{\nu})}{2}} \pi^{\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2} \right) \right] \det(\kappa_A \underline{V})^{\frac{N}{2}} \det(\kappa_\Sigma \underline{S})^{-\frac{\underline{\underline{\nu}}}{2}} \right\}^{-1} \times$$
$$\det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times \exp\left\{ -\frac{1}{2} tr\left[\Sigma^{-1} (A - \underline{A})' (\kappa_A \underline{V})^{-1} (A - \underline{A}) \right] \right\} \times \exp\left\{ -\frac{1}{2} tr\left[\Sigma^{-1} (\kappa_\Sigma \underline{S}) \right] \right\} \times$$
$$\Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2} \right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2} \right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \kappa_A^{-\frac{\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{ -\frac{1}{2} \frac{\underline{s}_{\kappa_A}}{\kappa_A} \right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2} \right) \right]^{-1} \det(\kappa_A \underline{V})^{-\frac{N}{2}} \det(\kappa_\Sigma \underline{S})^{\frac{\underline{\underline{\nu}}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$
$$\exp\left\{ -\frac{1}{2} tr\left[\Sigma^{-1} (A - \underline{A})' (\kappa_A \underline{V})^{-1} (A - \underline{A}) \right] \right\} \times \exp\left\{ -\frac{1}{2} tr\left[\Sigma^{-1} (\kappa_\Sigma \underline{S}) \right] \right\} \times$$
$$\Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2} \right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2} \right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \kappa_A^{-\frac{\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{ -\frac{1}{2} \frac{\underline{s}_{\kappa_A}}{\kappa_A} \right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2} \right) \right]^{-1} \det(\kappa_\Sigma \underline{S})^{\frac{\underline{\underline{\nu}}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$
$$\exp\left\{ -\frac{1}{2} tr\left[\Sigma^{-1} (\kappa_\Sigma \underline{S}) \right] \right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2} \right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2} \right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times \exp\left\{ -\frac{1}{2} tr\left[\Sigma^{-1} (A - \underline{A})' (\kappa_A \underline{V})^{-1} (A -$$
$$\underline{A}) \right] \right\} \det(\kappa_A \underline{V})^{-\frac{N}{2}} \kappa_A^{-\frac{\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{ -\frac{1}{2} \frac{\underline{s}_{\kappa_A}}{\kappa_A} \right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times \det\left(\underline{V}\right)^{-\frac{N}{2}} \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A - \underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A - \underline{A}\right)\right]\right\} \kappa_A^{-\frac{NK}{2}} \kappa_A^{-\frac{\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times \det\left(\underline{V}\right)^{-\frac{N}{2}} \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A - \underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A - \underline{A}\right)\right]\right\} \kappa_A^{-\left(\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}\right)} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times \det\left(\underline{V}\right)^{-\frac{N}{2}} \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left\{A'\left(\kappa_A \underline{V}\right)^{-1} A - A'\left(\kappa_A \underline{V}\right)^{-1}\underline{A} - \underline{A}'\left(\kappa_A \underline{V}\right)^{-1} A + \underline{A}'\left(\kappa_A \underline{V}\right)^{-1}\underline{A}\right\}\right]\right\} \kappa_A^{-\left(\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}\right)} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times \det\left(\underline{V}\right)^{-\frac{N}{2}} \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1} A' \underline{V}^{-1} \kappa_A^{-1} A - \Sigma^{-1} A' \underline{V}^{-1} \kappa_A^{-1} \underline{A} - \Sigma^{-1} \underline{A}' \underline{V}^{-1} \kappa_A^{-1} A + \Sigma^{-1} \underline{A}' \underline{V}^{-1} \kappa_A^{-1} \underline{A}\right]\right\} \kappa_A^{-\left(\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}\right)} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times \det\left(\underline{V}\right)^{-\frac{N}{2}} \exp\left\{-\frac{1}{2}\frac{1}{\kappa_A}\left[tr(\Sigma^{-1} A' \underline{V}^{-1} A) - tr(\Sigma^{-1} A' \underline{V}^{-1} \underline{A}) - tr(\Sigma^{-1} \underline{A}' \underline{V}^{-1} A) + tr(\Sigma^{-1} \underline{A}' \underline{V}^{-1} \underline{A})\right]\right\} \kappa_A^{-\left(\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}\right)} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A}}{\kappa_A}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times$$

$$\det\left(\underline{V}\right)^{-\frac{N}{2}} \kappa_A^{-\left(\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}\right)} \exp\left\{-\frac{1}{2}\left(\frac{\underline{s}_{\kappa_A} + \left[tr(\Sigma^{-1} A' \underline{V}^{-1} A) - tr(\Sigma^{-1} A' \underline{V}^{-1} \underline{A}) - tr(\Sigma^{-1} \underline{A}' \underline{V}^{-1} A) + tr(\Sigma^{-1} \underline{A}' \underline{V}^{-1} \underline{A})\right]}{\kappa_A}\right)\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\kappa_\Sigma \underline{S}\right)\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times$$

$$\det\left(\underline{V}\right)^{-\frac{N}{2}} \kappa_A^{-\left(\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}\right)} \exp\left\{-\frac{1}{2}\left(\frac{\underline{s}_{\kappa_A} + tr\left[\Sigma^{-1}\left(A'\underline{V}^{-1}A - A'\underline{V}^{-1}\underline{A} - \underline{A}'\underline{V}^{-1}A + \underline{A}'\underline{V}^{-1}\underline{A}\right)\right]}{\kappa_A}\right)\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\kappa_\Sigma \underline{S}\right)\right]\right\} \times \Gamma\left(\frac{\underline{\nu}_{\kappa_A}}{2}\right)^{-1} \left(\frac{\underline{s}_{\kappa_A}}{2}\right)^{\frac{\underline{\nu}_{\kappa_A}}{2}} \times$$

$$\det\left(\underline{V}\right)^{-\frac{N}{2}} \kappa_A^{-\frac{NK+\underline{\nu}_{\kappa_A}+2}{2}} \exp\left\{-\frac{1}{2}\frac{\underline{s}_{\kappa_A} + tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\underline{V}^{-1}\left(A-\underline{A}\right)\right]}{\kappa_A}\right\}$$

where

$$\overline{s}_{\kappa_A} = \underline{s}_{\kappa_A} + tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\underline{V}^{-1}\left(A-\underline{A}\right)\right]$$

$$\overline{\nu}_{\kappa_A} = NK + \underline{\nu}_{\kappa_A}$$

Therefore,

$$p(\kappa_A \,|Y, X, A, \Sigma, \kappa_\Sigma) \sim \mathcal{IG}2(\overline{s}_{\kappa_A}, \overline{\nu}_{\kappa_A})$$

The full conditional posterior distribution for the hyper-parameter κ_Σ is

$$p(\kappa_\Sigma \,|Y, X, A, \Sigma, \kappa_A) \propto p(A, \Sigma \,|\kappa_A, \kappa_\Sigma) p(\kappa_\Sigma)$$

$$= \left\{2^{\frac{N+(K+\underline{\nu})}{2}} \pi^{\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right] \det\left(\kappa_A \underline{V}\right)^{\frac{N}{2}} \det\left(\kappa_\Sigma \underline{S}\right)^{-\frac{\underline{\nu}}{2}}\right\}^{-1} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\kappa_\Sigma \underline{S}\right)\right]\right\} \times$$

$$\underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \kappa_\Sigma^{\underline{\nu}_{\kappa_\Sigma}-1} \exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A\underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(\kappa_\Sigma \underline{S}\right)\right]\right\} \det\left(\kappa_\Sigma \underline{S}\right)^{\frac{\underline{\nu}}{2}} \kappa_\Sigma^{\underline{\nu}_{\kappa_\Sigma}-1} \exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \det\left(\underline{S}\right)^{\frac{\underline{\nu}}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}(\kappa_\Sigma \underline{S})\right]\right\} \kappa_\Sigma^{\frac{N\underline{\nu}}{2}+\underline{\nu}_{\kappa_\Sigma}-1} \exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \det\left(\underline{S}\right)^{\frac{\underline{\nu}}{2}} \times$$

$$\exp\left\{-\frac{1}{2} \kappa_\Sigma tr\left[\Sigma^{-1} \underline{S}\right]\right\} \kappa_\Sigma^{\frac{N\underline{\nu}}{2}+\underline{\nu}_{\kappa_\Sigma}-1} \exp\left\{-\frac{\kappa_\Sigma}{\underline{s}_{\kappa_\Sigma}}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \det\left(\underline{S}\right)^{\frac{\underline{\nu}}{2}} \times$$

$$\kappa_\Sigma^{\frac{N\underline{\nu}+2\underline{\nu}_{\kappa_\Sigma}}{2}-1} \exp\left\{-\left(\kappa_\Sigma \frac{2+tr\left[\Sigma^{-1}\underline{S}\right]\underline{s}_{\kappa_\Sigma}}{2\underline{s}_{\kappa_\Sigma}}\right)\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \det\left(\underline{S}\right)^{\frac{\underline{\nu}}{2}} \times$$

$$\kappa_\Sigma^{\frac{N\underline{\nu}+2\underline{\nu}_{\kappa_\Sigma}}{2}-1} \exp\left\{-\kappa_\Sigma \div \frac{2\underline{s}_{\kappa_\Sigma}}{2+tr\left[\Sigma^{-1}\underline{S}\right]\underline{s}_{\kappa_\Sigma}}\right\}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \det\left(\underline{S}\right)^{\frac{\underline{\nu}}{2}} \bar{s}_{\kappa_\Sigma}^{\bar{\nu}_{\kappa_\Sigma}} \Gamma\left(\bar{\nu}_{\kappa_\Sigma}\right) \times$$

$$\frac{\kappa_\Sigma^{\frac{N\underline{\nu}+2\underline{\nu}_{\kappa_\Sigma}}{2}-1} \exp\left\{-\kappa_\Sigma \div \frac{2\underline{s}_{\kappa_\Sigma}}{2+tr\left[\Sigma^{-1}\underline{S}\right]\underline{s}_{\kappa_\Sigma}}\right\}}{\bar{s}_{\kappa_\Sigma}^{\bar{\nu}_{\kappa_\Sigma}} \Gamma\left(\bar{\nu}_{\kappa_\Sigma}\right)}$$

$$= 2^{-\frac{N+(K+\underline{\nu})}{2}} \pi^{-\frac{N(N+2K-1)}{4}} \left[\prod_{n=1}^{N} \Gamma\left(\frac{\underline{\nu}+1-n}{2}\right)\right]^{-1} \det\left(\kappa_A \underline{V}\right)^{-\frac{N}{2}} \times \det(\Sigma)^{-\frac{\underline{\nu}+N+K+1}{2}} \times$$

$$\exp\left\{-\frac{1}{2} tr\left[\Sigma^{-1}\left(A-\underline{A}\right)'\left(\kappa_A \underline{V}\right)^{-1}\left(A-\underline{A}\right)\right]\right\} \times \underline{s}_{\kappa_\Sigma}^{-\underline{\nu}_{\kappa_\Sigma}} \Gamma\left(\underline{\nu}_{\kappa_\Sigma}\right)^{-1} \det\left(\underline{S}\right)^{\frac{\underline{\nu}}{2}} \bar{s}_{\kappa_\Sigma}^{\bar{\nu}_{\kappa_\Sigma}} \Gamma\left(\bar{\nu}_{\kappa_\Sigma}\right) \times$$

$$\frac{\kappa_\Sigma^{\bar{\nu}_{\kappa_\Sigma}-1} \exp\left\{-\kappa_\Sigma \div \bar{s}_{\kappa_\Sigma}\right\}}{\bar{s}_{\kappa_\Sigma}^{\bar{\nu}_{\kappa_\Sigma}} \Gamma\left(\bar{\nu}_{\kappa_\Sigma}\right)}$$

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where

$$\bar{s}_{\kappa_\Sigma} = \frac{2\underline{s}_{\kappa_\Sigma}}{2 + tr\left[\Sigma^{-1}\underline{S}\right]\underline{s}_{\kappa_\Sigma}}$$

$$\bar{\nu}_{\kappa_\Sigma} = \frac{N\underline{\nu} + 2\underline{\nu}_{\kappa_\Sigma}}{2}$$

Therefore,

$$p(\kappa_\Sigma \,|Y, X, A, \Sigma, \kappa_A) \sim \mathcal{G}amma(\bar{\nu}_{\kappa_\Sigma}, \bar{s}_{\kappa_\Sigma})$$

Exercise 6 - Gibbs Sampler approach In this section we explain our Gibbs Sampler and the use of the full conditional posterior distributions.

Exercise 7 - Gibbs Sample R Function

Exercise 8 - Predictive Density approach

Exercise 9 - Predictive Density R Function

Exercise 10 - Parameter Estimates

Exercise 11 - Q4 2021 Forecast for 2022