Lecture 8
Online and Incremental Learning

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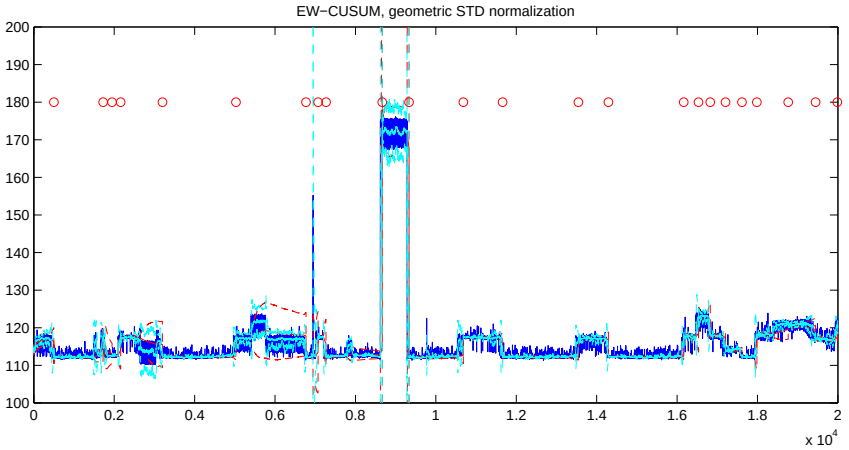
Advanced Topics in Machine Learning, 2012

Why Learning Online?

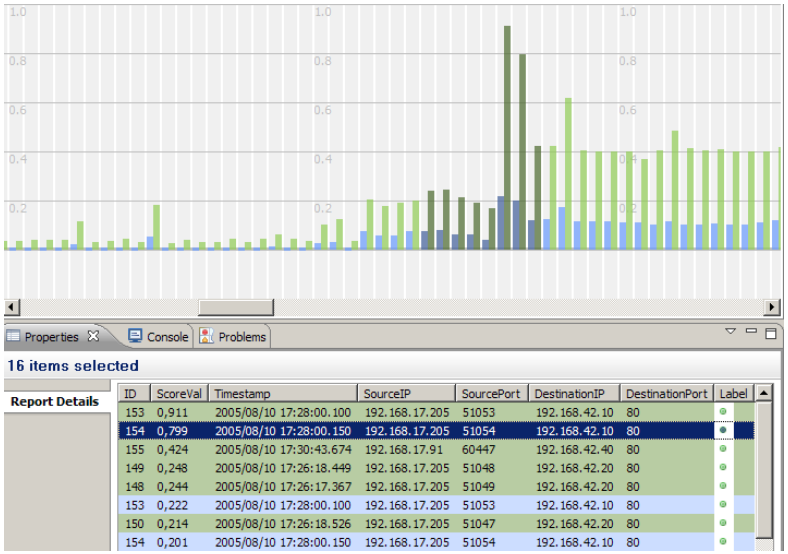
- Non-stationarity of real data
  - $\Rightarrow$ Update of models is needed when new data is available
  - $\Rightarrow$ Evaluation of relevance of past data
- Learning on a tight budget
  - $\Rightarrow$ Some data may be irrelevant
  - $\Rightarrow$ Cost-sensitive learning

Applications of Online Learning

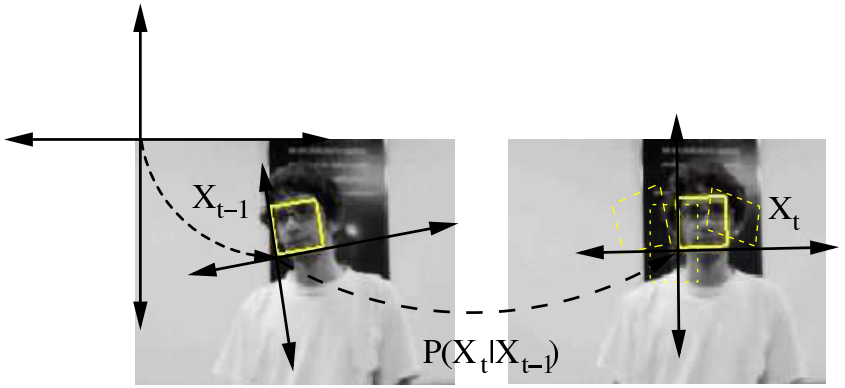
Time series analysis



Anomaly and change point detection



Security event monitoring



Object tracking

Computational Considerations

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- Ridge regression: $O(n^3)$ (matrix inversion)
- SVM: $O(n^2 \log n)$ (theoretical bound on feasible direction decomposition)

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How much are we willing to pay?

- An order of magnitude less: to match batch learning
- Constant or linear time: if we are really greedy!

Incremental SVM: Problem Setup

- Initial state: An SVM has been trained on a data set $X = \{(\mathbf{x_1}, y_1), \ldots, (\mathbf{x}_k, y_k)\}$, and the initial solution $\boldsymbol{\alpha}_0$ is available.

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- Key idea: Ensure that optimality conditions are enforced at every step of updating the solution from $\boldsymbol{\alpha}_0$ to $\boldsymbol{\alpha}_*$.
- Notation: All training data will be grouped into three sets according to the values of α:
  - set $\mathcal{S}$: $0 < \alpha_i < C$
  - set $\mathcal{O}$: $\alpha_i = 0$
  - set $\mathcal{E}$: $\alpha_i = C$

KKT Re-visited (once again ☺)

Consider the saddle-point formulation of the SVM training problem:

$$\max_{\mu} \min_{0 \leq \boldsymbol{\alpha} \leq C} W = -\mathbf{1}^\top \boldsymbol{\alpha} + \tfrac{1}{2} \boldsymbol{\alpha}^\top H \boldsymbol{\alpha} + \mu(\mathbf{y}^\top \boldsymbol{\alpha})$$

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The KKT conditions for this problem can be expressed as follows:

$$g_i = \frac{\partial W}{\partial \alpha_i} = \sum_j H_{ij}\alpha_j + y_i \mu - 1 \begin{cases} \geq 0, & i \in \mathcal{O} \\ = 0, & i \in \mathcal{S} \\ \leq 0, & i \in \mathcal{E} \end{cases}$$

$$\frac{\partial W}{\partial \mu} = \sum_j y_j \alpha_j = 0$$

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$$\frac{\partial W}{\partial \mu} = \sum_j y_j \alpha_j = 0$$

These conditions must be met for both $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_*$, i.e., before and after the inclusion of x_c, and at every single intermediate step!

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- If the last constraint is satified for $\alpha_c = 0$, then nothing needs to be done, and $\boldsymbol{\alpha}_* = \boldsymbol{\alpha}_0$.
- Otherwise we need to increase α_c and update all other α's so as to keep all conditions except for g_c satisfied.

KKT Conditions in Differential Form

Consider the difference between the optimality conditions in two different SVM states:

$$\Delta g_i = H_{ic}\Delta\alpha_c + \sum_{j\in\mathcal{S}} H_{ij}\Delta\alpha_j + y_i\Delta\mu, \quad \forall i \in D \cup \{c\}$$

$$0 = y_c\Delta\alpha_c + \sum_{j\in\mathcal{S}} y_j\Delta\alpha_j$$

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For $i \in \mathcal{S}$, $g_i = 0$; hence, these conditions can be written in matrix form:

$$\underbrace{\begin{bmatrix} 0 & y_{s_1} & \cdots & y_{s_k} \\ y_{s_1} & H_{s_1 s_1} & \cdots & H_{s_1 s_k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{s_k} & H_{s_k s_1} & \cdots & H_{s_k s_k} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \Delta\mu \\ \Delta\alpha_{s_1} \\ \vdots \\ \Delta\alpha_{s_k} \end{bmatrix}}_{\Delta\tilde{\boldsymbol{\alpha}}_{\mathcal{S}}} = -\underbrace{\begin{bmatrix} y_c \\ H_{s_1 c} \\ \vdots \\ H_{s_k c} \end{bmatrix}}_{\eta_c} \Delta\alpha_c$$

Sensitivity Conditions

- The conditions for $g_{\mathcal{S}}$ imply that

$$\Delta\tilde{\boldsymbol{\alpha}}_{\mathcal{S}} = \underbrace{-Q^{-1}\eta_c}_{\beta}\Delta\alpha_c$$

$\Rightarrow$ vector β measures sensitivity of $\tilde{\boldsymbol{\alpha}}$ with respect to $\Delta\alpha_c$.

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- Substituting the sensitivity of $\tilde{\boldsymbol{\alpha}}_{\mathcal{S}}$ back into the conditions for g, we obtain:

$$\Delta g_i = \gamma_i \Delta\alpha_c, \quad \forall i \in D \cup \{c\}$$
$$\text{where } \gamma_i = H_{ic} + \sum_{j\in\mathcal{S}} H_{ij}\beta_j + y_i\beta_0$$

  $\Rightarrow$ vector γ measures sensitivity of g_i, for $i \in \{\mathcal{O}, \mathcal{E}\}$, with respect to $\Delta\alpha_c$.

"Bookkeeping"

- Knowing the increment $\Delta\alpha_c$ we directly forecast:
  - the change $\Delta\tilde{\alpha}_{\mathcal{S}}$ of the weights for points in $\mathcal{S}$,
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- ☹ Unfortunately, these forecasts hold only as long as no structural changes occur between $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{E}$.
- Potential structural changes:
  - for $i \in \mathcal{S}, \alpha_i \to C$: i joins $\mathcal{E}$.
  - for $i \in \mathcal{S}, \alpha_i \to 0$: i joins $\mathcal{O}$.
  - for $i \in \mathcal{O}, g_i > 0 \to 0$: i joins $\mathcal{S}$.
  - for $i \in \mathcal{E}, g_i < 0 \to 0$: i joins $\mathcal{S}$.

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  - for $i \in \mathcal{O}, g_i > 0 \to 0$: i joins $\mathcal{S}$.
  - for $i \in \mathcal{E}, g_i < 0 \to 0$: i joins $\mathcal{S}$.
- Key idea: we can predict the smallest $\Delta\alpha_c$ such that for some i, the first structural change occurs.

Predicting Structural Changes

1. $i \in \mathcal{S}, \beta_i > 0$: α grows up to C.
   - find $\Delta\alpha_c^1 = \min_i \frac{C-\alpha_i}{\beta_i}$

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3. $i \in \mathcal{E}, \gamma_i > 0$ or $i \in \mathcal{O}, \gamma_i < 0$: g_i grows up / falls down to 0
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The smallest increment $\Delta\alpha_c$ among the 5 possible cases yields the first structural change.

Structural Change: Post-processing

The main operation to be performed after the structural change is the update of the inverse matrix Q^{-1}.

- Example k joins $\mathcal{S}$:
  - $\Rightarrow$ add a row and a column with zero entries to Q^{-1}
  - $\Rightarrow$ perform a recursive update (next slide)
- Example k leaves $\mathcal{S}$:
  - $\Rightarrow$ re-arrange Q^{-1} to put row and column k "at the end"
  - $\Rightarrow$ update the values (follows)

Recursive Expansion of Q^{-1}

Consider the specific structure of Q:

$$\tilde{Q}^{-1} = \begin{bmatrix} 0 & y_s^\top & y_k \\ y_s & H_{ss} & H_{sk} \\ y_k & H_{ks} & H_{kk} \end{bmatrix}^{-1} = \begin{bmatrix} Q & \eta_k \\ \eta_k^\top & H_{kk} \end{bmatrix}^{-1}$$

where

$$\eta_k^\top = \begin{bmatrix} y_k & H_{ks} \end{bmatrix}$$

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We will make use of the Sherman-Woodbury-Morrison formula:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BZCA^{-1} & -A^{-1}BZ \\ -ZCA^{-1} & Z \end{bmatrix}$$

where $Z = (D - CA^{-1}B)^{-1} = (H_{kk} - \eta_k^\top Q^{-1}\eta_k)^{-1}$ (scalar !)

Recursive Expansion of Q^{-1} (ctd.)

Putting everything together and using the fact that $\beta_k = -Q^{-1}\eta_k$ (see slide 9), we obtain:

$$\begin{bmatrix} 0 & y_s^\top & y_k \\ y_s & H_{ss} & H_{sk} \\ y_k & H_{ks} & H_{kk} \end{bmatrix}^{-1} = \begin{bmatrix} Q^{-1} + Z(Q^{-1}\eta_k)(Q^{-1}\eta_k)^\top & -Z(Q^{-1}\eta_k) \\ -Z(Q^{-1}\eta_k)^\top & Z \end{bmatrix}$$

$$= \begin{bmatrix} Q^{-1} & 0 \\ 0 & 0 \end{bmatrix} + Z \begin{bmatrix} \beta_k \\ 1 \end{bmatrix} \begin{bmatrix} \beta_k^\top & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} Q^{-1} & 0 \\ 0 & 0 \end{bmatrix} + Z \begin{bmatrix} \beta_k \\ 1 \end{bmatrix} \begin{bmatrix} \beta_k^\top & 1 \end{bmatrix}$$

Hence, expansion of Q^{-1} amounts to:

- extending Q^{-1} with a zero row and column ($O(s)$)
- computing Z ($O(s)^2$)
- computing and adding the outer product of $[\beta_k^\top \; 1]$ ($O(s)^2$)

Recursive Contraction of Q^{-1}

Before contraction of Q^{-1}, we have:

$$Q^{-1} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} \tilde{Q}^{-1} + Z\beta_k\beta_k^\top & Z\beta_k \\ Z\beta_k^\top & Z \end{bmatrix}$$

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Writing this relation component-wise, we obtain:

$$\begin{aligned} \tilde{Q}^{-1} &= q_{11} - Z\beta_k\beta_k^\top \\ q_{21}^\top = q_{12} &= Z\beta_k \\ q_{22} &= Z \end{aligned}$$

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Combination of the three constraints above yields:

$$\tilde{Q}^{-1} = q_{11} - \frac{q_{12}q_{21}}{q_{22}}$$

Incremental SVM: High-Level Algorithm

1: Read example x_c, compute g_c.
2: **while** $g_c < 0$ & $\alpha_c < C$ **do**
3: Compute β and γ.
4: Compute $\Delta\alpha_c^1, \ldots, \Delta\alpha_c^5$.
5: $\Delta\alpha_c^{\max} \leftarrow \min\{\Delta\alpha_c^1, \ldots, \Delta\alpha_c^5\}$
6: $\alpha_c \leftarrow \alpha_c + \Delta\alpha_c^{\max}$
7: $\alpha_s \leftarrow \beta\Delta\alpha_c^{\max}$
8: $g_{c,e,o} \leftarrow \gamma\Delta\alpha_c^{\max}$
9: Let k be the index of the example yielding the minimum in step 5.
10: **if** $k \in \mathcal{S}$ **then**
11: Move k from $\mathcal{S}$ to either $\mathcal{E}$ or $\mathcal{O}$.
12: **else if** $k \in \mathcal{E} \cup \mathcal{O}$ **then**
13: Move k from either $\mathcal{E}$ or $\mathcal{O}$ to $\mathcal{S}$.
14: **else**
15: $\{k = c$: do nothing, the algorithm terminates.$\}$
16: **end if**
17: Update Q^{-1} recursively.
18: **end while**

Incremental SVM: Summary

- The only general method to exactly update an existing SVM solution
- Can be reversed for incremental "unlearning"
- Can be extended for regression and anomaly detection
- Moderate performance:
  - $O(s^2)$ time and space (s being the size of the set $\mathcal{S}$)
  - suitable for working sets of up to 50,000 examples
- Further usage:
  - Computation of the leave-one-out error
  - Poisoning attacks against SVM

Online Learning

An online learning algorithm looks at every example exactly once.

- ☺ Easy update of existing solutions
- ☺ Low storage requirements
- ☺ (Sometimes) faster learning of batch datasets
- ☹ Inexact: may converge to a different solution
- ☹ Potentially slow convergence rate

LASVM: a Simple Online Extension of SMO

- Function PROCESS(k):
  1: $\alpha_k \leftarrow 0$, compute g_k, add k to $\mathcal{S}$.
  2: $j \leftarrow \text{argmin}_{s \in \mathcal{S}} y_s g_s$
  3: Bail out if (j, k) is not a τ-violating pair.
  4: Perform an SMO update step on (j, k).
- Function REPROCESS:
  1: $i \leftarrow \text{argmax}_{s \in \mathcal{S}} g_s$ with $\alpha_s < C$
  2: $j \leftarrow \text{argmin}_{s \in \mathcal{S}} g_s$ with $\alpha_s > 0$
  3: Bail out if (j, k) is not a τ-violating pair.
  4: Perform an SMO update step on (i, j).

LASVM: Main Algorithm and Analysis

- Initialization
  - Seed $\mathcal{S}$ with a few examples of each class and compute the initial gradient g
- Online iterations: repeat a predefined number of times:
  - Pick an example k_t
  - Run PROCESS(k_t)
  - Run REPROCESS once
- Finishing:
  - Repeat REPROCESS until $\delta \leq \tau$

☺ The online part brings us close enough to the optimal solution at low linear cost.

☹ Still no hard guarantees on the solution quality: the final stage has an uncontrollable number of iterations.

Stochastic Gradient Descent (SGD)

The Best Pill for an Easy Case

Consider a linear SVM in d-dimensional case. A large number of simple online algorithms implement the following simple strategy:

- Pick up a random example $\mathbf{x}_t$
- Update the weight as:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \gamma_t B_t \nabla_{\mathbf{w}} Q(\mathbf{x}_t, \mathbf{w}_t)$$

where γ_t is a learning rate and B_t is a scaling matrix.

Under mild regularity conditions and with appropriate learning rates (decreasing gains satisfying $\sum_t \gamma_t^2 < \infty$), the stochastic gradient descent converges to an optimal solution.

SGD Variants and their Convergence Rates

- First order, $B = \lambda^{-1} I$:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \frac{1}{\lambda(t + t_0)} g_t(\mathbf{w}_t)$$

  - Iterations to reach accuracy ρ: $\frac{\nu\kappa^2}{\rho} + O(\frac{1}{\rho})$, iteration cost: $O(d)$

- Second order, $B = H^{-1}$:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t - \frac{1}{t + t_0} H^{-1} g_t(\mathbf{w}_t)$$

  - Iterations to reach accuracy ρ: $\frac{\nu}{\rho} + O(\frac{1}{\rho})$, iteration cost: $O(d^2)$

Summary

- In contrast to incremental learning, online learning attempts to achieve fixed cost per example, potentially at the expense of learning accuracy.
- Theoretical analysis shows that high optimization accuracy is not always necessary for learning accuracy.
- Efficient algorithms for online learning exist mainly for linear learning problems.
- For non-linear methods, online learning currently cannot guarantee perfect covergence to the optimal solution.

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