Thompson Sampling for Cascading Bandits

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Abstract

We design and analyze TS-Cascade, a Thompson sampling algorithm for the cascading bandit problem. In TS-Cascade, Bayesian estimates of the click probability are constructed using a univariate Gaussian; this leads to a more efficient exploration procedure vis-à-vis existing UCB-based approaches. We also incorporate the empirical variance of each item's click probability into the Bayesian updates. These two novel features allow us to prove an expected regret bound of the form $O(\sqrt{KLT})$ where L and K are the number of ground items and the number of items in the chosen list respectively and $T \geq L$ is the number of Thompson sampling update steps. This matches the state-of-theart regret bounds for UCB-based algorithms. More importantly, it is the first theoretical guarantee on a Thompson sampling algorithm for any stochastic combinatorial bandit problem model with partial feedback. Empirical experiments demonstrate superiority of TS-Cascade compared to existing UCB-based procedures in terms of the expected cumulative regret and the time complexity.

Introduction 1

Online recommender systems seek to recommend a small list of items (such as movies or hotels) to users based on a larger ground set $[L] := \{1, \ldots, L\}$ of items. The model we consider in this paper is the cascading bandits model (Kveton et al., 2015a). In the standard cascade model of Craswell et al. (2008), which is used widely in information retrieval and online advertising, the user, upon seeing this set of items, scans through it in a sequential manner. She looks at the first item and if she is attracted by it, clicks on it. If not, she skips to the next item and clicks on it if she finds it attractive. This process stops when she clicks on one item in the list or when she comes to the end of the list, in which case she is *not attracted* by *any* of the items. The items that are in the ground set but not in the chosen list and those in the list that come after the attractive one are unobserved. Each item $i \in [L]$, which has a certain click probability $w(i) \in [0,1]$, attracts the user independently of other items. Under this assumption, the optimal solution is the list of items that maximizes the probability that the user finds an attractive item. This is precisely the list of the most attractive items.

In the multi-armed bandits version of the cascade model (Kveton et al., 2015a), the click probabilities w := $\{w(i)\}_{i=1}^{L}$ of the L items are unknown to the learning agent, and should be learned over time. If the user clicks on any item in the list, a reward of one is obtained by the learning agent. Otherwise, no reward is obtained. Based on the lists previously chosen and the rewards obtained thus far, the agent tries to learn the click probabilities (exploration) in order to adaptively and judiciously recommend other lists of items (exploitation) to maximize his overall reward after T time steps.

Main Contributions. We design and analyze TS-Cascade, a Thompson sampling algorithm (Thompson, 1933) for the cascading bandits problem. Our design involves the two novel features. First, the Bayesian estimates on the vector of latent click probabilities w are constructed by a univariate Gaussian distribution. Consequently, in each time step, Ts-Cascade conducts exploration in a suitably defined one-dimensional space. This leads to a more efficient exploration procedure than the existing Upper Confidence Bound (UCB) approaches, which conduct exploration in L-dimensional confidence hypercubes. Second, inspired by Audibert et al. (2009), we judiciously incorporate the empirical variance of each item's click probability in the Bayesian update. The allows efficient exploration on item i when w(i) is close to 0 or 1.

We establish a problem independent regret bound

for our proposed algorithm TS-CASCADE. Our regret bound matches the state-of-the-art regret bound for UCB algorithms on the cascading bandit model (Wang and Chen, 2017), up to a multiplicative logarithmic factor in the number of time steps T, when T > L. Our regret bound is the first theoretical guarantee on a Thompson sampling algorithm for the cascading bandit problem model, or for any stochastic combinatorial bandit problem model with partial feedback (see literature review). This addresses an open question on Thompson sampling raised by Zong et al. (2016), in the special case of no linear generalization. In our analysis, we successfully disentangle the statistical dependence between partial monitoring and Thompson sampling, by analyzing a suitably weighted version of the Thompson samples (Lemma 4.2). In addition, we reconcile the statistical inconsistency in using Gaussian random variables to model click probabilities by considering a certain truncated version of the Thompson samples (Lemma 4.4).

Literature Review. Our work is closely related to existing works on the class of stochastic combinatorial bandit (SCB) problems and Thompson sampling. In an SCB model, an arm corresponds to a subset of a ground set of items, each associated with a latent random variable. The corresponding reward depends on the constituent items' realized random variables. SCB models with semi-bandit feedback, where a learning agent observes all random variables of the items in a pulled arm, are extensively studied in existing works. Assuming semi-bandit feedback, Anantharam et al. (1987) study the case when the arms constitute a uniform matroid, Kveton et al. (2014) study the case of general matroids, Gai et al. (2010) study the case of permutations, and Gai et al. (2012), Chen et al. (2013), Combes et al. (2015), and Kveton et al. (2015b) investigate various general SCB problem settings. More general settings with contextual information (Li et al. (2010); Qin et al. (2014)) and linear generalization (Wen et al. (2015)) are also studied. All of the works listed above are based on the UCB idea.

Motivated by numerous applications in recommender systems and online advertisement placement, SCB models are studied under a more challenging setting of partial feedback, where a learning agent only observes the random variables for a subset the items in the pulled arm. A prime example of SCB model with partial feedback is the cascading bandit model, which is first introduced by Kveton et al. (2015a). Subsequently, Kveton et al. (2015c), Katariya et al. (2016), Lagrée et al. (2016) and Zoghi et al. (2017) study the cascading bandit model in various general settings. Cascading bandits with contextual information (Li et al. (2016)) and linear generalization

(Zong et al. (2016)) are also studied. Wang and Chen (2017) provide a general algorithmic framework on SCB models with partial feedback. All of the works listed above are also based on UCB.

On the one hand, UCB has been extensively applied for solving various SCB problems. On the other hand, Thompson sampling (Thompson, 1933; Chapelle and Li, 2011; Russo et al., 2018), an online algorithm based on Bayesian updates, has been shown to be empirically superior compared to UCB and ϵ -greedy algorithms in various bandit models. The empirical success has motivated a series of research works on the theoretical performance guarantees of Thompson sampling on multi-armed bandits (Agrawal and Goyal, 2012; Kaufmann et al., 2012; Agrawal and Goyal, 2013a, 2017), linear bandits (Agrawal and Goyal, 2013b), generalized linear bandits (Abeille and Lazaric, 2017), etc. Thompson sampling has also been studied for SCB problems with semi-bandit feedback. Komiyama et al. (2015) study the case when the combinatorial arms constitute a uniform matroid; Wang and Chen (2018) investigate the case of general matroids, and Gopalan et al. (2014) and Hüyük and Tekin (2018) consider settings with general reward functions. In addition, SCB problems with semi-bandit feedback are also studied in the Bayesian setting (Russo and Van Roy, 2014), where the latent model parameters are assumed to be drawn from a known prior distribution. Despite existing works, an analysis of Thompson sampling for an SCB problem in the more challenging case of partial feedback is yet to be done. Our work fills in this gap in the literature, and our analysis provides tools for handling the statistical dependence between Thompson sampling and partial feedback in the cascading bandit models.

2 Problem Setup

Let there be $L \in \mathbb{N}$ ground items, denoted as $[L] := \{1,\ldots,L\}$. Each item $i \in [L]$ is associated with a weight $w(i) \in [0,1]$, signifying the item's click probability. At each time step $t \in [T]$, the agent selects a list of $K \leq L$ items $S_t := (i_1^t,\ldots,i_K^t) \in \pi_K(L)$ to the user, where $\pi_K(L)$ denotes the set of all K-permutations of [L]. The user examines the items from i_1^t to i_K^t by examining each item one at a time until possibly all items are examined. For $1 \leq k \leq K$, $W_t(i_k^t) \sim \operatorname{Bern}(w(i_k^t))$ are i.i.d. and $W_t(i_k^t) = 1$ iff user clicks on i_k^t at time t.

The $instantaneous\ reward$ of the agent at time t is

$$R(S_t|\mathbf{w}) := 1 - \prod_{k=1}^{K} (1 - W_t(i_k^t)) \in \{0, 1\}.$$

In other words, the agent gets a reward of $R(S_t|\mathbf{w}) = 1$

Table 1: Upper bounds on the T-regret of TS-CASCADE, CUCB, CASCADEUCB1 and CASCADEKL-UCB and the lower bound of all Cascading bandits algorithms.

Algorithm	Reference	Bounds	Problem Indep.
TS-CASCADE	Present paper	$O(\sqrt{KLT}\log T + L\log^{5/2}T)$	
CUCB	Wang and Chen (2017)	$O(\sqrt{KLT\log T})$	$\sqrt{}$
CASCADEUCB1	Kveton et al. (2015a)	$O((\hat{L} - K)(\log T)/\Delta)$	×
CASCADEKL-UCB	Kveton et al. (2015a)	$O((L-K)\log(T/\Delta)/\Delta)$	×
Cascading Bandits	Kveton et al. (2015a)	$\Omega((L-K)(\log T)/\Delta)$ (Lower Bd)	×

if $W_t(i_k^t) = 1$ for some $1 \le k \le K$, and a reward of $R(S_t|\mathbf{w}) = 0$ if $W_t(i_k^t) = 0$ for all $1 \le k \le K$.

The feedback of the agent at time t is defined as

$$k_t := \min\{1 \le k \le K : W_t(i_k^t) = 1\},\$$

where we assume that the minimum over an empty set is ∞ . If $k_t < \infty$, then the agent observes $W_t(i_k^t) = 0$ for $1 \le k < k_t$, and also observes $W_t(i_k^t) = 1$, but does not observe $W_t(i_k^t)$ for $k > k_t$; otherwise, $k_t = \infty$, then the agent observes $W_t(i_k^t) = 0$ for $1 \le k \le K$.

As the agent aims to maximize the sum of rewards over all steps, a expected cumulative regret is defined to evaluate the performance of an algorithm. First, the expected instant reward is

$$r(S|\boldsymbol{w}) = \mathbb{E}[R(S|\boldsymbol{w})] = 1 - \prod_{i_k \in S} (1 - w(i_k)).$$

Note that the expected reward is permutation invariant, but the randomness in the set of observed items is not. Without loss of generality, we assume that $w(1) \geq w(2) \geq \ldots \geq w(L)$, then any permutation of $\{1,\ldots,K\}$ maximizes the mean reward. We let $S^*=(1,\ldots,K)$ be an optimal ordered K-subset for maximizing the expected reward; items in S^* as optimal items and others as suboptimal items. In T steps, we aim to minimize the expected cumulative regret:

$$\operatorname{Reg}(\mathbf{T}) := T \cdot r(S^* | \boldsymbol{w}) - \sum_{t=1}^{T} r(S_t | \boldsymbol{w}),$$

while the vector of click probabilities $\mathbf{w} \in [0, 1]^L$ is not known to the agent, and S_t is chosen online, i.e., dependent on previous choices and the previous rewards.

3 Algorithm

Our algorithm is presented in Algorithm 1. Intuitively, to minimize the expected cumulative regret, the agent aims to learn the true weight w(i) of each item $i \in [L]$ by exploring the space to identify S^* (i.e., exploitation) after a hopefully small number of steps. In our algorithm, we approximate the true

weight w(i) of each item i by an statistic $\theta_t(i)$ at each time step t. This statistic is known as the *Thompson sample*. To do so, first, we sample a one-dimensional standard Gaussian $Z_t \sim \mathcal{N}(0,1)$, define the empirical variance $\hat{\nu}_t(i) = \hat{\mu}_t(i)(1-\hat{\mu}_t(i))$ of the previously observed arms, and calculate $\theta_t(i)$. Secondly, we select $S_t = (i_1^t, i_2^t, \dots, i_K^t)$ such that $\theta_t(i_1^t) \geq \theta_t(i_2^t) \geq \dots \geq \theta_t(i_K^t) \geq \max_{j \notin S_t} \theta_t(j)$; this is reflected in Line 10 of Algorithm 1. Finally, we update the parameters for each observed item i in a standard manner by applying Bayes rule on the mean of the Gaussian (with conjugate prior being another Gaussian) in Line 13.

The algorithm results in the following theoretical guarantee. The proof is sketched in Section 4.

Theorem 3.1. Consider the cascading bandit problem. Algorithm TS-CASCADE, presented in Algorithm 1, incurs an expected regret at most

$$O(\sqrt{KLT}\log T + L\log^{5/2}T),$$

where the big O notation hides a constant factor that is independent of K, L, T, \mathbf{w} .

In practical applications, $T \gg L$ and so the regret bound is essentially $\tilde{O}(\sqrt{KLT})$. We elaborate on the main features of the algorithm and the guarantee.

Firstly, this is Thompson sampling (Thompson, 1933) applied to the cascading bandits problem with partial feedback. The algorithm, which only utilizes partial information is designed for real applications where a user stops to examine other items after observing an attractive one. Hence, the feedback from the user only reveals whether the examined items are attractive but no information about the un-examined ones.

Secondly, even though it is more natural to use a Beta-Bernoulli update to maintain a Bayesian estimate on the probability w(i) (Russo et al., 2018), we use the Gaussian distribution instead of the Beta distribution in our algorithm. The use of the Gaussian is useful, since it allows us to readily generalize the algorithm and analyses to the contextual setting (Li et al., 2010). This handles heterogeneity in the online setting (Li et al., 2016), as well as the linear bandits setting

Algorithm 1 TS-CASCADE, Thompson Sampling for Cascading Bandits with Gaussian Update

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1: Initialize \hat{\mu}_1(i) = 0, N_1(i) = 0 for all i \in [L].

2: for t = 1, 2, ... do

3: Sample a 1-dim r.v. Z_t \sim \mathcal{N}(0, 1).

4: for i \in [L] do

5: Calculate the empirical variance \hat{\nu}_t(i) = \hat{\mu}_t(i)(1 - \hat{\mu}_t(i)).
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6: Calculate std. dev. of the Thompson sample

$$\sigma_t(i) = \max\left\{\sqrt{\frac{\hat{\nu}_t(i)\log(t+1)}{N_t(i)+1}}, \frac{\log(t+1)}{N_t(i)+1}\right\}.$$

7: Construct the Thompson sample

$$\theta_t(i) = \hat{\mu}_t(i) + Z_t \sigma_t(i).$$

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8: end for
9: for k \in [K] do
10: Extract i_k^t \in \operatorname{argmax}_{i \in [L] \setminus \{i_1^t, \dots, i_{k-1}^t\}} \theta_t(i).
11: end for
12: Pull arm S_t = (i_1^t, i_2^t, \dots, i_K^t).
13: For each i \in [L], if W_t(i) is observed, define \hat{\mu}_{t+1}(i) = \frac{N_t(i)\hat{\mu}_t(i) + W_t(i)}{N_t(i) + 1}, N_{t+1}(i) = N_t(i) + 1.
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Otherwise, $\hat{\mu}_{t+1}(i) = \hat{\mu}_t(i), N_{t+1}(i) = N_t(i)$. 14: **end for**

(Zong et al., 2016) for handling a large L. We plan to study these extensions in a future work. However, the analysis of the Thompson sampling algorithm with the use of a Gaussian Thompson sample also comes with some difficulties as $\theta_t(i)$ is not in [0,1] with probability one. We perform a truncation of the Gaussian Thompson sample in the proof of Lemma 4.4 to show that this replacement of the Beta by the Gaussian does not incur any significant loss in terms of the regret and the analysis is not affected significantly.

Thirdly, Lines 5–7 indicate that the Thompson sample $\theta_t(i)$ is constructed to be a Gaussian random variable with mean $\hat{\mu}_t(i)$ and variance being the maximum of $\hat{\nu}_t(i) \log(t+1)/(N_t(i)+1)$ and $[\log(t+1)/(N_t(i)+1)]^2$. Note that $\hat{\nu}_t(i)$ is the variance of a Bernoulli distribution with mean $\hat{\mu}_t(i)$. In Thompson sampling algorithms, the choice of the variance is of crucial importance. The reason why we choose the variance in this manner is to (i) make the Bayesian estimates behave like Bernoulli random variables and to (ii) ensure that it is tuned so that the regret bound has a dependent

dence on \sqrt{K} (see Lemma 4.3) and does not depend on any pre-set parameters. We utilize a key result by Audibert et al. (2009) concerning the analysis of using the empirical variance in multi-arm bandit problems to achieve (i). In essence, in Lemma 4.3, the Thompson sample is shown to depend only on a single source of randomness, i.e., the Gaussian random variable Z_t (Line 3 of Algorithm 1). This shaves of a factor of \sqrt{K} vis-à-vis a more naïve analysis where the variance is pre-set in the relevant probability in Lemma 4.3 depends on K independent random variables.

Finally, in Table 1, we compare our regret bound for cascading bandits to those in the literature which are all based on the UCB idea (Wang and Chen, 2017; Kveton et al., 2015a). Note that the last column indicates whether or not the algorithm is problem dependent; being problem dependent means that the bound depends on the vector of click probabilities w. To present our results succinctly, for the problem dependent bounds, we assume that the optimal items have the same click probability w_1 and the suboptimal items also have the same click probability $w_2 < w_1$; note though that TS-CASCADE makes no such assumption. The gap $\Delta := w_1 - w_2$ is a measure of the difficulty of the problem. Table 1 implies that our upper bound grows like \sqrt{T} just like the others. Our bound also matches the state-of-the-art UCB bound (up to log factors) by Wang and Chen (2017), whose algorithm, when suitably specialized to the cascading bandits setting, is the same as CASCADEUCB1 in Kveton et al. (2015a). For the case in which $T \geq L$, our bound is a $\sqrt{\log T}$ factor worse than the problem independent bound in Wang and Chen (2017) but we are the first to analyze Thompson sampling for the cascading bandits problem.

4 Proof Sketch of Theorem 3.1

In this section, we prove a proof sketch of Theorem 3.1. We also provide the proofs of Lemmas 4.3 and 4.5. The remaining lemmas are proved in Appendix B.

During the iterations, we update $\hat{\mu}_{t+1}(i)$ such that it approaches w(i) eventually. To do so, we select a set S_t according to the order of $\theta_t(i)$'s at each time step. Hence, if $\hat{\mu}_{t+1}(i)$, $\theta_t(i)$ and w(i) are close enough, then we are likely to select the optimal set. This motivates us to define two "nice events" as follows:

$$\mathcal{E}_{\hat{\mu},t} := \{ \forall i \in [L] : |\hat{\mu}_t(i) - w(i)| \le g_t(i) \}, \\ \mathcal{E}_{\theta,t} := \{ \forall i \in [L] : |\theta_t(i) - \hat{\mu}_t(i)| < h_t(i) \},$$

where $\hat{\nu}_t(i)$ is defined in Line 5 of Algorithm 1, and

$$g_t(i) := \sqrt{\frac{16\hat{\nu}_t(i)\log(t+1)}{N_t(i)+1}} + \frac{24\log(t+1)}{N_t(i)+1},$$

$$h_t(i) := \sqrt{\log(t+1)}g_t(i).$$

Lemma 4.1. For each $t \in [T]$, we have

$$\Pr\left[\mathcal{E}_{\hat{\mu},t}\right] \ge 1 - \frac{3L}{(t+1)^3}, \quad \Pr\left[\mathcal{E}_{\theta,t}|\mathcal{E}_{\hat{\mu},t}\right] \ge 1 - \frac{1}{2(t+1)^2}.$$

Demonstrating that $\mathcal{E}_{\hat{\mu},t}$ has high probability requires the concentration inequality in Theorem A.1; this is a specialization of a result in Audibert et al. (2009) to Bernoulli random variables. Besides, demonstrating that $\mathcal{E}_{\hat{\theta},t}$ has high probability requires the concentration property of Gaussian random variables, as established in Theorem A.2.

To start our analysis, define

$$F(S,t) := \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j)) \right] (g_t(i_k) + h_t(i_k)). \quad (4.1)$$

Define the set

$$S_t := \left\{ S = (i_1, \dots, i_K) \in \pi_K(L) : \right.$$
$$F(S, t) \ge r(S^* | \boldsymbol{w}) - r(S | \boldsymbol{w}) \right\}.$$

Recall that $w(1) \geq w(2) \geq \ldots \geq w(L)$. As such, S_t is non-empty, since $S^* = (1, 2, \ldots, K) \in S_t$.

Intuitions behind set S_t and its complement \bar{S}_t . Ideally, we expect the user to click an item in S_t for every time step t. Recall that $g_t(i)$ and $h_t(i)$ are decreasing in $N_t(i)$, the number of time steps q's in $1, \ldots, t-1$ when we get to observe $W_q(i)$. Naively, arms in S_t can be thought of as arms that "lack observations", while arms in \bar{S}_t can be thought of as arms that are "observed enough", and are believed to be suboptimal. Note that $S^* \in S_t$ is a prime example of an arm that is under-observed.

To further elaborate, $g_t(i) + h_t(i)$ is the "statistical gap" between the Thompson sample $\theta_t(i)$ and the latent mean w(i). The gap shrinks with more observations of i. To balance exploration and exploitation, for any suboptimal item $i \in [L] \setminus [K]$ and any optimal item $k \in [K]$, we should have $g_t(i) + h_t(i) \ge w(k) - w(i)$. However, this is too much to hope for, and it seems that hoping for $S_t \in \mathcal{S}_t$ to happen would be more viable. (See the forthcoming Lemma 4.2.)

Further notations. In addition to set S_t , we define H_t as the collection of observations of the agent, from the beginning until the end of time t-1 (after everything during time t-1 has occurred). More precisely, we define $H_t := \{S_q\}_{q=1}^{t-1} \cup \{(i_k^q, W_q(i_k^q))_{k=1}^{\min\{k_t, \infty\}}\}_{q=1}^{t-1}$. Recall that $S_q \in \pi_K(L)$ is the arm pulled during time step q, and $(i_k^q, W_q(i_k^q))_{k=1}^{\min\{k_t, \infty\}}$ is the collection of observed items and their respective values

during time step q. At the start of time step t, the agent has observed everything in H_t , and determine the arm S_t to pull accordingly (see Algorithm 1). Note that event $\mathcal{E}_{\hat{\mu},t}$ is $\sigma(H_t)$ -measurable. For the convenience of discussion, we define $\mathcal{H}_{\hat{\mu},t} := \{H_t : \text{Event } \mathcal{E}_{\hat{\mu},t} \text{ is true in } H_t\}$. The first statement in Lemma 4.1 can thus be rephrased as $\Pr[H_t \in \mathcal{H}_{\hat{\mu},t}] \geq 1 - 2L/(t+1)^3$.

The performance of Algorithm 1 is analyzed using the following four Lemmas. To begin with, Lemma 4.2 quantifies a set of conditions on $\hat{\mu}_t$ and θ_t so that the pulled arm S_t belongs to S_t , the collection of arms that lack observations and should be explored. We recall from Lemma 4.1 that the events $\mathcal{E}_{\hat{\mu},t}$ and $\mathcal{E}_{\theta,t}$ hold with high probability. Subsequently, we will crucially use our definition of the Thompson sample θ_t to argue that inequality (4.2) holds with non-vanishing probability when t is sufficiently large.

Lemma 4.2. Consider a time step t. Suppose that events $\mathcal{E}_{\hat{\mu},t}$, $\mathcal{E}_{\theta,t}$ and inequality

$$\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \theta_t(k) \ge \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] w(k)$$
(4.2)

hold, then the event $\{S_t \in \mathcal{S}_t\}$ also holds.

In the following, we condition on H_t and show that θ_t is "typical" w.r.t. \boldsymbol{w} in the sense of (4.2). Due to the conditioning on H_t , the only source of randomness of the pulled arm S_t is from the Thompson sample. Thus, by analyzing a suitably weighted version of the Thompson samples in (4.2), we disentangle the statistical dependence between partial monitoring and Thompson sampling. Recall that θ_t is normal with $\sigma(H_t)$ -measurable mean and variance (Lines 5–7 in Algorithm 1).

Lemma 4.3. There exists an absolute constant $c \in (0,1)$ independent of \mathbf{w}, K, L, T such that, for any time step t and any historical observation $H_t \in \mathcal{H}_{\hat{\mu},t}$, the following inequality holds:

$$\Pr_{\theta_t} [\mathcal{E}_{\theta,t} \text{ and } (4.2) \text{ hold } | H_t] \ge c - \frac{1}{2(t+1)^3}.$$

Proof. We prove the Lemma by setting the absolute constant c to be $1/(4\sqrt{\pi}e^{8064}) > 0$.

For brevity, we define $\alpha(1) := 1$, and $\alpha(k) = \prod_{j=1}^{k-1} (1 - w(j))$ for $2 \le k \le K$. By the second part of Lemma 4.1, we know that $\Pr[\mathcal{E}_{\theta,t}|H_t] \ge 1 - 1/2(t+1)^3$, so to complete this proof, it suffices to show that

 $\Pr[(4.2) \text{ holds}|H_t] \geq c$. For this purpose, consider

$$\Pr_{\theta_{t}} \left[\sum_{k=1}^{K} \alpha(k)\theta_{t}(k) \geq \sum_{k=1}^{K} \alpha(k)w(k) \mid H_{t} \right] \\
= \Pr_{Z_{t}} \left[\sum_{k=1}^{K} \alpha(k) \left[\hat{\mu}_{t}(k) + Z_{t}\sigma_{t}(k) \right] \geq \sum_{k=1}^{K} \alpha(k)w(k) \mid H_{t} \right] \\
\geq \Pr_{Z_{t}} \left[\sum_{k=1}^{K} \alpha(k) \left[w(k) - g_{t}(k) \right] + Z_{t} \cdot \sum_{k=1}^{K} \alpha(k)\sigma_{t}(k) \right] \\
\geq \Pr_{Z_{t}} \left[\sum_{k=1}^{K} \alpha(k) \left[w(k) - g_{t}(k) \right] + Z_{t} \cdot \sum_{k=1}^{K} \alpha(k)\sigma_{t}(k) \right] \\
= \Pr_{Z_{t}} \left[Z_{t} \cdot \sum_{k=1}^{K} \alpha(k)w(k) \mid H_{t} \right] \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
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\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{7}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
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\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left[\sum_{k=1}^{K} \alpha(k)g_{t}(k) \right] \right\} \\
\geq \frac{1}{4\sqrt{\pi}} \exp \left\{ -\frac{$$

Step (4.3) is by the definition of $\{\theta_t(i)\}_{i\in L}$ in Line 7 in Algorithm 1. It is important to note that these samples share the *same* random seed Z_t . Next, step (4.4) is by the Lemma assumption that $H_t \in \mathcal{H}_{\hat{\mu},t}$, which means that $\hat{\mu}_t(k) \geq w_t(k) - g_t(k)$ for all $k \in [K]$. Step (4.5) is an application of the anti-concentration inequality of a normal random variable in Theorem A.2. Step (4.6) is by applying the definition of $g_t(i)$.

Combining Lemmas 4.2 and 4.3, we conclude that there exists an absolute constant c such that, for any time step t and any historical observation $H_t \in \mathcal{H}_{\hat{\mu},t}$,

$$\Pr_{\boldsymbol{\theta}_t} \left[S_t \in \mathcal{S}_t \mid H_t \right] \ge c - \frac{1}{2(t+1)^3}. \tag{4.7}$$

Equipped with (4.7), we are able to provide an upper bound on the regret of our Thompson sampling algorithm at every sufficiently large time step.

Lemma 4.4. Let c be an absolute constant such that Lemma 4.3 holds true. Consider a time step t that satisfies $c-1/(t+1)^3 > 0$. Conditional on an arbitrary

but fixed historical observation $H_t \in \mathcal{H}_{\hat{\mu},t}$, we have

$$\mathbb{E}_{\boldsymbol{\theta}_t}[r(S^*|\boldsymbol{w}) - r(S_t|\boldsymbol{w})|H_t]$$

$$\leq \left(1 + \frac{4}{c}\right) \mathbb{E}_{\boldsymbol{\theta}_t}\left[F(S_t, t) \mid H_t\right] + \frac{L}{2(t+1)^2}.$$

The proof of Lemma 4.4 relies crucially on truncating the original Thompson sample $\theta_t \in \mathbb{R}$ to $\tilde{\theta}_t \in [0,1]^L$. Under this truncation operation, S_t remains optimal under $\tilde{\theta}_t$ (as it was under θ_t) and $|\tilde{\theta}_t(i) - w(i)| \leq |\theta_t(i) - w(i)|$, i.e., the distance from the truncated Thompson sample to the ground truth is not increased.

For any t satisfying $c - 1/(t+1)^3 > 0$, define

$$\mathcal{F}_{i,t} := \{ \text{Observe } W_t(i) \text{ at } t \},$$

$$G(S_t, \mathbf{W}_t) := \sum_{k=1}^K \mathbb{1} \left(\mathcal{F}_{i_k^t, t} \right) \cdot \left(g_t(i_k^t) + h_t(i_k^t) \right) \right),$$

we unravel the upper bound in Lemma 4.4 to establish the expected regret at time step t:

$$\mathbb{E}\left\{r(S^{*}|\boldsymbol{w}) - r(S_{t}|\boldsymbol{w})\right\}
\leq \mathbb{E}\left[\mathbb{E}_{\boldsymbol{\theta}_{t}}\left[r(S^{*}|\boldsymbol{w}) - r(S_{t}|\boldsymbol{w}) \mid H_{t}\right] \cdot 1(H_{t} \in \mathcal{H}_{\hat{\mu},t})\right]
+ \mathbb{E}\left[1(H_{t} \notin \mathcal{H}_{\hat{\mu},t})\right]
\leq \left(1 + \frac{4}{c}\right) \mathbb{E}\left[\mathbb{E}_{\boldsymbol{\theta}_{t}}\left[F(S_{t},t) \mid H_{t}\right] 1(H_{t} \in \mathcal{H}_{\hat{\mu},t})\right]
+ \frac{1}{2(t+1)^{2}} + \frac{3L}{(t+1)^{3}}
\leq \left(1 + \frac{4}{c}\right) \mathbb{E}\left[F(S_{t},t)\right] + \frac{4L}{(t+1)^{2}}
= \left(1 + \frac{4}{c}\right) \mathbb{E}\left[\mathbb{E}_{\boldsymbol{W}_{t}}\left[G(S_{t},\boldsymbol{W}_{t}) \mid H_{t},S_{t}\right]\right] + \frac{4L}{(t+1)^{2}}
= \left(1 + \frac{4}{c}\right) \mathbb{E}\left[G(S_{t},\boldsymbol{W}_{t})\right] + \frac{4L}{(t+1)^{2}}, \tag{4.9}$$

where (4.8) follows by assuming t is sufficiently large.

We now bound the total regret by summing the per time step regret (4.9), and demonstrate the telescoping property of the summation in the following Lemma.

Lemma 4.5. For any realization of historical trajectory \mathcal{H}_{T+1} , we have

$$\sum_{t=1}^{T} G(S_t, \mathbf{W}_t) \le 6\sqrt{KLT} \log T + 144L \log^{5/2} T.$$

Note that here we prove a worst case bound, without needing the expectation operator.

Proof. Recall that for each $i \in [L]$ and $t \in [T+1]$, $N_t(i) = \sum_{s=1}^{t-1} 1(\mathcal{F}_{i,s})$ is the number of rounds in [t-1] when we get to observe the outcome for item i. Since

 $G(S_t, \mathbf{W}_t)$ involves $g_t(i) + h_t(i)$, we first bound this term. The definitions of $g_t(i)$ and $h_t(i)$ yield that

$$g_t(i) + h_t(i) \le \frac{12\log(t+1)}{\sqrt{N_t(i)+1}} + \frac{72\log^{3/2}(t+1)}{N_t(i)+1}.$$

Subsequently, we decompose $\sum_{t=1}^{T} G(S_t, W_t)$ according to its definition. For a fixed but arbitrary item i, consider the sequence $(U_t(i))_{t=1}^{T} = (1(\mathcal{F}_{i,t}) \cdot (g_t(i) + h_t(i)))_{t=1}^{T}$. Clearly, $U_t(i) \neq 0$ if and only if the decision maker observes the realization $W_t(i)$ of item i at t. Let $t = \tau_1 < \tau_2 < \ldots < \tau_{N_{T+1}}$ be the time steps when $U_t(i) \neq 0$. We assert that $N_{\tau_n}(i) = n-1$ for each n. Indeed, prior to time steps τ_n , item i is observed precisely in the time steps $\tau_1, \ldots, \tau_{n-1}$. Thus, we have

$$\sum_{t=1}^{T} 1(\mathcal{F}_{i,t}) \cdot (g_t(i) + h_t(i)) = \sum_{n=1}^{N_{T+1}(i)} (g_{\tau_n}(i) + h_{\tau_n}(i))$$

$$\leq \sum_{n=1}^{N_{T+1}(i)} \frac{12 \log T}{\sqrt{n}} + \frac{72 \log^{3/2} T}{n}. \tag{4.10}$$

Now we complete the proof as follows:

$$\sum_{t=1}^{T} \sum_{k=1}^{K} 1(\mathcal{F}_{i_{k}^{t},t}) \cdot (g_{t}(i_{k}^{t}) + h_{t}(i_{k}^{t}))$$

$$= \sum_{i \in [L]} \sum_{t=1}^{T} 1(\mathcal{F}_{i,t}) \cdot (g_{t}(i) + h_{t}(i))$$

$$\leq \sum_{i \in [L]} \sum_{n=1}^{N_{T+1}(i)} \frac{12 \log T}{\sqrt{n}} + \frac{72 \log^{3/2} T}{n}$$

$$\leq 6 \sum_{i \in [L]} \sqrt{N_{T+1}(i)} \log T + 72L \log^{3/2} T(\log T + 1)$$

$$\leq 6 \sqrt{L \sum_{i \in [L]} N_{T+1}(i)} \log T + 72L \log^{3/2} T(\log T + 1)$$

$$\leq 6 \sqrt{KLT} \log T + 144L \log^{5/2} T,$$

$$(4.12)$$

where (4.11) follows from (4.10), (4.12) follows from the Cauchy-Schwarz inequality, and (4.13) is because the decision maker can observe at most K items at each time step, hence $\sum_{i \in [L]} N_{T+1}(i) \leq KT$.

Finally, we bound the total regret from above by considering the time step $t_0 := \lceil 1/c^{1/3} \rceil$, and then bound the regret for the time steps before t_0 by 1 and the regret for time steps after by inequality (4.9), which

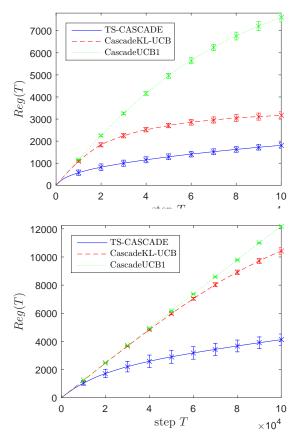


Figure 1: $\operatorname{Reg}(T)$ of TS-CASCADE, CASCADEKL-UCB and CASCADEUCB1 with $L \in \{64, 256\}$ (resp. top and bottom), K=2 and $\Delta=0.075$. Each line indicates the average $\operatorname{Reg}(T)$ (over 20 runs) and the length of each errorbar above and below each data point is the standard deviation.

holds for all $t > t_0$:

$$\operatorname{Reg}(T) \leq \left\lceil \frac{1}{c^{1/3}} \right\rceil + \sum_{t=t_0+1}^{T} \mathbb{E} \left\{ r(S^* | \boldsymbol{w}) - r(S_t | \boldsymbol{w}) \right\}$$
$$\leq \left\lceil \frac{1}{c^{1/3}} \right\rceil + \left(1 + \frac{4}{c} \right) \mathbb{E} \left[\sum_{t=1}^{T} G(S_t, \boldsymbol{W}_t) \right]$$
$$+ \sum_{t=t_0+1}^{T} \frac{4L}{(t+1)^2}.$$

It is clear that the third term is O(L), and by Lemma 4.5, the second term is $O(\sqrt{KLT}\log T + L\log^{5/2}T)$. Altogether, Theorem 3.1 is proved.

5 Experiments

In this section, we evaluate the performance of TS-CASCADE using numerical simulations. To demonstrate the effectiveness of our algorithm, we compare the expected cumulative regret of TS-

Table 2: The performances of TS-CASCADE, CASCADEKL-UCB and CASCADEUCB1 under 18 different settings. For each algorithm, the first column shows the mean and the standard deviation of Reg(T) and the second column shows the average running time in seconds. For each problem setting, the algorithm with smallest average Reg(T) or shortest running time is marked in bold.

L	K	Δ	TS-Cascade		CascadeKL-UCB		CascadeUCB1	
16	2	0.15	377.07 ± 11.67	3.16	$\textbf{359.35} \pm \textbf{26.42}$	54.3	1277.42 ± 25.88	2.82
16	4	0.15	294.55 ± 15.08	3.03	$\textbf{265.9}\pm\textbf{20.36}$	54.48	990.51 ± 31.72	2.84
16	8	0.15	$\textbf{138.85}\pm\textbf{9.81}$	3.51	148.36 ± 12.35	55.5	555.83 ± 14.41	3.17
32	2	0.15	$\textbf{738.19}\pm\textbf{19.23}$	3.41	764.42 ± 48.57	105.4	2711.44 ± 58.41	2.98
32	4	0.15	612.36 ± 10.66	3.55	619.68 ± 34.56	105.56	2237.77 ± 43.7	3.02
32	8	0.15	$\textbf{381.8}\pm\textbf{13.19}$	3.68	419.39 ± 19.59	105.64	1526.97 ± 24.48	3.14
32	2	0.075	1159 ± 63.43	3.49	1583.33 ± 104.04	106.62	4217.87 ± 129.08	3.95
32	4	0.075	$\textbf{1062.9}\pm\textbf{80.06}$	3.55	1208.06 ± 59.25	106.08	3301.44 ± 85.43	3.84
32	8	0.075	$\textbf{631.45}\pm\textbf{51.51}$	3.58	718.65 ± 32.27	106.51	1890.06 ± 47.8	3.97
64	2	0.075	1810.43 ± 126.74	4.74	3169.17 ± 156.98	207.31	7599.58 ± 199.99	4.24
64	4	0.075	1730.13 ± 128.09	4.88	2512.28 ± 106.85	208.08	6437.43 ± 239.96	5.04
64	8	0.075	1175.07 ± 46.91	4.7	1565.76 ± 72.98	208.34	3962.35 ± 87.61	4.77
128	2	0.075	2784.44 ± 185.08	5.36	6160.86 ± 300.48	414.45	11055.68 ± 156.27	5.17
128	4	0.075	2837.25 ± 239.41	4.76	5004.45 ± 188.68	412.55	11516.47 ± 227.48	4.7
128	8	0.075	2004.58 ± 122.26	4.87	3084.67 ± 105.78	413.6	7432.14 ± 129.24	4.61
256	2	0.075	4128.96 ± 400.88	8.35	10426.63 ± 249.33	816.52	12191.23 ± 39.69	7.22
256	4	0.075	4376.73 ± 373.99	7.49	9389.72 ± 251.5	818.07	15748.08 ± 131.08	7.56
256	8	0.075	${\bf 3258.24 \pm 238.91}$	7.24	6019.24 ± 145.95	820	12417.86 ± 160.53	7.83

CASCADE to CASCADEKL-UCB and CASCADEUCB1 in Kveton et al. (2015a). We reimplemented the latter two algorithms and checked that their performances are roughly the same as those in Table 1 of Kveton et al. (2015a).

We set the optimal items to have the same click probability w_1 and the suboptimal items to also have the same click probability $w_2 < w_1$. The gap $\Delta := w_1 - w_2$. We set $w_1 = 0.2$, $T = 10^5$ and vary L, K, and Δ . We conduct 20 independent simulations with each algorithm under each setting of L, K, and Δ . We calculate the average and standard deviation of Reg(T), and as well as the average running time of each experiment. Here we only present a subset of the results. More details are given in Appendix C.

In Table 2, we compare the performances of algorithms under 18 different settings. Since CascadeKL-UCB perfoms far better than CascadeUCB1, we mainly focus on the comparison between our method and CascadeKL-UCB. In most cases, the expected cumulative regret of our algorithm is significantly smaller than that of CascadeKL-UCB, especially when L is large and Δ is small. Note that a larger L means that the problem size is larger. A smaller Δ implies that the difference between optimal and sub-optimal arms are less pronounced. Hence, when L is large and Δ is small, the problem is "more difficult". However, the standard deviation of our algorithm is larger than that of CascadeKL-UCB in some cases. A possible ex-

planation is that Thompson sampling yields more randomness than UCB due to the additional randomness of the Thompson samples $\{\theta_t\}_{t\in[T]}$. In contrast, UCB-based algorithms do not have this source of randomness as each upper confidence bound is deterministically designed. Furthermore, Table 2 suggests that our algorithm is much faster than CASCADEKL-UCB and is just as fast as CASCADEUCB1. The reason why CASCADEKL-UCB is so slow is because an UCB has to be computed via an optimization problem for every $i \in [L]$. In contrast, TS-CASCADE in Algorithm 1 does not contain any computationally expensive steps.

In Figure 1, we plot $\operatorname{Reg}(T)$ as a function of T for TS-CASCADE, CASCADEKL-UCB and CASCADEUCB1 when $L \in \{64, 256\}$, K = 2 and $\Delta = 0.075$. It is clear that our method outperforms the two UCB algorithms. For the case where the number of ground items L = 256 is large, the UCB-based algorithms do not demonstrate the \sqrt{T} behavior even after $T = 10^5$ iterations. In contrast, $\operatorname{Reg}(T)$ for TS-CASCADE behaves as $O(\sqrt{T})$ which implies that the empirical performance corroborates the upper bound derived in Theorem 3.1. We have plotted $\operatorname{Reg}(T)$ for other settings of L, K and Δ in Appendix C and the same conclusion can be drawn.

6 Summary and Future work

This work presents the first theoretical analysis of Thompson sampling for cascading bandits. The expected regret matches the state-of-the-art based on UCB by Wang and Chen (2017) (which is identical to CASCADEUCB1 in Kveton et al. (2015a)). Empirical experiments, however, show the clear superiority of TS-CASCADE over CASCADEKL-UCB and CASCADEUCB1 in terms of regret and running time.

The following are avenues for further investigations. From Table 2, we see that a problem-independent lower bound is still not available. It is envisioned that a judicious construction of an adversarial bandit example, together with the information-theoretic technique of (Auer et al., 2002, Theorem 5.1) will lead to a lower bound of the form $\tilde{\Omega}(\sqrt{KLT})$, matching Theorem 3.1 here and Wang and Chen (2017). Next, we envision that a refinement of the proof techniques herein, especially the design of Thompson samples to be Gaussian, would be useful for generalization the contextual setting (Li et al., 2010; Qin et al., 2014; Li et al., 2016).

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A Useful theorems

Here are some basic facts from the literature that we will use:

Theorem A.1 ((Audibert et al., 2009), speicalized to Berounlli random variables). Consider N independently and identically distributed Bernoulli random variables $Y_1, \ldots, Y_N \in \{0, 1\}$, which have the common mean $m = \mathbb{E}[Y_1]$. In addition, consider their sample mean $\hat{\xi}$ and their sample variance \hat{V} :

$$\hat{\xi} = \frac{1}{N} \sum_{i=1}^{N} Y_i, \quad \hat{V} = \frac{1}{N} \sum_{i=1}^{N} (Y_i - \hat{\xi})^2 = \hat{\xi} (1 - \hat{\xi}).$$

For any $\delta \in (0,1)$, the following inequality holds:

$$\Pr\left(\left|\hat{\xi} - m\right| \le \sqrt{\frac{2\hat{V}\log(1/\delta)}{N}} + \frac{3\log(1/\delta)}{N}\right) \ge 1 - 3\delta.$$

Theorem A.2 ((Abramowitz and Stegun, 1964)). Let $Z \sim \mathcal{N}(\mu, \sigma^2)$. For any $z \geq 0$, the following inequalities hold:

$$\frac{1}{4\sqrt{\pi}}\exp\left(-\frac{7z^2}{2}\right) \le \Pr\left(|Z - \mu| > z\sigma\right) \le \frac{1}{2}\exp\left(-\frac{z^2}{2}\right).$$

B Proofs of main results

In this Section, we provide proofs of Lemmas 4.2, 4.4, as well as Lemmas 4.1, B.1.

B.1 Proof of Lemma 4.1

Lemma 4.1. For each $t \in [T]$, we have

$$\Pr\left[\mathcal{E}_{\hat{\mu},t}\right] \ge 1 - \frac{3L}{(t+1)^3}, \quad \Pr\left[\mathcal{E}_{\theta,t}|\mathcal{E}_{\hat{\mu},t}\right] \ge 1 - \frac{1}{2(t+1)^2}.$$

Proof. Bounding probability of event $\mathcal{E}_{\hat{\mu},t}$: We first consider a fixed non-negative integer N and a fixed item i. Let Y_1, \ldots, Y_N be i.i.d. Bernoulli random variables, with the common mean w(i). Denote $\hat{\xi}_N = \sum_{i=1}^N Y_i/N$ as the sample mean, and $\hat{V}_N = \hat{\xi}_N(1-\hat{\xi}_N)$ as the empirical variance. By applying Theorem A.1 with $\delta = 1/(t+1)^4$, we have

$$\Pr\left(\left|\hat{\xi}_{N} - w(i)\right| > \sqrt{\frac{8\hat{V}_{N}\log(t+1)}{N}} + \frac{12\log(t+1)}{N}\right) \le \frac{3}{(t+1)^{4}}.$$
(B.1)

By an abuse of notation, let $\hat{\xi}_N = 0$ if N = 0. Inequality (B.1) implies the following concentration bound when N is non-negative:

$$\Pr\left(\left|\hat{\xi}_N - w(i)\right| > \sqrt{\frac{16\hat{V}_N \log(t+1)}{N+1}} + \frac{24\log(t+1)}{N+1}\right) \le \frac{3}{(t+1)^4}.$$
(B.2)

Subsequently, we can establish the concentration property of $\hat{\mu}_t(i)$ by a union bound of $N_t(i)$ over $\{0, 1, \dots, t-1\}$:

$$\Pr\left(|\hat{\mu}_{t}(i) - w(i)| > \sqrt{\frac{16\hat{\nu}_{t}(i)\log(t+1)}{N_{t}(i)+1}} + \frac{24\log(t+1)}{N_{t}(i)+1}\right)$$

$$\leq \Pr\left(\left|\hat{\xi}_{N} - w(i)\right| > \sqrt{\frac{16\hat{V}_{N}\log(t+1)}{N+1}} + \frac{24\log(t+1)}{N+1} \text{ for some } N \in \{0, 1, \dots t-1\}\right)$$

$$\leq \frac{3}{(t+1)^{3}}.$$

Finally, taking union bound over all items $i \in L$, we know that event $\mathcal{E}_{\hat{\mu},t}$ holds true with probability at least $1 - 3L/(t+1)^3$.

Bounding probability of event $\mathcal{E}_{\theta,t}$, conditioned on event $\mathcal{E}_{\hat{\mu},t}$: Consider an observation trajectory H_t satisfying event $\mathcal{E}_{\hat{\mu},t}$. By the definition of the Thompson sample $\theta_t(i)$ (see Line 7 in Algorithm 1), we have

$$\Pr\left(\left|\theta_{t}(i) - \hat{\mu}_{t}(i)\right| > h_{t}(i) \text{ for all } i \in L|H_{\hat{\mu},t}\right) \\
= \Pr_{Z_{t}} \left(\left|Z_{t} \cdot \max\left\{\sqrt{\frac{\hat{\nu}_{t}(i)\log(t+1)}{N_{t}(i)+1}}, \frac{\log(t+1)}{N_{t}(i)+1}\right\}\right| > \\
\sqrt{\log(t+1)} \left[\sqrt{\frac{16\hat{\nu}_{t}(i)\log(t+1)}{N_{t}(i)+1}} + \frac{24\log(t+1)}{N_{t}(i)+1}\right] \text{ for all } i \in [L] \mid \hat{\mu}_{t}(i), N_{t}(i)\right) \\
\leq \Pr\left(\left|Z_{t} \cdot \max\left\{\sqrt{\frac{\hat{\nu}_{t}(i)\log(t+1)}{N_{t}(i)+1}}, \frac{\log(t+1)}{N_{t}(i)+1}\right\}\right| > \\
\sqrt{16\log(t+1)} \max\left\{\sqrt{\frac{\hat{\nu}_{t}(i)\log(t+1)}{N_{t}(i)+1}}, \frac{\log(t+1)}{N_{t}(i)+1}\right\} \text{ for all } i \in [L] \mid \hat{\mu}_{t}(i), N_{t}(i)\right) \\
\leq \frac{1}{2} \exp\left[-8\log(t+1)\right] \leq \frac{1}{2(t+1)^{3}}.$$
(B.3)

The inequality in (B.3) is by the concentration property of a Gaussian random variable, see Theorem A.2. Altogether, the lemma is proved.

B.2 Proof of Lemma 4.2

Proof. To start, we denote the shorthand $\theta_t(i)^+ = \max\{\theta_t(i), 0\}$. We demonstrate that, if events $\mathcal{E}_{\hat{\mu},t}, \mathcal{E}_{\theta,t}$ and inequality (4.2) hold, then for all $\bar{S} = (\bar{i}_1, \dots, \bar{i}_K) \in \bar{\mathcal{S}}_t$ we have:

$$\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\bar{i}_j)) \right] \cdot \theta_t(\bar{i}_k)^+ \stackrel{(\ddagger)}{<} \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \cdot \theta_t(k)^+ \stackrel{(\dagger)}{\leq} \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j^t)) \right] \cdot \theta_t(i_k^t)^+, \tag{B.4}$$

where we recall that $S_t = (i_1^t, \dots, i_K^t)$ in an optimal arm for $\boldsymbol{\theta}_t$, and $\theta_t(i_1^t) \geq \theta_t(i_2^t) \geq \dots \geq \theta_t(i_K^t) \geq \max_{i \in [L] \setminus \{i_1^t, \dots, i_K^t\}} \theta_t(i)$. The inequalities in (B.4) clearly implies that $S_t \in \mathcal{S}_t$. To justifies these inequalities, we proceed as follows:

Showing (†): This inequality is true even without requiring events $\mathcal{E}_{\hat{\mu},t}$, $\mathcal{E}_{\theta,t}$ and inequality (4.2) to be true. Indeed, we argue the following:

$$\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \cdot \theta_t(k)^+ \le \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \cdot \theta_t(i_k^t)^+$$
(B.5)

$$\leq \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j^t)) \right] \cdot \theta_t(i_k^t)^+.$$
 (B.6)

To justify inequality (B.5), consider function $f: \pi_K(L) \to \mathbb{R}$ defined as

$$f((i_k)_{k=1}^K) := \sum_{k=1}^K \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \cdot \theta_t(i_k)^+.$$

We assert that $S_t \in \operatorname{argmax}_{S \in \pi_K(L)} f(S)$. The assertion can be justified by the following two properties. First, by the choice of S_t , we know that $\theta_t(i_1^t)^+ \geq \theta_t(i_2^t)^+ \geq \ldots \geq \theta_t(i_K^t)^+ \geq \max_{i \in [L] \setminus S_t} \theta_t(i)^+$. Second, the linear coefficients in the function f are monotonic and non-negative, in the sense that $1 \geq 1 - w(1) \geq (1 - w(1))(1 - w(2)) \geq \ldots \geq \prod_{k=1}^{K-1} (1 - w(k)) \geq 0$. Altogether, we have $f(S_t) \geq f(S^*)$, hence inequality (B.5) is shown.

Next, inequality (B.6) clearly holds, since for each $k \in [K]$ we know that $\theta_t(i_k^t)^+ \geq 0$, and $\prod_{j=1}^{k-1} (1 - w(j)) \leq \prod_{j=1}^{k-1} (1 - w(i_j^t))$. The latter is due to the fact that $1 \geq w(1) \geq w(2) \geq \ldots \geq w(K) \geq \max_{i \in [L] \setminus [K]} w(i)$. Altogether, inequality (†) is established.

Showing (‡): The demonstration crucially hinges on events $\mathcal{E}_{\hat{\mu},t}$, $\mathcal{E}_{\theta,t}$ and inequality (4.2) being held true. For any $\bar{S} = (\bar{i}_1, \dots, \bar{i}_K) \in \bar{\mathcal{S}}_t$, we have

$$\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\bar{i}_{j})) \right] \theta_{t}(\bar{i}_{k})^{+} \leq \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\bar{i}_{j})) \right] (w(\bar{i}_{k}) + g_{t}(\bar{i}_{k}) + h_{t}(\bar{i}_{k}))$$

$$\leq \left\{ \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\bar{i}_{j})) \right] w(\bar{i}_{k}) \right\} + r(S^{*}|\boldsymbol{w}) - r(\bar{S}|\boldsymbol{w})$$

$$= r(S^{*}|\boldsymbol{w}) = \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] w(k)$$

$$\leq \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \theta_{t}(k) \leq \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(j)) \right] \theta_{t}(k)^{+}.$$
(B.9)

Inequality (B.7) is by the assumption that events $\mathcal{E}_{\hat{\mu},t}$, $\mathcal{E}_{\theta,t}$ are true, which means that for all $i \in [L]$ we have $\theta_t(i)^+ \leq \mu(i) + g_t(i) + h_t(i)$. Inequality (B.8) is by the fact that $S \in \bar{\mathcal{S}}_t$. Inequality (B.9) is by our assumption that inequality (4.2) holds.

Altogether, the inequalities (†, ‡) in (B.4) are shown, and the Lemma is established.

B.3 Proof of Lemma 4.4

Lemma 4.4. Let c be an absolute constant such that Lemma 4.3 holds true. Consider a time step t that satisfies $c - 1/(t+1)^3 > 0$. Conditional on an arbitrary but fixed historical observation $H_t \in \mathcal{H}_{\hat{\mu},t}$, we have

$$\mathbb{E}_{\boldsymbol{\theta}_t}[r(S^*|\boldsymbol{w}) - r(S_t|\boldsymbol{w})|H_t]$$

$$\leq \left(1 + \frac{4}{c}\right) \mathbb{E}_{\boldsymbol{\theta}_t}\left[F(S_t, t) \mid H_t\right] + \frac{L}{2(t+1)^2}.$$

The proof of Lemma 4.4 crucially uses the following lemma on the expression of the difference in expected reward between two arms:

Lemma B.1. [Implied by Zong et al. (2016)] Let $S = (i_1, \ldots, i_K)$, $S' = (i'_1, \ldots, i'_K)$ be two arbitrary ordered K-subsets of [L]. For any $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^L$, the following equalities holds:

$$\begin{split} r(S|\boldsymbol{w}) - r(S'|\boldsymbol{w}') &= \sum_{k=1}^K \left[\prod_{j=1}^{k-1} (1 - w(i_j)) \right] \cdot (w(i_k) - w'(i_k')) \cdot \left[\prod_{j=k+1}^K (1 - w'(i_j')) \right] \\ &= \sum_{k=1}^K \left[\prod_{j=1}^{k-1} (1 - w'(i_j')) \right] \cdot (w(i_k) - w'(i_k')) \cdot \left[\prod_{j=k+1}^K (1 - w(i_j)) \right]. \end{split}$$

While Lemma B.1 is folklore in the cascading bandit literature, we provide a proof in Appendix B.4 for the sake of completeness. Now, we proceed to the proof of Lemma 4.4:

Proof. In the proof, we always condition to the historical observation H_t stated in the Lemma. To proceed with the analysis, we define $\tilde{S}_t = (\tilde{i}_1^t, \dots, \tilde{i}_K^t) \in \mathcal{S}_t$ as an ordered K-subset that satisfies the following minimization criterion:

$$\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_j)) \right] (g_t(\tilde{i}_j) + h_t(\tilde{i}_j)) = \min_{S = (i_1, \dots, i_K) \in \mathcal{S}_t} \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j)) \right] (g_t(i_j) + h_t(i_j)). \tag{B.10}$$

We emphasize that both \tilde{S}_t and the left hand side of (B.10) are deterministic in the current discussion, where we condition on H_t . To establish tight bounds on the regret, we consider the truncated version, $\tilde{\theta}_t \in [0, 1]^L$, of the Thompson sample θ_t . For each $i \in L$, define

$$\tilde{\theta}_t(i) = \min\{1, \max\{0, \theta_t(i)\}\}.$$

The truncated version $\tilde{\theta}_t(i)$ serves as a correction of $\theta_t(i)$, in the sense that the Thompson sample $\theta_t(i)$, which serves as a Bayesian estimate of click probability w(i), should lie in [0, 1]. It is important to observe the following two properties hold under the truncated Thompson sample $\tilde{\theta}_t$:

Property 1 Our pulled arm S_t is still optimal under the truncated estimate $\tilde{\theta}_t$, i.e.

$$S_t \in \operatorname{argmax}_{S \in \pi_K(L)} r(S|\tilde{\boldsymbol{\theta}}_t).$$

Indeed, the truncated Thompson sample can be sorted in a descending order in the same way as for the original Thompson sample¹, i.e. $\tilde{\theta}_t(i_1^t) \geq \tilde{\theta}_t(i_2^t) \geq \ldots \geq \tilde{\theta}_t(i_K^t) \geq \max_{i \in [L] \setminus \{i_1^t, \ldots, i_K^t\}} \tilde{\theta}_t(i)$. The optimality of S_t thus follows.

Property 2 For any t, i, if it holds that $|\theta_t(i) - w(i)| \le g_t(i) + h_t(i)$, then it also holds that $|\tilde{\theta}_t(i) - w(i)| \le g_t(i) + h_t(i)$. Indeed, we know that $|\tilde{\theta}_t(i) - w(i)| \le |\theta_t(i) - w(i)|$.

Now, we use the ordered K-subset \tilde{S}_t and the truncated Thompson sample $\tilde{\theta}_t$ to decompose the conditionally expected round t regret as follows:

$$r(S^{*}|\boldsymbol{w}) - r(S_{t}|\boldsymbol{w}) = \left[r(S^{*}|\boldsymbol{w}) - r(\tilde{S}_{t}|\boldsymbol{w})\right] + \left[r(\tilde{S}_{t}|\boldsymbol{w}) - r(S_{t}|\boldsymbol{w})\right]$$

$$\leq \left[r(S^{*}|\boldsymbol{w}) - r(\tilde{S}_{t}|\boldsymbol{w})\right] + \left[r(\tilde{S}_{t}|\boldsymbol{w}) - r(S_{t}|\boldsymbol{w})\right] 1(\mathcal{E}_{\theta,t}) + 1(\neg \mathcal{E}_{\theta,t})$$

$$\leq \underbrace{\left[r(S^{*}|\boldsymbol{w}) - r(\tilde{S}_{t}|\boldsymbol{w})\right]}_{(\diamondsuit)} + \underbrace{\left[r(\tilde{S}_{t}|\tilde{\boldsymbol{\theta}}_{t}) - r(S_{t}|\tilde{\boldsymbol{\theta}}_{t})\right]}_{(\diamondsuit)} + \underbrace{\left[r(\tilde{S}_{t}|\boldsymbol{\theta}_{t}) - r(S_{t}|\boldsymbol{w})\right] 1(\mathcal{E}_{\theta,t})}_{(\heartsuit)} + \underbrace{\left[r(\tilde{S}_{t}|\boldsymbol{w}) - r(\tilde{S}_{t}|\tilde{\boldsymbol{\theta}}_{t})\right] 1(\mathcal{E}_{\theta,t})}_{(\diamondsuit)} + 1(\neg \mathcal{E}_{\theta,t}). \tag{B.11}$$

We bound $(\diamondsuit, \clubsuit, \heartsuit, \spadesuit)$ from above as follows:

Bounding (\diamondsuit): By the assumption that $\tilde{S}_t = (\tilde{i}_1^t, \dots, \tilde{i}_K^t) \in S_t$, with certainty we have

$$(\diamondsuit) \le \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_j^t)) \right] (g_t(\tilde{i}_j^t) + h_t(\tilde{i}_j^t)). \tag{B.12}$$

Bounding (4): By Property 1 of the truncated Thompson sample $\tilde{\theta}_t$, we know that $r(S_t|\tilde{\theta}_t) = \max_{S \in \pi_K(L)} r(S|\tilde{\theta}_t) \geq r(\tilde{S}_t|\tilde{\theta}_t)$. Therefore, with certainty we have

$$(\clubsuit) \le 0. \tag{B.13}$$

Recall that that $\theta_t(i_1^t) \ge \theta_t(i_2^t) \ge \ldots \ge \theta_t(i_K^t) \ge \max_{i \in [L] \setminus \{i_k^t\}_{k=1}^K} \theta_t(i)$ for the original Thompson sample $\boldsymbol{\theta}_t$.

Bounding (\heartsuit) : . We bound the term as follows:

$$1(\mathcal{E}_{\theta,t}) \left[r(S_{t} | \tilde{\boldsymbol{\theta}}_{t}) - r(S_{t} | \boldsymbol{w}) \right]$$

$$= 1(\mathcal{E}_{\theta,t}) \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_{j}^{t})) \right] \cdot (\tilde{\theta}_{t}(i_{k}^{t}) - w(i_{k}^{t})) \cdot \left[\prod_{j=k+1}^{K} (1 - \tilde{\theta}_{t}(i_{j}^{t})) \right]$$

$$\leq 1(\mathcal{E}_{\theta,t}) \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_{j}^{t})) \right] \cdot \left| \tilde{\theta}_{t}(i_{k}^{t}) - w(i_{k}^{t}) \right| \cdot \left[\prod_{j=k+1}^{K} \left| 1 - \tilde{\theta}_{t}(i_{j}^{t}) \right| \right]$$

$$\leq 1(\mathcal{E}_{\theta,t}) \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_{j}^{t})) \right] \cdot \left[g_{t}(i_{k}^{t}) + h_{t}(i_{k}^{t}) \right]$$
(B.15)

$$\leq \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j^t)) \right] \cdot \left[g_t(i_k^t) + h_t(i_k^t) \right]. \tag{B.16}$$

Equality (B.14) is by applying the second equality in Lemma B.1, with $S = S' = S_t$, as well as $\mathbf{w}' \leftarrow \mathbf{w}$, $\mathbf{w} \leftarrow \mathbf{\theta}_t$. Inequality (B.15) is by the following two facts: (1) By the definition of the truncated Thompson sample $\tilde{\boldsymbol{\theta}}$, we know that $\left|1 - \tilde{\theta}_t(i)\right| \leq 1$ for all $i \in [L]$; (2) By assuming event $\mathcal{E}_{\theta,t}$ and conditioning on H_t where event $\mathcal{E}_{\hat{\mu},t}$ holds true, **Property 2** implies that that $|\tilde{\theta}_t(i) - w(i)| \leq g_t(i) + h_t(i)$ for all i.

Bounding (\spadesuit): The analysis is similar to the analysis on (\heartsuit):

$$1(\mathcal{E}_{\theta,t}) \left[r(\tilde{S}_t | \boldsymbol{w}) - r(\tilde{S}_t | \tilde{\boldsymbol{\theta}}_t) \right]$$

$$= 1(\mathcal{E}_{\theta,t}) \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_j^t)) \right] \cdot (w(\tilde{i}_k^t) - \tilde{\theta}_t(\tilde{i}_k^t)) \cdot \left[\prod_{j=k+1}^{K} (1 - \tilde{\theta}_t(\tilde{i}_j^t)) \right]$$
(B.17)

$$\leq 1(\mathcal{E}_{\theta,t}) \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_j^t)) \right] \cdot \left[g_t(\tilde{i}_k^t) + h_t(\tilde{i}_k^t) \right]$$
(B.18)

$$\leq \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_{j}^{t})) \right] \cdot \left[g_{t}(\tilde{i}_{k}^{t}) + h_{t}(\tilde{i}_{k}^{t}) \right]. \tag{B.19}$$

Equality (B.17) is by applying the first equality in Lemma B.1, with $S = S' = \tilde{S}_t$, and $\boldsymbol{w} \leftarrow \boldsymbol{w}, \, \boldsymbol{w}' \leftarrow \boldsymbol{\theta}_t$. Inequality (B.18) follows the same logic as inequality (B.15).

Altogether, collating the bounds (B.12, B.13, B.16, B.19) for $(\diamondsuit, \clubsuit, \heartsuit, \spadesuit)$ respectively, we bound (B.11) from above (conditioned on H_t) as follows:

$$r(S^*|\boldsymbol{w}) - r(S_t|\boldsymbol{w}) \le 2\sum_{k=1}^K \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_j^t)) \right] (g_t(\tilde{i}_j^t) + h_t(\tilde{i}_j^t))$$

$$+ \sum_{k=1}^K \left[\prod_{j=1}^{k-1} (1 - w(i_j^t)) \right] \cdot \left[g_t(i_k^t) + h_t(i_k^t) \right] + 1(\neg \mathcal{E}_{\theta,t}).$$
(B.20)

Now, observe that

$$\mathbb{E}_{\boldsymbol{\theta}_{t}} \left[\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_{j}^{t})) \right] \left(g_{t}(i_{j}^{t}) + h_{t}(i_{j}^{t}) \right) \mid H_{t} \right]$$

$$\geq \mathbb{E}_{\boldsymbol{\theta}_{t}} \left[\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_{j}^{t})) \right] \left(g_{t}(i_{j}^{t}) + h_{t}(i_{j}^{t}) \right) \mid H_{t}, S_{t} \in \mathcal{S}_{t} \right] \Pr_{\boldsymbol{\theta}_{t}} \left[S_{t} \in \mathcal{S}_{t} \mid H_{t} \right]$$

$$\geq \left\{ \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(\tilde{i}_j)) \right] \left(g_t(\tilde{i}_j) + h_t(\tilde{i}_j) \right) \right\} \cdot \left(c - \frac{1}{2(t+1)^3} \right), \tag{B.21}$$

where we recall that $f(\lambda, t)$ is the probability lower bound defined in Equation (4.7). Thus, taking conditional expectation $\mathbb{E}_{\theta_t}[\cdot|H_t]$ on both sides in inequality (B.20) gives

$$\mathbb{E}_{\boldsymbol{\theta}_{t}}[R(S^{*}|\boldsymbol{w}) - R(S_{t}|\boldsymbol{w})|H_{t}]$$

$$\leq \left(1 + \frac{2}{c - \frac{1}{2(t+1)^{3}}}\right) \mathbb{E}_{\boldsymbol{\theta}_{t}} \left[\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_{j}^{t}))\right] \cdot \left[g_{t}(i_{k}^{t}) + h_{t}(i_{k}^{t})\right] \mid H_{t}\right] + \mathbb{E}_{\boldsymbol{\theta}_{t}}[1(\neg \mathcal{E}_{\boldsymbol{\theta}, t})|H_{t}].$$

Finally, the Lemma is proved by the assumption that $c > 1/(t+1)^3$, and noting from Lemma 4.1 that $\mathbb{E}_{\theta_t}[1(\neg \mathcal{E}_{\theta,t})|H_t] \leq 1/(2(t+1)^3)$.

B.4 Proof of Lemma B.1

Lemma B.1. [Implied by Zong et al. (2016)] Let $S = (i_1, \ldots, i_K)$, $S' = (i'_1, \ldots, i'_K)$ be two arbitrary ordered K-subsets of [L]. For any $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^L$, the following equalities holds:

$$r(S|\mathbf{w}) - r(S'|\mathbf{w}') = \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j)) \right] \cdot (w(i_k) - w'(i'_k)) \cdot \left[\prod_{j=k+1}^{K} (1 - w'(i'_j)) \right]$$
$$= \sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w'(i'_j)) \right] \cdot (w(i_k) - w'(i'_k)) \cdot \left[\prod_{j=k+1}^{K} (1 - w(i_j)) \right].$$

Proof. Observe that

$$\begin{split} &\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w(i_j)) \right] \cdot (w(i_k) - w'(i'_k)) \cdot \left[\prod_{j=k+1}^{K} (1 - w'(i'_j)) \right] \\ &= \sum_{k=1}^{K} \left\{ \left[\prod_{j=1}^{k-1} (1 - w(i_j)) \right] \cdot \left[\prod_{j=k}^{K} (1 - w'(i'_j)) \right] - \left[\prod_{j=1}^{k} (1 - w(i_j)) \right] \cdot \left[\prod_{j=k+1}^{K} (1 - w'(i'_j)) \right] \right\} \\ &= \prod_{k=1}^{K} (1 - w'(i'_k)) - \prod_{k=1}^{K} (1 - w(i_k)) = R(S|\mathbf{w}) - R(S'|\mathbf{w}'), \end{split}$$

and also that (actually we can also see this by a symmetry argument)

$$\sum_{k=1}^{K} \left[\prod_{j=1}^{k-1} (1 - w'(i'_j)) \right] \cdot (w(i_k) - w'(i'_k)) \cdot \left[\prod_{j=k+1}^{K} (1 - w(i_j)) \right]$$

$$= \sum_{k=1}^{K} \left\{ \left[\prod_{j=1}^{k} (1 - w'(i'_j)) \right] \cdot \left[\prod_{j=k+1}^{K} (1 - w(i_j)) \right] - \left[\prod_{j=1}^{k-1} (1 - w'(i'_j)) \right] \cdot \left[\prod_{j=k}^{K} (1 - w(i_j)) \right] \right\}$$

$$= \prod_{k=1}^{K} (1 - w'(i'_k)) - \prod_{k=1}^{K} (1 - w(i_k)) = R(S|\mathbf{w}) - R(S'|\mathbf{w}').$$

This completes the proof.

C Additional Numerical Results

We set $L \in \{16, 32, 64, 128, 256\}$, $K \in \{2, 4, 8\}$ and $\Delta \in \{0.15, 0.075\}$. This results in 30 parameter settings. We record all the results in Table 3. Here we can see that our algorithm clearly outperforms the other two algorithms when L is large, Δ is small. This superiority manifests itself in the expected regret and the time complexity.

Table 3: The performances of TS-CASCADE, CASCADEKL-UCB and CASCADEUCB1 under 30 different settings. For each algorithm, the first column shows the mean and the standard deviation of Reg(T) and the second column shows the average running time in seconds. For each problem setting, the algorithm with smallest average Reg(T) or shortest running time is marked in bold.

L	K	Δ	TS-Cascade		CascadeKL-UCB		CascadeUCB1	
16	2	0.15	377.07 ± 11.67	3.16	359.35 ± 26.42	54.3	1277.42 ± 25.88	2.82
16	4	0.15	294.55 ± 15.08	3.03	$\textbf{265.9}\pm\textbf{20.36}$	54.48	990.51 ± 31.72	2.84
16	8	0.15	138.85 ± 9.81	3.51	148.36 ± 12.35	55.5	555.83 ± 14.41	3.17
16	2	0.075	691.6 ± 58.39	2.98	736.08 ± 56.36	54.11	2028.56 ± 71.56	2.94
16	4	0.075	546.46 ± 40.78	3.15	526.93 ± 52.76	54.41	1485.14 ± 58.43	2.85
16	8	0.075	252.74 ± 20.52	3.44	261.76 ± 33.86	54.24	713.43 ± 46.93	2.9
32	2	0.15	$\textbf{738.19}\pm\textbf{19.23}$	3.41	764.42 ± 48.57	105.4	2711.44 ± 58.41	2.98
32	4	0.15	612.36 ± 10.66	3.55	619.68 ± 34.56	105.56	2237.77 ± 43.7	3.02
32	8	0.15	381.8 ± 13.19	3.68	419.39 ± 19.59	105.64	1526.97 ± 24.48	3.14
32	2	0.075	1159 ± 63.43	3.49	1583.33 ± 104.04	106.62	4217.87 ± 129.08	3.95
32	4	0.075	1062.9 ± 80.06	3.55	1208.06 ± 59.25	106.08	3301.44 ± 85.43	3.84
32	8	0.075	$\textbf{631.45}\pm\textbf{51.51}$	3.58	718.65 ± 32.27	106.51	1890.06 ± 47.8	3.97
64	2	0.15	1400.97 ± 45.61	4.62	1555.42 ± 44.88	208.48	5408.46 ± 83.34	4.13
64	4	0.15	1194.26 ± 21.69	5.47	1283.29 ± 49.22	208.3	4609.41 ± 84.2	4.17
64	8	0.15	812.1 ± 29.36	4.73	937.02 ± 30.52	208.03	3307.08 ± 43.78	4.74
64	2	0.075	1810.43 ± 126.74	4.74	3169.17 ± 156.98	207.31	7599.58 ± 199.99	4.24
64	4	0.075	1730.13 ± 128.09	4.88	2512.28 ± 106.85	208.08	6437.43 ± 239.96	5.04
64	8	0.075	1175.07 ± 46.91	4.7	1565.76 ± 72.98	208.34	3962.35 ± 87.61	4.77
128	2	0.15	2520.03 ± 74.04	5.06	3114.73 ± 74.62	416.05	10677.3 ± 193.72	4.18
128	4	0.15	$\bm{2216.26} \pm \bm{50.54}$	4.71	2602.08 ± 51.29	413.77	9163.15 ± 126.39	4.52
128	8	0.15	1591.75 ± 32.73	5.39	1916.45 ± 61.9	414.58	6589.88 ± 67.56	4.77
128	2	0.075	2784.44 ± 185.08	5.36	6160.86 ± 300.48	414.45	11055.68 ± 156.27	5.17
128	4	0.075	2837.25 ± 239.41	4.76	5004.45 ± 188.68	412.55	11516.47 ± 227.48	4.7
128	8	0.075	2004.58 ± 122.26	4.87	3084.67 ± 105.78	413.6	7432.14 ± 129.24	4.61
256	2	0.15	4386.43 ± 315.68	8.05	6255.14 ± 131.46	817.17	19088.19 ± 318.55	9.37
256	4	0.15	3998.61 ± 107.35	6.95	5209.96 ± 80.16	820.48	17287.79 ± 221.64	8.64
256	8	0.15	2934.38 ± 53.36	7.47	3786.36 ± 66.26	818.43	12519.56 ± 125.97	7.81
256	2	0.075	4128.96 ± 400.88	8.35	10426.63 ± 249.33	816.52	12191.23 ± 39.69	7.22
256	4	0.075	4376.73 ± 373.99	7.49	9389.72 ± 251.5	818.07	15748.08 ± 131.08	7.56
256	8	0.075	3258.24 ± 238.91	7.24	6019.24 ± 145.95	820	12417.86 ± 160.53	7.83

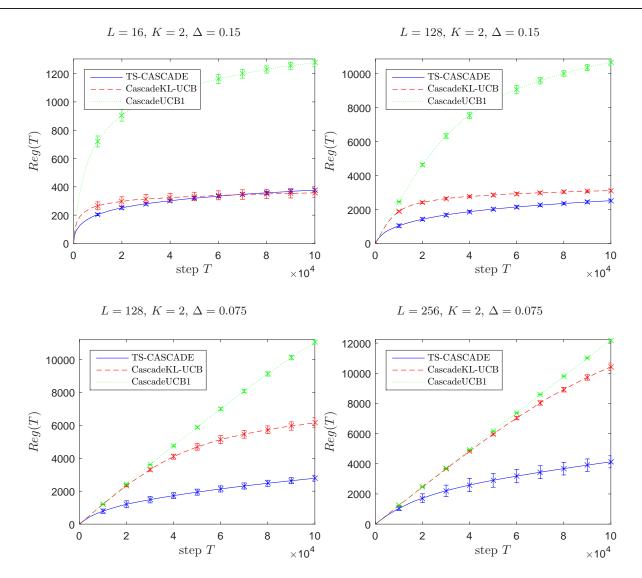


Figure 2: The T-step regret Reg(T) of TS-CASCADE, CASCADEKL-UCB and CASCADEUCB1 under 4 different parameter settings. Each line indicates the average Reg(T) of an algorithm and the length of each vertical error bar above and below each data point is the standard deviation.

To better understand the evolution of T-step regret $\operatorname{Reg}(T)$, we present four more plots in Figure 2. First of all, our algorithm clearly beats CASCADEUCB1 in all simulations. Even though our algorithm sometimes requires slightly more time to run than CASCADEUCB1, there is a significant improvement of TS-CASCADE over CASCADEUCB1. Comparing TS-CASCADE to CASCADEKL-UCB, we notice that when $L=16, K=2, \Delta=0.15$, CASCADEKL-UCB slightly outperforms our algorithm but requires more than ten times the computational time. Besides, when $L=128, K=2, \Delta=0.15$, our algorithm outperforms CASCADEKL-UCB in terms of regret and computational time. For the other two settings, our algorithm generates a T-step regret smaller than half of that of CASCADEKL-UCB, which confirms the superiority of our algorithm when the ground set is large.