6. CONTINUOUS PROBABILITY DISTRIBUTIONS [IT2110]

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Random Variables

- Discrete Random Variables
- Continuous Random Variables

Continuous Random Variables

- A random variable is said to be continuous, if it can take any value within a range.
- Continuous data are frequently measured in some way rather than counted.
- If X is a continuous random variable, $Pr(X = a) = 0$ for any value of a.

Examples

- Temperature
- Heart beat of a patient
- Rainfall
- Waiting time for a bus

PROBABILITY DISTRIBUTIONS

- For continuous random variables, the probability distribution cannot be presented in a tabular form.

- Probability distribution function of a continuous random variable is known as probability density function (*pdf*).

- The area under the p.d.f. gives probability values.

PDF - DEFINITION

- The function $f_X(x)$ is a probability density function for the continuous random variable X, defined over the set of real numbers $(\mathbb{R})$, if
  - $f_X(x) \geq 0$, for all $x \in R$
  - $\int_{-\infty}^{\infty} f_X(x)dx = 1$
  - $Pr(a<X<b) = \int_a^b f_X(x)dx$

Properties

- Let X be a continuous random variable with a p.d.f. ($f_X(x)$), defined over the set of real numbers ($\mathbb{R}$).
  - *The c.d.f.* $F_X(x) = Pr(X \leq x) = \int_{-\infty}^{x} f_X(x)dx$
  - $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$
  - $V[g(x)] = E[g(x)^2] - \{E[g(x)]\}^2$

PDF - Example

Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function,

$$f_X(x) = \begin{cases} cx^2 & -1 < x < 2 \\ 0 & otherwise \end{cases}$$

1) Find the value of c
2) Find $\Pr(0 < X \leq 1)$.
3) Find the expected value and the variance.
4) Find the c.d.f.

Continuous Probability Distributions

Continuous Probability Distributions

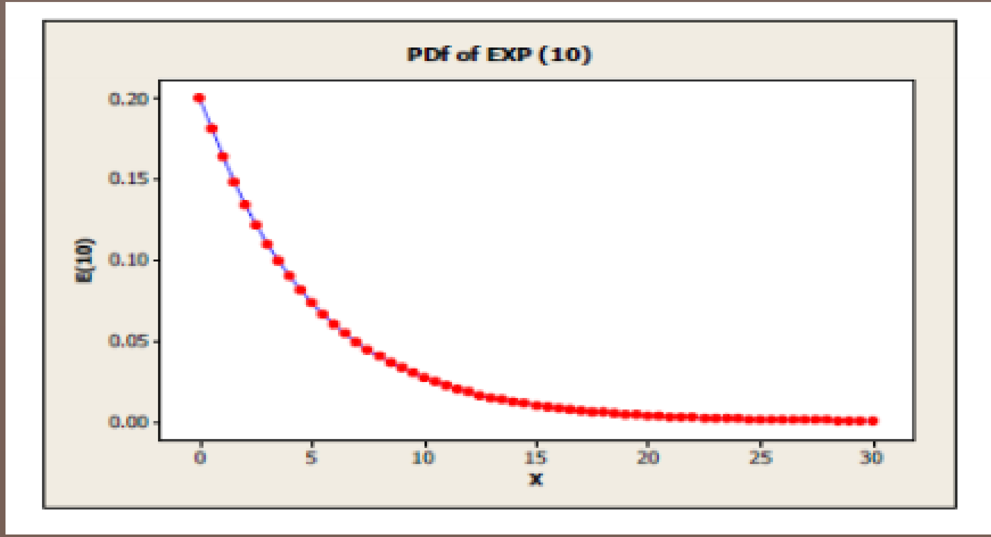
- *Exponential Distribution*
- *Normal Distribution*

Continuous Distributions

Exponential Distribution	Normal Distribution / Gaussian Distribution
The distribution is usually used to ***model life times***. (There is a link to the Poisson distribution)	This is most commonly used distribution. This is bell shaped distribution and perfectly symmetric around μ.
X ~ Exp(λ)	***X ~ N(μ,σ²)***
$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], -\infty < x < \infty$
$E(X) = 1/\lambda$	$E(X) = \mu$
$V(X) = 1/\lambda^2$	$V(X) = \sigma^2$

Exponential Distribution

- Widely used in waiting line (or queuing) theory to model the length of time between arrivals in process.
- **Examples:** duration between two customers at Bank ATMs, To model patients entering to an accident ward.



Exponential Distribution - Example

1) The time, in hours, during which an electrical generator is operational is a random variable that follows an exponential distribution with a mean of 160. What is the probability that a generator of this type will be operational for,

    a) Less than 40 hours?

    b) Between 60 and 160 hours?

    c) More than 200 hours?

Standard Normal Distribution

- Normal distribution with $\mu=0$ and $\sigma^2=1$ is known as the Standard Normal Distribution.
- Evaluating probabilities with Normal requires complex integration.
- To simplify the procedure, statistical tables are defined.
- But, tables for each combination of μ and σ^2 cannot be created.
- So, tables are only for the standard normal distribution.

Normal $\longrightarrow$ Standard Normal

If $X \sim N(\mu, \sigma^2)$, *Then*

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

Normal Distribution - Examples

1) For $Z \sim N(0, 1)$, calculate $Pr(Z \geq 1.13)$.
2) For $X \sim N(5, 4)$, calculate $Pr(-2.5 < X < 1.13)$.
3) The actual marks for FCS of Metro students revealed that they were normally distributed with a mean mark of 45 and a standard deviation of 22. What is the probability that a randomly chosen student will pass? (Assume that pass mark is 45)

Approximating Binomial Probabilities

- For $X \sim Bin(n, p)$ this approximation can be used if n is large and p is moderate.
- A general rule can be defined as, np and $n(1 - p)$ is greater than 5.
- Can be approximated with a r.v. with a distribution $N(np, np(1 - p))$.
- A continuity correction is needed because a discrete distribution is approximated with a continuous distribution.

Continuity Correction

- If $X \sim Bin(n, p)$ is approximated with a r.v. $Y \sim N(np, np(1 - p))$,
  - $Pr(X \leq a) = P(Y < a+0.5)$
  - $Pr(X \geq a) = P(Y > a-0.5)$
  - $Pr(X < a) = P(Y < a-0.5)$
  - $Pr(X > a) = P(Y > a+0.5)$
  - $Pr(X = a) = P(a-0.5 < Y < a+0.5)$

Example

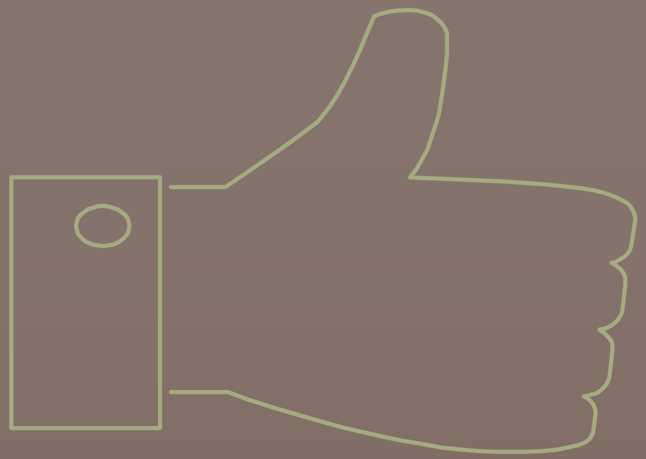
Suppose that a sample of $n = 1{,}600$ tires of the same type are obtained at random from an ongoing production process in which 8% of all such tires produced are defective. What is the probability that in such a sample 150 or fewer tires will be defective?

Approximating Poisson Probabilities

- If $X \sim Poisson(\lambda)$ then if λ is greater than 20, the approximation can be used.
- Can be approximated with a r.v. with a distribution $N(\lambda, \lambda)$.
- A continuity correction is needed because a discrete distribution is approximated with a continuous distribution (just as in the case of the Binomial to Normal approximation).

Example

The annual number of earthquakes registering at least 2.5 on the Richter Scale and having an epicenter within 40 miles of down town Memphis follows a Poisson distribution with mean 22.5. What is the probability that at least 25 such earthquakes will strike next year?



THANKS!

Any questions?