0.1 Architecture 1

0.1.1 Constraint analysis

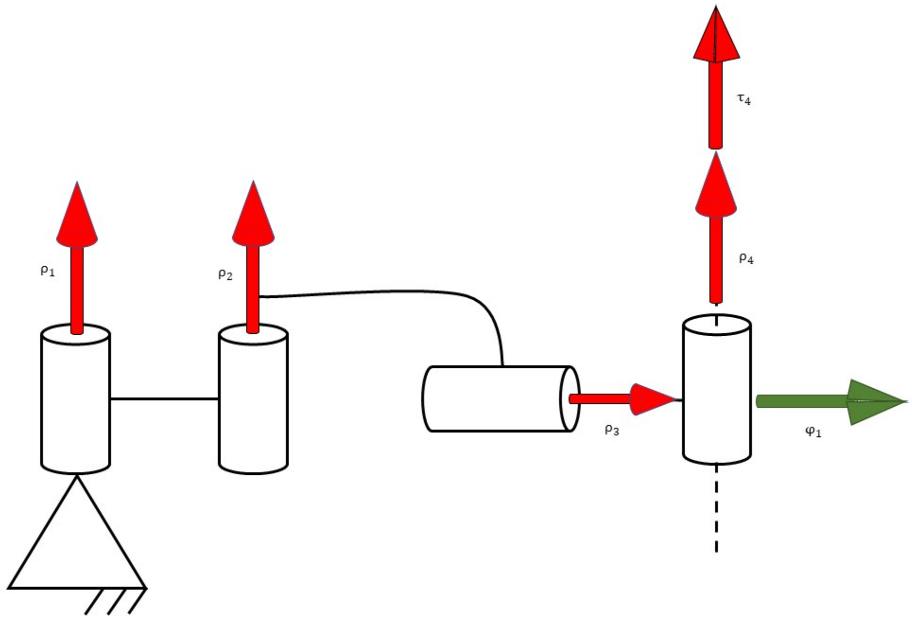

Figure 1: Kinematic chain 1, geometric constraint analysis

Considering the configuration of the kinematic chain 1, its constraint analysis is:

$dim(\mathcal{T}_1) = 5$
$dim(\mathcal{W}_1) = 1$
$Span(\mathcal{T}_1) = \rho_1, \rho_2, \rho_3, \rho_4, \tau_4$
$Span(\mathcal{W}_1) = \varphi_1$

The kinematic chain structural constraint is the wrench φ_1 (force), coplanar to the axis of ρ_1 and ρ_2, coplanar to ρ_3 and ρ_4 and orthogonal to τ_4.

To better show the geometric setup of the twists and wrenches, the picture [2] displays another configuration of the kinematic chain 1.

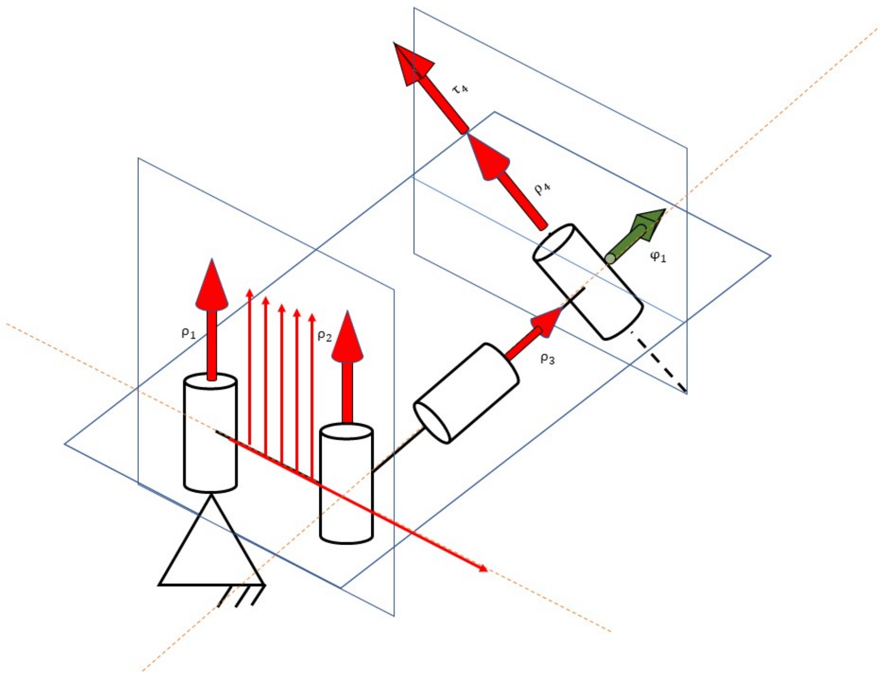

Figure 2: Kinematic chain 1, geometric constraint visualization

0.1.2 Statics, Kinematic analysis and gravity balancing

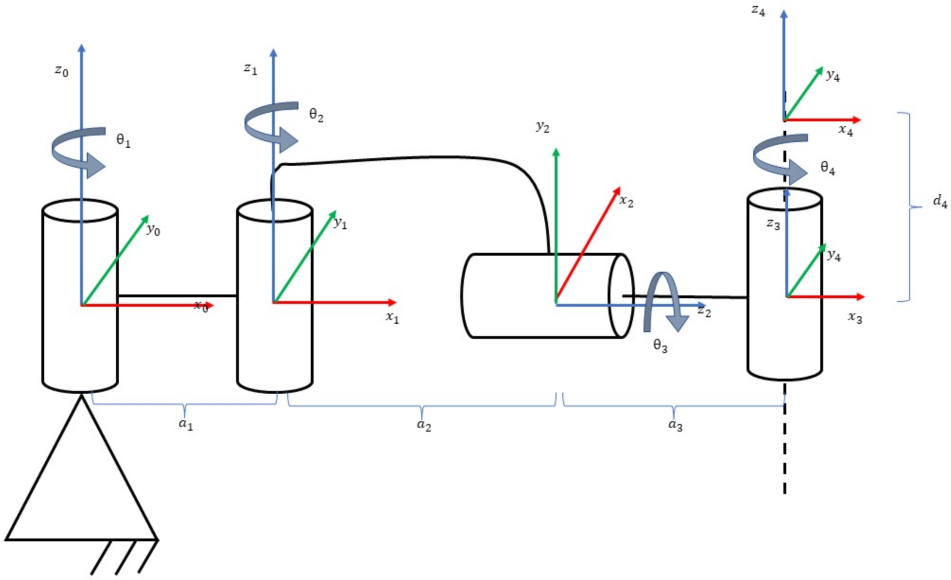

Figure 3: Kinematic chain 1, Denavit Hartenberg parametrization

According to figure 3, the following table [1] can be obtained. In table1 the parameters θ_1 and θ_2 are deliberately set to zero. This choice resulted convenient to show a lean kinematic evaluation and it was necessary for sake of brevity.

	JOINT 1	JOINT 2	JOINT 3	JOINT 4
TYPE	revolute	revolute	revolute	cylindrical
θ	0	0	θ_3	θ_4
d	0	0	0	d_4
a	a_1	a_2	a_3	0

Table 1: Table of geometric parameters for the kinematic chain 1

The first step to obtain the Jacobian matrix is to compute the rotation matrices of every reference frame with respect to the next one, starting from the base-frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_2^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_3^2 = \begin{pmatrix} 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$
$$R_4^3 = \begin{pmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 \\ \sin(\theta_4) & \cos(\theta_4) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second step is to compute the displacement vectors of each frame with respect to the previous one .

$$\mathrm{dv}_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{dv}_1^0 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{dv}_2^1 = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{dv}_3^2 = \begin{pmatrix} a_3 \cos(\theta_3) \\ a_3 \sin(\theta_3) \\ 0 \end{pmatrix}$$

$$\mathrm{dv}_4^3 = \begin{pmatrix} 0 \\ 0 \\ d_4 \end{pmatrix}$$

The third step is to concatenate the rotation matrices and the displacement vectors with respect to the previous reference frame to obtain the transformation matrices .

$$\mathrm{T}_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{T}_2^1 = \begin{pmatrix} 0 & 0 & 1 & a_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{T}_3^2 = \begin{pmatrix} 0 & \cos(\theta_3) & -\sin(\theta_3) & a_3 \cos(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) & a_3 \sin(\theta_3) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{T}_4^3 = \begin{pmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 & 0 \\ \sin(\theta_4) & \cos(\theta_4) & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth step is to multiply the transformation matrices obtained in order to obtain the transformation matrices with respect to the base frame.

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_2^0 = \begin{pmatrix} 0 & 0 & 1 & a_1 + a_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_3^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) & a_3 \cos(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) & a_3 \sin(\theta_3) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\mathrm{T}_4^0 = \begin{pmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 & a_1 + a_2 \\ \cos(\theta_3)\sin(\theta_4) & \cos(\theta_3)\cos(\theta_4) & -\sin(\theta_3) & a_3\cos(\theta_3) - d_4\sin(\theta_3) \\ \sin(\theta_3)\sin(\theta_4) & \cos(\theta_4)\sin(\theta_3) & \cos(\theta_3) & d_4\cos(\theta_3) + a_3\sin(\theta_3) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After the computation of the Transformation matrices with respect to the base frame, it is possible to extract the rotation matrices and the displacement vectors of every reference frame with respect to the base frame.

$$\mathrm{R}_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathrm{R}_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathrm{R}_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\mathrm{R}_3^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \end{pmatrix}$$

$$R_4^0 = \begin{pmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 \\ \cos(\theta_3)\sin(\theta_4) & \cos(\theta_3)\cos(\theta_4) & -\sin(\theta_3) \\ \sin(\theta_3)\sin(\theta_4) & \cos(\theta_4)\sin(\theta_3) & \cos(\theta_3) \end{pmatrix}$$
$$\mathrm{dv}_1^0 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{dv}_2^0 = \begin{pmatrix} a_1 + a_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{dv}_3^0 = \begin{pmatrix} a_1 + a_2 \\ a_3 \cos(\theta_3) \\ a_3 \sin(\theta_3) \end{pmatrix}$$

$$\mathrm{dv}_4^0 = \begin{pmatrix} a_1 + a_2 \\ a_3 \cos(\theta_3) - d_4 \sin(\theta_3) \\ d_4 \cos(\theta_3) + a_3 \sin(\theta_3) \end{pmatrix}$$

And finally the Jacobian matrix is obtained, following the table **??**

$$J = \begin{pmatrix} \sigma_1 & \sigma_1 & 0 & 0 \\ a_1 + a_2 & a_2 & -d_4 \cos(\theta_3) - a_3 \sin(\theta_3) & -\sin(\theta_3) \\ 0 & 0 & a_3 \cos(\theta_3) - d_4 \sin(\theta_3) & \cos(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin(\theta_3) \\ 1 & 1 & 0 & \cos(\theta_3) \end{pmatrix}$$

where

$$\sigma_1 = d_4 \sin(\theta_3) - a_3 \cos(\theta_3)$$

The gravitational component can be expressed as :

$$\zeta_e = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Transposing the Jacobian Matrix and multiplying it by the wrench $\zeta_e =$ (Principle of Virtual Work), we obtain the forces and torques of the joints to maintain the static balancing of the architecture.

$$\Phi = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg}\left(a_3 \cos\left(\theta_3\right) - d_4 \sin\left(\theta_3\right)\right) \\ -\mathrm{mg} \cos\left(\theta_3\right) \end{pmatrix}$$

0.2 Architecture 2

0.2.1 Constraint analysis

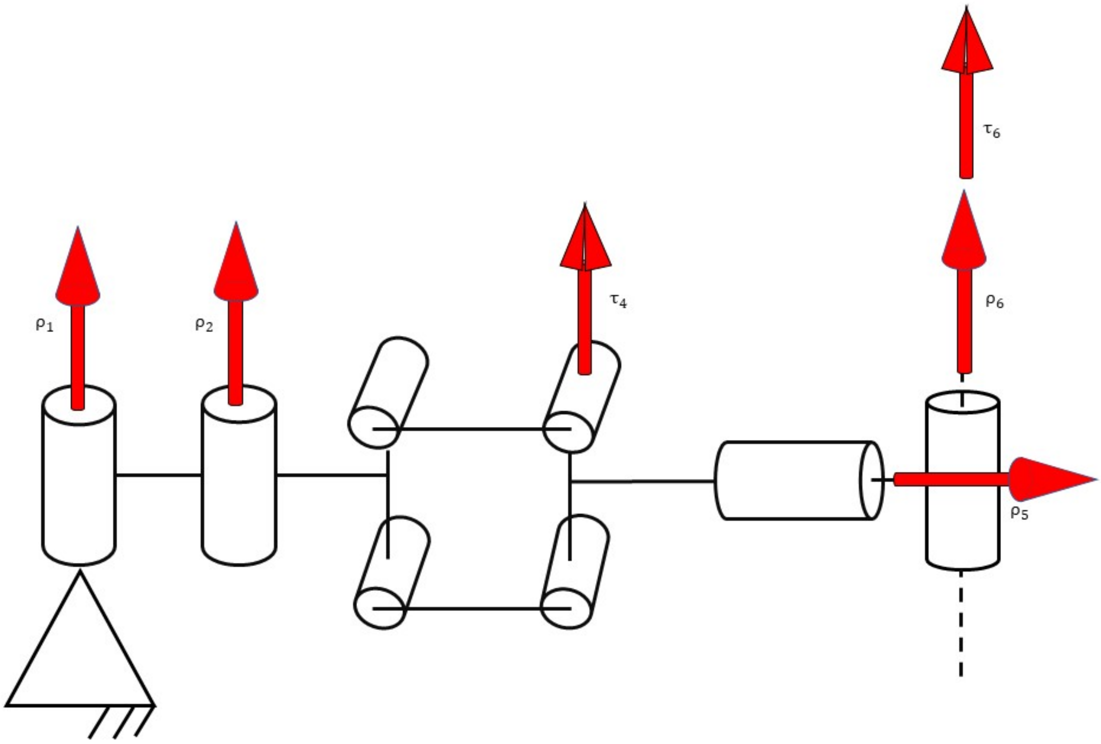

Figure 4: Kinematic chain 2, geometric constraint analysis

Considering the configuration of the kinematic chain 2, its constraint analysis is:

$dim(\mathcal{T}_1) = 6$
$dim(\mathcal{W}_1) = 0$
$Span(\mathcal{T}_1) = \rho_1, \rho_2, \tau_4, \rho_5, \rho_6, \tau_6$
$Span(\mathcal{W}_1) = \emptyset$

The kinematic chain has no structural constraint.

To better show the geometric setup of the twists and wrenches, the picture [5] displays another configuration of the kinematic chain 1. This configuration shows that is not possible to obtain a constraint wrench that could follow the reciprocity rule in a non-singular position.

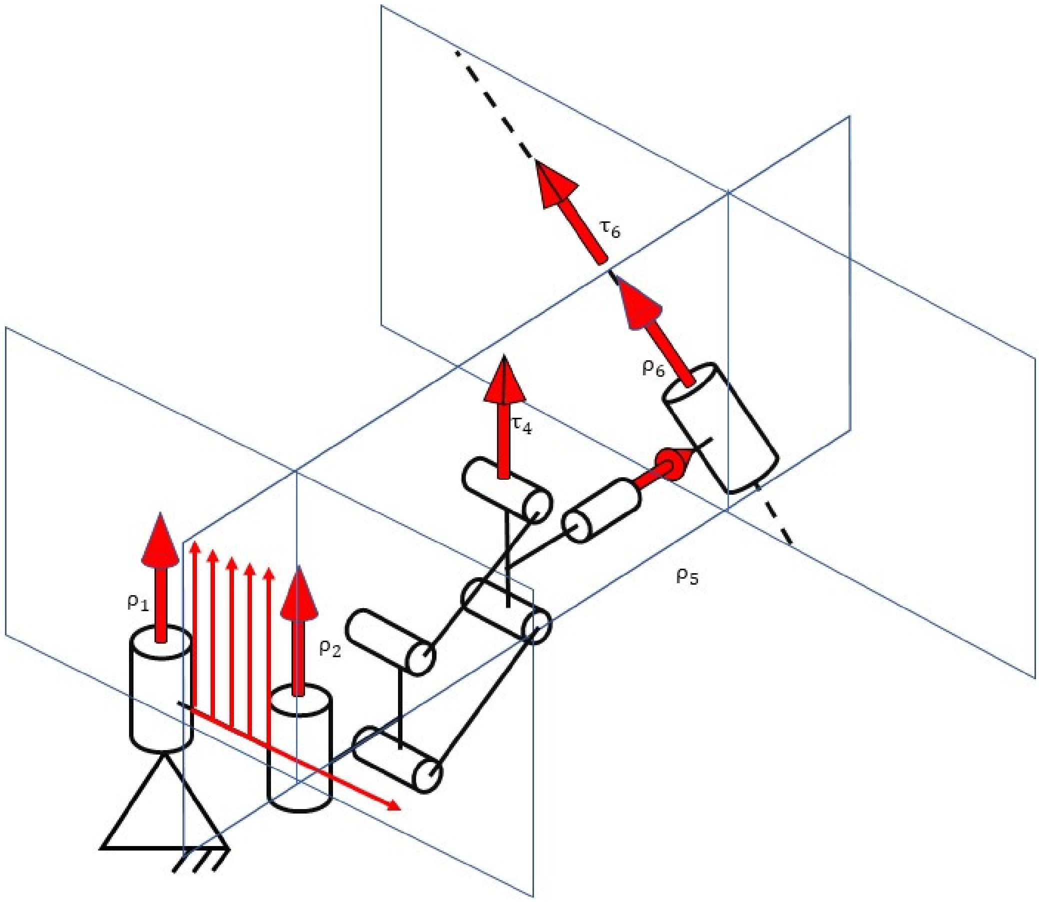

Figure 5: Kinematic chain 2, geometric constraint visualization

0.2.2 Statics, Kinematic analysis and gravity balancing

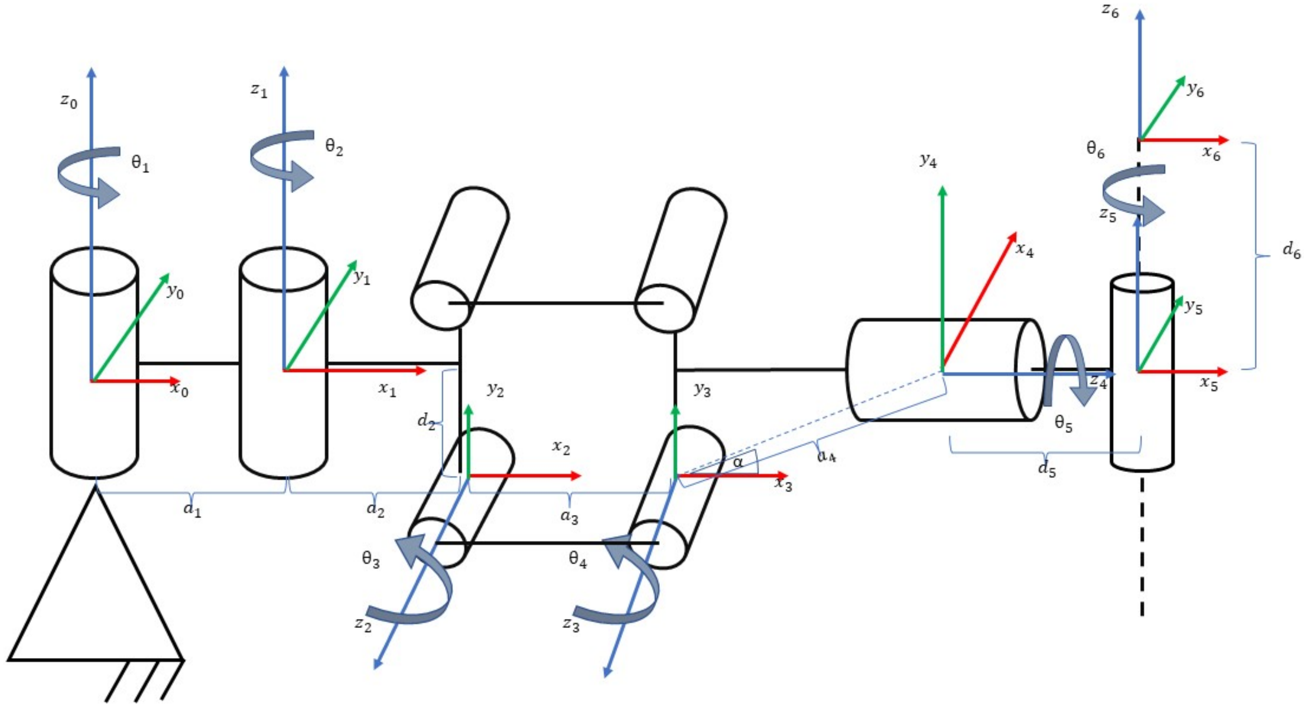

Figure 6: Kinematic chain 2, Denavit Hartenberg parametrization

According to figure 6, the following table [2] can be obtained. In table2 the parameters θ_1, θ_2, θ_3 and θ_4 are deliberately set to zero. This choice resulted convenient to show a lean kinematic evaluation and it was necessary for sake of brevity.

	JOINT 1	JOINT 2	JOINT 3	JOINT 4	JOINT 5	JOINT6
TYPE	revolute	revolute	revolute	revolute	revolute	cylindrical
θ	0	0	0	α	θ_5	θ_6
d	0	d_2	0	0	d_5	d_6
a	a_1	a_2	a_3	a_4	0	0

Table 2: Table of geometric parameters for the kinematic chain 2

As before, the first step displays the rotation matrices of every reference frame with respect to the next one, starting from the base-frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_4^3 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) \\ 0 & -1 & 0 \end{pmatrix}$$

$$R_5^4 = \begin{pmatrix} 0 & -\sin\theta_5 & -\cos\theta_5 \\ 0 & \cos(\theta_5) & -\sin(\theta_5) \\ 1 & 0 & 0 \end{pmatrix}$$

$$R_6^5 = \begin{pmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second step is to compute the displacement vectors of each frame with respect to the previous one .

$$dv_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_2^1 = \begin{pmatrix} a_2 \\ 0 \\ d_2 \end{pmatrix}$$

$$dv_3^2 = \begin{pmatrix} a_3 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_4^3 = \begin{pmatrix} a_4 \cos(\alpha) \\ a_4 \sin(\alpha) \\ 0 \end{pmatrix}$$

$$dv_5^4 = \begin{pmatrix} 0 \\ 0 \\ d_5 \end{pmatrix}$$

$$dv_6^5 = \begin{pmatrix} 0 \\ 0 \\ d_6 \end{pmatrix}$$

The third step is to concatenate the rotation matrices and the displacement vectors with respect to the previous reference frame to obtain the transformation matrices .

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^1 = \begin{pmatrix} 1 & 0 & 0 & a_2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^2 = \begin{pmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_4^3 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) & a_4 \cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) & a_4 \sin(\alpha) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_5^4 = \begin{pmatrix} 0 & -\sin(\theta_5) & -\cos(\theta_5) & 0 \\ 0 & \cos(\theta_5) & -\sin(\theta_5) & 0 \\ 1 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_6^5 = \begin{pmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth step is to multiply the transformation matrices obtained in order to obtain the transformation matrices with respect to the base frame.

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_2^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_3^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_4^0 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) & a_1 + a_2 + a_3 + a_4 \cos(\alpha) \\ 0 & 1 & 0 & 0 \\ -\cos(\alpha) & 0 & \sin(\alpha) & d_2 + a_4 \sin(\alpha) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_5^0 = \begin{pmatrix} \cos\alpha & -\sin\alpha \sin(\theta_5) & -\sin\alpha \cos(\theta_5) & a_1 + a_2 + a_3 + a_4 \cos\alpha + d_5 \cos\alpha \\ 0 & \cos(\theta_5) & -\sin(\theta_5) & 0 \\ \sin\alpha & \cos\alpha \sin(\theta_5) & \cos\alpha \cos(\theta_5) & d_2 + a_4 \sin\alpha + d_5 \sin\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_6^0 = \begin{pmatrix} C\alpha C\theta_6 - S\alpha S\theta_5 S\theta_6 & -C\alpha S\theta_6 - S\alpha C\theta_6 S\theta_5 & -S\alpha C\theta_5 & a_1 + a_2 + a_3 + a_4 C\alpha + d_5 C\alpha - d_6 S\theta_5 \\ C\theta_5 S\theta_6 & C\theta_5 C\theta_6 & -S\theta_5 & -d_6 S\theta_5 \\ S\alpha C\theta_6 + C\alpha S\theta_5 S\theta_6 & C\alpha C\theta_6 S\theta_5 - S\alpha S\theta_6 & C\alpha C\theta_5 & d_2 + a_4 S\alpha + d_5 S\alpha + d_6 C\alpha C\theta_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After the computation of the Transformation matrices with respect to the base frame, it is possible to extract the rotation matrices and the displacement vectors of every reference frame with respect to the base frame.

$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_2^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_3^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_4^0 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ 0 & 1 & 0 \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{pmatrix}$$
$$R_5^0 = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha)\sin(\theta_5) & -\sin(\alpha)\cos(\theta_5) \\ 0 & \cos(\theta_5) & -\sin(\theta_5) \\ \sin(\alpha) & \cos(\alpha)\sin(\theta_5) & \cos(\alpha)\cos(\theta_5) \end{pmatrix}$$

$$R_6^0 = \begin{pmatrix} \cos\alpha \cos(\theta_6) - \sin\alpha \sin\theta_5 \sin(\theta_6) & -\cos\alpha \sin(\theta_6) - \sin\alpha \cos(\theta_6) \sin\theta_5 & -\sin\alpha \cos\theta_5 \\ \cos\theta_5 \sin(\theta_6) & \cos\theta_5 \cos(\theta_6) & -\sin\theta_5 \\ \sin\alpha \cos(\theta_6) + \cos\alpha \sin\theta_5 \sin(\theta_6) & \cos\alpha \cos(\theta_6) \sin\theta_5 - \sin\alpha \sin(\theta_6) & \cos\alpha \cos\theta_5 \end{pmatrix}$$

$$dv_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$dv_2^0 = \begin{pmatrix} a_1 + a_2 \\ 0 \\ d_2 \end{pmatrix}$$
$$dv_3^0 = \begin{pmatrix} a_1 + a_2 + a_3 \\ 0 \\ d_2 \end{pmatrix}$$
$$dv_4^0 = \begin{pmatrix} a_1 + a_2 + a_3 + a_4 \cos(\alpha) \\ 0 \\ d_2 + a_4 \sin(\alpha) \end{pmatrix}$$
$$dv_5^0 = \begin{pmatrix} a_1 + a_2 + a_3 + a_4 \cos(\alpha) + d_5 \cos(\alpha) \\ 0 \\ d_2 + a_4 \sin(\alpha) + d_5 \sin(\alpha) \end{pmatrix}$$
$$dv_6^0 = \begin{pmatrix} a_1 + a_2 + a_3 + a_4 \cos(\alpha) + d_5 \cos(\alpha) - d_6 \sin(\alpha) \cos(\theta_5) \\ -d_6 \sin(\theta_5) \\ d_2 + a_4 \sin(\alpha) + d_5 \sin(\alpha) + d_6 \cos(\alpha) \cos(\theta_5) \end{pmatrix}$$

And finally the Jacobian matrix is obtained, following the table **??**

$$J = \begin{pmatrix} d_6 \sin(\theta_5) & d_6 \sin(\theta_5) & \sigma_4 & \sigma_4 & d_6 \sin(\alpha) \sin(\theta_5) & \sigma_2 \\ A21 & A22 & 0 & 0 & A25 & -\sin(\theta_5) \\ 0 & 0 & A33 & A34 & -d_6 \cos(\alpha) \sin(\theta_5) & \sigma_3 \\ 0 & 0 & 0 & 0 & \cos(\alpha) & \sigma_2 \\ 0 & 0 & -1 & -1 & 0 & -\sin(\theta_5) \\ 1 & 1 & 0 & 0 & \sin(\alpha) & \sigma_3 \end{pmatrix}$$

where

$$\begin{aligned} A21 &= a_1 + a_2 + a_3 + a_4 \cos(\alpha) + d_5 \cos(\alpha) - \sigma_1 \\ A22 &= a_2 + a_3 + a_4 \cos(\alpha) + d_5 \cos(\alpha) - \sigma_1 \\ A25 &= \sin(\alpha) \left(d_5 \cos(\alpha) - \sigma_1\right) - \cos(\alpha) \left(d_5 \sin(\alpha) + \sigma_5\right) \\ A33 &= a_3 + a_4 \cos(\alpha) + d_5 \cos(\alpha) - \sigma_1 \\ A34 &= a_4 \cos(\alpha) + d_5 \cos(\alpha) - \sigma_1 \\ \sigma_1 &= d_6 \sin(\alpha) \cos(\theta_5) \\ \sigma_2 &= -\sin(\alpha) \cos(\theta_5) \\ \sigma_3 &= \cos(\alpha) \cos(\theta_5) \\ \sigma_4 &= -a_4 \sin(\alpha) - d_5 \sin(\alpha) - \sigma_5 \\ \sigma_5 &= d_6 \cos(\alpha) \cos(\theta_5) \end{aligned}$$

The gravitational component can be expressed as :

$$\zeta_e = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Transposing the Jacobian Matrix and multiplying it by the vector ζ_e = (Principle of Virtual Work), we obtain the forces and torques of the joints to maintain the static balancing of the architecture.

$$\Phi = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg}\left(a_3 + a_4 \cos(\alpha) + d_5 \cos(\alpha) - d_6 \sin(\alpha)\, \cos(\theta_5)\right) \\ -\mathrm{mg}\left(a_4 \cos(\alpha) + d_5 \cos(\alpha) - d_6 \sin(\alpha)\, \cos(\theta_5)\right) \\ d_6\, \mathrm{mg} \cos(\alpha)\, \sin(\theta_5) \\ -\mathrm{mg} \cos(\alpha)\, \cos(\theta_5) \end{pmatrix}$$

0.3 Architecture 3

0.3.1 Constraint analysis

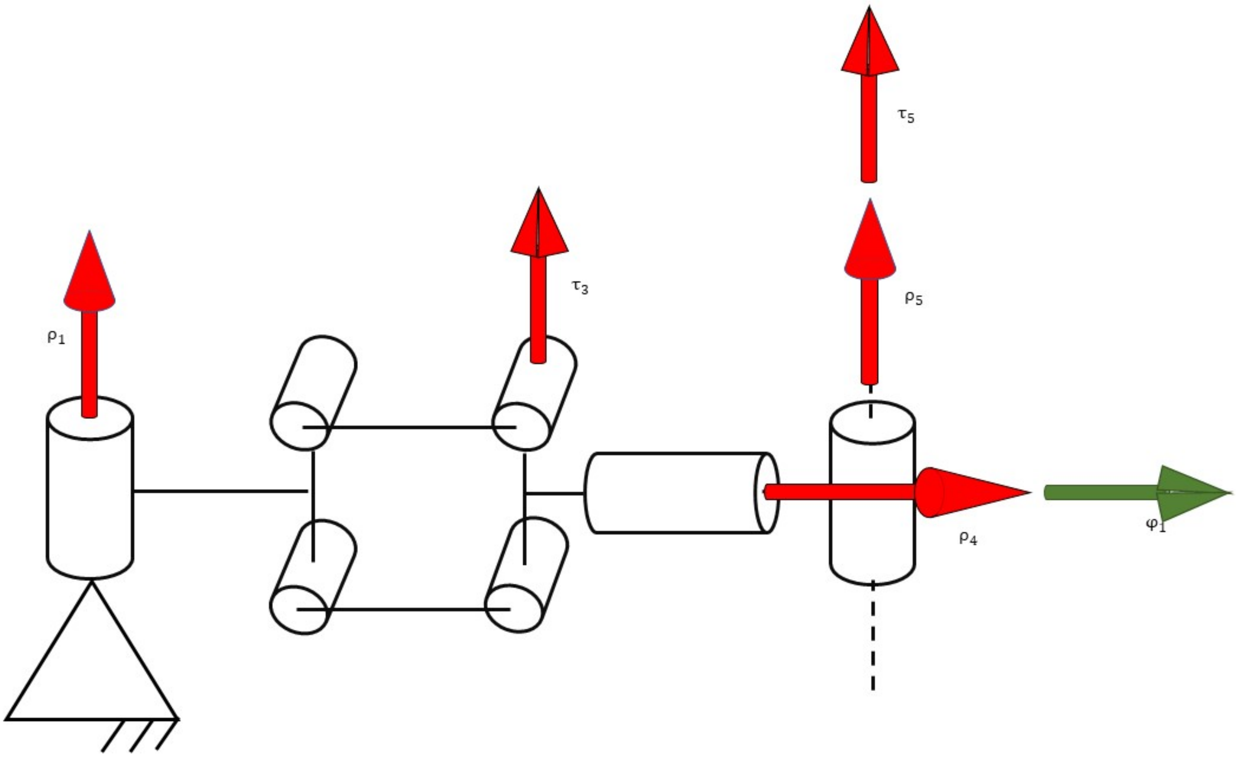

Figure 7: Kinematic chain 3, geometric constraint analysis

Considering the configuration of the kinematic chain 3, its constraint analysis is:

$dim(\mathcal{T}_1) = 5$
$dim(\mathcal{W}_1) = 1$
$Span(\mathcal{T}_1) = \rho_1, \tau_3, \rho_4, \rho_5\tau_5$
$Span(\mathcal{W}_1) = \varphi_1$

The kinematic chain structural constraint is the wrench φ_1 (force), coplanar to ρ_1,ρ_4 and ρ_5 and orthogonal to τ_3 and τ_5.

0.3.2 Statics, Kinematic analysis and gravity balancing

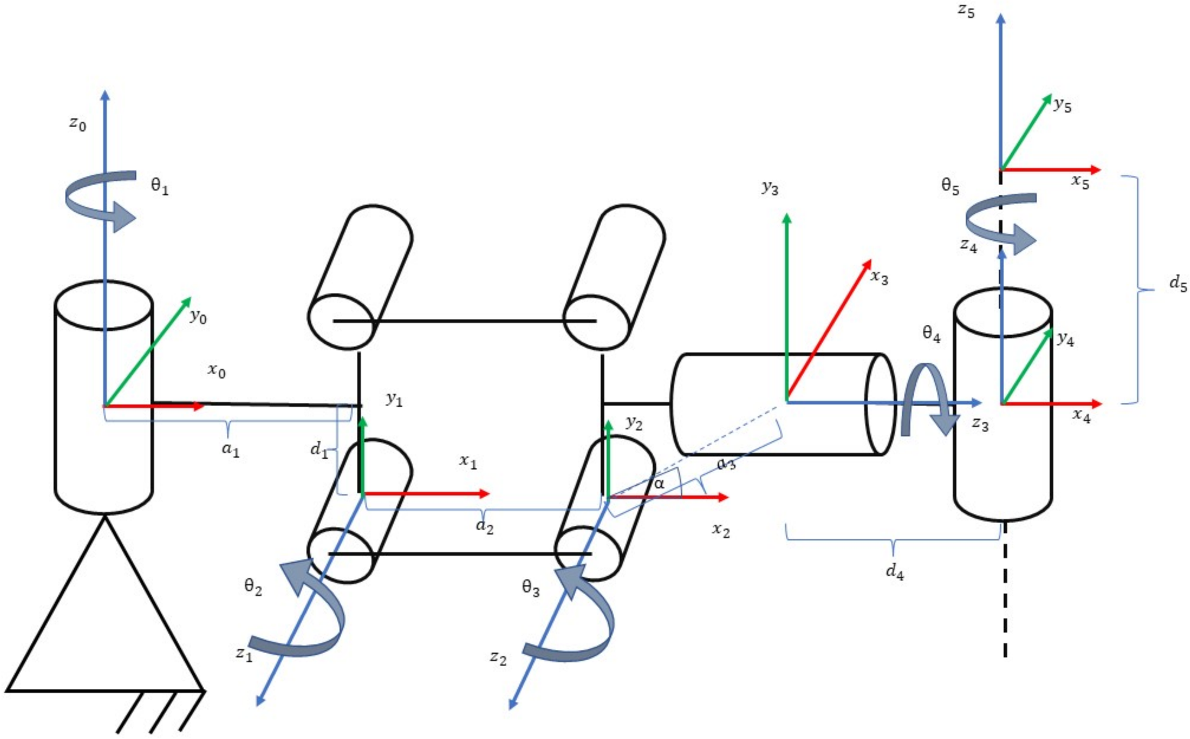

Figure 8: Kinematic chain 3, Denavit Hartenberg parametrization

According to figure 8, the following table [3] can be obtained. In table3 the parameters θ_1, θ_2, and θ_3 are deliberately set to zero. This choice resulted convenient to show a lean kinematic evaluation and it was necessary for sake of brevity.

	JOINT 1	JOINT 2	JOINT 3	JOINT 4	JOINT 5
TYPE	revolute	revolute	revolute	revolute	cylindrical
θ	0	0	α	θ_4	θ_5
d	d_1	0	0	d_4	d_5
a	a_1	a_2	a_3	0	0

Table 3: Table of geometric parameters for the kinematic chain 3

As before, the first step displays the rotation matrices of every reference frame with respect to the next one, starting from the base-frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_2^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_3^2 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) \\ 0 & -1 & 0 \end{pmatrix}$$
$$R_4^3 = \begin{pmatrix} 0 & -\sin(\theta_4) & -\cos(\theta_4) \\ 0 & \cos(\theta_4) & -\sin(\theta_4) \\ 1 & 0 & 0 \end{pmatrix}$$
$$R_5^4 = \begin{pmatrix} \cos(\theta_5) & -\sin(\theta_5) & 0 \\ \sin(\theta_5) & \cos(\theta_5) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second step is to compute the displacement vectors of each frame with respect to the previous one .

$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_2^1 = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_3^2 = \begin{pmatrix} a_3\,\cos(\alpha) \\ a_3\,\sin(\alpha) \\ 0 \end{pmatrix}$$
$$dv_4^3 = \begin{pmatrix} 0 \\ 0 \\ d_4 \end{pmatrix}$$
$$dv4_5 = \begin{pmatrix} 0 \\ 0 \\ d_5 \end{pmatrix}$$

The third step is to concatenate the rotation matrices and the displacement vectors with respect to the previous reference frame to obtain the transformation matrices .

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_2^1 = \begin{pmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_3^2 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) & a_3\,\cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) & a_3\,\sin(\alpha) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_4^3 = \begin{pmatrix} 0 & -\sin(\theta_4) & -\cos(\theta_4) & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) & 0 \\ 1 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_5^4 = \begin{pmatrix} \cos(\theta_5) & -\sin(\theta_5) & 0 & 0 \\ \sin(\theta_5) & \cos(\theta_5) & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth step is to multiply the transformation matrices obtained in order to obtain the transformation matrices with respect to the base frame.

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T_2^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^0 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) & a_1 + a_2 + a_3\,\cos(\alpha) \\ 0 & 1 & 0 & 0 \\ -\cos(\alpha) & 0 & \sin(\alpha) & d_1 + a_3\,\sin(\alpha) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_4^0 = \begin{pmatrix} \cos\alpha & -\sin\alpha\,\sin(\theta_4) & -\sin\alpha\,\cos(\theta_4) & a_1 + a_2 + a_3\,\cos\alpha + d_4\,\cos\alpha \\ 0 & \cos\theta_4 & -\sin\theta_4 & 0 \\ \sin\alpha & \cos\alpha\,\sin\theta_4 & \cos\alpha\,\cos\theta_4 & d_1 + a_3\,\sin\alpha + d_4\,\sin\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_5^0 = \begin{pmatrix} A11 & A12 & A13 & A14 \\ A21 & A22 & A23 & A24 \\ A31 & A32 & A33 & A34 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$where$

$A11 = \cos\alpha\,\cos\theta_5 - \sin\alpha\,\sin\theta_4\,\sin\theta_5$
$A12 = -\cos\alpha\,\sin\theta_5 - \sin\alpha\,\cos\theta_5\,\sin\theta_4$
$A13 = -\sin\alpha\,\cos\theta_4$
$A14 = a_1 + a_2 + a_3\,\cos\alpha + d_4\,\cos\alpha - d_5\,\sin\alpha\,\cos\theta_4$
$A21 = \cos\theta_4\,\sin\theta_5$
$A22 = \cos\theta_4\,\cos\theta_5$
$A23 = -\sin\theta_4$
$A24 = -d_5\,\sin\theta_4$
$A31 = \sin\alpha\,\cos\theta_5 + \cos\alpha\,\sin\theta_4\,\sin\theta_5$
$A32 = \cos\alpha\,\cos\theta_5\,\sin\theta_4 - \sin\alpha\,\sin\theta_5$
$A33 = \cos\alpha\,\cos\theta_4$
$A34 = d_1 + a_3\,\sin\alpha + d_4\,\sin\alpha + d_5\,\cos\alpha\,\cos\theta_4$

After the computation of the Transformation matrices with respect to the base frame, it is possible to extract the rotation matrices and the displacement vectors of every reference frame with respect to the base frame.

$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_2^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_3^0 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ 0 & 1 & 0 \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{pmatrix}$$

$$R_4^0 = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha)\,\sin(\theta_4) & -\sin(\alpha)\,\cos(\theta_4) \\ 0 & \cos(\theta_4) & -\sin(\theta_4) \\ \sin(\alpha) & \cos(\alpha)\,\sin(\theta_4) & \cos(\alpha)\,\cos(\theta_4) \end{pmatrix}$$

$$R_5^0 = \begin{pmatrix} A11 & A12 & A13 \\ A21 & A22 & A23 \\ A31 & A32 & A33 \end{pmatrix}$$

where

$$\begin{aligned}
A11 &= \cos(\alpha)\,\cos(\theta_5) - \sin(\alpha)\,\sin(\theta_4)\,\sin(\theta_5)\\
A12 &= -\cos(\alpha)\,\sin(\theta_5) - \sin(\alpha)\,\cos(\theta_5)\,\sin(\theta_4)\\
A13 &= -\sin(\alpha)\,\cos(\theta_4)\\
A21 &= \cos(\theta_4)\,\sin(\theta_5)\\
A22 &= \cos(\theta_4)\,\cos(\theta_5)\\
A23 &= -\sin(\theta_4)\\
A31 &= \sin(\alpha)\,\cos(\theta_5) + \cos(\alpha)\,\sin(\theta_4)\,\sin(\theta_5)\\
A32 &= \cos(\alpha)\,\cos(\theta_5)\,\sin(\theta_4) - \sin(\alpha)\,\sin(\theta_5)\\
A33 &= \cos(\alpha)\,\cos(\theta_4)
\end{aligned}$$

$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_2^0 = \begin{pmatrix} a_1 + a_2 \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_3^0 = \begin{pmatrix} a_1 + a_2 + a_3\,\cos(\alpha) \\ 0 \\ d_1 + a_3\,\sin(\alpha) \end{pmatrix}$$
$$dv_4^0 = \begin{pmatrix} a_1 + a_2 + a_3\,\cos(\alpha) + d_4\,\cos(\alpha) \\ 0 \\ d_1 + a_3\,\sin(\alpha) + d_4\,\sin(\alpha) \end{pmatrix}$$
$$dv_5^0 = \begin{pmatrix} a_1 + a_2 + a_3\,\cos(\alpha) + d_4\,\cos(\alpha) - d_5\,\sin(\alpha)\,\cos(\theta_4) \\ -d_5\,\sin(\theta_4) \\ d_1 + a_3\,\sin(\alpha) + d_4\,\sin(\alpha) + d_5\,\cos(\alpha)\,\cos(\theta_4) \end{pmatrix}$$

And finally the Jacobian matrix is obtained, following the table **??**

$$J = \begin{pmatrix}
d_5\,\sin(\theta_4) & \sigma_4 & \sigma_4 & d_5\,\sin(\alpha)\,\sin(\theta_4) & \sigma_2\\
A21 & 0 & 0 & A24 & -\sin(\theta_4)\\
0 & A32 & A33 & -d_5\,\cos(\alpha)\,\sin(\theta_4) & \sigma_3\\
0 & 0 & 0 & \cos(\alpha) & \sigma_2\\
0 & -1 & -1 & 0 & -\sin(\theta_4)\\
1 & 0 & 0 & \sin(\alpha) & \sigma_3
\end{pmatrix}$$

where

$$\begin{aligned}
A21 &= a_1 + a_2 + a_3\,\cos(\alpha) + d_4\,\cos(\alpha) - \sigma_1\\
A24 &= \sin(\alpha)\,(d_4\,\cos(\alpha) - \sigma_1) - \cos(\alpha)\,(d_4\,\sin(\alpha) + \sigma_5)\\
A32 &= a_2 + a_3\,\cos(\alpha) + d_4\,\cos(\alpha) - \sigma_1\\
A33 &= a_3\,\cos(\alpha) + d_4\,\cos(\alpha) - \sigma_1\\
\sigma_1 &= d_5\,\sin(\alpha)\,\cos(\theta_4)\\
\sigma_2 &= -\sin(\alpha)\,\cos(\theta_4)\\
\sigma_3 &= \cos(\alpha)\,\cos(\theta_4)\\
\sigma_4 &= -a_3\,\sin(\alpha) - d_4\,\sin(\alpha) - \sigma_5\\
\sigma_5 &= d_5\,\cos(\alpha)\,\cos(\theta_4)
\end{aligned}$$

The gravitational component can be expressed as :

$$\zeta_e = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Transposing the Jacobian Matrix and multiplying it by the vector $\zeta_e =$ (Principle of Virtual Work), we obtain the forces and torques of the joints to maintain the static balancing of the architecture.

$$\Phi = \begin{pmatrix} 0 \\ -\mathrm{mg}\,(a_2 + a_3\,\cos(\alpha) + d_4\,\cos(\alpha) - d_5\,\sin(\alpha)\,\cos(\theta_4)) \\ -\mathrm{mg}\,(a_3\,\cos(\alpha) + d_4\,\cos(\alpha) - d_5\,\sin(\alpha)\,\cos(\theta_4)) \\ d_5\,\mathrm{mg}\,\cos(\alpha)\,\sin(\theta_4) \\ -\mathrm{mg}\,\cos(\alpha)\,\cos(\theta_4) \end{pmatrix}$$

0.4 Architecture 4

0.4.1 Constraint analysis

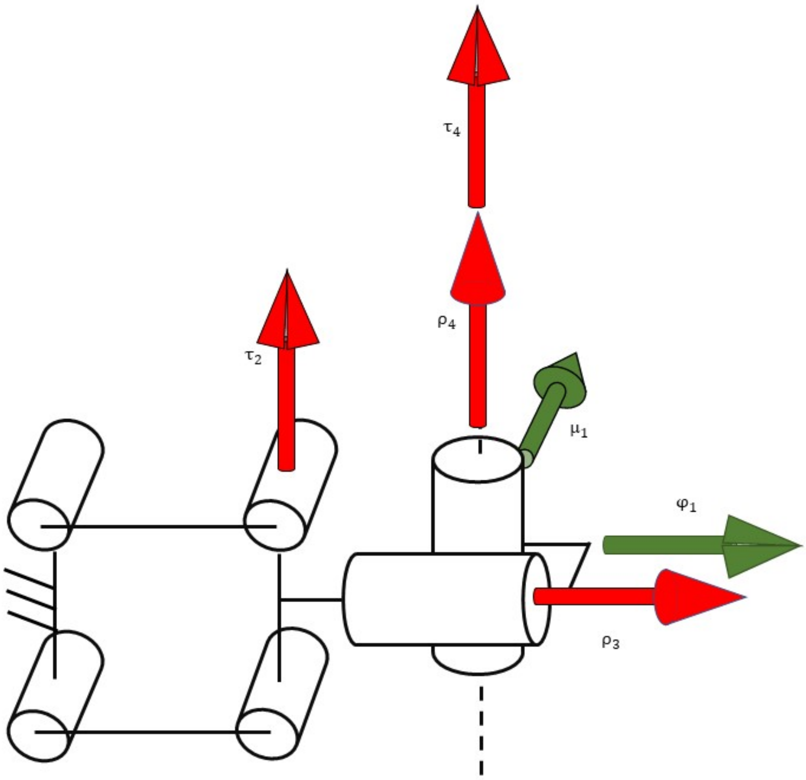

Figure 9: Kinematic chain 4, geometric constraint analysis

Considering the configuration of the kinematic chain 3, its constraint analysis is:

$dim(\mathcal{T}_1) = 4$
$dim(\mathcal{W}_1) = 2$
$Span(\mathcal{T}_1) = \tau_2, \rho_3, \rho_4, \tau_4$
$Span(\mathcal{W}_1) = \varphi_1, \mu_1$

The kinematic chain structural constraint are the wrench φ_1 (force), coplanar with ρ_2 and ρ_4 and orthogonal with τ_2 and τ_4, and the wrench μ_1 (couple) perpendicular to ρ_2 and ρ_4.

0.4.2 Statics, Kinematic analysis and gravity balancing

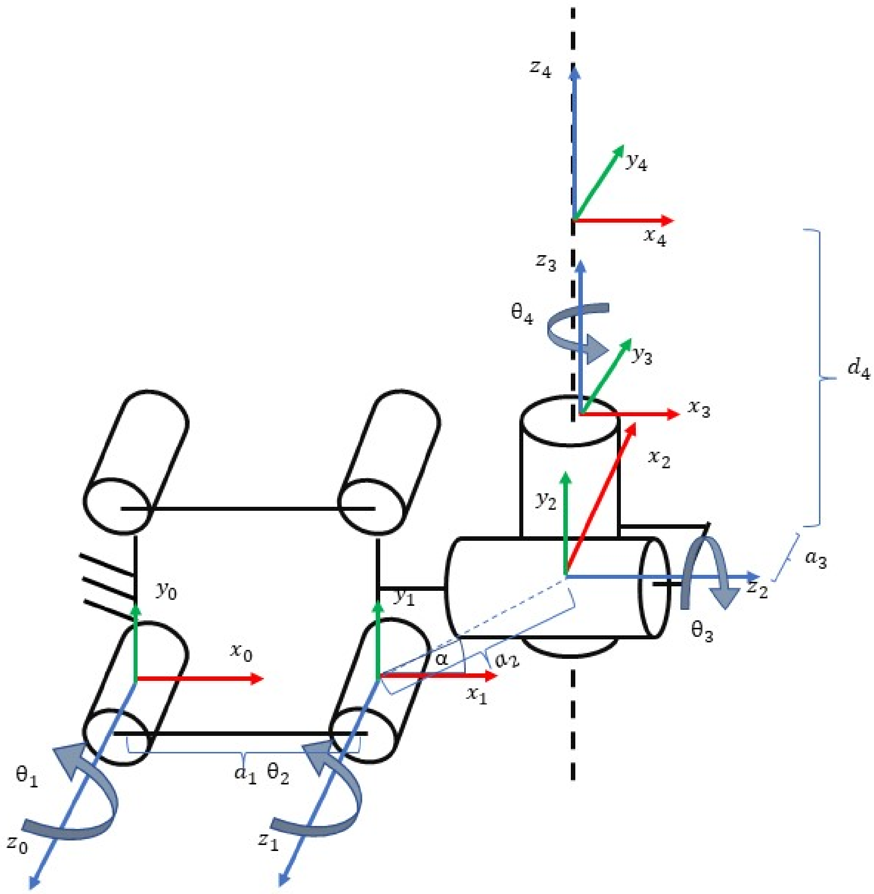

Figure 10: Kinematic chain 4, Denavit Hartenberg parametrization

According to figure 10, the following table [4] can be obtained. In table 4 the parameters θ_1, and θ_2 are deliberately set to zero. This choice resulted convenient to show a lean kinematic evaluation and it was necessary for sake of brevity.

	JOINT 1	JOINT 2	JOINT 3	JOINT 4
TYPE	revolute	revolute	revolute	cylindrical
θ	0	α	θ_3	θ_4
d	0	0	0	d_4
a	a_1	a_2	a_3	0

Table 4: Table of geometric parameters for the kinematic chain 4

As before, the first step displays the rotation matrices of every reference frame with respect to the next one, starting from the base-frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_2^1 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) \\ 0 & -1 & 0 \end{pmatrix}$$
$$R_3^2 = \begin{pmatrix} 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$
$$R_4^3 = \begin{pmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 \\ \sin(\theta_4) & \cos(\theta_4) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second step is to compute the displacement vectors of each frame with respect to the previous one .

$$dv_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_2^1 = \begin{pmatrix} a_2 \cos(\alpha) \\ a_2 \sin(\alpha) \\ 0 \end{pmatrix}$$

$$dv_3^2 \begin{pmatrix} a_3 \cos(\theta_3) \\ a_3 \sin(\theta_3) \\ 0 \end{pmatrix}$$

$$dv_4^3 = \begin{pmatrix} 0 \\ 0 \\ d_4 \end{pmatrix}$$

The third step is to concatenate the rotation matrices and the displacement vectors with respect to the previous reference frame to obtain the transformation matrices .

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^1 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) & a_2 \cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) & a_2 \sin(\alpha) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^2 = \begin{pmatrix} 0 & \cos(\theta_3) & -\sin(\theta_3) & a_3 \cos(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) & a_3 \sin(\theta_3) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_4^3 = \begin{pmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 & 0 \\ \sin(\theta_4) & \cos(\theta_4) & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth step is to multiply the transformation matrices obtained in order to obtain the transformation matrices with respect to the base frame

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T02 = \begin{pmatrix} \sin(\alpha) & 0 & \cos(\alpha) & a_1 + a_2 \cos(\alpha) \\ -\cos(\alpha) & 0 & \sin(\alpha) & a_2 \sin(\alpha) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T03 = \begin{pmatrix} \cos\alpha & \sin\alpha \cos\theta_3 & -\sin\alpha \sin\theta_3 & a_1 + a_2 \cos\alpha + a_3 \sin\alpha \cos\theta_3 \\ \sin\alpha & -\cos\alpha \cos\theta_3 & \cos\alpha \sin\theta_3 & a_2 \sin\alpha - a_3 \cos\alpha \cos\theta_3 \\ 0 & -\sin\theta_3 & -\cos\theta_3 & -a_3 \sin\theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T04 = \begin{pmatrix} A11 & A12 & A13 & A14 \\ A21 & A22 & A23 & A24 \\ A31 & A32 & A33 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$
\begin{aligned}
A11 &= \cos\alpha\,\cos(\theta_4) + \sin\alpha\,\cos\theta_3\,\sin(\theta_4)\\
A12 &= \sin\alpha\,\cos\theta_3\,\cos(\theta_4) - \cos\alpha\,\sin(\theta_4)\\
A13 &= -\sin\alpha\,\sin\theta_3\\
A14 &= a_1 + a_2\,\cos\alpha + a_3\,\sin\alpha\,\cos\theta_3 - d_4\,\sin\alpha\,\sin\theta_3\\
A21 &= \sin\alpha\,\cos(\theta_4) - \cos\alpha\,\cos\theta_3\,\sin(\theta_4)\\
A22 &= -\sin\alpha\,\sin(\theta_4) - \cos\alpha\,\cos\theta_3\,\cos(\theta_4)\\
A23 &= \cos\alpha\,\sin\theta_3\\
A24 &= a_2\,\sin\alpha - a_3\,\cos\alpha\,\cos\theta_3 + d_4\,\cos\alpha\,\sin\theta_3\\
A31 &= -\sin\theta_3\,\sin(\theta_4)\\
A32 &= -\cos(\theta_4)\,\sin\theta_3\\
A33 &= -\cos\theta_3\\
A34 &= -d_4\,\cos\theta_3 - a_3\,\sin\theta_3
\end{aligned}
$$

After the computation of the Transformation matrices with respect to the baseframe, it is possible to extract the rotation matrices and the displacement vectors of every reference frame with respect to the base frame.

$$
R_0^0 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}
$$

$$
R_1^0 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}
$$

$$
R_2^0 = \begin{pmatrix} \sin\alpha & 0 & \cos(\alpha)\\ -\cos(\alpha) & 0 & \sin(\alpha)\\ 0 & -1 & 0 \end{pmatrix}
$$

$$
R_3^0 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha)\,\cos(\theta_3) & -\sin(\alpha)\,\sin(\theta_3)\\ \sin(\alpha) & -\cos(\alpha)\,\cos(\theta_3) & \cos(\alpha)\,\sin(\theta_3)\\ 0 & -\sin(\theta_3) & -\cos(\theta_3) \end{pmatrix}
$$

$$
R_4^0 = \begin{pmatrix} A11 & \sin\alpha\,\cos\theta_3\,\cos\theta_4 - \cos\alpha\,\sin\theta_4 & -\sin\alpha\,\sin\theta_3\\ A21 & -\sin\alpha\,\sin\theta_4 - \cos\alpha\,\cos\theta_3\,\cos\theta_4 & \cos\alpha\,\sin\theta_3\\ A31 & -\cos\theta_4\,\sin\theta_3 & -\cos\theta_3 \end{pmatrix}
$$

where

$$
\begin{aligned}
A11 &= \cos\alpha\,\cos\theta_4 + \sin\alpha\,\cos\theta_3\,\sin\theta_4\\
A21 &= \sin\alpha\,\cos\theta_4 - \cos\alpha\,\cos\theta_3\,\sin\theta_4\\
A31 &= -\sin\theta_3\,\sin\theta_4
\end{aligned}
$$

$$
dv_0^0 = \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}
$$

$$
dv_1^0 = \begin{pmatrix} a_1\\ 0\\ 0 \end{pmatrix}
$$

$$
dv_2^0 = \begin{pmatrix} a_1 + a_2\,\cos(\alpha)\\ a_2\,\sin(\alpha)\\ 0 \end{pmatrix}
$$

$$dv_3^0 = \begin{pmatrix} a_1 + a_2\,\cos(\alpha) + a_3\,\sin(\alpha)\,\cos(\theta_3) \\ a_2\,\sin(\alpha) - a_3\,\cos(\alpha)\,\cos(\theta_3) \\ -a_3\,\sin(\theta_3) \end{pmatrix}$$

$$dv_4^0 = \begin{pmatrix} a_1 + a_2\,\cos(\alpha) + a_3\,\sin(\alpha)\,\cos(\theta_3) - d_4\,\sin(\alpha)\,\sin(\theta_3) \\ a_2\,\sin(\alpha) - a_3\,\cos(\alpha)\,\cos(\theta_3) + d_4\,\cos(\alpha)\,\sin(\theta_3) \\ -d_4\,\cos(\theta_3) - a_3\,\sin(\theta_3) \end{pmatrix}$$

And finally the Jacobian matrix is obtained, following the table **??**

$$J = \begin{pmatrix} \sigma_6 & \sigma_6 & -\sin(\alpha)\,\sigma_3 & \sigma_4 \\ A21 & A22 & \cos(\alpha)\,\sigma_3 & \sigma_5 \\ 0 & 0 & A33 & -\cos(\theta_3) \\ 0 & 0 & \cos(\alpha) & \sigma_4 \\ 0 & 0 & \sin(\alpha) & \sigma_5 \\ 1 & 1 & 0 & -\cos(\theta_3) \end{pmatrix}$$

where

$$\begin{aligned} A21 &= a_1 + a_2\,\cos(\alpha) + \sigma_2 - \sigma_1 \\ A22 &= a_2\,\cos(\alpha) + \sigma_2 - \sigma_1 \\ A33 &= -\cos(\alpha)\,(\sigma_8 - \sigma_7) - \sin(\alpha)\,(\sigma_2 - \sigma_1) \\ \sigma_1 &= d_4\,\sin(\alpha)\,\sin(\theta_3) \\ \sigma_2 &= a_3\,\sin(\alpha)\,\cos(\theta_3) \\ \sigma_3 &= d_4\,\cos(\theta_3) + a_3\,\sin(\theta_3) \\ \sigma_4 &= -\sin(\alpha)\,\sin(\theta_3) \\ \sigma_5 &= \cos(\alpha)\,\sin(\theta_3) \\ \sigma_6 &= \sigma_8 - a_2\,\sin(\alpha) - \sigma_7 \\ \sigma_7 &= d_4\,\cos(\alpha)\,\sin(\theta_3) \\ \sigma_8 &= a_3\,\cos(\alpha)\,\cos(\theta_3) \end{aligned}$$

The gravitational component can be expressed as :

$$\zeta_e = \begin{pmatrix} 0 \\ -\mathrm{mg} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Transposing the Jacobian Matrix and multiplying it by the vector $\zeta_e =$ (Principle of Virtual Work), we obtain the forces and torques of the joints to maintain the static balancing of the architecture.

$$\Phi = \begin{pmatrix} -\mathrm{mg}\,(a_1 + a_2\,\cos(\alpha) + \sigma_2 - \sigma_1) \\ -\mathrm{mg}\,(a_2\,\cos(\alpha) + \sigma_2 - \sigma_1) \\ -\mathrm{mg}\,\cos(\alpha)\,(d_4\,\cos(\theta_3) + a_3\,\sin(\theta_3)) \\ -\mathrm{mg}\,\cos(\alpha)\,\sin(\theta_3) \end{pmatrix}$$

where

$\sigma_1 = d_4 \, \sin(\alpha) \, \sin(\theta_3)$

$\sigma_2 = a_3 \, \sin(\alpha) \, \cos(\theta_3)$

0.5 Architecture 5

0.5.1 Constraint analysis

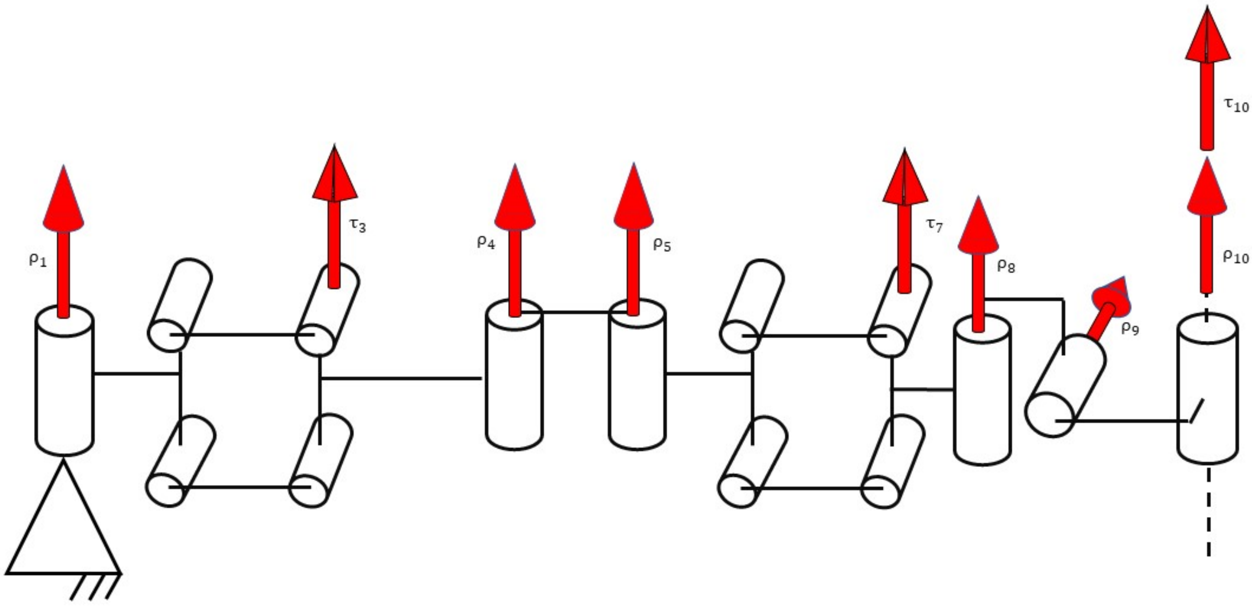

Figure 11: Kinematic chain 5, geometric constraint analysis

Considering the configuration of the kinematic chain 5, its constraint analysis is:

$dim(\mathcal{T}_1) = 9$
$dim(\mathcal{W}_1) = 0$
$Span(\mathcal{T}_1) = \rho_1, \tau_3, \rho_4, \rho_5, \tau_7, \rho_8, \rho_9, \rho_10, \tau_10$
$Span(\mathcal{W}_1) = \emptyset$

The kinematic chain has no structural constraint.

0.5.2 Statics, Kinematic analysis and gravity balancing

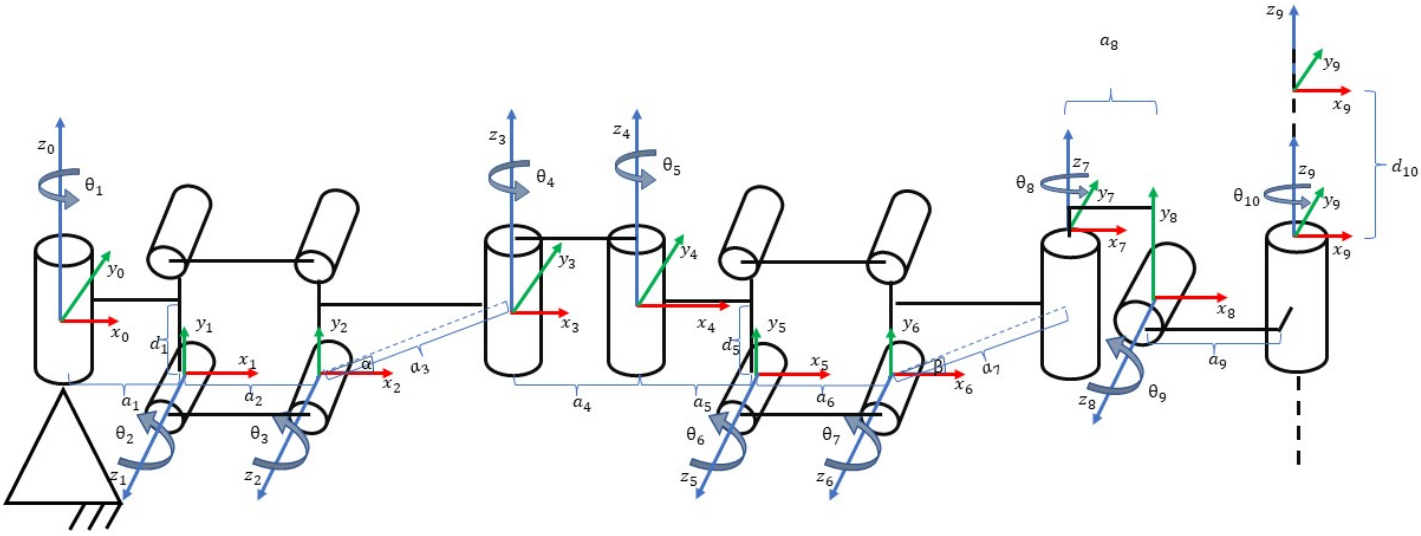

Figure 12: Kinematic chain 5, Denavit Hartenberg parametrization

According to figure 12, the following table [5] can be obtained. In table 5 the parameters from θ_1, to θ_8 are deliberately set to zero and the displacements between the frames 2 and 3 and the frames 6 and 7 are respectively approximated to $\cos(\alpha)$ and $\cos(\beta)$. This choice resulted convenient to show a lean kinematic evaluation and it was necessary for sake of brevity.

	J 1	J 2	J 3	J 4	J 5	J 6	J 7	J 8	J 9	J 10
TYPE	rev.	rev.	rev.	rev.	rev.	rev.	rev.	rev.	rev.	cyl.
θ	0	0	0	0	0	0	0	0	θ_9	θ_{10}
d	d_1	0	0	0	d_5	0	0	0	0	d_10
a	a_1	a_2	$a_3\cos(\alpha)$	a_4	a_5	a_6	$a_7\cos(\beta)$	a_8	a_9	0

Table 5: Table of geometric parameters for the kinematic chain 5

As before, the first step displays the rotation matrices of every reference frame with respect to the next one, starting from the base-frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_2^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$
$$R_4^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_5^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_6^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_7^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$
$$R_8^7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_9^8 = \begin{pmatrix} \cos(\theta_9) & 0 & -\sin(\theta_9) \\ \sin(\theta_9) & 0 & \cos(\theta_9) \\ 0 & -1 & 0 \end{pmatrix}$$
$$R_{10}^9 = \begin{pmatrix} \cos(\theta_{10}) & -\sin(\theta_{10}) & 0 \\ \sin(\theta_{10}) & \cos(\theta_{10}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second step is to compute the displacement vectors of each frame with respect to the previous one .

$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_2^1 = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_3^2 = \begin{pmatrix} a_3 \cos(\alpha) \\ 0 \\ 0 \end{pmatrix}$$
$$dv_4^3 = \begin{pmatrix} a_4 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_5^4 = \begin{pmatrix} a_5 \\ 0 \\ d_5 \end{pmatrix}$$
$$dv_6^5 = \begin{pmatrix} a_6 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_7^6 = \begin{pmatrix} a_7 + \cos(\beta) \\ 0 \\ 0 \end{pmatrix}$$
$$dv_8^7 = \begin{pmatrix} a_8 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_9^8 = \begin{pmatrix} a_9 \cos(\theta_9) \\ a_9 \sin(\theta_9) \\ 0 \end{pmatrix}$$
$$dv_{10}^9 = \begin{pmatrix} 0 \\ 0 \\ d_{10} \end{pmatrix}$$

The third step is to concatenate the rotation matrices and the displacement vectors with respect to the previous reference frame to obtain the transformation matrices .

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^1 = \begin{pmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^2 = \begin{pmatrix} 1 & 0 & 0 & a_3 \cos(\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_4^3 = \begin{pmatrix} 1 & 0 & 0 & a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_5^4 = \begin{pmatrix} 1 & 0 & 0 & a_5 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_6^5 = \begin{pmatrix} 1 & 0 & 0 & a_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_7^6 = \begin{pmatrix} 1 & 0 & 0 & a_7 + \cos(\beta) \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_8^7 = \begin{pmatrix} 1 & 0 & 0 & a_8 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_9^8 = \begin{pmatrix} \cos(\theta_9) & 0 & -\sin(\theta_9) & a_9 \cos(\theta_9) \\ \sin(\theta_9) & 0 & \cos(\theta_9) & a_9 \sin(\theta_9) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{10}^9 = \begin{pmatrix} \cos(\theta_{10}) & -\sin(\theta_{10}) & 0 & 0 \\ \sin(\theta_{10}) & \cos(\theta_{10}) & 0 & 0 \\ 0 & 0 & 1 & d_{10} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth step is to multiply the transformation matrices obtained in order to obtain the transformation matrices with respect to the base frame

$$T_1^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_3\ \cos(\alpha) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_4^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_4 + a_3\ \cos(\alpha) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_5^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_4 + a_5 + a_3\ \cos(\alpha) \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 + d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_6^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_4 + a_5 + a_6 + a_3\ \cos(\alpha) \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 + d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_7^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + \cos(\beta) + a_3\ \cos(\alpha) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & d_1 + d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_8^0 = \begin{pmatrix} 1 & 0 & 0 & a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & d_1 + d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_9^0 = \begin{pmatrix} \cos(\theta_9) & 0 & -\sin(\theta_9) & A14 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_9) & 0 & -\cos(\theta_9) & d_1 + d_5 - a_9\ \sin(\theta_9) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$A14 = a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9)$$

$$T_{10}^0 = \begin{pmatrix} \cos(\theta_9)\ \cos(\theta_{10}) & -\cos(\theta_9)\ \sin(\theta_{10}) & -\sin(\theta_9) & A14 \\ \sin(\theta_{10}) & \cos(\theta_{10}) & 0 & 0 \\ -\cos(\theta_{10})\ \sin(\theta_9) & \sin(\theta_9)\ \sin(\theta_{10}) & -\cos(\theta_9) & d_1 + d_5 - d_{10}\ \cos(\theta_9) - a_9\ \sin(\theta_9) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$A14 = a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)$$

After the computation of the Transformation matrices with respect to the baseframe, it is possible to extract the rotation matrices and the displacement vectors of every reference frame with respect to the base frame.

$$R_1^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_2^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_3^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_4^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$R_5^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_6^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$R_7^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$R_8^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$
$$R_9^0 = \begin{pmatrix} \cos(\theta_9) & 0 & -\sin(\theta_9) \\ 0 & 1 & 0 \\ -\sin(\theta_9) & 0 & -\cos(\theta_9) \end{pmatrix}$$
$$R_{10}^0 = \begin{pmatrix} \cos(\theta_9)\,\cos(\theta_{10}) & -\cos(\theta_9)\,\sin(\theta_{10}) & -\sin(\theta_9) \\ \sin(\theta_{10}) & \cos(\theta_{10}) & 0 \\ -\cos(\theta_{10})\,\sin(\theta_9) & \sin(\theta_9)\,\sin(\theta_{10}) & -\cos(\theta_9) \end{pmatrix}$$
$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_2^0 = \begin{pmatrix} a_1 + a_2 \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_3^0 = \begin{pmatrix} a_1 + a_2 + a_3\,\cos(\alpha) \\ 0 \\ d_1 \end{pmatrix}$$
$$dv_4^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_3\,\cos(\alpha) \\ 0 \\ d_1 \end{pmatrix}$$

$$dv_5^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_5 + a_3\ \cos(\alpha) \\ 0 \\ d_1 + d_5 \end{pmatrix}$$
$$dv_6^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_5 + a_6 + a_3\ \cos(\alpha) \\ 0 \\ d_1 + d_5 \end{pmatrix}$$
$$dv_7^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + \cos(\beta) + a_3\ \cos(\alpha) \\ 0 \\ d_1 + d_5 \end{pmatrix}$$
$$dv_8^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) \\ 0 \\ d_1 + d_5 \end{pmatrix}$$
$$dv_9^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) \\ 0 \\ d_1 + d_5 - a_9\ \sin(\theta_9) \end{pmatrix}$$
$$dv_{10}^0 = \begin{pmatrix} a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\ 0 \\ d_1 + d_5 - d_{10}\ \cos(\theta_9) - a_9\ \sin(\theta_9) \end{pmatrix}$$

And finally the Jacobian matrix is obtained, following the table **??**

$$J = \begin{pmatrix} 0 & \sigma_2 & \sigma_2 & 0 & 0 & \sigma_1 & \sigma_1 & 0 & \sigma_1 & -\sin(\theta_9) \\ A21 & 0 & 0 & A24 & A25 & 0 & 0 & A28 & 0 & 0 \\ 0 & A32 & A33 & 0 & 0 & A36 & A37 & 0 & A39 & -\cos(\theta_9) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin(\theta_9) \\ 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -\cos(\theta_9) \end{pmatrix}$$

where

$$\begin{aligned}
A21 &= a_1 + a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
A24 &= a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
A25 &= a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
A28 &= d_{10}\ \sin(\theta_9) - a_9\ \cos(\theta_9) - a_8 \\
A32 &= a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
A33 &= a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
A36 &= a_6 + a_7 + a_8 + \cos(\beta) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
A37 &= a_7 + a_8 + \cos(\beta) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9) \\
\sigma_1 &= d_{10}\ \cos(\theta_9) + a_9\ \sin(\theta_9) \\
\sigma_2 &= d_{10}\ \cos(\theta_9) - d_5 + a_9\ \sin(\theta_9)\ A39 = a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)
\end{aligned}$$

The gravitational component can be expressed as :

$$\zeta_e = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Transposing the Jacobian Matrix and multiplying it by the vector ζ_e = (Principle of Virtual Work), we obtain the forces and torques of the joints to maintain the static balancing of the architecture.

$$\Phi = \begin{pmatrix} 0 \\ -\mathrm{mg}\left(a_2 + a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)\right) \\ -\mathrm{mg}\left(a_4 + a_5 + a_6 + a_7 + a_8 + \cos(\beta) + a_3\ \cos(\alpha) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)\right) \\ 0 \\ 0 \\ -\mathrm{mg}\left(a_6 + a_7 + a_8 + \cos(\beta) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)\right) \\ -\mathrm{mg}\left(a_7 + a_8 + \cos(\beta) + a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)\right) \\ 0 \\ -\mathrm{mg}\left(a_9\ \cos(\theta_9) - d_{10}\ \sin(\theta_9)\right) \\ \mathrm{mg}\ \cos(\theta_9) \end{pmatrix}$$

0.6 Architecture 6

0.6.1 Constraint analysis

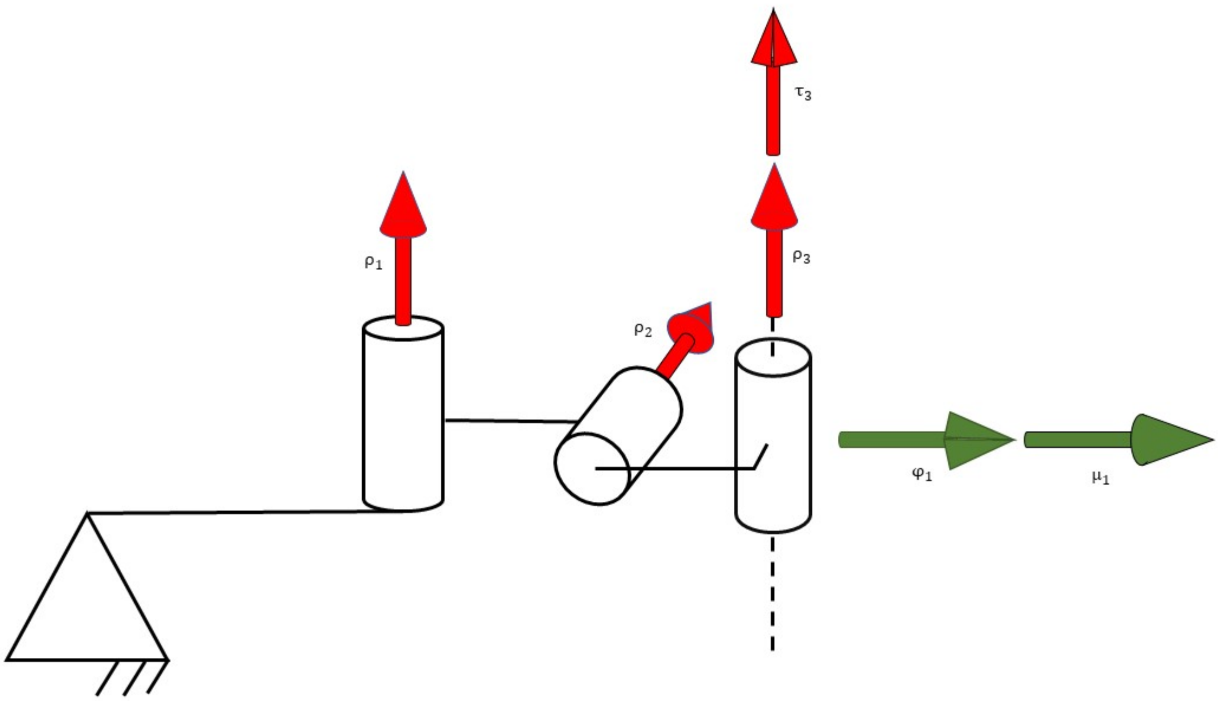

Figure 13: Kinematic chain 6, geometric constraint analysis

Considering the configuration of the kinematic chain 6, its constraint analysis is:

$dim(\mathcal{T}_1) = 4$
$dim(\mathcal{W}_1) = 2$
$Span(\mathcal{T}_1) = \rho_1, \rho_2, \rho_3, \tau_3$
$Span(\mathcal{W}_1) = \varphi_1, \mu_1$

The kinematic chain structural constraints are the wrench φ_1 (force), coplanar to ρ_1, ρ_2, and ρ_3 and orthogonal to τ_3, and the wrench μ_1 (couple) parallel to the axis of ρ_1 and ρ_3, and perpendicular to ρ_2 .

0.6.2 Statics, Kinematic analysis and gravity balancing

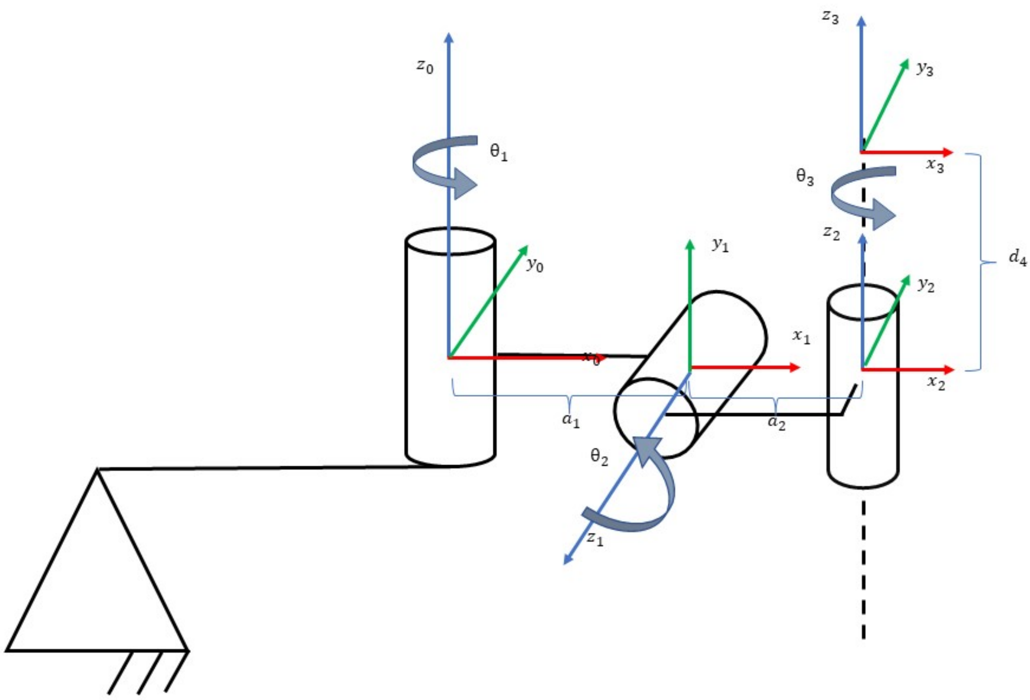

Figure 14: Kinematic chain 6, Denavit Hartenberg parametrization

According to figure 14, the following table [6] can be obtained. In table 6 the parameter θ_1 is deliberately set to zero. This choice resulted convenient to show a lean kinematic evaluation and it was necessary for sake of brevity.

	JOINT 1	JOINT 2	JOINT 3
TYPE	revolute	revolute	cylindrical
θ	0	θ_2	θ_3
d	0	0	d_3
a	a_1	a_2	0

Table 6: Table of geometric parameters for the kinematic chain 6

As before, the first step displays the rotation matrices of every reference frame with respect to the next one, starting from the base-frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R_2^1 = \begin{pmatrix} 0 & -\sin(\theta_2) & -\cos(\theta_2) \\ 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 1 & 0 & 0 \end{pmatrix}$$

$$R_3^2 = \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second step is to compute the displacement vectors of each frame with respect to the previous one .

$$dv_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_2^1 = \begin{pmatrix} a_2 \cos(\theta_2) \\ a_2 \sin(\theta_2) \\ 0 \end{pmatrix}$$

$$dv_3^2 = \begin{pmatrix} 0 \\ 0 \\ d_3 \end{pmatrix}$$

The third step is to concatenate the rotation matrices and the displacement vectors with respect to the previous reference frame to obtain the transformation matrices .

$$T_1^0 = \begin{pmatrix} 0 & 0 & 1 & a_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^1 = \begin{pmatrix} 0 & -\sin(\theta_2) & -\cos(\theta_2) & a_2\,\cos(\theta_2) \\ 0 & \cos(\theta_2) & -\sin(\theta_2) & a_2\,\sin(\theta_2) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^2 = \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The fourth step is to multiply the transformation matrices obtained in order to obtain the transformation matrices with respect to the base frame

$$T_1^0 = \begin{pmatrix} 0 & 0 & 1 & a_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T02 = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) & a_2\,\sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) & -a_2\,\cos(\theta_2) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T03 = \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_1 \\ \cos(\theta_2)\,\sin(\theta_3) & \cos(\theta_2)\,\cos(\theta_3) & -\sin(\theta_2) & a_2\,\sin(\theta_2) - d_3\,\sin(\theta_2) \\ \sin(\theta_2)\,\sin(\theta_3) & \cos(\theta_3)\,\sin(\theta_2) & \cos(\theta_2) & d_3\,\cos(\theta_2) - a_2\,\cos(\theta_2) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After the computation of the Transformation matrices with respect to the base frame, it is possible to extract the rotation matrices and the displacement vectors of every reference frame with respect to the base frame.

$$R_0^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R_2^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$

$$R_3^0 = \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \cos(\theta_2)\,\sin(\theta_3) & \cos(\theta_2)\,\cos(\theta_3) & -\sin(\theta_2) \\ \sin(\theta_2)\,\sin(\theta_3) & \cos(\theta_3)\,\sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$

$$dv_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$dv_1^0 = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix}$$
$$dv_2^0 = \begin{pmatrix} a_1 \\ a_2\, \sin(\theta_2) \\ -a_2\, \cos(\theta_2) \end{pmatrix}$$
$$dv_3^0 = \begin{pmatrix} a_1 \\ a_2\, \sin(\theta_2) - d_3\, \sin(\theta_2) \\ d_3\, \cos(\theta_2) - a_2\, \cos(\theta_2) \end{pmatrix}$$

And finally the Jacobian matrix is obtained, following the table **??**

$$J = \begin{pmatrix} d_3\, \sin(\theta_2) - a_2\, \sin(\theta_2) & 0 & 0 \\ a_1 & a_2\, \cos(\theta_2) - d_3\, \cos(\theta_2) & -\sin(\theta_2) \\ 0 & a_2\, \sin(\theta_2) - d_3\, \sin(\theta_2) & \cos(\theta_2) \\ 0 & 1 & 0 \\ 0 & 0 & -\sin(\theta_2) \\ 1 & 0 & \cos(\theta_2) \end{pmatrix}$$

The gravitational component can be expressed as :

$$\zeta_e = \begin{pmatrix} 0 \\ 0 \\ -\mathrm{mg} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Transposing the Jacobian Matrix and multiplying it by the vector $\zeta_e =$ (Principle of Virtual Work), we obtain the forces and torques of the joints to maintain the static balancing of the architecture.

$$\Phi = \begin{pmatrix} 0 \\ -\mathrm{mg}\,(a_2\, \sin(\theta_2) - d_3\, \sin(\theta_2)) \\ -\mathrm{mg}\, \cos(\theta_2) \end{pmatrix}$$