Relativistic Strings and Ehrenfest's Paradox

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Abstract

In this thesis, we have presented a new model for a relativistic string. We successfully determined that a three-dimensional relativistic N-coupled oscillator, parameterized by arc-length and coordinate time, behaves like the Nambu-Goto relativistic string. We also showed that in one-dimension this system does not look like a relativistic string. This finding is consistent with the notion that longitudinal motion on the Nambu-Goto string is unobservable, since the one-dimensional oscillator only describes longitudinal motion along itself. Additionally, we presented a relativistically consistent simulation of Ehrenfest's paradox. We observed that our simulation performed as expected at non-relativistic speeds. Interestingly, we observed that the length of a rod rotating at relativistic speeds was extended in comparison to one rotating at non-relativistic speeds.

Introduction

Physicists have theories that describe how subatomic particles move on small scales and separate theories that describe how massive bodies like planets or stars move on large scales. But these theories together cannot produce a unified theory of how small but massive objects move [5]. Or how big stars and the atoms that make them up can experience the same forces, yet gravity is more apparent for stars and electromagnetism is more prominent in atoms [11]. This incoherence leaves some of physics's greatest questions unanswered. Questions like: "What happens in the depths of a black hole?", or "What did the universe look like before the 'Big Bang'?" Proponents advertise that string theory is a candidate for the unification of all fundamental forces and all existing theories.

The aim of this thesis is not to talk about string theory in all of its 10 dimensional glory. It is to investigate a relatively simple, mechanical system which behaves the same as a string. This model should be somewhat accessible to an advanced introductory or sophomore-level level physics student. In order to compare this system to a string, the equation of motion for a relativistic string will first be developed, starting from the fundamentals of special relativity. Once this equation has been derived, we will discuss the parameter choices under which our model behaves the same as a string. This model contains an arbitrarily large number of masses that are connected by springs and move with a speed close to the speed of light. Throughout my thesis, this model will be called a relativistic N-coupled oscillator or a string. Hopefully a reader will learn from this work that there is more than one interpretation of a relativistic string.

Finally this thesis will conclude by considering the relativistic N-coupled oscillator in another context, namely, as a way of investigating Ehrenfest's paradox. When we think of a rigid body, we assume that no transformation of its shape takes place over time. But this paradox states that the length of a rigid rod at rest does not appear the same as the length of the rod in motion. Thus, Ehrenfest's paradox is attributed to the nonexistence of rigidity in special relativity. We will approximate a rotating rigid rod with a relativistic N-coupled oscillator of uniformly large spring constant. Finally, we will use a Verlet method to show how the position and momentum of each mass will change over time.

Chapter 1

Introduction to String Theory

This chapter provides a review of some elements of special relativity, followed by a derivation of the equation of motion for a relativistic string.

1.1 Special Relativity Dynamics

1.1.1 Length Minimization

In Newtonian mechanics, the separation between two events is simply the spatial distance they are away from each other. In special relativity, the distance between two events is called an invariant space-time interval and it takes into account not only the spatial separation between two events but also the amount of time occurring between them [12]. Observers of each event may see different spatial or temporal separations between two events, but special relativity says they will observe the same space-time interval. Before invariance in a relativistic setting is described, invariance will be described in a more intuitive sense for rotated coordinate systems.

Rotational Invariance

Say an object is at location (x, y) in one coordinate system and $(\bar{x}, \bar{y})$ in another coordinate system. The second coordinate system shares its origin with first, and is just rotated by an angle θ. (See Figure 1.1.) The coordinates in the rotated system can be written in terms of a rotation matrix and the non-rotated coordinates as such:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In the rotated coordinate system, the squared distance the object is from the origin is given by:

$$s^2 = \bar{x}^2 + \bar{y}^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = x^2 + y^2$$

In the non-rotated system, the squared distance the object is from the origin is also $s^2 = x^2 + y^2$. The distance s^2 is measured to be the same in either system; we

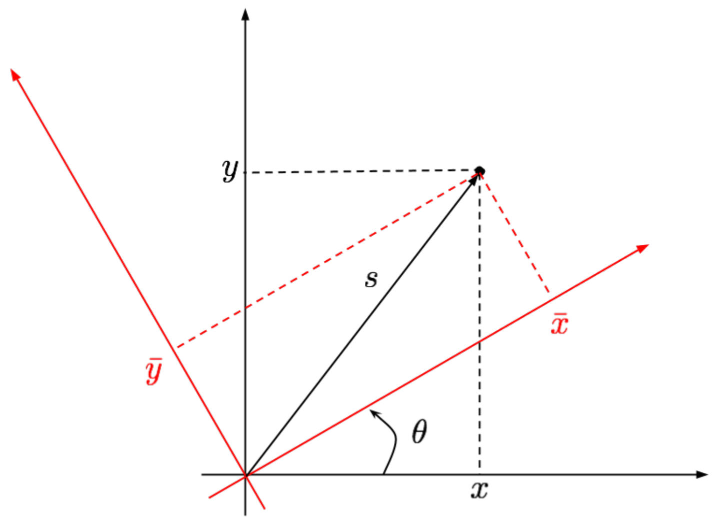

Figure 1.1: A point in two-dimensional space represented in two coordinate systems

would call it invariant. In special relativity a different invariant called the space-time interval is encountered. And in this case, instead of using the coordinates (x, y) to describe the location of an *object*, the coordinates (ct, λ) are used to describe the location of an *event*.

Space-time Invariance

Say the location of an event is (ct, λ) in one coordinate system and $(c\tau, \sigma)$ in another coordinate system where c is the speed of light, and λ and σ could be one, two or three-dimensional coordinates.[1] Using these coordinates, the space-time interval is defined as:

$$ds^2 = -(cdt)^2 + d\lambda^2 = -(cd\tau)^2 + d\sigma^2$$

where dt, $d\tau$, $d\lambda$, and $d\sigma$ are infinitesimal stretches in time or space in their respective coordinate systems.[2] This relationship can be used in order to figure out how the length or time observed in one coordinate system relates to those observed in another coordinate system.

Also using the space-time interval, a quantity called the relativistic free particle action can be introduced [2]. It assigns a single value to a trajectory and it is defined as such:

$$S = \alpha \int ds$$

where α is a constant such that the action has dimensions of [energy][time] (e.g. $\alpha = mc$). This quantity is useful since its minimization tells us the shortest path an

[1] Think of them like a single parameter to describe x, y, and z. Also, note that ct and $c\tau$ have dimensions of length similar to λ and σ.

[2] This quantity doesn't look exactly like the s^2 we found for rotations. We pick up a minus sign due to the Minkowski metric. See sources [9][2].

object can take and produces the equations of motion governing our object. Notice that the action could be written in terms of (λ, t) or (σ, τ). This choice of coordinates is what will be discussed next.

1.1.2 Choice of Coordinates

One could say the force on a mass due a spring is:

$$F = \frac{dp}{d\tau} = -kd\sigma \quad or \quad F = \frac{dp}{dt} = -kd\lambda$$

where typically $d\tau$ and $d\sigma$ are considered the time and stretch measured by the mass, and dt and $d\lambda$ the time and stretch measured by the laboratory. (We will refer to these perspectives throughout Chapter 2.) Here, $d\sigma$ and $d\lambda$ are defined to be strictly one-dimensional lengths. The parameter σ will be called proper length, τ proper time, and t coordinate time. The force written in terms of the proper time $dp/d\tau$ is often called the Minkowski force.

Before special relativity, one would say the lengths and times measured by both observers are the same, $d\tau = dt$ and $d\sigma = d\lambda$, but this is not the case. The force could be written using either parameterization. First, the relationships between times and lengths measured by the mass and those measured by the lab will be laid out. Then, the circumstances under which the time and lengths are observed to be nearly the same or different will be discussed.

Say we were in the (λ, t) frame and we wanted to compare our measurements of time and length to the masses in the (σ, τ) frame. Similar to the way the rotated coordinates were related to unrotated ones using a rotation matrix in the previous section, a matrix called the Lorentz boost matrix will relate the lengths a mass observes to the ones the lab observes:

$$\begin{pmatrix} cd\tau \\ d\sigma \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} cdt \\ d\lambda \end{pmatrix}$$

where $\beta = v/c$ for v is the velocity parallel to the motion (in one-dimension $d\lambda/dt$), and the Lorentz contraction factor is $\gamma = 1/\sqrt{1-\beta^2}$ [9]. We can chose a moment such that $cdt = 0$ in our frame. This would tell us that $d\sigma = \gamma d\lambda$.

Or say we chose the length in the mass's frame to be $d\sigma = 0$, meaning the mass is at rest in its frame. Then the Lorentz boost matrix could be inverted so as to solve for the time the lab observes in terms of the time a mass observes:

$$\begin{pmatrix} cdt \\ d\lambda \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} cd\tau \\ d\sigma \end{pmatrix}$$

This tells us that $dt = \gamma d\tau$.[3]

[3]Note that we chose $d\sigma$ and $d\lambda$ to be one-dimensional parameters here since the transformation of coordinates can be given by a simpler Lorentz boost matrix. In three spatial dimensions, a 4×4 matrix would be considered where the only factors which would pick up a Lorentz contraction factor are the ones parallel to the motion of an object.

Note that if we are in the (λ, t) coordinate system and we observed the mass moving relatively slowly, where $d\lambda/dt$ is not close to the speed of light, the two measurements of time and length are nearly the same: $d\tau \approx dt$ and $d\sigma \approx d\lambda$. But if we see the mass moving at close to the speed of light, where $\beta \to 1$ and therefore $\gamma \to \infty$, we would see $dt > d\tau$ and $d\sigma > d\lambda$. This means the time we observe appears longer than the time the mass observes. This effect is called time dilation. We also see the mass travels a shorter distances than the mass think it travels. This effect is called length contraction. What does this mean in the real world?

The Muon

Time dilation and length contraction have been verified experimentally by examining the dilated lifetime of a muon, a particle which is commonly detected and moves at about $.998c$ as it falls to Earth. According to the muon in the (σ, τ) coordinate system, the time before it decays into electrons is 2.197 μs. These particles travel from where they were created in the atmosphere (through interactions with cosmic rays) to some detector on Earth in the (λ, t) coordinate system [7].

According to classical physics, the distance traveled by the muon divided by its velocity equals the time it travels. If that distance was from its starting point in the atmosphere to a detector on the highest mountain in the world, it would take 26.74 μs. This time is longer than the lifetime of the muon, which implies that no muons should be detected on Earth. They would have all decayed by the time they reach our detectors. But muons are commonly detected on Earth, so what's happening here?

Using relativistic physics, we find that the time required to reach a detector on Earth would actually be 1.69 μs in the (σ, τ) coordinate system, within the muon's lifetime. The muon observes this scenario slightly differently. According to the muon, it is stationary and the Earth is traveling to it at .998c. The muon sees the distance the Earth travels towards it to be length contracted [7]. Thus the Earth still gets to the muon within the muon's lifetime.

Example of Difficulty in Translation

We have just explained how in theory, relationships between t and τ, or λ and σ can be found, but in practice this is not necessarily so easy. Say we are given the position of an object in terms of the coordinates λ and t:

$$\lambda(t) = a \sin(\omega t)$$

where a is a constant with units of position. The space-time interval can be used to try to figure out what the relationship between τ and t is:

$$-(cdt)^2 + d\lambda^2 = -(cd\tau)^2 + d\sigma^2 \to \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{1}{c^2}\left(\frac{d\lambda}{dt}\right)^2\right) = 1 - \frac{1}{c^2}\left(\frac{d\sigma}{d\tau}\right)^2$$

where $d\lambda/dt$ was rewritten using the chain rule.[4] Note that according to the object in the (σ, τ) coordinate system, it is at rest and therefore has no velocity ($d\sigma/d\tau = 0$). Thus, $dt/d\tau$ becomes:

$$\left(\frac{dt}{d\tau}\right) = 1 \Big/ \sqrt{1 - \frac{1}{c^2}\left(\frac{d\lambda}{dt}\right)^2} \rightarrow \left(\frac{dt}{d\tau}\right)^2 = 1 \Big/ \sqrt{1 - \frac{a^2\omega^2}{c^2}\cos^2(\omega t)}$$

where the time derivative of $\lambda(t)$ was plugged in. This differential equation is not an easy one to solve and often it won't be.

1.2 Area Minimization and the Nambu-Goto Relativistic String

In this section, drawing upon the definition of the action that we provided earlier, the equation governing a relativistic string will be derived. This derivation will draw largely from Franklin's *Advanced Mechanics and General Relativity* but we will try to explain the process in a more accessible way to an introductory student [2].[5] If you'd just like to see what the equation of motion for a relativistic string is, then feel free to skip the lengthy discussion and jump to the final result at the end of this section. We would like to say (before you skip ahead) that though this section appears dense, the derivation of the equation governing a relativistic string is really all about area minimization. This concept is very present in our everyday life. The simplest example of this principle at work is for soap bubbles, which minimize their surface area for a given volume of air [1]. Now let's get to the derivation.

The vector pointing from a three-dimensional origin to a point on the relativistic string's surface will be called $\mathbf{X}$,[6] and it will be defined to be a function of time τ and some distance parameter σ that tells us how far along the string we are.[7] For a small change $d\sigma$ or $d\tau$ in the parameters, a parallelogram is spanned (see Figure 1.2).

The lengths of the sides of the parallelogram are:

$$\partial\mathbf{X}_\sigma \equiv \mathbf{X}(\sigma + d\sigma, \tau) - \mathbf{X}(\sigma, \tau) \cong \frac{\partial \mathbf{X}}{\partial \sigma} d\sigma \;\; \& \;\; \partial\mathbf{X}_\tau \equiv \mathbf{X}(\sigma, \tau + d\tau) - \mathbf{X}(\sigma, \tau) \cong \frac{\partial \mathbf{X}}{\partial \tau} d\tau$$

where the definition of the derivative was used.[8] The magnitude of the cross product of two vectors is the area spanned by those vectors. So the squared area spanned by

[4]Explicitly:

$$\frac{d\lambda}{d\tau} = \frac{d\lambda}{dt}\frac{dt}{d\tau}$$

[5]A more advanced physics student could refer to Zwiebach's *First Course in String Theory* in addition to Franklin's text [12].

[6]We will be using boldface to denote vector quantities throughout my thesis.

[7]Note that this is a choice; we could have instead written $\mathbf{X}$ as a function of λ and t and continued from there.

[8]Specifically,

$$f'(x) = \lim_{dx \to 0} (f(x + dx) - f(x))/dx$$

.

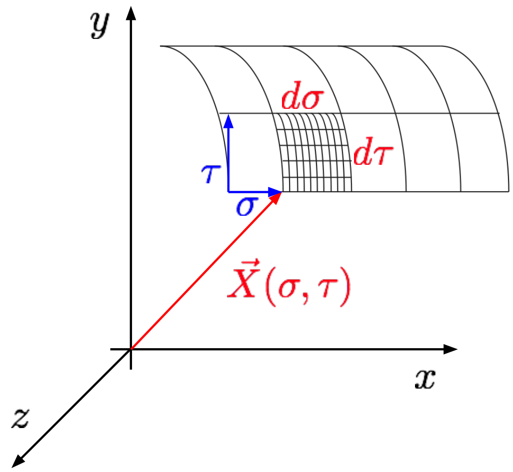

Figure 1.2: The surface area $d\sigma d\tau$ spanned by a tiny patch in our relativistic string

these vectors is:

$$dA^2 = [(\mathbf{X}(\sigma+d\sigma,\tau)-\mathbf{X}(\sigma,\tau))\times(\mathbf{X}(\sigma,\tau+d\tau)-\mathbf{X}(\sigma,\tau))]^2 = (\partial\mathbf{X}_\sigma\times\partial\mathbf{X}_\tau)\cdot(\partial\mathbf{X}_\sigma\times\partial\mathbf{X}_\tau)$$

Using a vector calculus identity,[9] the squared area can be rewritten as:

$$dA^2 = (\partial\mathbf{X}_\sigma\cdot\partial\mathbf{X}_\sigma)(\partial\mathbf{X}_\tau\cdot\partial\mathbf{X}_\tau) - (\partial\mathbf{X}_\sigma\cdot\partial\mathbf{X}_\tau)^2$$

Thus the area spanned by the motion of the string is:

$$dA = \sqrt{(\partial\mathbf{X}_\sigma\cdot\partial\mathbf{X}_\sigma)(\partial\mathbf{X}_\tau\cdot\partial\mathbf{X}_\tau) - (\partial\mathbf{X}_\sigma\cdot\partial\mathbf{X}_\tau)^2}$$

Or written in terms of spatial and temporal derivatives:

$$dA = \sqrt{\left(\frac{\partial\mathbf{X}}{\partial\sigma}\cdot\frac{\partial\mathbf{X}}{\partial\sigma}\right)\left(\frac{\partial\mathbf{X}}{\partial\tau}\cdot\frac{\partial\mathbf{X}}{\partial\tau}\right) - \left(\frac{\partial\mathbf{X}}{\partial\sigma}\cdot\frac{\partial\mathbf{X}}{\partial\tau}\right)^2}d\sigma d\tau$$

Why do we even care about this parallelogram? Because similar to the way the path length was used to define an action, an area will be used to define an action. Then the action can be used to find the equation of motion that minimizes that area. The Nambu-Goto action is defined as such:

$$S = -\frac{T_0}{c}\int dA$$

The constants T_0 and c are introduced such that the action has dimensions of [energy][time]. Before dA is plugged in, note that the argument under the square root is negative. Thus, in order for dA to be a real value, the negative of the terms inside the square root must be considered [2][12]. The action becomes:

$$S = -\frac{T_0}{c}\int\int\sqrt{\left(\frac{\partial\mathbf{X}}{\partial\sigma}\cdot\frac{\partial\mathbf{X}}{\partial\tau}\right)^2 - \left(\frac{\partial\mathbf{X}}{\partial\sigma}\cdot\frac{\partial\mathbf{X}}{\partial\sigma}\right)\left(\frac{\partial\mathbf{X}}{\partial\tau}\cdot\frac{\partial\mathbf{X}}{\partial\tau}\right)}d\sigma d\tau \tag{1.1}$$

[9]For vectors $\mathbf{a}$ and $\mathbf{b}$: $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$

The inside of this integral is called the Lagrange density, denoted by $\mathcal{L}$. The derivatives will be called $\mathbf{X}' = \frac{\partial \mathbf{X}}{\partial \sigma}$ and $\dot{\mathbf{X}} = \frac{\partial \mathbf{X}}{\partial \tau}$. We want to find the unique area $\mathbf{X}(\sigma, \tau)$ that minimizes S. In order to do so an arbitrary perturbation to $\mathbf{X}$ called $\boldsymbol{\eta}$ will be introduced and it will be required to go to zero at the endpoints of the path. With this perturbation, a small change in the value of S looks like:

$$\delta S = \int\int \left[\mathcal{L}(\mathbf{X}' + \eta', \dot{\mathbf{X}} + \dot{\eta}) - \mathcal{L}(\mathbf{X}', \dot{\mathbf{X}})\right] d\sigma d\tau$$

$$\delta S \approx \int\int \left[\left(\mathcal{L}(\mathbf{X}', \dot{\mathbf{X}}) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'} \cdot \boldsymbol{\eta}' + \frac{\mathcal{L}}{\partial \dot{\mathbf{X}}} \cdot \dot{\boldsymbol{\eta}}\right)\right) - \mathcal{L}(\mathbf{X}', \dot{\mathbf{X}})\right] d\sigma d\tau$$

$$\delta S = \int\int \left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'} \cdot \boldsymbol{\eta}' + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}} \cdot \dot{\boldsymbol{\eta}}\right) d\sigma d\tau$$

up to order $\dot{\boldsymbol{\eta}}^2$ and $\boldsymbol{\eta}'^2$ for small perturbation $\boldsymbol{\eta}$.[10]

To find the minimum area, $\mathbf{X}(\sigma, \tau)$ must be found such that $\delta S = 0$. But we don't know what $\boldsymbol{\eta}$ is and we don't want to specify anything about it, so first the inside of δS can be rewritten such that a common term of $\boldsymbol{\eta}$ factors out of both terms, leaving just derivatives of the Lagrange density. First, note the product rule for derivatives says that:

$$\frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'} \cdot \boldsymbol{\eta}\right) = \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'}\right) \cdot \boldsymbol{\eta} + \frac{\partial \mathcal{L}}{\partial \mathbf{X}'} \cdot \boldsymbol{\eta}'$$

Thus the first term of dS can be rewritten as:

$$\int\int \left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'} \cdot \boldsymbol{\eta}'\right) d\sigma d\tau = \int\int \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'} \cdot \boldsymbol{\eta}\right) d\sigma d\tau - \int\int \left(\frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'}\right) \cdot \boldsymbol{\eta}\right) d\sigma d\tau$$

$$= -\int\int \left(\frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'}\right) \cdot \boldsymbol{\eta}\right) d\sigma d\tau$$

where the first term went to zero because we defined the perturbation $\boldsymbol{\eta}$ to vanish at the temporal and spatial endpoints. Similarly, the second term in the action can be rewritten as:

$$\int\int \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}} \cdot \dot{\boldsymbol{\eta}}\right) d\sigma d\tau = -\int\int \left(\frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}}\right) \cdot \boldsymbol{\eta}\right) d\sigma d\tau$$

Plugging both of these into δS

$$\delta S = -\int \left(\left(\frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}}\right) + \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'}\right)\right) \cdot \boldsymbol{\eta}\right) d\sigma d\tau$$

Now for $\delta S = 0$ and for any arbitrary perturbation $\boldsymbol{\eta}$, the argument inside the parentheses must be zero.

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}}\right) + \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'}\right) = 0 \tag{1.2}$$

[10]Basically we are doing a Taylor series expansion of a $\mathcal{L}(\mathbf{X}', \dot{\mathbf{X}})$. For an arbitrary function $f(x, y)$, $f(x + dx, y + dx) \approx f(x, y) + \frac{\partial f(x,y)}{\partial x}((x + dx) - x) + \frac{\partial f(x,y)}{\partial y}((y + dy) - y) + \mathcal{O}(dx^2)$. In our case, dx is $\boldsymbol{\eta}'$ and dy is $\dot{\boldsymbol{\eta}}$.

Note $d\mathcal{L}/d\mathbf{X}'$ and $d\mathcal{L}/d\dot{\mathbf{X}}$ are quite complicated terms, so though the Euler-Lagrange equation looks quite simple in this form, the full equation is complex. Plugging in the actual Lagrangian from Equation (1.1) [2][12]:

$$\frac{\partial L}{\partial \dot{\mathbf{X}}} = \frac{-T_0}{c} \frac{\left(\left(\mathbf{X}' \cdot \dot{\mathbf{X}}\right)\mathbf{X}' - \left(\mathbf{X}' \cdot \mathbf{X}'\right)\dot{\mathbf{X}}\right)}{\sqrt{\left(\mathbf{X}' \cdot \dot{\mathbf{X}}\right)^2 - \left(\mathbf{X}' \cdot \mathbf{X}'\right)\left(\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}\right)}} \tag{1.3}$$

$$\frac{\partial L}{\partial \mathbf{X}'} = \frac{-T_0}{c} \frac{\left(\left(\mathbf{X}' \cdot \dot{\mathbf{X}}\right)\dot{\mathbf{X}} - \left(\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}\right)\mathbf{X}'\right)}{\sqrt{\left(\mathbf{X}' \cdot \dot{\mathbf{X}}\right)^2 - \left(\mathbf{X}' \cdot \mathbf{X}'\right)\left(\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}\right)}} \tag{1.4}$$

These terms can be simplified significantly if a few characteristics of the parameters σ and τ are specified. We are free to do this; it's called gauge freedom. By fixing parameters we can better understand the physical characteristics of the system we are talking about [2].

For the classical relativistic string, our temporal parameter will be set equal to coordinate time. Note that $\mathbf{X}$ is a four-vector $\{c\tau, x, y, z\}$ also written as $\{c\tau, \mathbf{x}\}$ where $\mathbf{x}$ vector refers to spatial part of $\mathbf{X}$. So $\mathbf{X}'$ and $\dot{\mathbf{X}}$ in coordinate time ($d\tau = dt$) can be written as:

$$\mathbf{X}' = \frac{\partial}{\partial\sigma}\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\partial \mathbf{x}}{\partial \sigma} \end{pmatrix} \quad \& \quad \dot{\mathbf{X}} = \frac{\partial}{\partial t}\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} c \\ \frac{\partial \mathbf{x}}{\partial t} \end{pmatrix}$$

Thus, all references to $d\tau$ become dt and $\dot{\mathbf{X}} \cdot \dot{\mathbf{X}} = -c^2 + \left(\frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial t}\right)$.[11] Equation (1.1) becomes:

$$S = -\frac{T_0}{c} \int \sqrt{\left(\frac{\partial \mathbf{x}}{\partial \sigma} \cdot \frac{\partial \mathbf{x}}{\partial t}\right)^2 - \left(-c^2 + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial t}\right)\left(\frac{\partial \mathbf{x}}{\partial \sigma} \cdot \frac{\partial \mathbf{x}}{\partial \sigma}\right)}\, d\sigma dt \tag{1.5}$$

Now we want to better define the string's spatial derivative $\partial\mathbf{x}/\partial\sigma$. The temporal derivative $\partial\mathbf{x}/\partial t$ describes how a string moves in time along lines of constant σ [12]. The spatial derivative is a bit trickier. This is subtle, but when looking at the motion of a string at two nearby times, it isn't possible to say that a point on the string moved from point A to point B. To measure distance between points on a string, a definition of σ is required but we know this parameterization is arbitrary since our system is reparameterization invariant. Thus longitudinal motion exists for points on a string, but parameterizing it is not unique, and therefore longitudinal motion is not "physically meaningful"[12]. The only observable velocity on a string is the transverse velocity, pointing in a direction orthogonal to the string. It is defined as such:

$$\mathbf{v}_\perp = \frac{\partial \mathbf{x}}{\partial t} - \left(\frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial \mathbf{x}}{\partial s}\right)\frac{\partial \mathbf{x}}{\partial s}$$

[11] The minus sign in front of the c^2 term comes from the definition the Minkowski metric. (See Footnote 2.)

using the definition of a perpendicular vector[12] and where s could be any parameter such that $\partial\mathbf{x}/\partial s$ is a unit vector pointing along the direction of the string. Here it is the arc-length parameter, which satisfies two conditions [12]:

$$\frac{\partial \mathbf{x}}{\partial s}\cdot\frac{\partial \mathbf{x}}{\partial s} = 1 \;\;\&\;\; \frac{\partial \mathbf{x}}{\partial s}\cdot\frac{\partial \mathbf{x}}{\partial t} = 0.$$

Using arc-length parameterization ($d\sigma = ds$) and writing the action in terms of $\mathbf{v}_\perp$, Equation (1.5) becomes much simpler:

$$S = -\frac{T_0}{c}\int\sqrt{\left(c^2 - \frac{\partial \mathbf{x}}{\partial t}\cdot\frac{\partial \mathbf{x}}{\partial t}\right)}dsdt = -T_0\int\sqrt{1-\left(\frac{v_\perp}{c}\right)^2}dsdt \tag{1.6}$$

In the current gauge, Equations (1.3) and (1.4) become:

$$\frac{\partial L}{\partial \dot{\mathbf{X}}} = \frac{T_0}{c^2}\frac{\dot{\mathbf{X}}}{\sqrt{1-\left(\frac{v_\perp}{c}\right)^2}}$$

$$\frac{\partial L}{\partial \mathbf{X}'} = -T_0\sqrt{1-\left(\frac{v_\perp}{c}\right)^2}\mathbf{X}'$$

And these can be plugged into Equation (1.2):

$$\frac{\partial}{\partial t}\left(\frac{T_0}{c^2}\frac{\dot{\mathbf{X}}}{\sqrt{1-\left(\frac{v_\perp}{c}\right)^2}}\right) + \frac{\partial}{\partial s}\left(-T_0\sqrt{1-\left(\frac{v_\perp}{c}\right)^2}\mathbf{X}'\right) = 0 \tag{1.7}$$

Before simplifying, note that for the first component of the four-vector $\mathbf{X}$, the Euler-Lagrange equation gives us the following condition:

$$\frac{\partial}{\partial t}\left(\frac{T_0}{c}\frac{1}{\sqrt{1-\left(\frac{v_\perp}{c}\right)^2}}\right) = 0$$

Thus a few factors in the first term of Equation (1.7) can be pulled out since they are time independent. This leaves us with the equation of motion for the Nambu-Goto relativistic string.

$$\boxed{\frac{T_0}{c^2\sqrt{1-\frac{v_\perp^2}{c^2}}}\ddot{\mathbf{X}} - \frac{\partial}{\partial s}\left(T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\mathbf{X}'\right) = 0} \tag{1.8}$$

Now that we have derived the equation of motion for a particular parameterization of a relativistic string, we can present a much more intuitive, physical system whose equation of motion models this in Chapter 2. This system has an arbitrary number of masses, attached by springs, moving at relativistic speeds.

[12] "For any vector $\mathbf{u}$, its component perpendicular to a unit vector $\mathbf{n}$ is $\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$" [12]. Here the unit vector is $\frac{d\mathbf{X}}{ds}$, and the component of $\frac{d\mathbf{X}}{dt}$ perpendicular to it is given above.

Chapter 2

Simple Model of a String

The goal of this chapter is to show that under specific parameter choices, a three-dimensional relativistic N-coupled oscillator obeys the same equation of motion as the Nambu-Goto relativistic string (Equation 1.8). First, the system's behavior in one dimension will be considered. This analysis will examine both the perspective of a mass along the oscillator and the perspective of the laboratory. The equations of motion for this system will be found by first stating what forces acting on the masses must be, and then the same equations will be found by considering the Lagrangian of the system. It is not until we introduce our N-coupled oscillator in three-dimensions though that the relativistic N-coupled oscillator is found to behave like a relativistic string.

2.1 One-Dimensional Motion

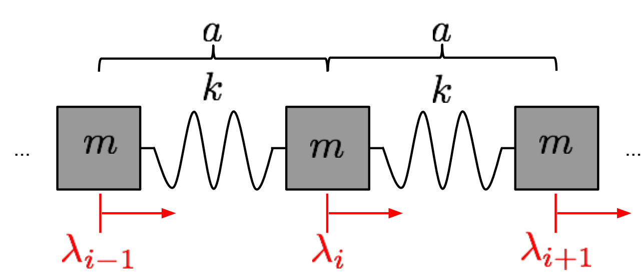

Figure 2.1: Close up on three internal bodies of a harmonic oscillator

Consider 3 masses attached by 2 springs with spring constant k and equilibrium length a. The total force on the central mass can be written as the following, for a parameter λ in the lab's frame of reference [4]:

$$F_0 = -k((\lambda_0 - \lambda_{-1}) - a) + k((\lambda_{+1} - \lambda_0) - a)$$
$$m\ddot{\lambda}_0 = k(\lambda_{-1} - 2\lambda_0 + \lambda_{+1}) \tag{2.1}$$

Notice there is no reference to the equilibrium length of the spring for an internal mass. This simple model shows that the motion of an internal mass only depends on the relative displacements of the masses to the left and right of itself, with no reference to the equilibrium length. This finding still holds for an arbitrarily large number of masses N and springs $(N-1)$.

Now, Equation (2.1) will be written in terms of a displacement function $\phi(\lambda_i, t)$. The displacement of the ith mass in our N-coupled oscillator from its equilibrium location is $\phi(\lambda_i, t) = \lambda_i - ia$, and the displacements of the masses on either side are $\phi(\lambda_{i\pm1}, t)$. Thus the force on the ith mass is:

$$F_i = -k[\phi(\lambda_i, t) - \phi(\lambda_{i-1}, t)] + k[\phi(\lambda_{i+1}, t) - \phi(\lambda_i, t)]$$

$$m\ddot{\phi}(\lambda_i, t) = k[\phi(\lambda_{i+1}, t) - 2\phi(\lambda_i, t) + \phi(\lambda_{i-1}, t)]$$

This is the discretized formula for the force on an internal mass where the grid spacing is the equilibrium length a.

In the continuum limit, the grid spacing will be minimized (that is to say, the distances between masses will shrink). In this regime, the displacement function will be redefined in terms of grid spacing $a = d\lambda$, where $d\lambda$ is the typical small differential element. And now, Newton's second law can be written in terms of λ, a continuous parameter. That is, $\phi(\lambda, t)$ becomes the displacement from equilibrium for some internal mass and $\phi(\lambda \pm d\lambda, t)$ becomes the displacement for the masses to the left and right of it.

$$m\ddot{\phi}(\lambda, t) = k[\phi(\lambda + d\lambda, t) - 2\phi(\lambda, t) + \phi(\lambda - d\lambda, t)] \tag{2.2}$$

Note that a finite difference approximation[1] can be used to simplify Equation (2.2):

$$\frac{m}{d\lambda}\ddot{\phi}(\lambda, t) = kd\lambda\phi''(\lambda, t)$$

to order $\mathcal{O}(d\lambda^3)$. Notice that in order for the terms $kd\lambda$ and $m/d\lambda$ to approach reasonable limits as $d\lambda \to 0$, the masses would need to shrink and the spring constant would need to increase at the same rate as the decrease in $d\lambda$. These terms will be considered be constants of our system. They will be interpreted as a sort of tension force K and a mass density μ respectively, and will be referred to throughout the following sections [2].

2.1.1 Factors with Length Contraction

As described in Chapter 1, when masses oscillate at relativistic speeds, the distances that the lab observes the masses travel may appear different than the distances the masses observe. The spring constant k will also experience a transformation. This will be shown by examining the force and energy for a single mass in the system. In the previous chapter, a Lorentz boost matrix was used to express the relationship

[1] Specifically,

$$f''(x) = \lim_{dx \to 0} \frac{f(x + dx) - 2f(x) + f(x - dx)}{dx^2}$$

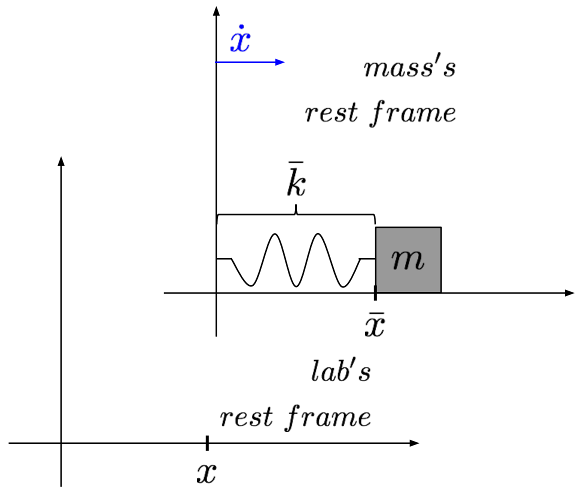

Figure 2.2: Two reference frames - one of the spring moving with velocity $\dot{x}$, the other of the laboratory at rest.

between a length $d\sigma$ observed in the moving reference frame of the mass, and the length $d\lambda$ observed in the laboratory's frame. For the moment, let's rename $d\sigma$ to be $\bar{x}$ and $d\lambda$ to be x.[2] From the previous chapter, the relationship between lengths and times measured by each observer is:

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

If the force is measured in the spring's frame, we can measure the displacement $\bar{x}$ instantaneously, at $\bar{t} = 0$. The measurement of the force in the lab's frame x can be found by plugging in $\bar{t} = 0$ and inverting the above matrix:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \bar{x} \end{pmatrix}$$

Thus, $x = \gamma\bar{x}$, or in terms of the coordinates we will soon use, this is $d\lambda = \gamma d\sigma$.[3]

Now we'd like to require that forces in a direction parallel to the boost be unchanged between each reference frame. In other words, we'd like to say that the force measured according to the lab and the mass are equal. Therefore, a relationship between the spring constant measured in one frame to another is found to be:

$$\bar{F} = F \to \bar{k}\bar{x} = kx \to \bar{k}\frac{1}{\gamma}x = kx \to \bar{k} = k\gamma$$

[2] We won't hold onto this barred notation for long. We are just using it here since seems easier to convey that the terms I am about to introduce: $\bar{k}$, $\bar{F}$, $\bar{E}$, and etc., refer to one frame and k, F, and E to another. This is as opposed to using many different Greek letters to describe a frame's counterpart.

[3] Notice this relationship is different than the one we obtained in Chapter 1, since here we are measuring lengths in the spring's frame. That's why we can set $d\bar{t} = 0$ here. As opposed to measuring them in the lab's frame, where we set $dt = 0$.

From an energy perspective, the same constraint can be found. Using the same Lorentz boost matrix to relate energies and momenta in either frame:

$$\begin{pmatrix} \bar{E}/c \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}$$

In the mass's frame, it is at rest, thus $\bar{p} = 0$. This can be plugged into the above matrix and then inverted to solve for the energy in the laboratory's frame:

$$\begin{pmatrix} E/c \\ p \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \bar{E}/c \\ 0 \end{pmatrix}$$

Thus, $E = \gamma\bar{E}$. Then the energy according to the laboratory E and the energy according to the mass $\bar{E}$ can be plugged in to find the same relationship between $\bar{k}$ and k:

$$E = \gamma\bar{E} \rightarrow \frac{mc^2}{\sqrt{1-(\dot{x}/c)^2}} + \frac{1}{2}kx^2 = \gamma(mc^2 + \frac{1}{2}\bar{k}\bar{x}^2)$$

$$\frac{mc^2}{\sqrt{1-(\dot{x}/c)^2}} + \frac{1}{2}kx^2 = \frac{mc^2}{\sqrt{1-(\dot{x}/c)^2}} + \gamma\frac{1}{2}\bar{k}\frac{x^2}{\gamma^2} \rightarrow \bar{k} = k\gamma$$

Now the length contraction occurring for $d\sigma$ will be discussed from the four different perspectives one could have on our N-coupled oscillator's equations of motion.

2.2 Four Different Perspectives on One-Dimensional Motion

We could measure the force on an internal mass from four different perspectives. Either we could consider the force in terms of the displacement measured in the lab's frame $F(k, \phi(\lambda))$, or we could consider the force as a function of the displacement of the mass in its own frame of reference $F(\bar{k}, \phi(\sigma))$. After considering those two perspectives, we could also consider either force and write them as $dp/d\tau$ or dp/dt.

2.2.1 The Laboratory's Perspective

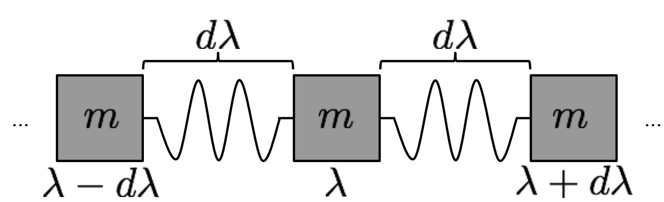

Figure 2.3: One-dimensional oscillator with stretch according to the lab

First, we will consider the force measured in the lab's frame. From Equation (2.2):

$$F = k[\phi(\lambda + d\lambda, t) - 2\phi(\lambda, t) + 2\phi(\lambda - d\lambda, t)]$$

where k is the ordinary spring constant and λ is the length as it appears to the laboratory. The same finite difference approximation that was applied at the beginning of this chapter can be applied here for the second derivative of $\phi(\lambda, t)$. This reduces Equation (2.2) to:

$$F = k[\phi(\lambda + d\lambda, t) - 2\phi(\lambda, t) + \phi(\lambda - d\lambda, t)] \approx k\phi''(\lambda, t)d\lambda^2$$

Next, applying the definition of relativistic momentum

$$F = \frac{dp}{dt} = \frac{d}{dt}\left[\frac{m\dot{\phi}(\lambda, t)}{\sqrt{1 - \dot{\phi}(\lambda, t)^2/c^2}}\right],$$

setting the RHS's of the two equations above equal, and combining all constants ($K/\mu = kd\lambda^2/m$), the force in the laboratory's rest frame becomes:[4]

$$\frac{d}{dt}\left[\frac{\dot{\phi}}{\sqrt{1 - \dot{\phi}^2/c^2}}\right] - \frac{K}{\mu}\phi''(\lambda, t) = 0$$

$$\boxed{\frac{\ddot{\phi}}{(1 - \dot{\phi}^2/c^2)^{3/2}} - \frac{K}{\mu}\phi''(\lambda, t) = 0} \tag{2.3}$$

If we instead take this force to be the Minkowski force, that is $F = dp/d\tau$, then Equation (2.3) becomes:

$$\frac{dt}{d\tau}\frac{d}{dt}\left[\frac{\dot{\phi}}{\sqrt{1 - \dot{\phi}^2/c^2}}\right] - \frac{K}{\mu}\phi''(\lambda, t) = 0$$

$$\frac{1}{\sqrt{1 - \dot{\phi}^2/c^2}}\frac{d}{dt}\left[\frac{\dot{\phi}}{\sqrt{1 - \dot{\phi}^2/c^2}}\right] - \frac{K}{\mu}\phi''(\lambda, t) = 0$$

$$\boxed{\frac{\ddot{\phi}}{(1 - \dot{\phi}^2/c^2)^{2}} - \frac{K}{\mu}\phi''(\lambda, t) = 0} \tag{2.4}$$

where the relation between the mass's time $d\tau$ and the lab's time dt from Chapter 1 was used.

2.2.2 The Mass's Perspective

Now let us consider $F(\bar{k}, \phi(\sigma))$, the force from a mass's perspective. The mass will appear at rest from its perspective, so Equation (2.2) should be re-written to account

[4] We have dropped the "$\approx$" signs from here on out since we will be making more approximations and they are cumbersome to carry around. The leading order error for this approximation is $\mathcal{O}(d\lambda^3)$.

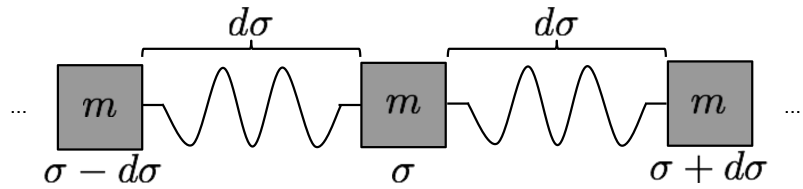

Figure 2.4: One-dimensional oscillator with stretch according to an internal mass

for the transformation of the stretch observed by the lab. Though before this is done, Equation (2.2) can be re-written using a finite difference approximation[5] for the first derivative, so that the stretching of the spring to the left becomes:

$$\phi(\sigma + d\sigma, t) - \phi(\sigma, t) = \phi((\sigma + \frac{1}{2}d\sigma) + \frac{1}{2}d\sigma, t) - \phi((\sigma + \frac{1}{2}d\sigma) - \frac{1}{2}d\sigma, t)$$

$$\phi(\sigma + d\sigma, t) - \phi(\sigma, t) = \phi'(\sigma + \frac{1}{2}d\sigma, t)d\sigma$$

to order $\mathcal{O}(d\sigma^2)$. Similarly, the stretching of the string to the right can be approximated by the same finite difference method:

$$\phi(\sigma - d\sigma, t) - \phi(\sigma, t) = \phi'(\sigma - \frac{1}{2}d\sigma, t)d\sigma$$

also to $\mathcal{O}(d\sigma^2)$. These approximations can be plugged into Equation (2.2) to get:

$$F = \bar{k}[\phi'(\sigma + \frac{1}{2}d\sigma, t)d\sigma - \phi'(\sigma - \frac{1}{2}d\sigma, t)d\sigma]$$

The relationship between the lab's observed stretch $d\lambda$ and the mass's observed stretch $d\sigma$ was found in the previous chapter: $d\lambda = \gamma d\sigma$. In order to take into account this length contraction observed in the laboratory's frame of reference, a different factor of γ needs to be introduced:

$$\gamma_{\pm 1/2} = 1/\sqrt{1 - \dot{\phi}(\lambda \pm \frac{1}{2}d\lambda, t)^2/c^2}$$

which moves $d\sigma = d\lambda/\gamma_{\pm 1/2}$ and $\phi(\sigma \pm \frac{1}{2}d\sigma, t) = \phi(\lambda \pm \frac{1}{2}d\lambda, t)/\gamma_{\pm\frac{1}{2}}$, but keeps $\phi'(\sigma \pm \frac{1}{2}d\sigma, t)$ the same because of the equal transformations of $d\sigma$ and ϕ. That is:

$$\phi'(\sigma \pm \frac{1}{2}d\sigma, t) = \frac{d\phi(\sigma_{\pm 1/2}, t)}{d\sigma} = \frac{d\phi(\lambda_{\pm 1/2}, t)/\gamma_{\pm 1/2}}{d\lambda/\gamma_{\pm 1/2}} = \frac{d\phi(\lambda_{\pm 1/2}, t)}{d\lambda}.$$

After these length contractions are accounted for, the force measured in the lab becomes:

$$F = \bar{k}\left[\phi'(\lambda + \frac{1}{2}d\lambda, t)\frac{d\lambda}{\gamma_{+1/2}} - \phi'(\lambda - \frac{1}{2}d\lambda, t)\frac{d\lambda}{\gamma_{-1/2}}\right] \tag{2.5}$$

[5] Specifically

$$f'(x) = \lim_{dx \to 0} \frac{f(x + dx) - f(x - dx)}{2dx} \approx \frac{f(x + dx) - f(x - dx)}{2dx}$$

The right-hand side of Equation (2.5) looks reminiscent of another finite difference approximation.[6] Thus, this line can be simplified even further as:[7]

$$F = \bar{k}\frac{d}{d\lambda}\left[\phi'(\lambda,t)/\gamma\right]d\lambda^2 \tag{2.6}$$

Then, with relativistic momentum and the transformation of $\bar{k}$ taken into account, and all constants absorbed into K/μ (same definition as in Section 2.1.2), the equation for an internal mass of the oscillator from the laboratory's perspective becomes:

$$\frac{d}{dt}\left[\frac{\dot\phi}{\sqrt{1-\dot\phi^2/c^2}}\right] - \frac{K/\mu}{\sqrt{1-\dot\phi^2/c^2}}\frac{d}{d\lambda}\left[\phi'\sqrt{1-\dot\phi^2/c^2}\right] = 0$$

$$\boxed{\frac{\ddot\phi}{(1-\dot\phi^2/c^2)} - \frac{K}{\mu}\frac{d}{d\lambda}\left[\phi'\sqrt{1-\dot\phi^2/c^2}\right] = 0} \tag{2.7}$$

If the Minkowski force is considered, then the equation of motion becomes:

$$\frac{dt}{d\tau}\frac{d}{dt}\left[\frac{\dot\phi}{\sqrt{1-\dot\phi^2/c^2}}\right] - \frac{K/\mu}{\sqrt{1-\dot\phi^2/c^2}}\frac{d}{d\lambda}\left[\phi'\sqrt{1-\dot\phi^2/c^2}\right] = 0$$

$$\frac{1}{\sqrt{1-\dot\phi^2/c^2}}\frac{d}{dt}\left[\frac{\dot\phi}{\sqrt{1-\dot\phi^2/c^2}}\right] - \frac{K/\mu}{\sqrt{1-\dot\phi^2/c^2}}\frac{d}{d\lambda}\left[\phi'\sqrt{1-\dot\phi^2/c^2}\right] = 0$$

$$\boxed{\frac{\ddot\phi}{(1-\dot\phi^2/c^2)^{3/2}} - \frac{K}{\mu}\frac{d}{d\lambda}\left[\phi'\sqrt{1-\dot\phi^2/c^2}\right] = 0} \tag{2.8}$$

These four different perspectives on a relativistic N-coupled oscillator each give us four different equations of motion (Equations (2.3-8)). None of these produce the equation of motion for a relativistic string (Equation 1.8), nor could they since they describe longitudinal oscillation. Remember in Chapter 1, we saw that the only motion along the string that could be observed is transverse to the movement of the string. Next these equations will be arrived at via a different method.

2.2.3 Lagrangian Approach

Now, that the equations of motion have been found via Newton's second law, we will double-check that the same equations of motion are found via the Euler-Lagrange equations of motion. We will only consider the force parameterized by the laboratory,

[6]Similar to Footnote 5, except here instead of $f(x+dx) = \phi(\lambda + (d\lambda/2), t)$, we have $f(x+dx) = \phi(\lambda + (d\lambda/2), t)\sqrt{1 - (\dot\phi(\lambda + (d\lambda/2), t)/c)^2}$

[7]Note that with the force and displacement already written in terms of the lab's observed length λ, we did not need to introduce any new factors of γ arising from the finite difference approximation.

meaning we are only going to verify Equation's (2.3) and (2.4), but this approach could also be considered for the force from the spring's perspective, to verify Equations (2.7) and (2.8). The potential energy is what one would expect for a spring and the relativistic kinetic energy of each mass transforms as described by source [9]. Thus the Lagrangian for our N-coupled oscillator is:

$$L = -mc^2 \sum_{i=1}^{N} \sqrt{1 - \dot{\phi}(\lambda_i, t)^2/c^2} - \frac{1}{2}k \sum_{i=1}^{N-1} (\phi(\lambda_i, t) - \phi(\lambda_{i+1}, t))^2 \tag{2.9}$$

For small $d\lambda$, a finite difference approximation of the potential energy term can be made. That is, $(\phi(\lambda_i, t) - \phi(\lambda_{i+1}, t))^2 \approx (-\phi'(\lambda_i, t)d\lambda)^2$.[8] So the Lagrangian becomes:

$$L = -mc^2 \sum_{i=1}^{N} \sqrt{1 - \dot{\phi}(\lambda_i, t)^2/c^2} - \frac{1}{2}k \sum_{i=1}^{N-1} \phi'(\lambda_i, t)^2 d\lambda^2$$

Two factors of $d\lambda$ can be distributed in preparation for integration with respect to this $d\lambda$.

$$L = -\frac{m}{d\lambda}c^2 \sum_{i=1}^{N} \sqrt{1 - \dot{\phi}(\lambda_i, t)^2/c^2}d\lambda - \frac{1}{2}kd\lambda \sum_{i=1}^{N-1} \phi'(\lambda_i, t)^2 d\lambda$$

Now, the continuum limit of this equation, that is when $d\lambda$ is small, leads the discrete sums to become integrals over the whole length of the N-coupled oscillator.

$$L = \int_0^L \left(-\mu c^2 \sqrt{1 - (\dot{\phi}/c)^2} + \frac{1}{2}K\phi'^2\right) d\lambda \tag{2.10}$$

where $\mu = m/d\lambda$ and $K = kd\lambda$ have been re-introduced. The action in the continuum limit is:

$$S[\phi] = \int_0^T \int_0^L \mathcal{L}(\phi, \dot{\phi}, \phi') d\lambda dt \approx \int_0^T \left[\int_0^L \left(-\mu c^2 \sqrt{1 - (\dot{\phi}/c)^2} + \frac{1}{2}K\phi'^2\right) d\lambda\right] dt \tag{2.11}$$

where the Lagrange density $\mathcal{L}$, the quantity in parenthesis on the RHS of Equations (2.10) & (2.11), is integrated. The Euler-Lagrange field equations for this system are found by varying our action with respect to an arbitrary function $\partial\phi$ that vanishes on spatial and temporal boundaries.[9] Under this variation, the action becomes:

$$S[\phi + \partial\phi] = \int_0^T \int_0^L \mathcal{L}(\phi + d\phi, \dot{\phi} + d\dot{\phi}, \phi' + d\phi') d\lambda dt$$

$$S[\phi + \partial\phi] \approx \int_0^T \int_0^L \mathcal{L}(\phi, \dot{\phi}, \phi') d\lambda dt + \int_0^T \int_0^L \left[\frac{\partial \mathcal{L}}{\partial \phi}\partial\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\partial\dot{\phi} + \frac{\partial \mathcal{L}}{\partial \phi'}\partial\phi'\right] d\lambda dt$$

The first piece of the integral is the original action $S[\phi]$, and the second is the variation $\partial S[\phi]$. The Euler-Lagrange equations are found by considering $\partial S[\phi]/\partial\phi$. So the

[8] The same approximation from Footnote 2.

[9] The same way the action was varied in the previous chapter in order to get the equation of motion for the Nambu-Goto string. Earlier we called our perturbation $\boldsymbol{\eta}$ as opposed to $d\phi$ here.

variation $\partial S[\phi]$ needs to be rewritten so as to more easily factor out a $\partial\phi$ from each term. The first term already has a factor of $\partial\phi$, but factors of $\partial\phi$ from the other two terms will need to be coaxed out using integration by parts. Note that because $\partial\phi$ vanishes on the spatial boundaries:

$$\int_0^T \int_0^L \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\partial\phi\right) d\lambda dt = \int_0^T \frac{\partial\mathcal{L}}{\partial\phi'}\partial\phi\bigg|_0^L d\lambda dt = 0.$$

And by the product rule, the LHS can be written as:

$$\int_0^T \int_0^L \left[\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\partial\phi + \frac{\partial\mathcal{L}}{\partial\phi'}\partial\phi'\right] d\lambda dt = 0$$

$$\int_0^T \int_0^L \left[\frac{\partial\mathcal{L}}{\partial\phi'}\partial\phi'\right] d\lambda dt = -\int_0^T \int_0^L \left[\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\partial\phi\right] d\lambda dt.$$

Using this trick for the $\partial\dot{\phi}$ term also (since $\partial\phi$ also vanishes on the temporal boundaries), $\partial S[\phi]$ simplifies to:

$$dS[\phi] = \int_0^T \int_0^L \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\phi'}\right]\partial\phi d\lambda dt$$

Thus the Euler-Lagrange field equations for this system are [2][9]:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\phi'} = 0$$

Plugging in the Lagrange density and its derivatives from Equation (2.10):

$$\frac{d}{dt}\left[\frac{\mu\dot{\phi}}{\sqrt{1-(\dot{\phi}/c)^2}}\right] - \frac{d}{d\lambda}\left[K\phi'\right] = 0 \tag{2.12}$$

The constants μ and K can be pulled out of their respective derivatives, so that Equation (2.12) becomes:

$$\frac{\ddot{\phi}}{(1-(\dot{\phi}/c)^2)^{3/2}} - \frac{K}{\mu}\phi''(\lambda,t) = 0$$

This last line is exactly the same as Equation (2.3), where the force is measured according to the lab. If the Minkowski force was considered, then we would have written our Lagrangian as a function of $\phi(\sigma,\tau)$, and found our Euler-Lagrange equation to have a derivative with respect to τ. If this derivative was transformed into coordinate time by a change of variables, then Equation (2.4) would have been retrieved.

If we instead considered the forces as they appear to the mass, Equation (2.7) would have been retrieved. And then if the Minkowski force was considered in that regime, Equation (2.8) would be retrieved.

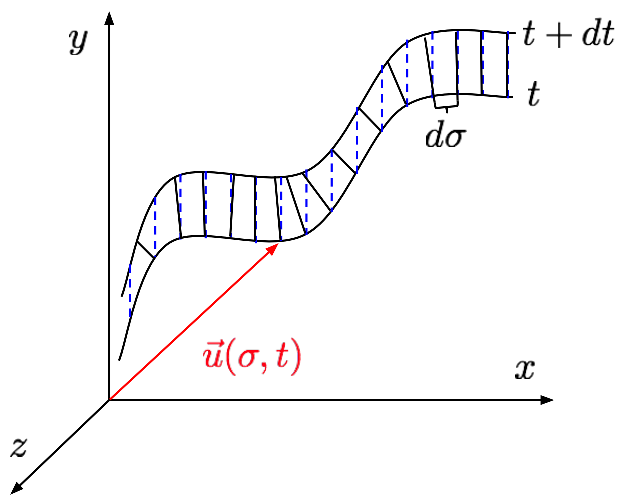

Figure 2.5: Three dimensional oscillator with two different parameterizations shown

2.3 Three-Dimensional Motion

In three dimensions, the motion of a mass inside our N-coupled oscillator will be described by a vector function, $\mathbf{u}(\sigma, t)$. This displacement vector is drawn from the origin of the laboratory's rest frame to a point on the string, unlike the way $\phi(\sigma, t)$ was defined as the relative displacement between the exact position and the equilibrium length of a mass in the N-coupled oscillator (see Figure 2.5). Here, dλ and dσ are lengths in three dimensions. This is as opposed to the parameterization considered earlier with just stretch along the single direction of motion. The relativistic form of Newton's second law for this system is:

$$\frac{d}{dt}\left[\frac{m\dot{\mathbf{u}}}{\sqrt{1-\|\dot{\mathbf{u}}\|^2/c^2}}\right] = \bar{k}\left[\mathbf{u}(\sigma+d\sigma,t) - 2\mathbf{u}(\sigma,t) + \mathbf{u}(\sigma-d\sigma,t)\right]$$

To simplify things, the finite difference trick used in Section 2.2.1 can be used here:

$$\frac{d}{dt}\left[\frac{m\dot{\mathbf{u}}}{\sqrt{1-\|\dot{\mathbf{u}}\|^2/c^2}}\right] = \bar{k}\frac{d}{d\sigma}\left[\frac{d\mathbf{u}}{d\sigma}\right]d\sigma^2$$

We want to write this force in terms of the laboratory's observed length $d\lambda$ so the $d\sigma$ derivatives can be turned into $d\lambda$ ones as such:

$$\frac{d}{dt}\left[\frac{m\dot{\mathbf{u}}}{\sqrt{1-\|\mathbf{u}\|^2/c^2}}\right] = \bar{k}d\sigma\frac{d\lambda}{d\sigma}\frac{d}{d\lambda}\left[\frac{d\mathbf{u}}{d\lambda}\frac{d\lambda}{d\sigma}\right]d\sigma$$

Note that for the spring force to be measured the same in each system, similar to how we defined this k earlier, we will fix that $\bar{k}d\sigma = kd\lambda$. Also, m will be rewritten in terms of the mass density $\mu d\lambda$. So for three-dimensional motion, Newton's second law is:

$$\frac{d}{dt}\left[\frac{(\mu d\lambda)\dot{\mathbf{u}}}{\sqrt{1-\|\dot{\mathbf{u}}\|^2/c^2}}\right] = kd\lambda\frac{d\lambda}{d\sigma}\frac{d}{d\lambda}\left[\frac{d\mathbf{u}}{d\lambda}\frac{d\lambda}{d\sigma}\right]d\sigma \qquad (2.13)$$

In three dimensions, the relationship between $d\sigma$ and $d\lambda$ is not necessarily the same as what was previously described in one dimension. In one dimension, γ is expected to be a function of only the velocity parallel $\dot{\mathbf{u}}_{\parallel}$ to the length contraction. If the same was true in three dimensions, then Equation (2.13) would contain both $\dot{\mathbf{u}}$ and $\dot{\mathbf{u}}_{\parallel}$ terms. In the next section, we will describe the freedom in parameterization we have and under what conditions the equation of motion for a relativistic string is retrieved from this model.

2.3.1 Discussion of Parameterization

We are at liberty to choose how we would like to parameterize the three-dimensional curve our string carves out. So the parameterization chosen to define the motion of our string according to the laboratory will be an arc-length parameter s (meaning $\lambda = s$) and coordinate time t. (This arc-length parameterization is shown in black in Figure 2.5.) These same choices in parameterization were made in Chapter 1 when deriving the relativistic string solution. This parametrization meets two conditions:

$$\left\|\frac{d\mathbf{u}}{ds}\right\|^2 = \frac{d\mathbf{u}}{ds}\cdot\frac{d\mathbf{u}}{ds} = 1 \quad \& \quad \frac{d\mathbf{u}}{ds}\cdot\frac{d\mathbf{u}}{dt} = 0$$

The first condition is due to the definition of arc-length parameterization. The second, is by our choice of parameterization that the velocity along the string $d\mathbf{u}/ds$ can be defined to be perpendicular to $d\mathbf{u}/dt$.

Now, say we wanted to parameterize our string in terms of σ and τ from the spring's reference frame instead of the arc-length parameter s and t from earlier. That is, $\mathbf{u}(s,t)$ is now $\mathbf{u}(\sigma,\tau)$. Under this new parameterization the two conditions above become:

$$\left\|\frac{d\mathbf{u}}{d\sigma}\right\|^2 = \frac{d\mathbf{u}}{d\sigma}\cdot\frac{d\mathbf{u}}{d\sigma} = \left(\frac{d\mathbf{u}}{ds}\right)^2\left(\frac{ds}{d\sigma}\right)^2 = \left(\frac{ds}{d\sigma}\right)^2 \quad \& \quad \frac{d\mathbf{u}}{d\sigma}\cdot\frac{d\mathbf{u}}{d\tau} = \left(\frac{d\mathbf{u}}{ds}\frac{ds}{d\sigma}\right)\cdot\left(\frac{d\mathbf{u}}{dt}\frac{dt}{d\tau}\right) = 0$$

First, we can define the parameter $d\sigma$ to be independent of time. If $d\sigma$ is independent of time, it can be pulled into the time derivative and Equation (2.13) becomes:

$$\frac{d}{dt}\left[\frac{\mu(ds/d\sigma)\dot{\mathbf{u}}}{\sqrt{1-\|\dot{u}\|^2/c^2}}\right] = kds\frac{ds}{d\sigma}\frac{d}{ds}\left[\frac{d\mathbf{u}}{ds}\frac{ds}{d\sigma}\right] \tag{2.14}$$

Next, the relationship between $d\sigma$ and ds can be found by choosing that the area and length covered by our relativistic N-coupled oscillator is the same regardless of parameterization (see Figure 2.6). This area invariance means [12]:

$$cd\tau\cdot d\sigma = cdt\cdot ds \implies \frac{d\sigma}{ds} = \frac{dt}{d\tau}$$

And according to the definition of proper time from Chapter 1:

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\|\dot{\mathbf{u}}\|^2/c^2}} \implies \frac{d\sigma}{ds} = \frac{1}{\sqrt{1-\|\dot{\mathbf{u}}\|^2/c^2}}.$$

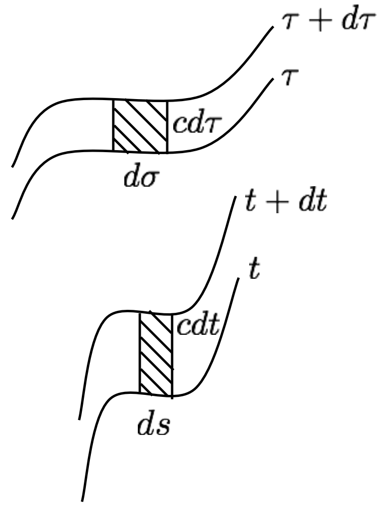

Figure 2.6: The area measured in the (s, t) and (σ, τ) parameterizations.

Placing this relationship inside Equation (2.14):

$$\frac{d}{dt}\left[\mu\dot{\mathbf{u}}\right] - kds\sqrt{1 - \|\dot{\mathbf{u}}\|^2/c^2}\frac{d}{ds}\left[\frac{d\mathbf{u}}{ds}\sqrt{1 - \|\dot{\mathbf{u}}\|^2/c^2}\right] = 0$$

$$\frac{\ddot{\mathbf{u}}}{\sqrt{1 - \|\dot{\mathbf{u}}\|^2/c^2}} - \frac{K}{\mu}\frac{d}{ds}\left[\frac{d\mathbf{u}}{ds}\sqrt{1 - \|\dot{\mathbf{u}}\|^2/c^2}\right] = 0 \tag{2.15}$$

where $K = kd\lambda = kds$. Amazingly, this is the equation of motion for a relativistic string up to a few constants (Equation 1.8) [2].[10]

[10] Note that our constant K/μ have dimensions of $([\text{length}]/[\text{time}])^2$ consistent with c^2, the constant from Equation 1.8. The constant T_0 factors out of both terms.

Chapter 3

Ehrenfest's Paradox

Now we will take a step back from this model with all its complicated choices of parameterization, and just study a relativistic N-coupled oscillator. It will be used to investigate an interesting paradox in special relativity.

3.1 Ehrenfest's Paradox

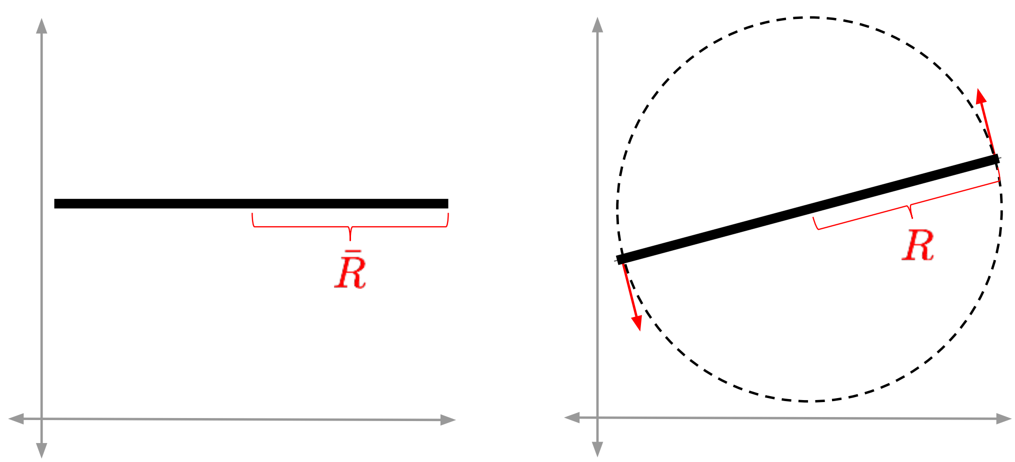

Figure 3.1: On the left, the rod at rest and on the right, the rod in motion

Imagine a rigid rod spinning about its center of mass with an angular velocity ω. The endpoints of the rod will move faster than any other points along the rod. In fact they will move with a linear velocity $\omega\bar{R}$ where $\bar{R}$ is the radius of the rod at rest. One could imagine that if we spun our rod with a fast enough frequency, or if we extended our rod far enough, the endpoints could have a linear velocity exceeding the speed of light ($\omega\bar{R} > c$). Special relativity tells us that this cannot be allowed; no mass can move with a speed faster than the speed of light. Thus in order to meet the demands of special relativity, such a rotating rod must have some different radius

R when it is in motion, such that the endpoints travel with a linear velocity ωR less than the speed of light ($\omega R < c$). This must mean that the radius of the bar at rest and in motion, $\bar{R}$ and R, are different - in fact, $\bar{R} > R$. If we consider rigidity to mean that our rod has the same radius at rest as it does in motion, then Ehrenfest's paradox implies that there does not exist rigidity in special relativity [6][10][8].

What we aim to do with this chapter is to use a relativistic coupled oscillator with uniformly large spring constant (such that our string is basically a rod) to test whether or not a rotating rigid string will lose it's rigidity. Given no torque, the system should rotate rigidly at its initial low frequency. But with a torque applied about the center or the rod, this system will experience angular acceleration and may lose rigidity as its endpoints approach the speed of light.

3.2 Verlet Method

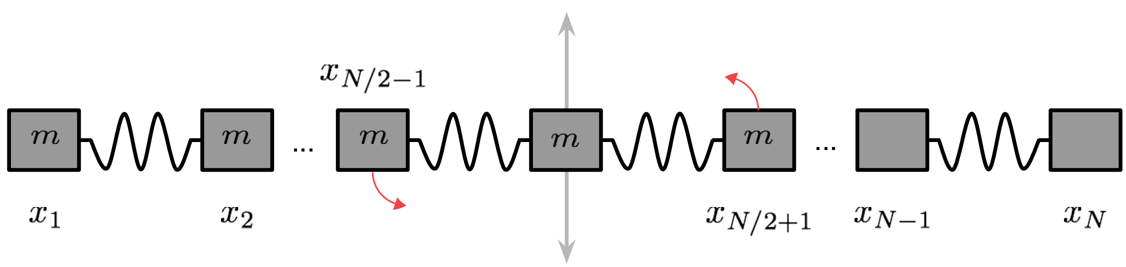

Figure 3.2: Our approximation to a rotating rigid rod

In order to compute how the positions and momenta of each mass in the harmonic oscillator system evolve over time, a Verlet method will be used. In order to do this, an update map in time for position and momentum will need to be found [3].

First, we will set up a grid in space, where masses in the rod have index $j \in [1, N_s]$. Now discretized relationships for $\mathbf{p}_j$ and $\mathbf{x}_j$ will written. The change in $\mathbf{p}_j$ describes the forces acting on our system. For a central mass ($j \in [2, N_s/2-2]$ or $j \in [N_s/2+2, N_s-1]$):

$$\frac{d\mathbf{p}_j}{dt} = -k(||\mathbf{x}_j - \mathbf{x}_{j+l}|| - a)\frac{\mathbf{x}_j - \mathbf{x}_{j+1}}{||\mathbf{x}_j - \mathbf{x}_{j+1}||} + k(||\mathbf{x}_{j-1} - \mathbf{x}_j|| - a)\frac{\mathbf{x}_{j-1} - \mathbf{x}_j}{||\mathbf{x}_{j-1} - \mathbf{x}_j||} \tag{3.1}$$

The two masses on either side of the pivot point ($j = N_s/2-1$ or $j = N_s/2+1$) pick up an additional term due to the torque enforced when we accelerate the rod from its center. Their net force looks like:

$$\begin{aligned}\frac{d\mathbf{p}_j}{dt} = -k(||\mathbf{x}_j - \mathbf{x}_{j+1}|| - a)\frac{\mathbf{x}_j - \mathbf{x}_{j+1}}{||\mathbf{x}_j - \mathbf{x}_{j+1}||} \\ + k(||\mathbf{x}_{j-1} - \mathbf{x}_j|| - a)\frac{\mathbf{x}_{j-1} - \mathbf{x}_j}{||\mathbf{x}_{j-1} - \mathbf{x}_j||} \pm \frac{||\boldsymbol{\tau}(t)||}{||\mathbf{x}_j||}\frac{\boldsymbol{\tau}(t) \times \mathbf{x}_j}{||\boldsymbol{\tau}(t) \times \mathbf{x}_j||}\end{aligned} \tag{3.2}$$

where $\boldsymbol{\tau}(t)$ is a time-dependent torque and $\boldsymbol{\tau} = \mathbf{x} \times \mathbf{F}$ was used to solve for $\mathbf{F}$. Notice how in order to get a force due to torque to point in our two-dimensional plane, perpendicular to $\mathbf{x}_j$, the torque needs to point in the $\hat{\mathbf{z}}$-direction. The forces on the endpoints ($j = 1$ and N_s) are:

$$\frac{d\mathbf{p}_1}{dt} = -k(||\mathbf{x}_1 - \mathbf{x}_2|| - a)\frac{\mathbf{x}_1 - \mathbf{x}_2}{||\mathbf{x}_1 - \mathbf{x}_2||}$$
$$\& \quad \frac{d\mathbf{p}_{N_s}}{dt} = +k(||\mathbf{x}_{N_s-1} - \mathbf{x}_{N_s}|| - a)\frac{\mathbf{x}_{N_s-1} - \mathbf{x}_{N_s}}{||\mathbf{x}_{N_s-1} - \mathbf{x}_{N_s}||} \tag{3.3}$$

Lastly, the change in $\mathbf{x}_j$ can be found by considering the definition of relativistic momentum.

$$\mathbf{p}_j = \frac{m\mathbf{v}_j}{\sqrt{1 - \frac{\mathbf{v}_j^2}{c^2}}} \rightarrow m_j\frac{d\mathbf{x}_j}{dt} = \frac{\mathbf{p}_j}{\sqrt{1 + \frac{||\mathbf{p}_j||^2}{m_j c^2}}} \tag{3.4}$$

Next, we will set up a grid in time, where steps in time have index $i \in [1, N_t]$ and width Δt. Using finite difference approximations, each derivative with respect to time can be written as:

$$\frac{d\mathbf{x}_j^i}{dt} \approx \frac{\mathbf{x}_j^{i+1} - \mathbf{x}_j^{i-1}}{2\Delta t} \quad \& \quad \frac{d\mathbf{p}_j^i}{dt} \approx \frac{\mathbf{p}_j^{i+1} - \mathbf{p}_j^{i-1}}{2\Delta t}$$

to $\mathcal{O}(\Delta t^2)$. Temporal update maps for $\mathbf{x}_j^i$ and $\mathbf{p}_j^i$ can be found from rearranging these relationships and plugging in Equations (3.1-4):

$$\mathbf{x}_j^{i+1} = \mathbf{x}_j^{i-1} + \frac{2\Delta t}{m_j}\frac{\mathbf{p}_j^i}{\sqrt{1 + \frac{||\mathbf{p}_j^i||^2}{m_j c^2}}} \tag{3.5}$$

$$\mathbf{p}_j^{i+1} = \mathbf{p}_j^{i-1} + 2\Delta t\frac{d\mathbf{p}_j^i}{dt} \tag{3.6}$$

where $d\mathbf{p}_j/dt$ is the relevant definition for the force on the jth mass (Equations (3.1-3)).

Notice that in order to start using these maps, not only are position and momentum for each mass initially required, but also both of these values at one time step before that. In order to get the position and momentum prior to $t = 0$, different finite difference approximations for $d\mathbf{p}_j^i/dt$ and $d\mathbf{x}_j^i/dt$ can be solved:

$$\frac{d\mathbf{p}_j^i}{dt} = \frac{\mathbf{p}_j^i - \mathbf{p}_j^{i-1}}{\Delta t} \;\&\; \frac{d\mathbf{x}_j^i}{dt} = \frac{\mathbf{x}_j^i - \mathbf{x}_j^{i-1}}{\Delta t} \implies \mathbf{p}_j^{i-1} = \mathbf{p}_j^i - \Delta t\frac{d\mathbf{p}_j^i}{dt} \;\&\; \mathbf{x}_j^{i-1} = \mathbf{x}_j^i - \Delta t\frac{d\mathbf{x}_j^i}{dt}$$

to $\mathcal{O}(\Delta t)$. Equations (3.1-4) can be used to find initial momentum and velocity, $d\mathbf{p}_j^0/dt$ and $d\mathbf{x}_j^0/dt$, so that they can then be to plugged into the above equations to get $\mathbf{p}_j^{-1}$ and $\mathbf{x}_j^{-1}$.

3.2.1 Initial Conditions

We place the central mass at the origin and assume symmetry about the axis of rotation. That is, the $N_s/2$ masses on the right half of the string will have equal but

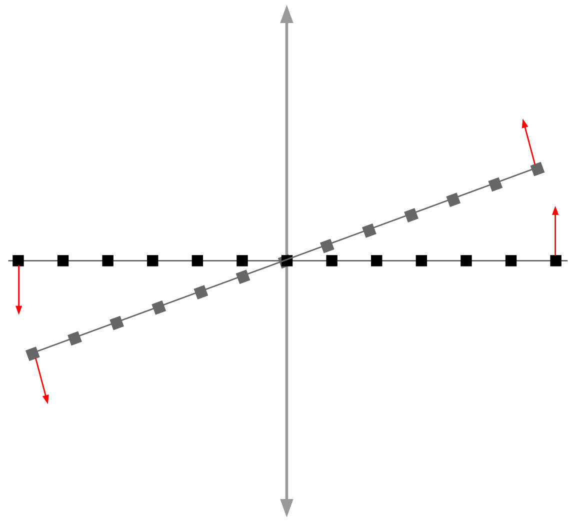

Figure 3.3: Rotating rigid rod placed initially along the x-axis with equal but opposite initial velocity at the endpoints

opposite initial position, velocity, momentum and force from the masses on the left (see Figure 3.3).

The position, velocity, and acceleration of the jth mass on our rod can be expressed as:

$$\mathbf{x}_j = x_j \cos(\omega t)\hat{\mathbf{x}} + x_j \sin(\omega t)\hat{\mathbf{y}}$$

$$\dot{\mathbf{x}}_j = \omega(-x_j \sin(\omega t)\hat{\mathbf{x}} + x_j \cos(\omega t)\hat{\mathbf{y}})$$

$$\ddot{\mathbf{x}}_j = \omega^2(-x_j \cos(\omega t)\hat{\mathbf{x}} - x_j \cos(\omega t)\hat{\mathbf{y}}) = -\omega^2 \mathbf{x}_j$$

Note $\dot{\mathbf{x}}_j^2 = \dot{\mathbf{x}}_j \cdot \dot{\mathbf{x}}_j = \omega^2(x_j^2 \sin^2(\omega t) + x_j^2 \cos^2(\omega t)) = \omega^2 x_j^2$. Before special relativity, one would say the force due to uniform circular motion equalled $m\omega^2 x_j$. The relativistic form of the centripetal force is:

$$\frac{d}{dt}\left[\frac{m_j \dot{\mathbf{x}}_j}{\sqrt{1-(\dot{\mathbf{x}}_j/c)^2}}\right] = \frac{m_j \ddot{\mathbf{x}}_j}{\sqrt{1-(\omega x_j/c)^2}} = \frac{-m_j \omega^2 \mathbf{x}_j}{\sqrt{1-(\omega x_j/c)^2}}$$

Initially all the masses are lined up along the x-axis, meaning the jth mass has some position, $\mathbf{x}_j^0 = \left\{x_j^0, 0\right\}$. We will set the initial spring forces on the jth mass equal to its centripetal acceleration. This will maintain that initially the motion of the rod is rigid, that is, it obeys uniform circular motion.

$$\frac{-m_j \omega^2 x_j^0}{\sqrt{1-(\omega x_j^0/c)^2}} = -k((x_j^0 - x_{j-1}^0) - a) + k((x_{j+1}^0 - x_j^0) - a) = k(x_{j-1}^0 - 2x_j^0 + x_{j+1}^0)$$

$$x_{j+1}^0 = \left(2 - \frac{(m_j \omega^2/k)}{\sqrt{1-(\omega x_j^0/c)^2}}\right) x_j^0 - x_{j-1}^0$$

The same argument can be made for each subsequent mass. The equilibrium length can be fixed by looking at the force acting on the mass at the very end:

$$\frac{-m_j\omega^2 x^0_{N_s}}{\sqrt{1-(\omega x^0_{N_s}/c)^2}} = -k((x^0_{N_s} - x^0_{N_{s-1}}) - a) \implies a = \left(\frac{-m_j\omega^2/k}{\sqrt{1-(\omega x^0_{N_s}/c)^2}} + 1\right) x^0_{N_s} - x^0_{N_{s-1}}$$

The initial momentum of the jth mass is:

$$\mathbf{p}^0_j = \frac{m_j \dot{\mathbf{x}}^0_j}{\sqrt{1-(\dot{\mathbf{x}}^0_j/c)^2}} = \frac{m_j \omega x^0_j}{\sqrt{1-(\omega x^0_j/c)^2}}\hat{\mathbf{y}}$$

where the momenta point in the $\hat{\mathbf{y}}$-direction (since $\dot{\mathbf{x}}^0_j = \omega(-x^0_j \sin(0)\hat{\mathbf{x}} + x^0_j \cos(0)\hat{\mathbf{y}}) = \omega x^0_j \hat{\mathbf{y}}$). Note the momenta to the right of the origin should be equal in magnitude but opposite in sign to the momenta to the left of the origin (since $\mathbf{x}$ is similarly antisymmetric).

3.2.2 Energy & Angular Frequency

We would also like to see how the energy and angular frequency of our system changes over time. The energy of our system at any time is:

$$E = \sum_{j=0}^{Ns} \sqrt{(m_j c^2)^2 + (\mathbf{p}_j c)^2} + \sum_{j=1}^{Ns-1} \frac{1}{2} k(||\mathbf{x}_{j+1} - \mathbf{x}_j|| - a)^2 \tag{3.7}$$

by the definition of the energy for a relativistic point particle and potential energy due to a spring force. The angular frequency of my system is:

$$\boldsymbol{\omega}_j = \frac{\mathbf{x}_j \times \left(\frac{d\mathbf{x}_j}{dt}\right)}{||\mathbf{x}_j||^2} \rightarrow ||\boldsymbol{\omega}_j||^2 = \left|\left|\frac{\mathbf{x}_j \times (\mathbf{p}_j/m_j)}{||\mathbf{x}_j||^2\sqrt{1+||\mathbf{p}_j||^2/(m_j c^2)}}\right|\right|^2 \tag{3.8}$$

where the definition of the velocity of a point in terms of its momentum has been used (Equation 3.4).

Our findings should show that when no torque is applied, the energy of the system is approximately conserved, and when a time-dependent torque is applied, the energy is not conserved. The standard deviation of the angular frequency divided by the average angular frequency for each mass will be considered as a measurement of how rigid our system is - since for rigid body motion, every mass along our rod should move with the same angular velocity. This measurement is called the “coefficient of variation” in statistics, and it gives us a better idea the variance of our data with respect to a changing mean.

3.3 Non-dimensionalization

Before we compute these results, we would like to non-dimensionalize Equations (3.1-3), (3.4), and (3.7-8). This process is helpful because in some cases it reduces the number of terms in an equation. We will do this by redefining terms like $\mathbf{x}$ by $x_0\widetilde{\mathbf{x}}$, where $\mathbf{x}$'s dimensions of length are held by the constant x_0, such that $\widetilde{\mathbf{x}}$ is a dimensionless quantity. Our full set of transformations[1] from dimension-full to dimensionless quantities is:

$$\mathbf{x} = x_0\widetilde{\mathbf{x}} \quad m = m_0\widetilde{m} \quad t = T_0\widetilde{t} \quad \omega = (1/T_0)\widetilde{\omega} \quad a = x_0\widetilde{a}$$

$$k = (m_0/T_0^2)\widetilde{k} \quad \boldsymbol{\tau} = (m_0x_0^2/T_0^2)\widetilde{\boldsymbol{\tau}} \quad E = (m_0x_0^2/T_0^2)\widetilde{E} \quad \mathbf{p} = (m_0x_0/T_0)\widetilde{\mathbf{p}}$$

Equation (3.1) can be fully non-dimensionalized without defining the dimension-full parameters. In other words, when this equation is non-dimensionalized we get:

$$\frac{m_0x_0}{T_0^2}\frac{d\widetilde{\mathbf{p}}_j}{d\widetilde{t}} = -\frac{m_0x_0}{T_0^2}\widetilde{k}\left[(||\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_{j+l}|| - \widetilde{a})\frac{\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_{j+1}}{||\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_{j+1}||} + (||\widetilde{\mathbf{x}}_{j-1} - \widetilde{\mathbf{x}}_j|| - a)\frac{\widetilde{\mathbf{x}}_{j-1} - \widetilde{\mathbf{x}}_j}{||\widetilde{\mathbf{x}}_{j-1} - \widetilde{\mathbf{x}}_j||}\right]$$

$$\frac{d\widetilde{\mathbf{p}}_j}{d\widetilde{t}} = -\widetilde{k}(||\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_{j+l}|| - \widetilde{a})\frac{\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_{j+1}}{||\widetilde{\mathbf{x}}_j - \widetilde{\mathbf{x}}_{j+1}||} + \widetilde{k}(||\widetilde{\mathbf{x}}_{j-1} - \widetilde{\mathbf{x}}_j|| - a)\frac{\widetilde{\mathbf{x}}_{j-1} - \widetilde{\mathbf{x}}_j}{||\widetilde{\mathbf{x}}_{j-1} - \widetilde{\mathbf{x}}_j||}$$

Equation (3.3) follows similarly. The only difference between Equation (3.1) and (3.2) is the addition of torque. The term due to torque carries around the same dimension-full constants as $d\mathbf{p}_j/dt$ though, so Equation (3.2) is also fully non-dimensionalized without requiring anything about the dimension-full parameters.

$$\frac{m_0x_0}{T_0^2}\frac{d\widetilde{\mathbf{p}}_j}{d\widetilde{t}} = ... + \frac{m_0x_0}{T_0^2}\frac{||\widetilde{\boldsymbol{\tau}}(t)||}{||\widetilde{\mathbf{x}}_j||}\frac{\widetilde{\boldsymbol{\tau}}(t) \times \widetilde{\mathbf{x}}_j}{||\widetilde{\boldsymbol{\tau}}(t) \times \widetilde{\mathbf{x}}_j||}$$

$$\frac{d\widetilde{\mathbf{p}}_j}{d\widetilde{t}} = ... + \frac{||\widetilde{\boldsymbol{\tau}}(t)||}{||\widetilde{\mathbf{x}}_j||}\frac{\widetilde{\boldsymbol{\tau}}(t) \times \widetilde{\mathbf{x}}_j}{||\widetilde{\boldsymbol{\tau}}(t) \times \widetilde{\mathbf{x}}_j||}$$

Next, let's consider Equation (3.4):

$$\frac{m_0x_0}{T_0}\widetilde{m}_j\frac{d\widetilde{\mathbf{x}}_j}{d\widetilde{t}} = \frac{m_0x_0}{T_0}\frac{\widetilde{\mathbf{p}}_j}{\sqrt{1 + (\frac{m_0x_0}{T_0})^2\frac{1}{m_0^2}\frac{||\widetilde{\mathbf{p}}_j||^2}{\widetilde{m}_jc^2}}}$$

$$\widetilde{m}_j\frac{d\widetilde{\mathbf{x}}_j}{d\widetilde{t}} = \frac{\widetilde{\mathbf{p}}_j}{\sqrt{1 + (\frac{x_0}{T_0})^2\frac{||\widetilde{\mathbf{p}}_j||^2}{\widetilde{m}_jc^2}}}$$

We are free to choose how we define our dimension-full parameters, so let's choose our parameters x_0 and T_0 such that $(x_0/T_0) = c$ and the denominator of the above expression simplifies a bit.

Now let's consider non-dimensionalizing Equation (3.7). The first term on the LHS becomes:

$$\sum_{j=1}^{Ns}\sqrt{(m_0^2)(\widetilde{m}_jc^2)^2 + (m_0x_0/T_0)^2(\widetilde{\mathbf{p}}_jc)^2} = m_0c^2\sum_{j=1}^{Ns}\sqrt{(\widetilde{m}_j)^2 + (x_0/T_0)^2(\widetilde{\mathbf{p}}_j/c)^2}$$

[1] These relationships come from what we know about the dimensions of each quantity.

Again, from this relationship we see that it is helpful to define $(x_0/T_0) = c$ to simplify terms under the square root. When we non-dimensionalize the potential energy term from Equation (3.7), we find:

$$\sum_{j=1}^{Ns-1} \frac{1}{2} k(||\mathbf{x}_{j+1} - \mathbf{x}_j|| - a)^2 = \sum_{j=1}^{Ns-1} \frac{1}{2} \frac{m_0 x_0}{T_0^2} \widetilde{k}(||\widetilde{\mathbf{x}}_{j+1} - \widetilde{\mathbf{x}}_j|| - \widetilde{a})^2$$

The RHS of Equation (3.8) transforms like:

$$\frac{m_0 x_0^2}{T_0^2} \widetilde{E}$$

The potential energy terms carry the same constants out front as E does and again we see that if $(x_0/T_0) = c$, we will be dimensionally consistent throughout this equation.

Finally, let's consider Equation (3.8). Its non-dimensionalized form is:

$$\frac{1}{T_0} \widetilde{\boldsymbol{\omega}}_j = \frac{x_0 \widetilde{\mathbf{x}}_j \times \left(\frac{x_0}{T_0} \frac{d\widetilde{\mathbf{x}}_j}{dt}\right)}{x_0^2 ||\widetilde{\mathbf{x}}_j||^2}$$

$$\widetilde{\boldsymbol{\omega}}_j = \frac{\widetilde{\mathbf{x}}_j \times \left(\frac{d\widetilde{\mathbf{x}}_j}{dt}\right)}{||\widetilde{\mathbf{x}}_j||^2}$$

Thus Equation (3.8) is also fully non-dimensionalized without specifying the dimension-full parameters.

The fixing of x_0/T_0, means that the linear speed presented soon in our results section are scaled such that they represent a fraction of the speed of light. Explicitly, the velocity must scale like $v = x_0 \widetilde{v}/T_0$, which implies the non-dimensionalized velocity we will use in our calculation is $\widetilde{v} = v/c$.

3.4 Comparison of the Length of the Rod in Relativistic and Non-Relativistic Settings

To determine how the length of a relativistic rigid rod compares to the length of a non-relativistic rigid rod, we will just consider how the distance between two masses changes depending on whether we are using the non-relativistic or relativistic version of Newton's second law. Imagine these two masses are placed symmetrically about the origin, each a distance x away. The distance between those two masses, if they are not moving at relativistic speeds, is $L = 2x$. This distance can be found by setting the force on a single mass equal to its non-relativistic centripetal force.

$$F = k(2x - a) = m\omega^2 x \rightarrow L = 2x = \frac{a}{1 - \left(\frac{m\omega^2}{2k}\right)} \tag{3.9}$$

If the two masses are rotating at relativistic speeds, the distance between the two masses is $\bar{L} = 2\bar{x}$ and the force on a single mass is equal to the relativistic version of its centripetal force.

$$\bar{F} = k(2\bar{x} - a) = \frac{m\omega^2\bar{x}}{\sqrt{1 - \left(\frac{\omega\bar{x}}{c}\right)^2}} \rightarrow (\bar{L} - a) = \frac{m\omega^2\bar{L}/2k}{\sqrt{1 - \left(\frac{\omega\bar{L}}{2c}\right)^2}} \tag{3.10}$$

where instead of solving this equation for $\bar{x}$ to find $\bar{L}$, $\bar{x} = \bar{L}/2$ was plugged in so that we found an equation in terms of $\bar{L}$.

Before we talk about how L and $\bar{L}$ compare, we will non-dimensionalize these relations using the transformations we described in the section above plus $L = x_0\widetilde{L}$. That is, Equations (3.9) and (3.10) become:

$$x_0\widetilde{L} = \frac{x_0\widetilde{a}}{1 - \left(\frac{m_0\widetilde{m}}{2\widetilde{k}}\frac{T_0^2}{m_0}\frac{\widetilde{\omega}^2}{T_0^2}\right)} \rightarrow \widetilde{L} = \frac{\widetilde{a}}{1 - \left(\frac{\widetilde{m}\widetilde{\omega}^2}{2\widetilde{k}}\right)} \tag{3.11}$$

$$(x_0\widetilde{\bar{L}} - x_0\widetilde{a}) = \frac{T_0^2}{2m_0\widetilde{k}}\frac{(m_0\widetilde{m})(\widetilde{\omega}/T_0)^2(x_0\widetilde{\bar{L}})}{\sqrt{1 - \left(\frac{(\widetilde{\omega}/T_0)(x_0\widetilde{\bar{L}})}{2c}\right)^2}} \rightarrow (\widetilde{\bar{L}} - \widetilde{a}) = \frac{\widetilde{m}\widetilde{\omega}^2\widetilde{\bar{L}}/2\widetilde{k}}{\sqrt{1 - \frac{1}{4}\left(\frac{x_0}{T_0c}\right)^2\left(\widetilde{\omega}\widetilde{\bar{L}}\right)^2}} \tag{3.12}$$

Equation (3.11) says that for a physically realistic length of the rod in the non-relativistic setting $\widetilde{L} > 0$, it must be true that $1 > \left(\frac{\widetilde{m}\widetilde{\omega}^2}{2\widetilde{k}}\right)$. Also, we are free to choose to define x_0 and T_0 such that $(x_0/T_0) = c$, so we will do so (as we did above). So that Equation (3.12) becomes:

$$(\widetilde{\bar{L}} - \widetilde{a}) = \frac{\widetilde{m}\widetilde{\omega}^2\widetilde{\bar{L}}/2\widetilde{k}}{\sqrt{1 - \left(\widetilde{\omega}\widetilde{\bar{L}}/2\right)^2}} \tag{3.13}$$

What do these relationships tell us? Well let's consider $\widetilde{k} = \widetilde{m}\widetilde{w}^2$, that makes Equation (3.11): $\widetilde{L} = 2a$. Then, if we set the speed at the endpoints to be some small fraction of the speed of light $\widetilde{\omega}(\widetilde{\bar{L}}/2) = \epsilon$, Equation (3.13) can simplified to solve for $\widetilde{\bar{L}}$:

$$(\widetilde{\bar{L}} - \widetilde{a}) = \frac{\widetilde{\bar{L}}/2}{\sqrt{1 - \epsilon^2}} \rightarrow \widetilde{\bar{L}} = \frac{2\widetilde{a}}{2 - \frac{1}{\sqrt{1-\epsilon^2}}} = \frac{\widetilde{L}}{2 - \frac{1}{\sqrt{1-\epsilon^2}}}$$

Thus $\widetilde{\bar{L}} > \widetilde{L}$ for small epsilon.

3.5 Results

The following results were produced for $N_s = 20$ masses. The initial angular velocity was 0.1. We ran both the systems with torque and with no torque over $N_t = 200,000$ steps. Both considered steps in time of width $\Delta t = 0.01$. We chose a torque which was linear in time so that we could start out with rigid rotation and see how the presence of an increasing torque changed the rod's motion. The time-dependent torque function that we considered started at $\widetilde{\tau}(t) = 0$ and ramped up to $\widetilde{\tau}(t) = 0.33$ by the end of the computation. We assumed all masses have the same value and we set both the speed of light and each mass to 1 for convenience.

First we will vary $\widetilde{k}$ and show how the energy, maximum linear speed, average angular velocity, and coefficient of variation for the angular velocity change. Then we will show the time-evolution of the rigid rod and non-rigid rod for a relatively high spring constant $\widetilde{k} = 1000$.

3.5.1 Dependence on $\widetilde{k}$

All springs have the same spring constant value, which we have chosen to take as $\widetilde{k} = 10$, $\widetilde{k} = 100$, and $\widetilde{k} = 1000$. Figures 3.4-7 show the system with no torque in red, and the system with a time-dependent torque in blue. Figures 3.8-9 show the time evolution for the rotation of a rigid rod with and without the torque. In the following sections, we discuss the results shown in Figure 3.4-9.

Energy

From Figure 3.4, you can see that the energy of the rod without any torque is approximately constant at $\widetilde{E} = 21.08$ over time (with a standard deviation of 4.26×10^{-8}). This is exactly what we would expect. We are adding no external time-dependent force so the energy of our system should theoretically stay the same. This is the case over all values of $\widetilde{k}$ shown. Also, as we would expect, we see that the energy of the system is not conserved for the time-dependent torque case. The energy of the torqued and torque-free systems start at about the same $\widetilde{E}$, but the energy of the torqued system increases to $\widetilde{E} = 48.12$, 70.12, to 80.73 for increasing $\widetilde{k}$ over time. This makes sense since the torque we are providing starts at zero and increases over time.

Maximum Speed

From Figure 3.5, we see for the torque-free system the maximum linear speed of any point along the rod is a constant low value of $\widetilde{v} = 0.10$ (with a standard deviation of 2.11×10^{-8}). This is consistent with what we would expect to see for a rigid rod experiencing no angular acceleration due to torque. Considering the fastest point on the rod will be the endpoints that are located about a distance 1 from the center and the initial angular velocity is 0.1, we should see that the velocity of the endpoints is about 0.1. This is consistent with Figure 3.5.

We see that our torqued rod has a maximum linear speed which starts at the same velocity as torque-free system, but increases more steeply for increasing $\widetilde{k}$. We have scaled the system such that $c = 1$, so Figure 3.5 shows that the maximum linear speed approaches the speed of light, $\widetilde{v} = 1.00$.

Average Angular Velocity

In Figure 3.6, we see that the average angular velocity of the torque-free system is constant at $\widetilde{\omega} = 0.1$ (with a standard deviation of 3.90×10^{-8}), which was the initial angular velocity we provided. This is as expected since with no torque, no angular acceleration should be introduced.

We also see that for our system with time-dependent torque, the average angular velocity steeply increases more significantly for larger values of $\widetilde{k}$. Also, for all three values of $\widetilde{k}$, the angular velocity begins to decrease at a point in time. This result suggests that the length of the rod is increasing over time. If you consider the endpoints to be the fastest point along our rod, then as their velocity (which we know to be ωR) remains constant but their average angular velocity ω decreases, then the radius R should increase. This is consistent with our analysis from Section 3.4. We will return to discuss this more when we consider the time-evolution of the accelerating rod in Section 3.5.2.

Coefficient of Variation for Angular Velocity

As you can see in Figure 3.7, our system with no torque has very little variation in angular velocity across the rod. The coefficient of variation in angular velocity is 5.22×10^{-4}. This stays constant over time as we would expect. The system where torque has been added shows that the standard deviation is at first a bit scattered, likely because the system is trying to equilibrate as the torque begins to be applied. The steepness of the curves shown in Figure 3.7 seem to be affected by increasing $\widetilde{k}$. The larger coefficient of variation for the torqued system, approximately 0.114, suggests that the system is less rigid than the system where no torque has been applied.

3.5.2 Time Evolution of the Motion

In Figure 3.8, we see that the rod rotates while maintaining its rigidity. In Figure 3.9, we can see that initially the rod loses rigidity due to the torque being applied. This is particularly visible in the uppermost center frame. This result is consistent with the initial change in standard deviation shown in Figure 3.7. For the rest of the evolution the motion of the system appears fairly rigid, even though the coefficient of variation for the angular velocity of our system is relatively large. This discrepancy could be due to the extension of the length of the rod over time.

In Figure 3.8, we can see that the length of the rod stays constant throughout the rotation. We can particularly see this from the uppermost left frame and the bottommost left frame. This is not true for the system where torque is present. This is particularly visible in the uppermost left frame and the bottommost left frame of Figure 3.9. This result is consistent with our findings from Section 3.4. The length of the rod in the torqued system, where linear velocities approach the speed of light, is extended in comparison to the length of the rod in the torque free system, where all linear velocities are not close to the speed of light.

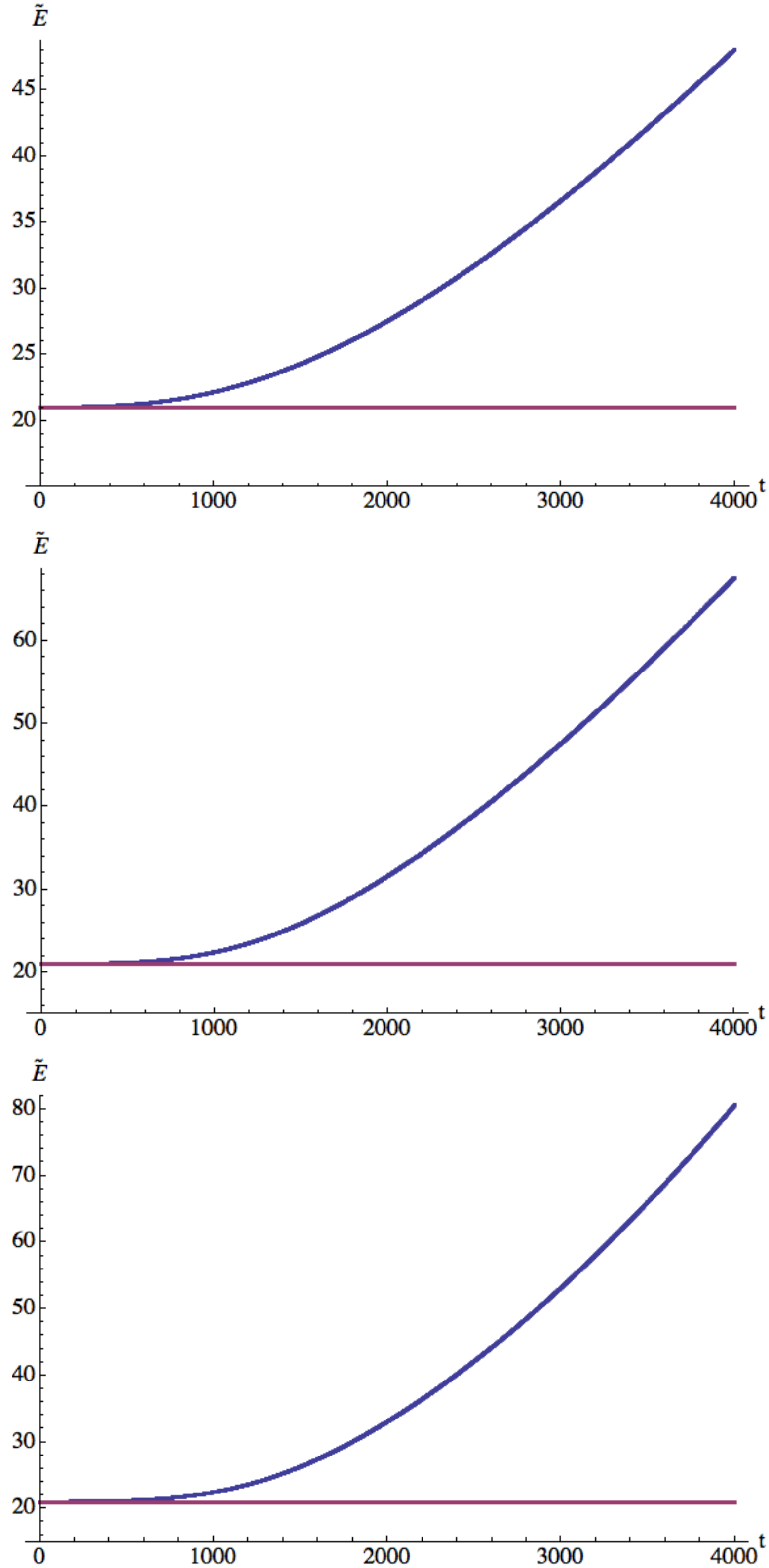

Figure 3.4: Energy of the no torque (red) and time-dependent torque (blue) systems for varying $\widetilde{k}$. From the top to the bottom, $\widetilde{k} = 10$, $\widetilde{k} = 100$, and $\widetilde{k} = 1000$.

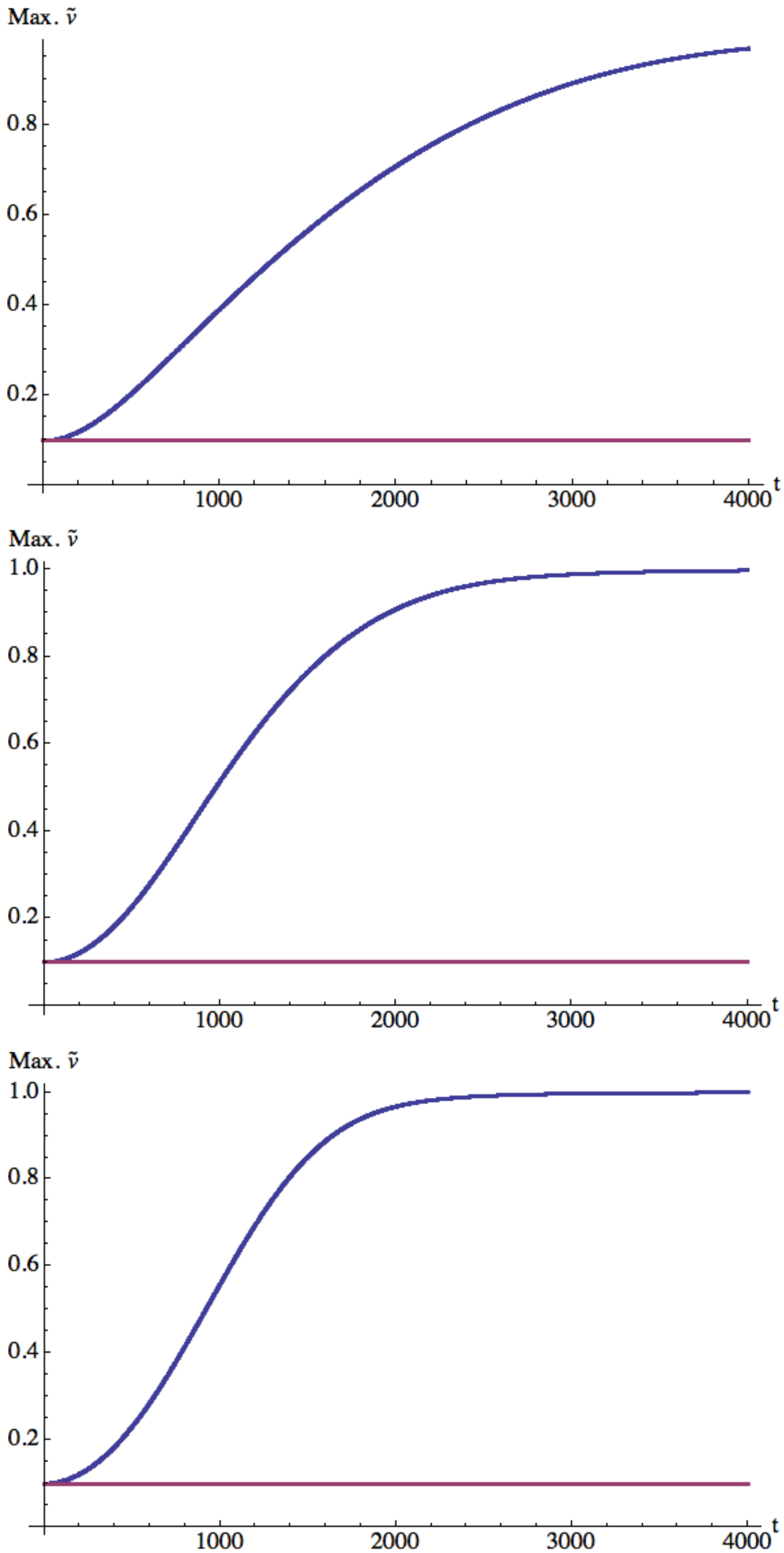

Figure 3.5: Maximum linear speed of a point on the rod for the no torque (red) and time-dependent torque (blue) systems for varying $\widetilde{k}$. From the top left to the bottom right, $\widetilde{k} = 10$, $\widetilde{k} = 100$, and $\widetilde{k} = 1000$.

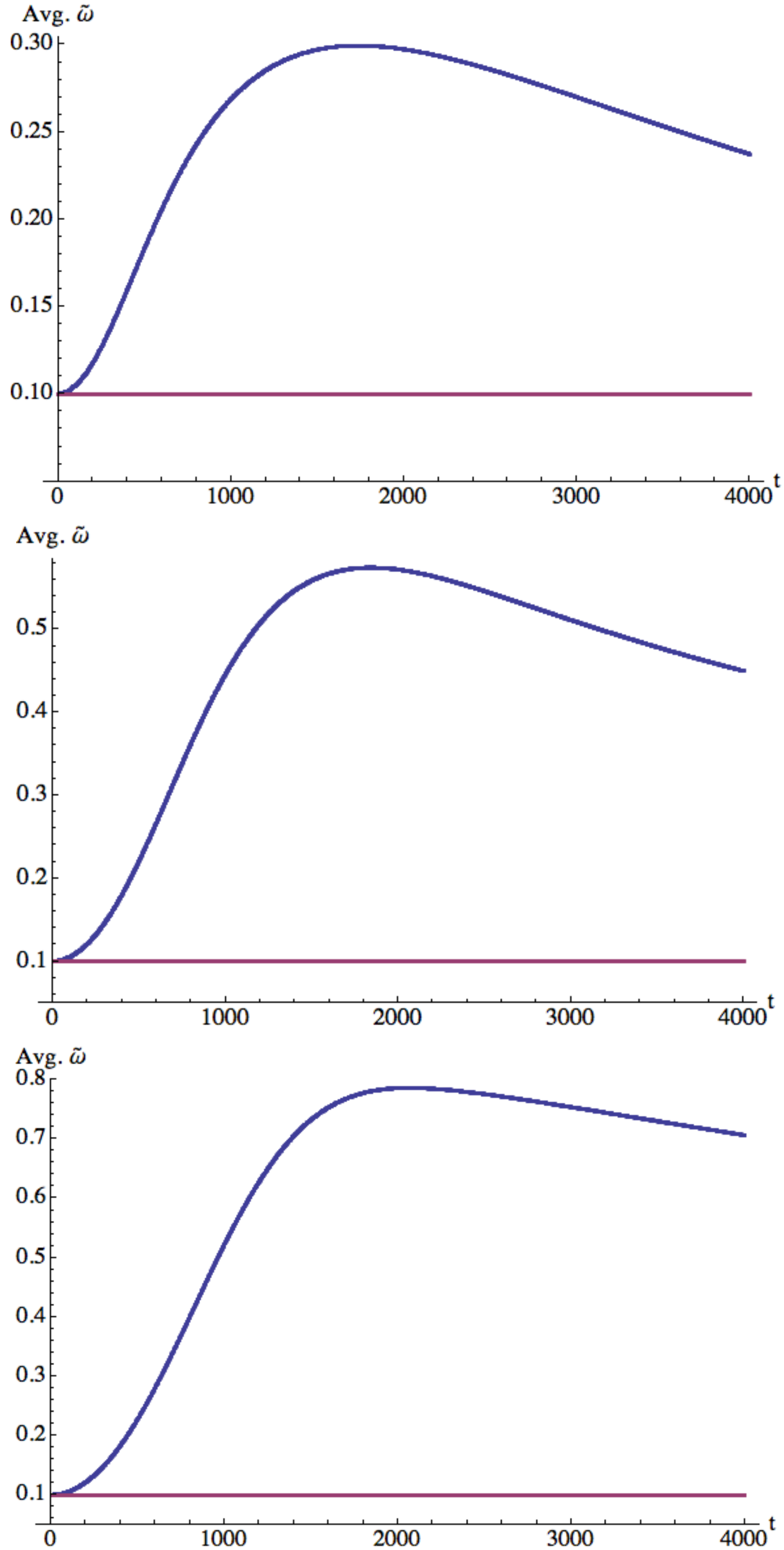

Figure 3.6: Average angular velocity of all points on the rod for the no torque (red) and time-dependent torque (blue) systems for varying $\widetilde{k}$. From the top to the bottom, $\widetilde{k} = 10$, $\widetilde{k} = 100$, and $\widetilde{k} = 1000$.

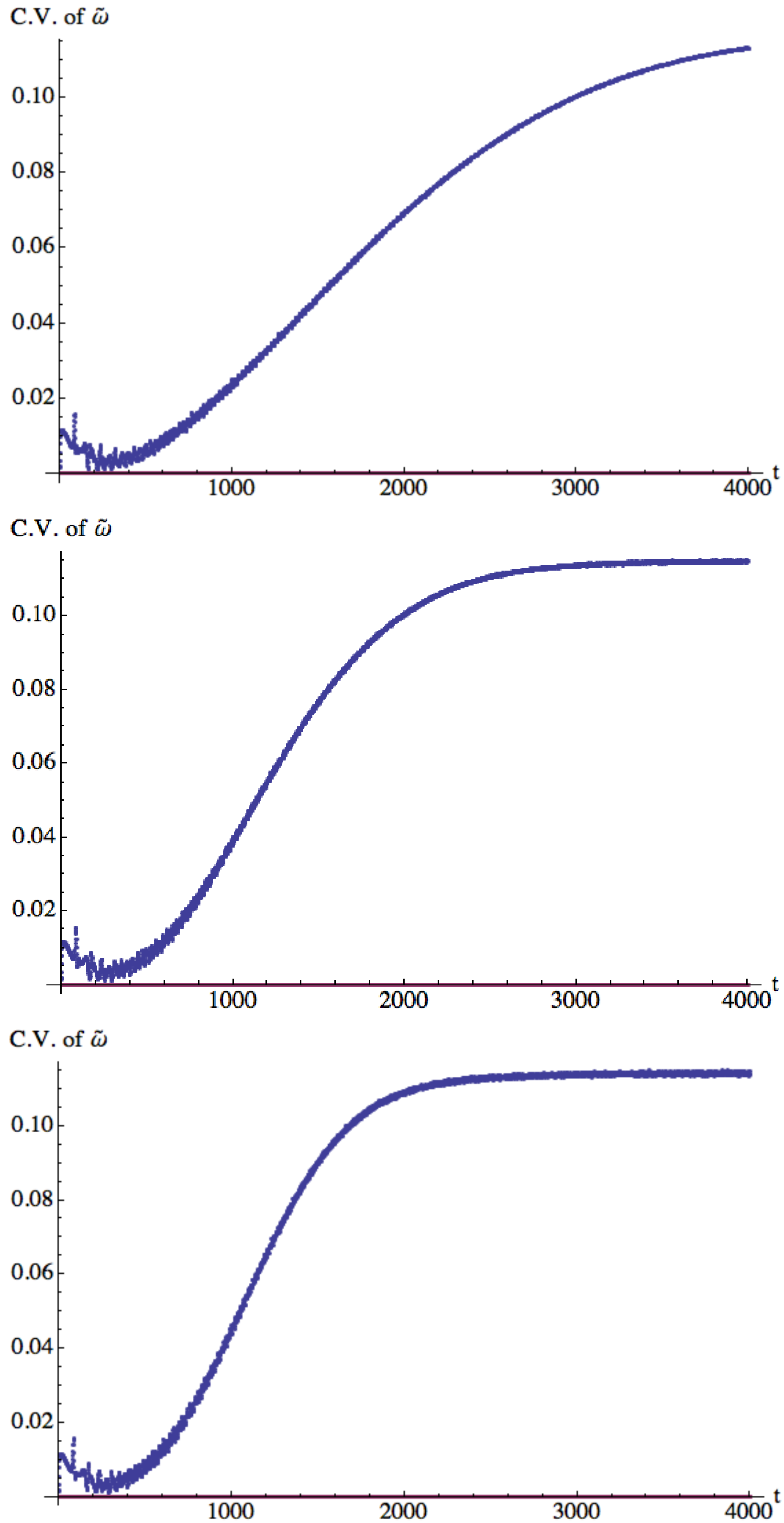

Figure 3.7: The standard deviation in angular velocity over the average angular velocity for all points on the rod for the no torque (red) and time-dependent torque (blue) systems for varying $\widetilde{k}$. From the top to bottom, $\widetilde{k} = 10$, $\widetilde{k} = 100$, and $\widetilde{k} = 1000$.

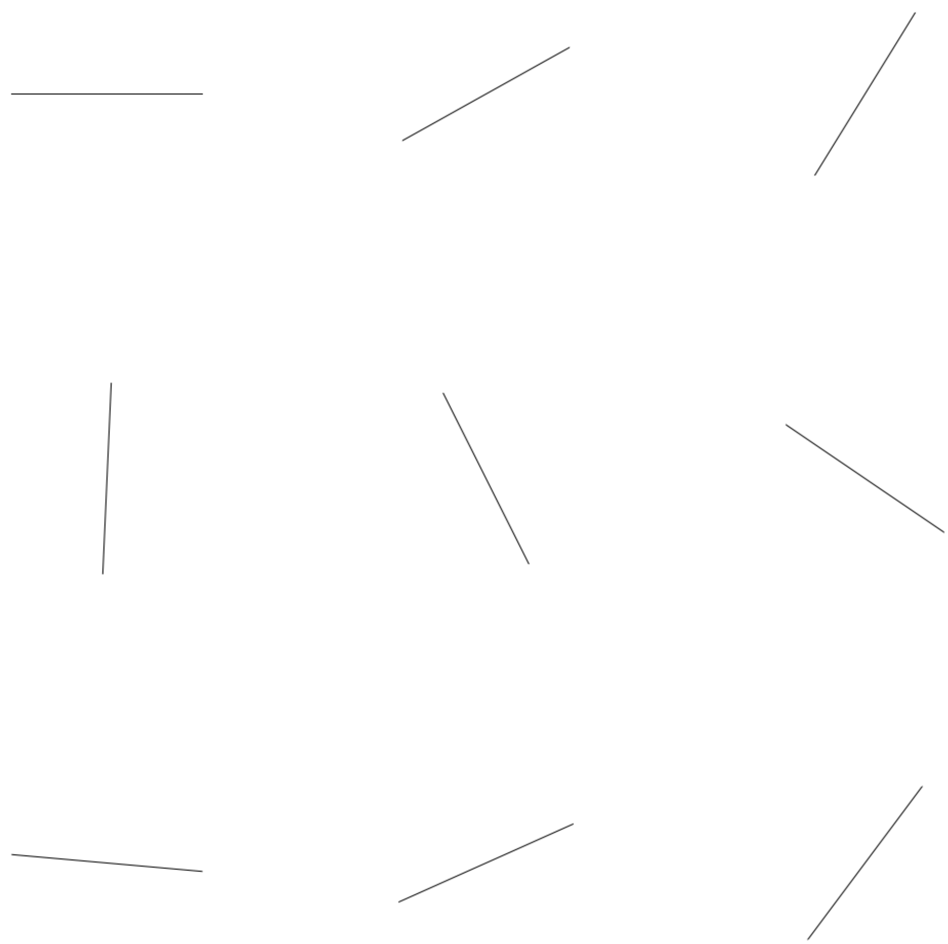

Figure 3.8: Evenly spaced snapshots describing the motion of our torque-free system.

Figure 3.9: Evenly spaced snapshots describing the motion of our torqued system

Conclusion

In this thesis we set out to investigate the relativistic N-coupled oscillator in two different contexts. In Chapters 1 and 2, the equation of motion for a relativistic string was derived, and it was shown that our model in three-dimensions, parameterized by arc length and coordinate time, looks like a relativistic string. The fact that the one-dimensional model, with purely longitudinal motion, did not behave like a string makes sense because our derivation in Chapter 1 showed that there was no way to observe longitudinal motion along a relativistic string. The implication of this work is that there is more than one way to think about the motion of a relativistic string. The second context in which we explored relativistic N-coupled oscillators was in consideration of Ehrenfest's paradox. In Chapter 3, we developed a relativistically-consistent model of a rotating rigid rod, and saw that our numerical findings for position, momenta, energy, linear velocity, and angular velocity were as expected. At non-relativistic velocities, our rigid rod maintained rigidity, and at relativistic velocities, our rod lost its rigidity as its length extended. Future work could involve considering the effect on rigidity due to varying the number of masses in our system. Also more work could be done varying the spring constant over a wider range of values, or considering a different time-dependent torque.

Appendix A

Important Pieces of Code

A.1 Energy

```
Energy[xout_, pout_] := Module[{energy, j},
  energy = 0.0;
  For[j = 2, j ≤ Length[pout] - 1, j = j + 1,
   energy = energy + Sqrt[1 + pout[[j]].pout[[j]]] +
      (1 / 2) k
       ( Sqrt[(xout[[j + 1]] - xout[[j]]).(xout[[j + 1]] - xout[[j]])] - a) ^2 +
      (1 / 2) k
       ( Sqrt[(xout[[j]] - xout[[j - 1]]).(xout[[j]] - xout[[j - 1]])] - a) ^2;
  ];
  j = 1;
  energy = energy + Sqrt[1 + pout[[j]].pout[[j]]] +
    (1 / 2) k ( Sqrt[(xout[[j + 1]] - xout[[j]]).(xout[[j + 1]] - xout[[j]])] - a) ^
      2;
  j = Length[xout];
  energy = energy + Sqrt[1 + pout[[j]].pout[[j]]] +
    (1 / 2) k ( Sqrt[(xout[[j]] - xout[[j - 1]]).(xout[[j]] - xout[[j - 1]])] - a) ^
      2;
  Return[energy];
 ]
```

A.2 Angular Frequency

```
Omega[xout_, pout_] := Module[{ω, xoutm, poutm, ll, i, j},
  xoutm = Delete[xout, 11];
  poutm = Delete[pout, 11];
  ll = Table[Cross[xoutm[[j]] / Sqrt[xoutm[[j]].xoutm[[j]]], poutm[[j]]],
    {j, 1, Length[xoutm]}];
  ω = Table[Sqrt[ll[[j]].ll[[j]]] / Sqrt[xoutm[[j]].xoutm[[j]]] /
     Sqrt[1 + ll[[j]].ll[[j]]], {j, 1, Length[xoutm]}];
  Return[ω];
 ]
```

A.3 Verlet

```
Verlet[x0_, p0_, dt_, Nt_] :=
 Module[{xout, pout, vout, x0prev, v0, Forces, xnext, pnext, xcur,
   pcur, xprev, pprev, t, index, jndex, rn, rp},
  xout = Table[0., {j, 1, Nt}];
  pout = Table[0., {j, 1, Nt}];
  vout = Table[0., {j, 1, Nt}];

  xcur = x0;
  pcur = p0;
  v0 = Table[p0[[j]] / Sqrt[1 + p0[[j]].p0[[j]]], {j, 1, Length[x0]}];

  xprev = x0 - dt * v0;
  pprev = p0 - dt * Fout[x0, 0];
  xnext = x0;
  pnext = p0;
  t = 0.0;

  For[index = 1, index ≤ Nt, index = index + 1,
   jndex = 1;
   Forces = Fout[xcur, t];
   xnext[[jndex]] =
    xprev[[jndex]] +
     (2 * dt) * (pcur[[jndex]] / Sqrt[1 + (pcur[[jndex]].pcur[[jndex]])]);
   rn = xcur[[jndex]] - xcur[[jndex + 1]];
   pnext[[jndex]] = pprev[[jndex]] + 2 dt Forces[[jndex]];

   For[jndex = 2, jndex ≤ (Length[x0] - 1) , jndex = jndex + 1 ,
    xnext[[jndex]] =
     xprev[[jndex]] +
      (2 * dt) * (pcur[[jndex]] / Sqrt[1 + (pcur[[jndex]].pcur[[jndex]])]);
    pnext[[jndex]] = pprev[[jndex]] + 2 dt Forces[[jndex]];
   ];
   jndex = Length[x0];
   xnext[[jndex]] =
    xprev[[jndex]] +
     (2 * dt) * (pcur[[jndex]] / Sqrt[1 + (pcur[[jndex]].pcur[[jndex]])]);
   pnext[[jndex]] = pprev[[jndex]] + 2 dt Forces[[jndex]];
   xout[[index]] = xcur;
   pout[[index]] = pcur;
   vout[[index]] = Table[pcur[[j]] / Sqrt[1 + pcur[[j]].pcur[[j]]],
     {j, 1, Length[pcur]}];

   xprev = xcur;
   xcur = xnext;
   pprev = pcur;
   pcur = pnext;

   t = t + dt;
  ];
  Return[{xout, pout, vout}];
 ]
```

A.4 Forces

```
Fout[xin_, tin_] := Module[{Fout, index, jndex, rn, rp, Fhat},
  Fout = xin;
  jndex = 1;
  rn = xin[[jndex]] - xin[[jndex + 1]];
  Fout[[jndex]] = (k) * (- (Sqrt[rn.rn] - a) rn / Sqrt[rn.rn]);

  For[jndex = 2, jndex ≤ (Length[x0] - 1) , jndex = jndex + 1 ,
   rn = xin[[jndex]] - xin[[jndex + 1]];
   rp = xin[[jndex - 1]] - xin[[jndex]];
   Fout[[jndex]] =
    (k) * (- (Sqrt[rn.rn] - a) rn / Sqrt[rn.rn] +
       (Sqrt[rp.rp] - a) rp / Sqrt[rp.rp]);

   If[(jndex == Floor[Length[x0] / 2] || jndex == Ceiling[Length[x0] / 2] + 1),
    Fhat = Cross[{0, 0, 1.}, xin[[jndex]]];
    Fhat = Fhat / Sqrt[Fhat.Fhat];
    Fout[[jndex]] =
     Fout[[jndex]] + (Torque[tin] / Sqrt[xin[[jndex]].xin[[jndex]]] ) Fhat;
   ];

  ];
  jndex = Length[xin];
  rp = xin[[jndex - 1]] - xin[[jndex]];
  Fout[[jndex]] = (k) * ((Sqrt[rp.rp] - a) rp / Sqrt[rp.rp]);

  Return[Fout];
 ]
```

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