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Magnetic Helicity Fluxes from Triple Correlators

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Abstract

Fluxes of the magnetic helicity density play an important role in large-scale turbulent dynamos, allowing the growth of large-scale magnetic fields while overcoming catastrophic quenching. We show here, analytically, how several important types of magnetic helicity fluxes can arise from terms involving triple correlators of fluctuating fields in the helicity density evolution equation. For this, we assume incompressibility and weak inhomogeneity, and use a quasi-normal closure approximation: fourth-order correlators are replaced by products of second-order ones, and the effect of the fourth-order cumulants on the evolution of the third moments is modeled by a strong damping term. First, we show how a diffusive helicity flux, until now only measured in simulations, arises from the triple correlation term. This is accompanied by what we refer to as a *random advective flux*, which predominantly transports magnetic helicity along the gradients of the random fields. We also find that a new helicity flux contribution, in some aspects similar to that first proposed by Vishniac, can arise from the triple correlator. This contribution depends on the gradients of the random magnetic and kinetic energies along the large-scale vorticity, and thus arises in any rotating, stratified system, even if the turbulence is predominantly non-helical. It can source a large-scale dynamo by itself while spatially transporting magnetic helicity within the system.

Unified Astronomy Thesaurus concepts: Galaxy magnetic fields (604); Cosmic magnetic fields theory (321); Magnetohydrodynamics (1964); Astrophysical magnetism (102)

1. Introduction

Dynamo theory studies the spontaneous amplification and maintenance of magnetic fields by electromagnetic induction due to the motion of a conducting fluid. Astrophysical magnetic fields in stars and galaxies are observed to be ordered on scales much larger than the integral scale of the fluid turbulence that is partially responsible for their maintenance (Brandenburg & Subramanian 2005; Beck et al. 2019). Large-scale (or mean-field) turbulent dynamo theory attempts to understand the mechanisms behind the generation and sustenance of such fields, which typically involves some form of breaking of mirror symmetry. In the standard picture, differential rotation generates toroidal magnetic fields from poloidal ones, while helical motions regenerate the poloidal field from the toroidal field, by what is referred to as the α-effect (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005; Rincon 2019; Shukurov & Subramanian 2021; Tobias 2021).

Dynamos saturate when Lorentz forces due to the generated magnetic fields' backreaction on the velocity fields driving the dynamo. In mean-field helical dynamos, this saturation is strongly constrained by the near conservation of magnetic helicity (a measure of links, twists, and writhing of the field). In such dynamos, the generation of a large-scale magnetic field and the associated magnetic helicity is accompanied by the concurrent transfer of an equal amount of magnetic helicity of the opposite sign to small scales. This transfer is accomplished by the turbulent electromotive force (EMF), the cross correlation between the fluctuating velocity and magnetic fields (see below). Such a *bihelical* field has indeed been found in observations and simulations (Blackman & Brandenburg 2003; Singh et al. 2018).

Although equal and oppositely signed magnetic helicities accumulate at large and small scales as the mean field grows, the Lorentz force is dominated by smaller scales. In simple closures, one can show that this leads to a small-scale current helicity that opposes the effect of helical motions by decreasing the kinetic α-effect. A drastic suppression of the kinematic α-effect, which would also suppress mean-field growth, is referred to as *catastrophic* quenching (Cattaneo & Vainshtein 1991; Vainshtein & Cattaneo 1992). This was attributed to helicity conservation by Gruzinov & Diamond (1994); Bhattacharjee & Yuan (1995). Thus, one requires some mechanism for shedding the accumulated small-scale helicity. Resistive dissipation is too slow. However, fluxes of unwanted small-scale helicity out of the dynamo active region can allow the mean-field dynamo to grow on timescales much smaller than the ohmic timescale (Blackman & Field 2000; Brandenburg & Subramanian 2005).

A number of such fluxes have been derived by simplifying various terms in the evolution equation for the small-scale magnetic helicity (Kleeorin et al. 2000; Vishniac & Cho 2001; Subramanian & Brandenburg 2004, 2006; Vishniac & Shapovalov 2014; Shukurov & Subramanian 2021; Kleeorin & Rogachevskii 2022). They are used in dynamical-quenching models, where the mean-field dynamo equation is solved together with the evolution equation for the current helicity, to show how quenching of the large-scale dynamo can be alleviated (Kleeorin et al. 2000; Sur et al. 2007; Chamandy et al. 2013a, 2015). The helicity fluxes considered so far mostly require either large-scale outflows, or a large-scale magnetic field, to operate. Astrophysical systems need not host such outflows. Further, a flux that depends on the mean magnetic field is not expected to be important in the initial stages of dynamo action. There is thus a need to explore the possibility of other fluxes of magnetic helicity that can alleviate catastrophic quenching or even lead to new generative effects.

One helicity flux term that has so far not been studied as systematically as other terms involves a triple correlation of small-scale fields. Its simplification requires a closure approximation, and its study is the main focus of this paper. We show that simplification of this triple correlation term leads to several important helicity fluxes. First, it leads to a diffusive flux of the magnetic helicity, so far only postulated (as a flux of current helicity) based on heuristic arguments (e.g., Covas et al. 1998; Kleeorin et al. 2002) and measured in direct numerical simulations (Mitra et al. 2010). We find that this flux comes together with an advective flux that depends on the gradients of the random fields. We also show that a helicity flux contribution, similar to that first proposed by Vishniac (2012), and depending only on small-scale field gradients and large-scale vorticity, can arise from this triple correlator term. Indeed, this latter flux could potentially drive a large-scale dynamo by itself, even without the presence of kinetic helicity.

In Section 2, we set up the mean-field formalism and describe the evolution equation for the small-scale magnetic helicity. In Section 3, we simplify the triple correlator using a closure approximation, presenting a partially simplified expression (which nevertheless already displays all the relevant effects) in Section 3.1. In Section 3.2, we apply a further approximation to simplify some nonlocal terms involving the pressure and the scalar potential. The detailed expressions for many of the correlators involved are left to the appendices. In Section 4, we discuss some physical implications of the fluxes we have obtained. Finally, Section 5 presents our conclusions.

2. Magnetic Helicity: Evolution and Fluxes

Let us consider an incompressible fluid, and use both overbars and angle brackets to denote an average that follows Reynolds rules. The magnetic field, $\boldsymbol{B}$, evolves according to the induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}), \tag{1}$$

where $\boldsymbol{V}$ is the velocity and η is the resistivity. We now average Equation (1) after splitting all quantities into their mean and fluctuating parts (e.g., $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$ with $\langle \boldsymbol{b} \rangle = 0$). Here, fluctuating quantities are denoted by lowercase letters. The evolution equation for the mean magnetic field is then

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times (\overline{\boldsymbol{V}} \times \overline{\boldsymbol{B}} + \mathcal{E} - \eta \nabla \times \overline{\boldsymbol{B}}), \tag{2}$$

where $\mathcal{E} \equiv \langle \boldsymbol{v} \times \boldsymbol{b} \rangle$ is called the turbulent EMF.

Expressing $\mathcal{E}$ in terms of the mean fields themselves is a closure problem. Assuming a sufficient scale separation between the mean and fluctuating fields, $\mathcal{E}$ is expanded in terms of the mean magnetic field and its derivative. For statistically homogeneous and isotropic small-scale fields, one then gets

$$\mathcal{E} = \alpha \overline{\boldsymbol{B}} - \eta_t \nabla \times \overline{\boldsymbol{B}}. \tag{3}$$

One can use the quasilinear approximation to show that α is related to the kinetic helicity of the fluctuating velocity field when Lorentz forces are negligible, while η_t, the turbulent resistivity, depends on its energy density (Moffatt 1978). A nonzero α allows a mean magnetic field to grow and be sustained even without a mean velocity field.

Subtracting Equation (2) from Equation (1), we obtain the evolution equation for the fluctuating magnetic field:

$$\frac{\partial b_i}{\partial t} = -\epsilon_{ijk} \partial_j e_k, \tag{4}$$

where

$$e_k \equiv -\epsilon_{klm} \overline{V}_l b_m - \epsilon_{klm} v_l \overline{B}_m - \epsilon_{klm} v_l b_m + \mathcal{E}_k + \eta \epsilon_{klm} \partial_l b_m, \tag{5}$$

so that the fluctuating electric field is given by $\boldsymbol{e}/c$. The fluctuating vector potential (which we denote by $\boldsymbol{a}$) then evolves according to

$$\frac{\partial a_k}{\partial t} = -e_k + \partial_k \varphi, \tag{6}$$

where φ is a scalar potential (our definition differs from the usual one by a negative sign). The evolution of the random velocity field $\boldsymbol{v}$ is given by

$$\frac{\partial v_i}{\partial t} = -\overline{V}_j \partial_j v_i - v_j \partial_j \overline{V}_i - v_j \partial_j v_i + \langle v_j \partial_j v_i \rangle - \frac{\partial_i p}{\rho} + \frac{\overline{B}_j \partial_j b_i}{4\pi\rho} + \frac{b_j \partial_j \overline{B}_i}{4\pi\rho} + \frac{b_j \partial_j b_i}{4\pi\rho} - \frac{\langle b_j \partial_j b_i \rangle}{4\pi\rho} + \nu \partial_j \partial_j v_i. \tag{7}$$

Here, ρ is the density (taken as constant assuming an incompressible flow), ν is the viscosity, and p is the pressure determined using $\nabla \cdot \boldsymbol{v} = 0$ in Equation (7).

We define the magnetic helicity density as $h^b \equiv \langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$, and work in the Coulomb gauge, with $\nabla \cdot \boldsymbol{a} = 0$. This implies that $\nabla^2 \varphi = \partial_k e_k$. Using Equations (4) and (6), we can write the time derivative of the magnetic helicity density as

$$\frac{\partial h^b}{\partial t} = -2\langle e_i b_i \rangle + \langle b_i \partial_i \varphi \rangle - \partial_j \langle \epsilon_{jki} e_k a_i \rangle. \tag{8}$$

Substituting Equation (5) into the above, we write

$$\begin{aligned}\frac{\partial h^b}{\partial t} = &-2\,\mathcal{E}\cdot\overline{\boldsymbol{B}} - 2\,\eta\langle \boldsymbol{b}\cdot(\nabla\times\boldsymbol{b})\rangle + \langle b_i\partial_i\varphi\rangle + \overbrace{\partial_j(\langle \epsilon_{jki}\epsilon_{klm}b_m a_i\rangle \overline{V}_l)}^{\text{advective}} \\ &+ \underbrace{\partial_j(\langle \epsilon_{jki}\epsilon_{klm}v_l a_i\rangle \overline{B}_m)}_{\text{A+VC}} + \underbrace{\partial_j\langle \epsilon_{jki}\epsilon_{klm}v_l b_m a_i\rangle}_{\text{triple}-\text{correlator}} - \partial_j(\eta\langle \epsilon_{jki}a_i\epsilon_{klm}\partial_l b_m\rangle).\end{aligned} \tag{9}$$

The first term above transfers magnetic helicity between the mean and fluctuating fields, while the second describes resistive dissipation of magnetic helicity density. The term marked *advective* corresponds to the advection of the magnetic helicity by the mean velocity field in the isotropic limit (Subramanian & Brandenburg 2006). The term marked A+VC can be split into two parts, which are known as the *antisymmetric* (Kleeorin et al. 2000) and Vishniac–Cho (VC) fluxes (Vishniac & Cho 2001; Subramanian & Brandenburg 2006). These are dependent on the mean magnetic field, and are thus unimportant when the mean magnetic field is weak (i.e., in the initial stages of dynamo action). The term marked triple correlator requires a closure hypothesis to evaluate, and has not yet been studied. The last term in Equation (9) is a resistive flux, expected to be negligible at high magnetic Reynolds numbers, Rm. Finally, $\langle b_i \varphi \rangle$ is a nonlocal term that gives contributions similar to all the above fluxes.

Suppose the turbulent quantities are statistically steady, and $\partial h^b / \partial t = 0$, in a time-averaged sense. Then Equation (9) implies

$$\mathcal{E}\cdot\overline{\boldsymbol{B}} = -\eta\langle \boldsymbol{b}\cdot(\nabla\times\boldsymbol{b})\rangle - \frac{1}{2}\nabla\cdot(\text{fluxes}). \tag{10}$$

In the absence of magnetic helicity fluxes, $\mathcal{E}\cdot\overline{\boldsymbol{B}}$ is dominated by the resistive term, and becomes negligible at high Rm. The mean-field dynamo then becomes catastrophically quenched in the absence of helicity fluxes. Thus, helicity fluxes are crucial for the growth of the mean magnetic field over timescales shorter than the resistive one.

In this paper, we will focus on the hitherto unevaluated contribution to Equation (9) marked as a triple correlator, which arises from the $-\boldsymbol{v}\times\boldsymbol{b}$ part of $\boldsymbol{e}$, and also similar terms that arise from $b_i\varphi$.

3. Evaluation of the Flux Due to the Triple Correlator

3.1. Simplification of the Local Terms

The term in Equation (5) for the small-scale electric field $\boldsymbol{e}$ that we are interested in is $-\boldsymbol{v}\times\boldsymbol{b}$. Substituting this in Equation (8), the triple correlator contribution to helicity density evolution is given by

$$\left(\frac{\partial h^b}{\partial t}\right)_{\text{triple}} = \underbrace{\partial_j\langle v_i b_j a_i\rangle}_{I_1^{\text{triple}}} - \underbrace{\partial_j\langle v_j b_i a_i\rangle}_{I_2^{\text{triple}}}. \tag{11}$$

The two terms I_1^{triple} and I_2^{triple} contain triple correlators involving the random velocity, magnetic, and vector potential fields. These are evaluated in Appendix B by the following steps:

1. We write the evolution equations for the triple correlators by taking their time derivatives. These contain fourth-order correlators along with triple correlators involving the pressure or the scalar potential.
2. We assume that the fourth-order correlators can be written in terms of second-order correlators (which would be the case if the turbulent fields were all Gaussian) plus a damping term with some characteristic timescale τ, that models the irreducible fourth-order cumulant. The fourth-order correlator also involves the product of third- and first-order correlators, but this vanishes as fluctuations have zero mean. This is similar to the eddy damped quasi-normal closure (Lesieur 2008). We further assume that this damping timescale is much smaller than the slower evolution timescales of various averaged quantities we are interested in, so that we can drop the time derivatives in the evolution equations for the triple correlators.
3. Contributions to the triple correlation flux involving the EMF and the mean magnetic field are neglected. These terms are negligible in the initial stages of dynamo action, when an appreciable large-scale field has not yet developed. Terms involving η or ν are neglected since the magnetic and fluid Reynolds numbers are high in most astrophysical systems.

4. At this stage, one has an expression for the triple correlators in terms of the double correlators of various fields (except for nonlocal terms that involve p or φ). To simplify these double correlators, we use the expressions in Appendix A.1, which are from Roberts & Soward (1975). These correspond to assuming incompressibility and weak inhomogeneity. The resulting double correlators are listed in Appendix A.2. They match those derived using an alternative method proposed by Vainshtein & Zel'dovich (1972).

Substituting the final expressions for I_1^{triple} from Equation (B7) and I_2^{triple} from Equation (B8) into Equation (11), we obtain Equation (B9). There are diffusive and advective fluxes, along with terms similar to a flux previously proposed by Vishniac (2012, 2015). The physical interpretation of these fluxes will be discussed in Section 4. There are also some unsimplified terms involving the pressure or scalar potential. These terms may modify the coefficients and properties of the fluxes, and so we now deal with them.

3.2. Simplification of the Nonlocal Terms

The evaluation of these terms is more involved, as they depend on an integral operator (the inverse of the Laplacian) acting on the correlators between fields at spatially separated points. We approximate these terms by replacing nonlocal integrals with appropriate powers of an eddy correlation length, which we denote by λ (Vishniac & Cho 2001). Also, for simplicity, we take the same constant λ for all correlators. For example, to estimate a term of the form $-\langle \partial_a[\nabla^{-2}(f(\boldsymbol{x}))]g(\boldsymbol{x})\rangle$, one could write it as

$$\int \mathrm{d}\boldsymbol{y}\, \frac{y_a - x_a}{4\pi|\boldsymbol{x} - \boldsymbol{y}|^3}\langle f(\boldsymbol{y})\, g(\boldsymbol{x})\rangle = \int \mathrm{d}\boldsymbol{r}\, \frac{r_a}{4\pi|\boldsymbol{r}|^3}\langle f(\boldsymbol{x} + \boldsymbol{r})\, g(\boldsymbol{x})\rangle \tag{12}$$

$$\approx \int \mathrm{d}\boldsymbol{r}\, \frac{r_a}{4\pi|\boldsymbol{r}|^3}\left\langle g(\boldsymbol{x})\left(f(\boldsymbol{x}) + r_b\frac{\partial f(\boldsymbol{x})}{\partial r_b} + \frac{r_a r_b}{2}\frac{\partial^2 f}{\partial r_a \partial r_b} + \dots\right)\right\rangle \approx \frac{\lambda\delta_{ab}}{3}\left\langle g(\boldsymbol{x})\frac{\partial f(\boldsymbol{x})}{\partial x_b}\right\rangle + \mathcal{O}(\lambda^3). \tag{13}$$

This approximation is applied in Appendix C.1.1 and Appendix C.2 to evaluate the nonlocal terms in Equation (B9) involving the pressure and scalar potential, respectively. The VC approximation is also used to evaluate the nonlocal term $\langle b_i\varphi\rangle$ in Appendix C.2.2. For the latter, we keep only contributions to $\boldsymbol{e}$ from $-\overline{\boldsymbol{V}} \times \boldsymbol{b}$ and $\boldsymbol{v} \times \boldsymbol{b}$, neglecting again the terms depending on the mean magnetic field and resistivity. The calculation involves evaluating several nonlocal triple correlators, and the result is given by Equation (C27). Some of the terms here are similar to those resulting from the advective flux, which itself is calculated in Appendix D.

We use $-\partial_j(F_j^T)$ to denote the total contributions to $\partial h^b/\partial t$ (Equation (9)) from triple correlations, the $\langle b_i\phi\rangle$ and advection terms. This expression is quite cumbersome, and is given in Appendix E (except for terms that are higher-order, $\mathcal{O}(\tau^2)$, corrections to the advective flux). To make this expression more tractable, we replace $\lambda^2\langle\omega^2\rangle = \langle v^2\rangle$, $\lambda^2\langle j^2\rangle = \langle b^2\rangle$, $\lambda^2\langle b^2\rangle = \langle a^2\rangle$, $\lambda^2\int \mathrm{d}k\, 8\pi k^4 N(k, \boldsymbol{R}) = h^c$, and $\lambda^2\int \mathrm{d}k\, 8\pi k^4 F(k, \boldsymbol{R}) = h^v$. This assumes that there is a dominant scale λ in all the spectra. If we are interested only in regimes where the kinetic α-effect contributions in the helicity flux are minimal, we can drop terms involving h^v. Then, we get a total helicity flux $\boldsymbol{F}^T$, where

$$\begin{aligned}
-\partial_j(F_j^T) = \partial_j\Bigg[& \overbrace{\frac{7\tau}{27}\langle v^2\rangle\partial_j h^b + \frac{7\tau}{27}\frac{\langle b^2\rangle}{4\pi\rho}\partial_j h^b}^{\text{diffusion}} - \overbrace{\frac{\tau}{18}h^b\partial_j\langle v^2\rangle - \frac{7\tau}{27}\frac{1}{4\pi\rho}h^b\partial_j\langle b^2\rangle}^{\text{random advection}} \\
& - \overbrace{\frac{13\tau^2}{135}\frac{1}{4\pi\rho}\epsilon_{jkl}\overline{V}_l\partial_k(h^b h^c) - \frac{\tau^2}{18}\frac{1}{4\pi\rho}\epsilon_{jkl}\overline{V}_l h^c\partial_k h^b}^{\text{random advection}} + \overbrace{\frac{7\tau^2}{45}\frac{1}{4\pi\rho}\epsilon_{jkl}\overline{V}_l\langle b^2\rangle\partial_k\langle b^2\rangle}^{\text{NV}} \\
& - \underbrace{\frac{203\tau^2}{5400}\epsilon_{jkl}\overline{V}_l\langle b^2\rangle\partial_k\langle v^2\rangle + \frac{403\tau^2}{8100}\epsilon_{jkl}\overline{V}_l\langle v^2\rangle\partial_k\langle b^2\rangle - \frac{\lambda^2}{6}\epsilon_{jkl}\overline{V}_l\partial_k\langle b^2\rangle}_{\text{NV}} - \overline{V}_j h^b\Bigg].
\end{aligned} \tag{14}$$

4. Implications of the Obtained Fluxes

We write the total helicity flux $\boldsymbol{F}_{\mathrm{T}}$ in Equation (14) as a sum of a diffusive flux $\boldsymbol{F}_{\mathrm{D}}$, random advective flux $\boldsymbol{F}_{\mathrm{RA}}$, advective flux $\boldsymbol{F}_{\mathrm{A}}$, and NV flux $\boldsymbol{F}_{\mathrm{NV}}$. The terms which involve $h^b h^c$ also contribute to these fluxes. We note that the terms that involve the density ρ arise from the small-scale Lorentz force. We now discuss these fluxes in turn.

4.1. The Diffusive and Advective Fluxes

Diffusive fluxes of the magnetic helicity have been used in models of the mean-field dynamo, which include a dynamical equation for the current helicity of the small-scale fields (Covas et al. 1998; Kleeorin et al. 2002). Such a flux has been measured in direct numerical simulations (Mitra et al. 2010). In some situations, it has been found that the inclusion of such a diffusive term in a mean-field model helps overcome quenching of the dynamo (Brandenburg et al. 2009). This flux is derived here for the first time, and results from the triple correlation contribution to the helicity fluxes. In many physical contexts, random motions can lead to not only diffusion, but also advection due to their gradients. This is also true here, where we see from Equation (14) an advective flux that depends on gradients of $\langle v^2\rangle$ and $\langle b^2\rangle$, along with a term involving $\boldsymbol{V}$ and h^c. The explicit form of these two fluxes is

$$\boldsymbol{F}_{\rm D} = -\left(\frac{7\tau}{27}\langle v^2\rangle + \frac{7\tau}{27}\frac{\langle b^2\rangle}{4\pi\rho}\right)\nabla h^b,\quad \boldsymbol{F}_{\rm RA} = h^b\left(\frac{\tau}{18}\nabla\langle v^2\rangle + \frac{7\tau}{27}\frac{1}{4\pi\rho}\nabla\langle b^2\rangle + \frac{\tau^2}{18}\frac{1}{4\pi\rho}\left[\overline{\boldsymbol{V}}\times\nabla h^c - \frac{41}{15}h^c\nabla\times\overline{\boldsymbol{V}}\right]\right). \tag{15}$$

The random velocity contributes a diffusion coefficient of order η_t (where $\eta_t = \tau\langle v^2\rangle/3$) in $\boldsymbol{F}_{\rm D}$; at equipartition, the random magnetic field doubles this value. Near equipartition, we also find that the magnetic term dominates the random advection. Comparing $\boldsymbol{F}_{\rm RA}$ with advective flux ($\boldsymbol{F}_{\rm A} = \overline{\boldsymbol{V}}h^b$, the last term of Equation (14)), we note that even when $\overline{\boldsymbol{V}} = 0$, the helicity is advected by an effective velocity which is related to the gradients of the fluctuating fields.

4.2. The New Vishniac Flux

The terms marked NV (stands for the new Vishniac flux) are, in some aspects, similar to a flux previously proposed by Vishniac (2012, 2015).[1] These can be rewritten as $-\nabla\cdot\boldsymbol{F}_{\rm NV}$, where

$$\boldsymbol{F}_{\rm NV} = (\nabla\times\overline{\boldsymbol{V}})\left[C_1\frac{\tau^2}{8\pi\rho}(\langle b^2\rangle)^2 + C_2\tau^2\langle v^2\rangle\langle b^2\rangle + C_4\lambda^2\langle b^2\rangle\right] + \tau^2(C_3 - C_2)(\overline{\boldsymbol{V}}\times\langle v^2\rangle\nabla\langle b^2\rangle), \tag{16}$$

and the constants are given by $(C_1, C_2, C_3, C_4) = (7/45, -203/5400, 403/8100, -1/6)$. We also get

$$\begin{aligned} -\nabla\cdot\boldsymbol{F}_{\rm NV} = &-(\nabla\times\overline{\boldsymbol{V}})\cdot\left[C_1\tfrac{1}{4\pi\rho}\langle b^2\rangle\tau^2\nabla\langle b^2\rangle + C_2\langle b^2\rangle\tau^2\nabla\langle v^2\rangle + C_3\langle v^2\rangle\tau^2\nabla\langle b^2\rangle + C_4\lambda^2\nabla\langle b^2\rangle\right]\\ &-\tau^2(C_3 - C_2)\overline{\boldsymbol{V}}\cdot(\nabla\langle b^2\rangle\times\nabla\langle v^2\rangle). \end{aligned} \tag{17}$$

The expression in Equation (16) is our final result for the NV flux, and Equation (17) is its contribution to the time evolution of the magnetic helicity density. The NV flux $\boldsymbol{F}_{\rm NV}$ has even parity, while $\nabla\cdot\boldsymbol{F}_{\rm NV}$ is explicity parity odd (like h^b).

We note in passing that Kleeorin & Rogachevskii (2022, Equation (18)) calculated helicity fluxes *assuming* all triple correlations combined to contribute a purely diffusive flux governed by η_t. Thus, they do not get any of the other diffusive, RA, or NV terms in Equation (17) that we obtain from triple correlator contributions. They do get an NV flux term similar to the C_4 term that arises from double correlators in the $\langle b_i\phi\rangle$ and advective terms of Equation (9). They retain the effect of the mean magnetic field, and all their fluxes (except the C_4 term) depend on this being strong. Similarly, Pipin (2008, Equation (51)) also obtained a term similar to the C_4 term as a flux of the current helicity.

The second line in Equation (17) can be neglected if we assume all the turbulent correlators vary predominantly in one direction. This may be the case in a disk galaxy or in an accretion disk, where we expect a predominant stratification perpendicular to the disk, and in stars where we expect it to be radial. The first line in Equation (17) is nonzero in the presence of mean vorticity and stratification, and will lead to the generation of magnetic helicity even if the initial random velocity and magnetic fields are non-helical. This corresponds to a current helicity $h^c = h^b/\lambda^2$, which in simple closures leads to a magnetic-α effect $\alpha_m = (\tau/3)(h^c/4\pi\rho)$ (Pouquet et al. 1976; Gruzinov & Diamond 1994; Blackman & Field 2002; Rädler et al. 2003; Brandenburg & Subramanian 2005). Such terms could be particularly interesting in magnetically dominated turbulence. Rotation and stratification can also lead to the generation of kinetic helicity, whose source has a similar mathematical form and in turn results in a kinetic α-effect. Such an α-effect is susceptible to suppression as it results in oppositely signed α_m due to the generation of volume (first) term in Equation (9).

Interestingly, the NV flux and its divergence are nonzero even in the absence of large-scale magnetic fields; however, it crucially depends on having nonzero small-scale magnetic fields. These small-scale fields could grow due to the action of a fluctuation dynamo in a turbulent flow over eddy turnover timescales, even if the random motions are non-helical (Kazantsev 1968; Haugen et al. 2004; Schekochihin et al. 2004; Bhat & Subramanian 2013). Moreover, random magnetic fields can also arise due to the magnetorotational instability (MRI) in systems like accretion disks (Balbus & Hawley 1998).

Consider a system like a disk galaxy or an accretion disk, rotating with angular velocity $\Omega(r)$, i.e $\overline{\boldsymbol{V}} = r\Omega(r)\hat{\boldsymbol{\phi}}$. Its vorticity can be written as $\nabla\times\overline{\boldsymbol{V}} = \hat{z}(2\Omega + rd\Omega/dr) = \hat{z}\Omega(2-q)$ for $\Omega\propto r^{-q}$. We then find that $\boldsymbol{F}_{\rm NV}$ is predominantly in the z-direction. Different parts of $\boldsymbol{\nabla}\cdot\boldsymbol{F}_{\rm NV}$ can add or cancel depending on the sign of the gradients of n$\langle v^2\rangle$ and $\langle b^2\rangle$, with the overall sign and magnitude depending on their relative importance. Direct simulations suggest that $\langle v^2\rangle$ increases while $\langle b^2\rangle$ decreases away from the disk midplane in both disk galaxies (Abhijit Bendre, private communication using data from Bendre et al. 2015) and accretion disks (Prasun Dhang, private communication). Then, Equation (17) gives

[1] The general form of this flux has been discussed by Vishniac in several talks (and in private communications) but not yet published, apart from the abstracts cited (Vishniac 2012, 2015). So, unfortunately, a detailed comparison is not possible.

$$\frac{\partial h^b}{\partial t} \approx \frac{\langle b^2 \rangle \Omega (2-q)}{H} [(v_A \tau)^2 C_1 + \langle v^2 \rangle \tau^2 (C_3 - C_2) + C_4 \lambda^2]. \tag{18}$$

Here, we have defined the Alfvén velocity $v_A \equiv (\langle b^2 \rangle / 4\pi\rho)^{1/2}$, and estimated the gradient as division by the disk scale height H with the sign appropriate for $z > 0$. Let us consider the cases of disk galaxies and accretion disks in turn.

4.2.1. Application to Disk Galaxies

In the context of disk galaxies, supernovae drive turbulence in the interstellar medium, and random magnetic fields are generated due to the fluctuation dynamo. Current simulations at modest values of the magnetic Reynolds number $\mathrm{Rm} \sim 10^3$ suggest $v_A \sim 0.5$–$0.7v$ at saturation, with $v \equiv \sqrt{\langle v^2 \rangle}$ (see Haugen et al. 2004; Schekochihin et al. 2004; Cho et al. 2009; Bhat & Subramanian 2013; Sur et al. 2014; Bhat et al. 2016; Federrath 2016; Seta et al. 2020; Seta & Federrath 2021), while $v_A \sim v$ is obtained by Eyink et al. (2013).[2] Observations indicate a random magnetic field a few times larger than the mean field (Beck et al. 2019; Shukurov & Subramanian 2021), but it is not clear which of these should be of the order of the equipartition value. For our estimates of the right-hand side (RHS) of Equation (18), we adopt $v_A = v$. We also take $\lambda = v\tau = l$, where l is the eddy scale; and $q = 1$, corresponding to a flat rotation curve. Then $\partial h^b / \partial t \approx 0.076 \langle b^2 \rangle \alpha_0$, where $\alpha_0 = (\Omega l^2 / H)$ is the standard estimate for the α-effect in disk galaxies (see Shukurov & Subramanian 2021). Assuming that the random field has saturated, neglecting other flux terms, and integrating to get h^b and hence $h^c = h^b / \lambda^2$, this gives $\alpha_m = (\tau/3) h^c / (4\pi\rho) \approx 0.025 (t/\tau) \alpha_0$. Thus, the NV flux alone can build up a significant magnetic α-effect in about $40\tau \sim 4e8$ yr, where we have taken $\tau \sim e7$ yr for galactic turbulence.

In order to obtain more reliable estimates of α_m and its consequences for the growth of the galactic large-scale field, one needs a dynamical-quenching model, solving the helicity evolution equation, including other fluxes, the mean-field equation, and an equation for the small-scale field simultaneously. There are also uncertainties associated with the approximations that have been employed to simplify the triple correlation flux. Nevertheless, the above estimate makes this a promising mechanism for the generation of a large-scale field just due to the influence of helicity fluxes.

4.2.2. Application to Accretion Disks

For MRI-driven turbulence in an accretion disk (Balbus & Hawley 1998), we expect the random magnetic field to grow on a timescale of $\sim \Omega^{-1}$, and saturate with v_A a fraction, say f_A of the speed of sound c_s, that is $v_A \sim f_A c_s$. The correlation time is expected to be $\tau \sim 1/\Omega$ and thus $v_A \tau \sim f_A c_s / \Omega \sim f_A H$, where we have used the relation $H \sim c_s / \Omega$ for such disks. Numerical simulations also show that the magnetic energy dominates the kinetic energy by a factor of order 2, is correlated with a scale $\lambda \sim f_\lambda H$, and decreases with $|z|$ (Prasun Dhang, private communication). Adopting $f_A = 0.2, f_\lambda = 0.5$ it turns out that the C_4 term dominates the generation of magnetic helicity (with partial cancellation by other terms). Taking $q = 3/2$ as appropriate for accretion disks and integrating Equation (18), we get $h^c \sim -0.07 (\langle b^2 \rangle / H) \Omega t$, leading to an estimate of $\alpha_m \approx -0.023 f_A^2 c_s (\Omega t) \sim -10^{-3} (\Omega t) \Omega H$.

Moreover, v_A is seen to increase with height in simulations, and so the positive contributions due to the C_1, C_2, and C_3 terms gain in importance at larger heights. For example, now taking a larger $f_A = 0.5 = f_\lambda$, $v^2 \sim v_A^2 / 2$ in Equation (18), we now get $\alpha_m \sim 1.4 \times 10^{-3} (\Omega t) \Omega H$. These estimates again need to be firmed up in a detailed calculation, but for $\Omega t \sim 1$–10, look promising to explain both the magnitude and the sign of the α profile in simulations of MRI, which also lead to a large-scale dynamo (Brandenburg et al. 1995; Brandenburg & Sokoloff 2002; Davis et al. 2010; Gressel & Pessah 2015; Hogg & Reynolds 2018; Dhang & Sharma 2019; Dhang et al. 2020).

5. Conclusions

We have presented a detailed calculation of several potentially important types of magnetic helicity fluxes that can arise from triple correlators in the helicity density evolution equation, in the presence of random magnetic and velocity fields. For this we have assumed the velocity field to be incompressible, the random fields to be weakly inhomogeneous, and used a quasi-normal closure approximation with strong cumulant-induced damping. To begin with, we have been able to obtain a diffusive flux of the magnetic helicity from these hitherto unsimplified terms involving triple correlators of the fluctuating fields. Such a flux has indeed been measured in direct simulations (Mitra et al. 2010), and proves useful in dynamical-quenching models of the large-scale dynamo, which attempt to solve the catastrophic dynamo quenching problem. We find that the diffusive flux is always accompanied by an advective flux due to gradients in the random velocity and magnetic fields.

A helicity flux similar to that first suggested by Vishniac (2012) has also been obtained from the triple correlators. This flux arises even in the absence of a large-scale magnetic field, and depends only on the random magnetic and velocity fields apart from the mean flow. Its divergence depends on the gradients of the random fields predominantly along the large-scale vorticity. It can lead, by itself, to a turbulent EMF and to a parity-odd magnetic-α effect, by merely redistributing small-scale magnetic helicity within the system. Given that such a flux only requires rotation and stratification of random fields, it could be important to generate large-scale magnetic fields in disk galaxies, accretion disks, or stars. More detailed calculations solving dynamical-quenching models, including the NV flux, are required to firm up our conclusions. Moreover, direct numerical simulations of such stratified rotating systems are important, both to measure such helicity flux contributions and to confirm if they can indeed lead to large-scale dynamo action as envisaged here.

[2] See http://turbulence.pha.jhu.edu/Forced_MHD_turbulence.aspx.

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Appendix A
Double Correlators in a Weakly Inhomogeneous System

A.1. Roberts–Soward Expressions

We define the Fourier transform as

$$f(\boldsymbol{k}) \equiv \int \frac{\mathrm{d}\boldsymbol{x}}{(2\pi)^3}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}). \tag{A1}$$

To study a two-point correlator $\langle f(\boldsymbol{x}^{(1)}) f(\boldsymbol{x}^{(2)})\rangle$, we define $\boldsymbol{R} \equiv (\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)})/2$ and $\boldsymbol{r} \equiv \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}$. In homogeneous turbulence, the correlator would be independent of $\boldsymbol{R}$. Instead, we assume that the turbulence is *weakly inhomogeneous*, i.e., that the correlator varies with $\boldsymbol{R}$ much more slowly than with $\boldsymbol{r}$. Taking a Fourier transform ($\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{k}^{(1)}, \boldsymbol{k}^{(2)}$), we find that $\boldsymbol{R} \to \boldsymbol{K} = \boldsymbol{k}^{(1)} + \boldsymbol{k}^{(2)}$ while $\boldsymbol{r} \to \boldsymbol{k} = (\boldsymbol{k}^{(1)} - \boldsymbol{k}^{(2)})/2$. Since the turbulence is weakly inhomogeneous, we expand the correlator as a Taylor series in $\boldsymbol{K}$, assume the lowest-order terms are isotropic, and discard the $\mathcal{O}(K^2)$ terms. Since $\boldsymbol{v}$ and $\boldsymbol{b}$ are divergenceless, their double correlators are given by Roberts & Soward (1975) as

$$\begin{aligned}\left\langle b_i(\tfrac{1}{2}\boldsymbol{K} + \boldsymbol{k}) b_j(\tfrac{1}{2}\boldsymbol{K} - \boldsymbol{k})\right\rangle &= \mathrm{P}_{ij}(\boldsymbol{k}) M(k, \boldsymbol{K}) - \frac{i}{k^2}\epsilon_{ijc} k_c N(k, \boldsymbol{K}) - \frac{1}{2k^2}(k_i K_j M(k, \boldsymbol{K}) - k_j K_i M(k, \boldsymbol{K}))\\ &\quad + \frac{i}{2k^4}(k_i \epsilon_{jcd} + k_j \epsilon_{icd}) k_c K_d N(k, \boldsymbol{K}),\end{aligned} \tag{A2}$$

$$\begin{aligned}\left\langle v_i(\tfrac{1}{2}\boldsymbol{K} + \boldsymbol{k}) v_j(\tfrac{1}{2}\boldsymbol{K} - \boldsymbol{k})\right\rangle &= \mathrm{P}_{ij}(\boldsymbol{k}) E(k, \boldsymbol{K}) - \frac{i}{k^2}\epsilon_{ijc} k_c F(k, \boldsymbol{K}) - \frac{1}{2k^2}(k_i K_j E(k, \boldsymbol{K}) - k_j K_i E(k, \boldsymbol{K}))\\ &\quad + \frac{i}{2k^4}(k_i \epsilon_{jcd} + k_j \epsilon_{icd}) k_c K_d F(k, \boldsymbol{K}),\end{aligned} \tag{A3}$$

where $4\pi k^2 E(\boldsymbol{k}, \boldsymbol{R})$, $8\pi k^2 F(\boldsymbol{k}, \boldsymbol{R})$, $k^2 M(\boldsymbol{k}, \boldsymbol{R})$, and $8\pi N(\boldsymbol{k}, \boldsymbol{R})$ are the kinetic energy, kinetic helicity, magnetic energy, and magnetic helicity spectra, respectively. The current helicity spectrum is $8\pi k^2 N(\boldsymbol{k}, \boldsymbol{R})$, while $\langle j^2\rangle = \int \mathrm{d}k\; 8\pi k^4 M(k, \boldsymbol{R})$. Note that the double correlators are symmetric under the simultaneous interchange of $i \leftrightarrow j$ and $\boldsymbol{k} \to -\boldsymbol{k}$, which is not satisfied by a term in the corresponding expressions of Rädler et al. (2003) and Brandenburg & Subramanian (2005).

A.2. Correlators in Real Space

To evaluate correlators of interest in real space, we use Equations (A2) and (A3) along with the following angular integrals:

$$\int \mathrm{d}\Omega\, k_i = 0, \quad \int \mathrm{d}\Omega\, k_i k_j = \frac{4\pi k^2}{3}\delta_{ij}, \quad \int \mathrm{d}\Omega\, k_i k_j k_m = 0, \quad \int \mathrm{d}\Omega\, k_i k_j k_m k_n = \frac{4\pi k^4}{15}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{A4}$$

where $\mathrm{d}\Omega$ is the angular part of the integral over $\boldsymbol{k}$, such that $\mathrm{d}\boldsymbol{k} = k^2\,\mathrm{d}\Omega\,\mathrm{d}k$. We also use the fact that $a_i(\boldsymbol{k}) = \epsilon_{ijk}\, i k_j\, b_k(\boldsymbol{k})/k^2$. We then find

$$\langle v_i v_j\rangle = \frac{1}{3}\delta_{ji}\langle v^2\rangle + \mathcal{O}(\partial^2), \tag{A5}$$

$$\langle v_i \partial_k v_j\rangle = \frac{1}{6}\left(\delta_{ij}\partial_k\langle v^2\rangle + \frac{1}{2}\delta_{ik}\partial_j\langle v^2\rangle - \frac{1}{2}\delta_{jk}\partial_i\langle v^2\rangle - h^v\epsilon_{ijk}\right) + \mathcal{O}(\partial^2), \tag{A6}$$

$$\langle b_i b_j\rangle = \frac{1}{3}\delta_{ij}\langle b^2\rangle + \mathcal{O}(\partial^2), \tag{A7}$$

$$\langle b_i \partial_k b_j\rangle = \frac{1}{6}\left(\delta_{ij}\partial_k\langle b^2\rangle + \frac{1}{2}\delta_{ik}\partial_j\langle b^2\rangle - \frac{1}{2}\delta_{jk}\partial_i\langle b^2\rangle - h^c\epsilon_{ijk}\right) + \mathcal{O}(\partial^2), \tag{A8}$$

$$\langle a_i b_j\rangle = \frac{1}{3}\delta_{ij} h^b + \frac{1}{12}\epsilon_{ijm}\partial_m\langle a^2\rangle + \mathcal{O}(\partial^2), \tag{A9}$$

$$\langle (\partial_j b_i) a_k \rangle = \delta_{ik} \frac{7}{30} \partial_j h^b + \delta_{jk} \frac{1}{15} \partial_i h^b - \frac{1}{10} \delta_{ij} \partial_k h^b + \epsilon_{kji} \frac{1}{6} \langle b^2 \rangle + \mathcal{O}(\partial^2), \tag{A10}$$

$$\begin{aligned} \langle (\partial_q \partial_j b_i) a_k \rangle = & \frac{7}{60} \epsilon_{kqi} \partial_j \langle b^2 \rangle + \frac{7}{60} \epsilon_{kji} \partial_q \langle b^2 \rangle - \frac{1}{60} \epsilon_{kli} \delta_{qj} \partial_l \langle b^2 \rangle + \frac{1}{30} (\epsilon_{klj} \delta_{iq} + \epsilon_{klq} \delta_{ij}) \partial_l \langle b^2 \rangle \\ & + \frac{1}{30} (\delta_{qk} \delta_{ij} + \delta_{jk} \delta_{iq} - 4 \delta_{ik} \delta_{qj}) h^c + \mathcal{O}(\partial^2), \end{aligned} \tag{A11}$$

$$\begin{aligned} \langle b_i \partial_j \partial_k b_l \rangle = & -\frac{1}{12} \partial_k \epsilon_{ilj} h^c - \frac{1}{12} \partial_j \epsilon_{ilk} h^c - \frac{1}{30} (4 \delta_{jk} \delta_{il} - \delta_{ji} \delta_{kl} - \delta_{jl} \delta_{ki}) \langle j^2 \rangle \\ & - \frac{1}{30} (\delta_{ji} \epsilon_{lkd} + \delta_{ki} \epsilon_{ljd} + \delta_{jl} \epsilon_{ikd} + \delta_{kl} \epsilon_{ijd}) \frac{1}{2} \partial_d h^c + \mathcal{O}(\partial^2), \end{aligned} \tag{A12}$$

$$\begin{aligned} \langle v_i \partial_j \partial_k v_l \rangle = & -\frac{1}{12} \partial_k \epsilon_{ilj} h^v - \frac{1}{12} \partial_j \epsilon_{ilk} h^v - \frac{1}{30} (4 \delta_{jk} \delta_{il} - \delta_{ji} \delta_{kl} - \delta_{jl} \delta_{ki}) \langle \omega^2 \rangle \\ & - \frac{1}{30} (\delta_{ji} \epsilon_{lkd} + \delta_{ki} \epsilon_{ljd} + \delta_{jl} \epsilon_{ikd} + \delta_{kl} \epsilon_{ijd}) \frac{1}{2} \partial_d h^v + \mathcal{O}(\partial^2), \end{aligned} \tag{A13}$$

$$\langle b_i \partial_e \partial_j \partial_k b_l \rangle = \frac{1}{30} (\delta_{ej} \epsilon_{ilk} + \delta_{ek} \epsilon_{ilj} + \delta_{jk} \epsilon_{ile}) \int \mathrm{d}k \; 8\pi k^4 N(k, \boldsymbol{R}) + \mathcal{O}(\partial), \tag{A14}$$

$$\langle b_i \partial_e \nabla^2 b_l \rangle = \frac{1}{6} \epsilon_{ile} \int \mathrm{d}k \; 8\pi k^4 N(k, \boldsymbol{R}) - \left(\frac{3}{10} \delta_{il} \partial_e - \frac{7}{60} \delta_{el} \partial_i - \frac{1}{20} \delta_{ei} \partial_l \right) \langle j^2 \rangle + \mathcal{O}(\partial^2), \tag{A15}$$

$$\langle v_i \partial_e \nabla^2 v_l \rangle = \frac{1}{6} \epsilon_{ile} \int \mathrm{d}k \; 8\pi k^4 F(k, \boldsymbol{R}) - \left(\frac{3}{10} \delta_{il} \partial_e - \frac{7}{60} \delta_{el} \partial_i - \frac{1}{20} \delta_{ei} \partial_l \right) \langle \omega^2 \rangle + \mathcal{O}(\partial^2), \tag{A16}$$

$$\langle a_i (\partial_d \partial_e \partial_b b_i) \rangle = 0 + \mathcal{O}(\partial), \tag{A17}$$

where $\mathcal{O}(\partial^n)$ denotes that we have neglected terms with more than n large-scale derivatives.

Appendix B
The Triple Correlators

Using Equations (7), (4), and (6), we write

$$\begin{aligned} \frac{\partial}{\partial t} \langle v_a b_b a_c \rangle = & -\overline{V}_d \langle (\partial_d v_a) b_b a_c \rangle - (\partial_d \overline{V}_a) \langle v_d b_b a_c \rangle - \langle v_d (\partial_d v_a) b_b a_c \rangle + \langle v_d (\partial_d v_a) \rangle \langle b_b a_c \rangle - \left\langle \frac{\partial_a p'}{\rho} b_b a_c \right\rangle \\ & + \frac{\overline{B}_d}{4\pi\rho} \langle (\partial_d b_a) b_b a_c \rangle + \frac{\partial_d \overline{B}_a}{4\pi\rho} \langle b_d b_b a_c \rangle + \left\langle \frac{b_d (\partial_d b_a)}{4\pi\rho} b_b a_c \right\rangle - \frac{\langle b_d (\partial_d b_a) \rangle}{4\pi\rho} \langle b_b a_c \rangle \\ & + \nu \langle (\partial_d \partial_d v_a) b_b a_c \rangle + (\partial_d \overline{V}_b) \langle v_a b_d a_c \rangle - \overline{V}_d \langle v_a (\partial_d b_b) a_c \rangle + \overline{B}_d \langle v_a (\partial_d v_b) a_c \rangle - (\partial_d \overline{B}_b) \langle v_a v_d a_c \rangle \\ & + \langle v_a b_d (\partial_d v_b) a_c \rangle - \langle v_a v_d (\partial_d b_b) a_c \rangle - \epsilon_{bde} (\partial_d \mathcal{E}_e) \langle v_a a_c \rangle + \eta \langle v_a (\partial_d \partial_d b_b) a_c \rangle + \epsilon_{cde} \overline{V}_d \langle v_a b_b b_e \rangle \\ & + \epsilon_{cde} \overline{B}_e \langle v_a b_b v_d \rangle + \epsilon_{cde} \langle v_a b_b v_d b_e \rangle - \mathcal{E}_c \langle v_a b_b \rangle - \eta \epsilon_{cde} \langle v_a b_b \partial_d b_e \rangle + \langle v_a b_b (\partial_c \varphi) \rangle. \end{aligned} \tag{B1}$$

In what follows, we drop terms dependent on $\overline{\boldsymbol{B}}$, ν, η, or mixed correlators of the form $\langle vb\rangle$ or $\langle va\rangle$. Assuming the fourth-order correlators above can be expressed as products of second-order correlators along with a damping term, we write

$$\begin{aligned}\frac{\partial}{\partial t}\langle v_a b_b a_c\rangle = &-\overline{V}_d\langle(\partial_d v_a)b_b a_c\rangle - (\partial_d\overline{V}_a)\langle v_d b_b a_c\rangle - \left\langle\frac{\partial_a p'}{\rho}b_b a_c\right\rangle + \langle b_d b_b\rangle\left\langle\frac{(\partial_d b_a)}{4\pi\rho}a_c\right\rangle\\ &+ \langle b_d a_c\rangle\left\langle\frac{(\partial_d b_a)}{4\pi\rho}b_b\right\rangle + (\partial_d\overline{V}_b)\langle v_a b_d a_c\rangle - \overline{V}_d\langle v_a(\partial_d b_b)a_c\rangle + \langle v_a(\partial_d v_b)\rangle\langle b_d a_c\rangle\\ &- \langle v_a v_d\rangle\langle(\partial_d b_b)a_c\rangle + \epsilon_{cde}\overline{V}_d\langle v_a b_b b_e\rangle + \epsilon_{cde}\langle v_a v_d\rangle\langle b_b b_e\rangle + \langle v_a b_b(\partial_c\varphi)\rangle - \frac{1}{\tau}\langle v_a b_b a_c\rangle.\end{aligned} \tag{B2}$$

In the steady state,

$$\begin{aligned}\langle v_a b_b a_c\rangle = \tau\Bigg[&-\overline{V}_d\langle(\partial_d v_a)b_b a_c\rangle - (\partial_d\overline{V}_a)\langle v_d b_b a_c\rangle - \left\langle\frac{\partial_a p'}{\rho}b_b a_c\right\rangle + \langle b_d b_b\rangle\left\langle\frac{(\partial_d b_a)}{4\pi\rho}a_c\right\rangle\\ &+ \langle b_d a_c\rangle\left\langle\frac{(\partial_d b_a)}{4\pi\rho}b_b\right\rangle + (\partial_d\overline{V}_b)\langle v_a b_d a_c\rangle - \overline{V}_d\langle v_a(\partial_d b_b)a_c\rangle + \langle v_a(\partial_d v_b)\rangle\langle b_d a_c\rangle\\ &- \langle v_a v_d\rangle\langle(\partial_d b_b)a_c\rangle + \epsilon_{cde}\overline{V}_d\langle v_a b_b b_e\rangle + \epsilon_{cde}\langle v_a v_d\rangle\langle b_b b_e\rangle + \langle v_a b_b(\partial_c\varphi)\rangle\Bigg].\end{aligned} \tag{B3}$$

Note that above, $\langle v_a b_b a_c\rangle$ appears on both the left-hand side (LHS) and the RHS. Moreover, the RHS contains other triple correlators such as $\langle v_a b_b b_e\rangle$. We can obtain a series for $\langle v_a b_b a_c\rangle$ by the method of successive approximations, thereby obtaining a series in τ. For our purposes, we can discard $\mathcal{O}(\tau^3)$ contributions to $\langle v_a b_b a_c\rangle$.

Following exactly the same procedure, we can write

$$\begin{aligned}\langle v_a b_b b_c\rangle = \tau\Bigg\{&-\left\langle\frac{\partial_a p'}{\rho}b_b b_c\right\rangle + \frac{1}{4\pi\rho}\langle b_d b_b\rangle\langle(\partial_d b_a)b_c\rangle + \frac{1}{4\pi\rho}\langle b_d b_c\rangle\langle(\partial_d b_a)b_b\rangle + \langle v_a(\partial_d v_b)\rangle\langle b_d b_c\rangle\\ &- \langle v_a v_d\rangle\langle(\partial_d b_b)b_c\rangle + \langle v_a(\partial_d v_c)\rangle\langle b_b b_d\rangle - \langle v_a v_d\rangle\langle b_b(\partial_d b_c)\rangle\Bigg\} + \mathcal{O}(\tau^2),\end{aligned} \tag{B4}$$

$$\begin{aligned}\langle(\partial_d v_a)b_b a_c\rangle = \tau\Bigg[&-\left\langle\frac{\partial_a\partial_d p'}{\rho}b_b a_c\right\rangle + \frac{1}{4\pi\rho}\langle(\partial_d b_j)b_b\rangle\langle(\partial_j b_a)a_c\rangle + \frac{1}{4\pi\rho}\langle(\partial_d b_j)a_c\rangle\langle(\partial_j b_a)b_b\rangle\\ &+ \frac{1}{4\pi\rho}\langle b_j b_b\rangle\langle(\partial_d\partial_j b_a)a_c\rangle + \frac{1}{4\pi\rho}\langle b_j a_c\rangle\langle(\partial_d\partial_j b_a)b_b\rangle + \langle(\partial_d v_a)(\partial_j v_b)\rangle\langle b_j a_c\rangle\\ &- \langle(\partial_d v_a)v_j\rangle\langle(\partial_j b_b)a_c\rangle + \epsilon_{clm}\langle(\partial_d v_a)v_l\rangle\langle b_b b_m\rangle + \langle(\partial_d v_a)b_b(\partial_c\varphi)\rangle\Bigg] + \mathcal{O}(\tau^2),\end{aligned} \tag{B5}$$

$$\begin{aligned}\langle v_a(\partial_d b_b)a_c\rangle = \tau\Bigg[&-\left\langle\frac{\partial_a p'}{\rho}(\partial_d b_b)a_c\right\rangle + \frac{1}{4\pi\rho}\langle b_j(\partial_d b_b)\rangle\langle(\partial_j b_a)a_c\rangle + \frac{1}{4\pi\rho}\langle b_j a_c\rangle\langle(\partial_j b_a)(\partial_d b_b)\rangle\\ &+ \langle v_a(\partial_j v_b)\rangle\langle(\partial_d b_j)a_c\rangle + \langle v_a(\partial_d\partial_j v_b)\rangle\langle b_j a_c\rangle - \langle v_a(\partial_d v_j)\rangle\langle(\partial_j b_b)a_c\rangle\\ &- \langle v_a v_j\rangle\langle(\partial_d\partial_j b_b)a_c\rangle + \epsilon_{clm}\langle v_a v_l\rangle\langle(\partial_d b_b)b_m\rangle + \langle v_a(\partial_d b_b)\partial_c\varphi\rangle\Bigg] + \mathcal{O}(\tau^2).\end{aligned} \tag{B6}$$

In Appendix A.2, we have listed expressions for the various double correlators appearing in Equations (B3)–(B6), assuming weakly inhomogeneous turbulence. Using these, discarding terms with more than one large-scale derivative, and substituting Equations (B4)–(B6) into Equation (B3), a lengthy but straightforward calculation gives us the terms I_1^{triple} and I_2^{triple} (which appear in the evolution equation for the helicity density, Equation (11)):

$$
\begin{aligned}
I_1^{\text{triple}} = \partial_j\Bigg[& -\tau\left\langle \frac{\partial_i p'}{\rho} b_j a_i \right\rangle + \tau\frac{2}{9}\frac{\langle b^2\rangle}{4\pi\rho}\partial_j h^b - \tau\frac{1}{36}\frac{1}{4\pi\rho}h^c\partial_j\langle a^2\rangle \\
& + \tau\frac{1}{9}h^b\partial_j\langle v^2\rangle + \tau\frac{1}{36}h^v\partial_j\langle a^2\rangle - \tau\frac{1}{9}\langle v^2\rangle\partial_j h^b + \tau\langle v_i b_j(\partial_i\varphi)\rangle \\
& + \overline{V}_d\tau^2\Bigg\{\left\langle a_i\partial_d\left(\frac{\partial_i p'}{\rho}b_j\right)\right\rangle + \epsilon_{jdi}\frac{1}{36}\frac{h^c}{4\pi\rho}\partial_i h^b - \epsilon_{jdi}\frac{1}{24}\frac{1}{4\pi\rho}\langle b^2\rangle\partial_i\langle b^2\rangle \\
& + \delta_{dj}\frac{1}{18}\frac{1}{4\pi\rho}\langle b^2\rangle h^c + \epsilon_{jdi}\frac{11}{72}\langle b^2\rangle\partial_i\langle v^2\rangle - \delta_{jd}\frac{1}{9}\langle b^2\rangle h^v - \langle(\partial_i\varphi)\partial_d(v_i b_j)\rangle \\
& - \epsilon_{jdi}\frac{1}{36}h^v\partial_i h^b + \delta_{jd}\frac{1}{18}h^v\langle b^2\rangle + \epsilon_{jdi}\frac{1}{18}\langle v^2\rangle\partial_i\langle b^2\rangle - \epsilon_{ide}\left\langle\frac{\partial_i p'}{\rho}b_j b_e\right\rangle\Bigg\} \\
& - (\partial_d\overline{V}_i)\tau^2\Bigg\{-\left\langle\frac{\partial_d p'}{\rho}b_j a_i\right\rangle + \epsilon_{ijd}\frac{1}{18}\frac{1}{4\pi\rho}\langle b^2\rangle\langle b^2\rangle + \epsilon_{jid}\frac{1}{18}\frac{1}{4\pi\rho}h^b h^c \\
& + \epsilon_{dij}\frac{1}{18}h^v h^b + \epsilon_{idj}\frac{1}{18}\langle v^2\rangle\langle b^2\rangle + \langle v_d b_j(\partial_i\varphi)\rangle\Bigg\} \\
& + (\partial_d\overline{V}_j)\tau^2\Bigg\{-\left\langle\frac{\partial_i p'}{\rho}b_d a_i\right\rangle + \langle v_i b_d(\partial_i\varphi)\rangle\Bigg\}\Bigg] + \mathcal{O}(\tau^3),
\end{aligned}
\tag{B7}
$$

and

$$
\begin{aligned}
I_2^{\text{triple}} = -\partial_j\Bigg[& -\tau\left\langle \frac{\partial_j p'}{\rho} b_i a_i \right\rangle + \tau\frac{1}{9}\frac{\langle b^2\rangle}{4\pi\rho}\partial_j h^b + \tau\frac{1}{9}\frac{1}{4\pi\rho}h^b\partial_j\langle b^2\rangle \\
& + \tau\frac{1}{36}\frac{1}{4\pi\rho}h^c\partial_j\langle a^2\rangle - \tau\frac{1}{36}h^v\partial_j\langle a^2\rangle - \tau\frac{2}{9}\langle v^2\rangle\partial_j h^b + \tau\langle v_j b_i(\partial_i\varphi)\rangle \\
& + \overline{V}_d\tau^2\Bigg\{\left\langle a_i\partial_d\left(\frac{\partial_j p'}{\rho}b_i\right)\right\rangle - \epsilon_{jdi}\frac{1}{36}\frac{h^c}{4\pi\rho}\partial_i h^b + \epsilon_{jdi}\frac{1}{72}\frac{1}{4\pi\rho}\langle b^2\rangle\partial_i\langle b^2\rangle \\
& + \frac{1}{18}\frac{1}{4\pi\rho}\langle b^2\rangle\delta_{dj}h^c - \langle(\partial_i\varphi)\partial_d(v_j b_i)\rangle - \epsilon_{jdi}\frac{1}{72}\langle b^2\rangle\partial_i\langle v^2\rangle + \epsilon_{jdi}\frac{1}{36}h^v\partial_i h^b \\
& - \delta_{jd}\frac{1}{18}h^v\langle b^2\rangle - \epsilon_{jdi}\frac{1}{18}\langle v^2\rangle\partial_i\langle b^2\rangle - \epsilon_{ide}\left\langle\frac{\partial_j p'}{\rho}b_i b_e\right\rangle\Bigg\} \\
& - (\partial_d\overline{V}_j)\tau^2\Bigg\{-\left\langle\frac{\partial_d p'}{\rho}b_i a_i\right\rangle + \langle v_d b_i(\partial_i\varphi)\rangle\Bigg\} \\
& + (\partial_d\overline{V}_i)\tau^2\Bigg\{-\left\langle\frac{\partial_j p'}{\rho}b_d a_i\right\rangle + \epsilon_{idj}\frac{1}{18}\frac{1}{4\pi\rho}\langle b^2\rangle\langle b^2\rangle + \epsilon_{dij}\frac{1}{18}\frac{1}{4\pi\rho}h^b h^c \\
& + \epsilon_{jid}\frac{1}{18}h^v h^b + \epsilon_{ijd}\frac{1}{18}\langle v^2\rangle\langle b^2\rangle + \langle v_j b_d(\partial_i\varphi)\rangle\Bigg\}\Bigg] + \mathcal{O}(\tau^3).
\end{aligned}
\tag{B8}
$$

Substituting Equations (B7) and (B8) into Equation (11), we obtain

$$\begin{aligned}\left(\frac{\partial h^b}{\partial t}\right)_{\text{triple}} = \partial_j\Bigg[& \overbrace{\frac{\tau}{9}\left(\langle v^2\rangle + \frac{\langle b^2\rangle}{4\pi\rho}\right)\partial_j h^b}^{\text{diffusion}} + \tau\frac{1}{18}\left(h^v - \frac{h^c}{4\pi\rho}\right)\partial_j\langle a^2\rangle + \overbrace{\frac{\tau}{9}h^b\partial_j\left(\langle v^2\rangle - \frac{\langle b^2\rangle}{4\pi\rho}\right)}^{\text{random advection}}\\
& - \tau\left\langle\frac{\partial_i p'}{\rho}b_j a_i\right\rangle + \tau\left\langle\frac{\partial_j p'}{\rho}b_i a_i\right\rangle + \tau\langle v_i b_j(\partial_i\varphi)\rangle - \tau\langle v_j b_i(\partial_i\varphi)\rangle\\
& + \overline{V}_d\tau^2\Bigg\{\overbrace{-\epsilon_{jdi}\frac{1}{18}\frac{1}{4\pi\rho}\langle b^2\rangle\partial_i\langle b^2\rangle + \epsilon_{jdi}\frac{1}{6}\langle b^2\rangle\partial_i\langle v^2\rangle + \epsilon_{jdi}\frac{1}{9}\langle v^2\rangle\partial_i\langle b^2\rangle}^{\text{NV}}\\
& + \epsilon_{jdi}\frac{1}{18}\frac{h^c}{4\pi\rho}\partial_i h^b - \epsilon_{jdi}\frac{1}{18}h^v\partial_i h^b + \left\langle a_i\partial_d\left(\frac{\partial_i p'}{\rho}b_j\right)\right\rangle - \epsilon_{ide}\left\langle\frac{\partial_i p'}{\rho}b_j b_e\right\rangle\\
& - \left\langle a_i\partial_d\left(\frac{\partial_j p'}{\rho}b_i\right)\right\rangle + \langle(\partial_i\varphi)\partial_d(v_j b_i)\rangle - \langle(\partial_i\varphi)\partial_d(v_i b_j)\rangle + \epsilon_{ide}\left\langle\frac{\partial_j p'}{\rho}b_i b_e\right\rangle\Bigg\}\\
& - (\partial_d\overline{V}_i)\tau^2\left(-\left\langle\frac{\partial_d p'}{\rho}b_j a_i\right\rangle + \langle v_d b_j(\partial_i\varphi)\rangle - \left\langle\frac{\partial_j p'}{\rho}b_d a_i\right\rangle + \langle v_j b_d(\partial_i\varphi)\rangle\right)\\
& + (\partial_d\overline{V}_j)\tau^2\left(-\left\langle\frac{\partial_i p'}{\rho}b_d a_i\right\rangle + \langle v_i b_d(\partial_i\varphi)\rangle - \left\langle\frac{\partial_d p'}{\rho}b_i a_i\right\rangle + \langle v_d b_i(\partial_i\varphi)\rangle\right)\Bigg].\end{aligned} \tag{B9}$$

Appendix C
The Nonlocal Terms

C.1. Correlators Involving the Pressure

C.1.1. $\langle\frac{\partial_a p'}{\rho}b_b a_c\rangle$

If we drop the viscosity and mean magnetic field terms in Equation (7), take its divergence, and solve for the pressure, we obtain

$$p' = \rho\nabla^{-2}\partial_i\left(-\overline{V}_j\partial_j v_i - v_j\partial_j\overline{V}_i - v_j\partial_j v_i + \langle v_j\partial_j v_i\rangle + \frac{b_j\partial_j b_i}{4\pi\rho} - \frac{\langle b_j\partial_j b_i\rangle}{4\pi\rho}\right), \tag{C1}$$

where ∇^{-2} denotes the integral operator that is the inverse of the Laplacian, given by

$$\nabla^{-2}f(\boldsymbol{x}) = -\int d^3y\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}f(\boldsymbol{y}). \tag{C2}$$

Using Equations (C1) and (C2), we write

$$\begin{aligned}\left\langle\frac{\partial_a p'}{\rho}b_b a_c\right\rangle = & -\int d^3y\frac{x_a - y_a}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\frac{\partial\overline{V}_e(\boldsymbol{y})}{\partial y_d}\left\langle b_b(\boldsymbol{x})a_c(\boldsymbol{x})\frac{\partial v_d(\boldsymbol{y})}{\partial y_e}\right\rangle\\
& - \int d^3y\frac{x_a - y_a}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\frac{\partial\overline{V}_d(\boldsymbol{y})}{\partial y_e}\left\langle b_b(\boldsymbol{x})a_c(\boldsymbol{x})\frac{\partial v_e(\boldsymbol{y})}{\partial y_d}\right\rangle\\
& - \int d^3y\frac{x_a - y_a}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\left\langle b_b(\boldsymbol{x})a_c(\boldsymbol{x})\frac{\partial v_e(\boldsymbol{y})}{\partial y_d}\frac{\partial v_d(\boldsymbol{y})}{\partial y_e}\right\rangle\\
& + \int d^3y\frac{x_a - y_a}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\langle b_b(\boldsymbol{x})a_c(\boldsymbol{x})\rangle\left\langle\frac{\partial v_e(\boldsymbol{y})}{\partial y_d}\frac{\partial v_d(\boldsymbol{y})}{\partial y_e}\right\rangle\\
& + \int d^3y\frac{x_a - y_a}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\frac{1}{4\pi\rho}\left\langle b_b(\boldsymbol{x})a_c(\boldsymbol{x})\frac{\partial b_e(\boldsymbol{y})}{\partial y_d}\frac{\partial b_d(\boldsymbol{y})}{\partial y_e}\right\rangle\\
& - \int d^3y\frac{x_a - y_a}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\frac{1}{4\pi\rho}\langle b_b(\boldsymbol{x})a_c(\boldsymbol{x})\rangle\left\langle\frac{\partial b_e(\boldsymbol{y})}{\partial y_d}\frac{\partial b_d(\boldsymbol{y})}{\partial y_e}\right\rangle.\end{aligned} \tag{C3}$$

Using the quasi-normal approximation for the fourth-order correlators and discarding correlators of the form $\langle vb\rangle$ gives

$$\left\langle \frac{\partial_a p'}{\rho} b_b a_c \right\rangle = -2\int \mathrm{d}^3y \frac{x_a - y_a}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|^3} \frac{\partial \overline{V}_e(\boldsymbol{y})}{\partial y_d} \frac{\partial}{\partial y_e} \langle b_b(\boldsymbol{x}) a_c(\boldsymbol{x}) v_d(\boldsymbol{y})\rangle + 2\int \mathrm{d}^3y \frac{x_a - y_a}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|^3} \frac{1}{4\pi\rho} \frac{\partial}{\partial y_d} \langle b_b(\boldsymbol{x}) b_e(\boldsymbol{y})\rangle \frac{\partial}{\partial y_e} \langle a_c(\boldsymbol{x}) b_d(\boldsymbol{y})\rangle. \tag{C4}$$

Using the VC approximation, we write the above as

$$\left\langle \frac{\partial_a p'}{\rho} b_b a_c \right\rangle \approx \frac{2\lambda^2}{3} \frac{\partial \overline{V}_e(\boldsymbol{x})}{\partial x_d} \left\langle b_b(\boldsymbol{x}) a_c(\boldsymbol{x}) \frac{\partial^2 v_d(\boldsymbol{x})}{\partial x_e \partial x_a} \right\rangle - \frac{2\lambda^2}{3} \frac{1}{4\pi\rho} \left\langle b_b(\boldsymbol{x}) \frac{\partial^2 b_e(\boldsymbol{x})}{\partial x_d \partial x_a} \right\rangle \left\langle a_c(\boldsymbol{x}) \frac{\partial b_d(\boldsymbol{x})}{\partial x_e} \right\rangle - \frac{2\lambda^2}{3} \frac{1}{4\pi\rho} \left\langle b_b(\boldsymbol{x}) \frac{\partial b_e(\boldsymbol{x})}{\partial x_d} \right\rangle \left\langle a_c(\boldsymbol{x}) \frac{\partial^2 b_d(\boldsymbol{x})}{\partial x_e \partial x_a} \right\rangle. \tag{C5}$$

If we substitute Equations (A8), (A10), (A11), and (A12) for the various correlators in the above equation, we obtain

$$\left\langle \frac{\partial_a p'}{\rho} b_b a_c \right\rangle \approx \frac{2\lambda^2}{3} \frac{\partial \overline{V}_e(\boldsymbol{x})}{\partial x_d} \left\langle b_b(\boldsymbol{x}) a_c(\boldsymbol{x}) \frac{\partial^2 v_d(\boldsymbol{x})}{\partial x_e \partial x_a} \right\rangle + \frac{13\lambda^2}{675} \frac{1}{4\pi\rho} \langle j^2\rangle \delta_{ca} \partial_b h^b + \frac{\lambda^2}{1350} \frac{1}{4\pi\rho} \langle j^2\rangle \delta_{cb} \partial_a h^b - \frac{\lambda^2}{150} \frac{1}{4\pi\rho} \langle j^2\rangle \delta_{ab} \partial_c h^b + \frac{17\lambda^2}{270} \frac{1}{4\pi\rho} \delta_{bc} \langle b^2\rangle \partial_a h^c - \frac{\lambda^2}{90} \frac{1}{4\pi\rho} \delta_{ac} \langle b^2\rangle \partial_b h^c + \frac{\lambda^2}{135} \frac{1}{4\pi\rho} \delta_{ba} \langle b^2\rangle \partial_c h^c + \frac{\lambda^2}{54} \frac{1}{4\pi\rho} \epsilon_{cba} \langle b^2\rangle \langle j^2\rangle - \frac{\lambda^2}{54} \frac{1}{4\pi\rho} \epsilon_{bac} h^c h^c. \tag{C6}$$

Now, the term in Equation (C6) containing the mean velocity is required only for the index choices $a = c$ or $b = c$. Using Equation (C34) and then Equations (A11)–(A14), and (A16), we write Equation (C6) for $a = c$ as

$$\left\langle \frac{\partial_i p'}{\rho} b_j a_i \right\rangle \approx \frac{2\tau\lambda^2}{3} (\partial_d \overline{V}_e) \Bigg[-\frac{1}{4\pi\rho} \frac{1}{36} \epsilon_{ejd} \langle j^2\rangle \langle b^2\rangle - \frac{1}{4\pi\rho} \epsilon_{jde} \frac{1}{36} h^c h^c + \frac{1}{4\pi\rho} \frac{1}{18} \epsilon_{jde} h^b \int \mathrm{d}k\ 8\pi k^4 N(k, \boldsymbol{R}) - \frac{1}{18} \epsilon_{jde} h^b \int \mathrm{d}k\ 8\pi k^4 F(k, \boldsymbol{R}) - \frac{1}{36} \epsilon_{edj} \langle \omega^2\rangle \langle b^2\rangle \Bigg] + \frac{7\lambda^2}{135} \frac{1}{4\pi\rho} \langle j^2\rangle \partial_j h^b + \frac{\lambda^2}{27} \frac{1}{4\pi\rho} h^c \partial_j \langle b^2\rangle. \tag{C7}$$

Similarly, we can use Equation (C35) and then Equations (A12) and (A11) to write Equation (C6) for $b = c$ as

$$\left\langle \frac{\partial_j p'}{\rho} b_i a_i \right\rangle \approx \frac{2\lambda^2}{135} \frac{1}{4\pi\rho} \langle j^2\rangle \partial_j h^b + \frac{2\lambda^2}{27} \frac{1}{4\pi\rho} \langle b^2\rangle \partial_j h^c + \frac{\lambda^2}{9} \frac{1}{4\pi\rho} h^c \partial_j \langle b^2\rangle. \tag{C8}$$

C.1.2. $\langle \frac{\partial_a p'}{\rho} b_b b_c \rangle$

This correlator appears in Equation (B9) multiplied by a τ^2 factor. Similar to Equation (C4), we write

$$\left\langle \frac{\partial_a p'}{\rho} b_b b_c \right\rangle = 2\int \mathrm{d}^3y \frac{x_a - y_a}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|^3} \frac{1}{4\pi\rho} \left\langle b_b(\boldsymbol{x}) \frac{\partial b_e(\boldsymbol{y})}{\partial y_d} \right\rangle \left\langle b_c(\boldsymbol{x}) \frac{\partial b_d(\boldsymbol{y})}{\partial y_e} \right\rangle. \tag{C9}$$

Using the VC approximation, this becomes

$$\left\langle \frac{\partial_a p'}{\rho} b_b b_c \right\rangle \approx -\frac{2\lambda^2}{3} \frac{1}{4\pi\rho} \langle b_b \partial_d b_e\rangle \langle b_c \partial_a \partial_e b_d\rangle - \frac{2\lambda^2}{3} \frac{1}{4\pi\rho} \langle b_b \partial_a \partial_d b_e\rangle \langle b_c \partial_e b_d\rangle. \tag{C10}$$

Examining Equation (B9), we see that the correlator we need is $\epsilon_{ide} \left\langle \frac{\partial_i p'}{\rho} b_j b_e \right\rangle$. Using Equations (A8) and (A12) along with the above, we get

$$\epsilon_{ide} \left\langle \frac{\partial_i p'}{\rho} b_j b_e \right\rangle = \frac{\lambda^2}{18} \frac{1}{4\pi\rho} \epsilon_{jid} h^c \partial_i h^c. \tag{C11}$$

C.1.3. $\langle a_a \partial_d(\frac{\partial_b p'}{\rho} b_c)\rangle$

Note that this correlator appears in Equation (B9) multiplied by τ^2. We need to keep up to one large-scale derivative in it, but we can discard terms involving the mean velocity. We write

$$\left\langle a_a \partial_d\left(\frac{\partial_b p'}{\rho} b_c\right)\right\rangle = \partial_d\left\langle \frac{\partial_b p'}{\rho} b_c a_a\right\rangle - \left\langle \frac{\partial_b p'}{\rho} b_c(\partial_d a_a)\right\rangle \tag{C12}$$

Following steps similar to those in Appendix (C.1.1), the second correlator on the RHS is

$$\begin{aligned}\left\langle \frac{\partial_b p'}{\rho} b_c(\partial_d a_a)\right\rangle \approx &-\frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle b_c(\partial_f\partial_b b_e)\rangle\,\partial_d\langle a_a(\partial_e b_f)\rangle + \frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle b_c(\partial_f\partial_b b_e)\rangle\langle a_a(\partial_d\partial_e b_f)\rangle\\ &-\frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle b_c(\partial_f b_e)\rangle\,\partial_d\langle a_a(\partial_e\partial_b b_f)\rangle + \frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle b_c(\partial_f b_e)\rangle\langle a_a(\partial_d\partial_e\partial_b b_f)\rangle.\end{aligned} \tag{C13}$$

Using the above along with Equation (C5), we write

$$\begin{aligned}\left\langle a_a \partial_d\left(\frac{\partial_b p'}{\rho} b_c\right)\right\rangle = &-\frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle a_a(\partial_e b_f)\rangle\,\partial_d\langle b_c(\partial_f\partial_b b_e)\rangle - \frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle a_a(\partial_e\partial_b b_f)\rangle\,\partial_d\langle b_c(\partial_f b_e)\rangle\\ &-\frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle b_c(\partial_f\partial_b b_e)\rangle\langle a_a(\partial_d\partial_e b_f)\rangle - \frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\langle b_c(\partial_f b_e)\rangle\langle a_a(\partial_d\partial_e\partial_b b_f)\rangle.\end{aligned} \tag{C14}$$

Now, we only need the values of the above correlator for $b = a$ and $c = a$. They are

$$\left\langle a_i \partial_d\left(\frac{\partial_i p'}{\rho} b_c\right)\right\rangle = -\frac{2\lambda^2}{3}\frac{1}{4\pi\rho}\frac{4}{30}\langle j^2\rangle\frac{1}{3}\delta_{dc}h^c - \frac{\lambda^2}{270}\frac{1}{4\pi\rho}\langle j^2\rangle\,\epsilon_{idc}\partial_i\langle b^2\rangle - \frac{\lambda^2}{30}\frac{1}{4\pi\rho}h^c\epsilon_{cdf}\partial_f h^c \tag{C15}$$

and

$$\left\langle a_i \partial_d\left(\frac{\partial_b p'}{\rho} b_i\right)\right\rangle = \frac{2\lambda^2}{135}\frac{1}{4\pi\rho}\epsilon_{idb}\langle j^2\rangle\partial_i\langle b^2\rangle - \frac{4\lambda^2}{135}\frac{1}{4\pi\rho}\delta_{db}\langle j^2\rangle h^c - \frac{2\lambda^2}{135}\frac{1}{4\pi\rho}h^c\epsilon_{dbg}\partial_g h^c. \tag{C16}$$

C.2. Correlators Involving the Scalar Potential

Taking the divergence of the evolution equation for the vector potential, Equation (6), we obtain

$$\partial_k e_k = \nabla^2\varphi \quad\Longrightarrow\quad \varphi = \nabla^{-2}\partial_k e_k, \tag{C17}$$

where ∇^{-2} is the inverse of the Laplacian, given in Equation (C2).

On the other hand, taking the divergence of both sides of Equation (5) gives us

$$\partial_k e_k = -\epsilon_{klm}\partial_k(\overline{V}_l b_m) - \epsilon_{klm}\partial_k(v_l b_m), \tag{C18}$$

where we have dropped terms dependent on $\overline{\boldsymbol{B}}$ and $\mathcal{E}$.

The scalar potential is then

$$\varphi = \nabla^{-2}[-\epsilon_{klm}\partial_k(\overline{V}_l b_m) - \epsilon_{klm}\partial_k(v_l b_m)]. \tag{C19}$$

C.2.1. $\langle v_a b_b(\partial_c\phi)\rangle$

Using Equation (C19), we write

$$\begin{aligned}\langle v_a b_b(\partial_c\phi)\rangle = &\ \epsilon_{klm}\int d^3r\frac{r_c}{4\pi\,|\boldsymbol{r}|^3}\frac{\partial\overline{V}_l(\boldsymbol{y})}{\partial y_k}\langle v_a(\boldsymbol{x})b_b(\boldsymbol{x})b_m(\boldsymbol{x}+\boldsymbol{r})\rangle\\ &+\epsilon_{klm}\int d^3r\frac{r_c}{4\pi\,|\boldsymbol{r}|^3}\overline{V}_l(\boldsymbol{x}+\boldsymbol{r})\frac{\partial}{\partial r_k}\langle v_a(\boldsymbol{x})b_b(\boldsymbol{x})b_m(\boldsymbol{x}+\boldsymbol{r})\rangle\\ &+\epsilon_{klm}\int d^3r\frac{r_c}{4\pi\,|\boldsymbol{r}|^3}\frac{\partial}{\partial r_k}\langle v_a(\boldsymbol{x})b_b(\boldsymbol{x})v_l(\boldsymbol{x}+\boldsymbol{r})b_m(\boldsymbol{x}+\boldsymbol{r})\rangle.\end{aligned} \tag{C20}$$

Following a procedure similar to that used to derive Equation (C4) and then using the VC approximation, we write

$$\langle v_a b_b (\partial_c \phi)\rangle \approx \epsilon_{klm}\frac{\lambda^2}{3}(\partial_k \overline{V}_l)\langle v_a b_b \partial_c b_m\rangle + \epsilon_{klm}\frac{\lambda^2}{3}\overline{V}_l\langle v_a b_b \partial_k \partial_c b_m\rangle + \epsilon_{klm}\frac{\lambda^2}{3}(\partial_c \overline{V}_l)\langle v_a b_b \partial_k b_m\rangle$$
$$+ \epsilon_{klm}\frac{\lambda^2}{3}\langle v_a \partial_c \partial_k v_l\rangle\langle b_b b_m\rangle + \epsilon_{klm}\frac{\lambda^2}{3}\langle v_a \partial_c v_l\rangle\langle b_b \partial_k b_m\rangle + \epsilon_{klm}\frac{\lambda^2}{3}\langle v_a \partial_k v_l\rangle\langle b_b \partial_c b_m\rangle$$
$$+ \epsilon_{klm}\frac{\lambda^2}{3}\langle v_a v_l\rangle\langle b_b \partial_c \partial_k b_m\rangle. \tag{C21}$$

Using Equations (A13), (A7), (A6), (A8), (A5), and (A12), we keep up to one large-scale derivative and write the part of the correlator that does not involve $\overline{V}$ as

$$\langle v_a b_b(\partial_c\phi)\rangle = -\frac{\lambda^2}{9}\langle b^2\rangle\left[\frac{1}{10}\delta_{bc}\partial_a h^v - \frac{7}{30}\delta_{ba}\partial_c h^v - \frac{1}{15}\delta_{ca}\partial_b h^v + \frac{1}{6}\epsilon_{cab}\langle\omega^2\rangle\right] + \frac{\lambda^2}{108}\delta_{bc}h^c\partial_a\langle v^2\rangle$$
$$+ \frac{\lambda^2}{216}\delta_{cb}h^v\partial_a\langle b^2\rangle + \frac{5\lambda^2}{216}\delta_{ab}h^v\partial_c\langle b^2\rangle - \frac{5\lambda^2}{216}\delta_{ab}h^c\partial_c\langle v^2\rangle - \frac{\lambda^2}{216}\delta_{ac}h^c\partial_b\langle v^2\rangle - \frac{\lambda^2}{108}\delta_{ac}h^v\partial_b\langle b^2\rangle$$
$$- \frac{\lambda^2}{9}\langle v^2\rangle\left[\frac{7}{30}\delta_{ab}\partial_c h^c - \frac{1}{10}\delta_{ac}\partial_b h^c + \frac{1}{15}\delta_{cb}\partial_a h^c + \frac{1}{6}\epsilon_{cab}\langle j^2\rangle\right] + \frac{2\lambda^2}{54}\epsilon_{abc}h^v h^c. \tag{C22}$$

For $a = c$, we can use Equations (C33) and (C36) along with Equation (C22) to write

$$\langle v_c b_b(\partial_c\phi)\rangle = \tau\epsilon_{klb}\frac{\lambda^2}{54}(\partial_k\overline{V}_l)\frac{1}{4\pi\rho}h^c h^c + \tau\epsilon_{blm}\frac{\lambda^2}{54}\overline{V}_l\frac{1}{4\pi\rho}\langle b^2\rangle\partial_m\langle j^2\rangle - \tau\epsilon_{blj}\frac{\lambda^2}{216}\overline{V}_l\frac{1}{4\pi\rho}\langle j^2\rangle\partial_j\langle b^2\rangle$$
$$+ \tau\epsilon_{klb}\frac{\lambda^2}{54}\overline{V}_l\frac{1}{4\pi\rho}h^c\partial_k h^c - \frac{\tau\lambda^2}{54}\frac{1}{4\pi\rho}\overline{V}_b h^c\langle j^2\rangle + \tau\epsilon_{jlb}\frac{\lambda^2}{18}\overline{V}_l\langle j^2\rangle\partial_j\langle v^2\rangle + \tau\epsilon_{kbl}\frac{\lambda^2}{108}\overline{V}_l h^v\partial_k h^c$$
$$+ \tau\epsilon_{klb}\frac{\lambda^2}{54}\overline{V}_l\langle v^2\rangle\partial_k\langle j^2\rangle - \tau\epsilon_{bli}\frac{\lambda^2}{216}\overline{V}_l\langle\omega^2\rangle\partial_i\langle b^2\rangle + \tau\epsilon_{bkl}\frac{\lambda^2}{27}\overline{V}_l h^c\partial_k h^v - \tau\frac{\lambda^2}{54}\overline{V}_b h^c\langle\omega^2\rangle$$
$$- \tau\epsilon_{blm}\frac{\lambda^2}{54}\overline{V}_l\langle b^2\rangle\partial_m\langle\omega^2\rangle + \tau\frac{1}{4\pi\rho}\epsilon_{klb}\frac{\lambda^2}{54}(\partial_k\overline{V}_l)h^c h^c + \tau\frac{1}{4\pi\rho}\epsilon_{blc}\frac{\lambda^2}{54}(\partial_c\overline{V}_l)\langle j^2\rangle\langle b^2\rangle$$
$$- \tau\epsilon_{blc}\frac{\lambda^2}{54}(\partial_c\overline{V}_l)\langle\omega^2\rangle\langle b^2\rangle - \tau\frac{1}{90}\epsilon_{klm}\frac{\lambda^2}{3}(\partial_c\overline{V}_l)5\delta_{bk}\delta_{cm}\langle\omega^2\rangle\langle b^2\rangle + \frac{\lambda^2}{27}\langle b^2\rangle\partial_b h^v$$
$$- \frac{\lambda^2}{36}h^c\partial_b\langle v^2\rangle. \tag{C23}$$

For $b = c$, we can use Equations (C33) and (C37) along with Equation (C22) to write

$$\langle v_a b_c(\partial_c\phi)\rangle = -\tau\epsilon_{kla}\frac{\lambda^2}{27}(\partial_k\overline{V}_l)\frac{1}{4\pi\rho}h^c h^c + \tau\epsilon_{kla}\frac{\lambda^2}{18}(\partial_k\overline{V}_l)\frac{1}{4\pi\rho}\langle j^2\rangle\langle b^2\rangle - \tau\epsilon_{kla}\frac{\lambda^2}{18}(\partial_k\overline{V}_l)\langle\omega^2\rangle\langle b^2\rangle$$
$$- \tau\frac{\lambda^2}{54}\overline{V}_l\epsilon_{ila}\frac{1}{4\pi\rho}\langle b^2\rangle\partial_i\langle j^2\rangle - \tau\frac{\lambda^2}{27}\overline{V}_a\frac{1}{4\pi\rho}\langle b^2\rangle\int \mathrm{d}k\ 8\pi k^4 N(k, \boldsymbol{R})$$
$$+ \tau\frac{7\lambda^2}{180}\overline{V}_l\epsilon_{kla}\frac{1}{4\pi\rho}\langle b^2\rangle\partial_k\langle j^2\rangle - \tau\frac{\lambda^2}{72}\overline{V}_l\epsilon_{alj}\frac{1}{4\pi\rho}\langle j^2\rangle\partial_j\langle b^2\rangle + \tau\frac{\lambda^2}{54}\overline{V}_l\epsilon_{kal}\frac{1}{4\pi\rho}h^c\partial_k h^c$$
$$+ \tau\frac{\lambda^2}{54}\overline{V}_a\frac{1}{4\pi\rho}h^c\langle j^2\rangle - \tau\frac{\lambda^2}{24}\overline{V}_l\epsilon_{jla}\langle\omega^2\rangle\partial_j\langle b^2\rangle + \tau\frac{\lambda^2}{27}\overline{V}_a\langle b^2\rangle\int \mathrm{d}k\ 8\pi k^4 F(k, \boldsymbol{R})$$
$$- \tau\frac{7\lambda^2}{180}\overline{V}_l\epsilon_{kla}\langle b^2\rangle\partial_k\langle\omega^2\rangle - \tau\frac{\lambda^2}{108}\overline{V}_l\epsilon_{kla}h^c\partial_k h^v + \tau\frac{\lambda^2}{54}\overline{V}_a\langle\omega^2\rangle h^c + \frac{\lambda^2}{36}h^v\partial_a\langle b^2\rangle$$
$$- \frac{\lambda^2}{27}\langle v^2\rangle\partial_a h^c. \tag{C24}$$

C.2.2. $\langle b_j\phi\rangle$

Using Equation (C19), we can write this correlator as

$$\langle b_j\phi\rangle = \epsilon_{klm}\int \mathrm{d}^3r\frac{1}{4\pi|\boldsymbol{r}|}\overline{V}_l(\boldsymbol{x}+\boldsymbol{r})\left\langle b_j(\boldsymbol{x})\frac{\partial b_m(\boldsymbol{x}+\boldsymbol{r})}{\partial r_k}\right\rangle + \epsilon_{klm}\int \mathrm{d}^3r\frac{1}{4\pi|\boldsymbol{r}|}\frac{\partial\overline{V}_l(\boldsymbol{y})}{\partial y_k}\langle b_j(\boldsymbol{x})b_m(\boldsymbol{x}+\boldsymbol{r})\rangle$$
$$+ \epsilon_{klm}\int \mathrm{d}^3r\frac{1}{4\pi|\boldsymbol{r}|}\frac{\partial}{\partial r_k}\langle b_j(\boldsymbol{x})v_l(\boldsymbol{x}+\boldsymbol{r})b_m(\boldsymbol{x}+\boldsymbol{r})\rangle. \tag{C25}$$

Using the VC approximation, we write

$$\begin{aligned}\langle b_j\phi\rangle = \epsilon_{klm}\lambda^2\overline{V}_l\langle b_j\partial_k b_m\rangle + \epsilon_{klm}\lambda^2(\partial_k\overline{V}_l)\langle b_j b_m\rangle \\ + \epsilon_{klm}\lambda^2\partial_k\langle v_l b_j b_m\rangle - \epsilon_{klm}\lambda^2\langle v_l b_m(\partial_k b_j)\rangle.\end{aligned} \tag{C26}$$

For the correlator $\langle vbb\rangle$, we can use Equation (C31), while for the correlator $\langle vb\partial b\rangle$, we can use Equation (C32). Using these along with Equations (A7) and (A8) and keeping only up to one large-scale derivative, the above can be written as

$$\begin{aligned}\langle b_j\phi\rangle = \lambda^2\frac{1}{12}\epsilon_{jkl}\overline{V}_l\partial_k\langle b^2\rangle - \overbrace{\lambda^2\frac{1}{3}h^c\overline{V}_j}^{\text{part of the advective flux}} + \lambda^2\frac{1}{3}\langle b^2\rangle\epsilon_{jkl}(\partial_k\overline{V}_l) + \tau^2\lambda^2\frac{1}{18}\frac{1}{4\pi\rho}h^c h^c\epsilon_{jdk}(\partial_d\overline{V}_k) \\ - \tau\lambda^2\frac{1}{4\pi\rho}\frac{1}{6}h^c\partial_j\langle b^2\rangle + \tau\lambda^2\frac{1}{4\pi\rho}\frac{1}{9}\langle b^2\rangle\partial_j h^c - \tau\lambda^2\frac{1}{12}h^v\partial_j\langle b^2\rangle - \tau\lambda^2\frac{5}{36}h^c\partial_j\langle v^2\rangle + \tau\lambda^2\frac{1}{9}\langle v^2\rangle\partial_j h^c.\end{aligned} \tag{C27}$$

We note that if one is willing to make the identification $\lambda^2 h^c = h^b$, the second term above is just a part of the already-known advective flux (we show this in Appendix D).

C.2.3. $\langle(\partial_a\phi)\partial_d(v_b b_c)\rangle$

Here, we do not need to keep terms containing the mean velocity, but we need to keep up to one large-scale derivative. Using Equation (C19) and following a procedure similar to that used to obtain Equation (C4), we write

$$\begin{aligned}\langle(\partial_a\phi)\partial_d(v_b b_c)\rangle = \epsilon_{klm}\int\frac{d^3r}{4\pi}\left(-\frac{\delta_{ka}}{r^3} + \frac{3r_k r_a}{r^5}\right)\left\langle\frac{\partial v_b(\boldsymbol{x})}{\partial x_d}v_l(\boldsymbol{x}-\boldsymbol{r})\right\rangle\langle b_c(\boldsymbol{x})b_m(\boldsymbol{x}-\boldsymbol{r})\rangle \\ + \epsilon_{klm}\int\frac{d^3r}{4\pi}\left(-\frac{\delta_{ka}}{r^3} + \frac{3r_k r_a}{r^5}\right)\langle v_b(\boldsymbol{x})v_l(\boldsymbol{x}-\boldsymbol{r})\rangle\left\langle\frac{\partial b_c(\boldsymbol{x})}{\partial x_d}b_m(\boldsymbol{x}-\boldsymbol{r})\right\rangle.\end{aligned} \tag{C28}$$

Using the VC approximation, we write, for $a = b$ and $a = c$, respectively,

$$\begin{aligned}\langle(\partial_a\phi)\partial_d(v_a b_c)\rangle \approx -\lambda^2\frac{1}{36}\epsilon_{cdk}h^c\partial_k h^v + \lambda^2\frac{1}{360}\epsilon_{dac}\langle\omega^2\rangle\partial_a\langle b^2\rangle + \lambda^2\frac{1}{90}\delta_{cd}h^c\langle\omega^2\rangle \\ - \epsilon_{ldc}\lambda^2\frac{7}{1080}\langle j^2\rangle\partial_l\langle v^2\rangle - \lambda^2\frac{1}{540}\epsilon_{cdg}h^v\partial_g h^c - \lambda^2\frac{1}{270}\delta_{cd}h^v\langle j^2\rangle \\ + \lambda^2\frac{1}{135}\delta_{cd}\langle b^2\rangle\int dk\ 8\pi k^4 F(k,\boldsymbol{R}) + \lambda^2\frac{13}{1350}\epsilon_{dac}\langle b^2\rangle\partial_a\langle\omega^2\rangle,\end{aligned} \tag{C29}$$

$$\begin{aligned}\langle(\partial_a\phi)\partial_d(v_b b_a)\rangle \approx \epsilon_{dbm}\lambda^2\frac{7}{1080}\langle\omega^2\rangle\partial_m\langle b^2\rangle + \lambda^2\frac{1}{36}\epsilon_{lbd}h^v\partial_l h^c + \lambda^2\frac{1}{360}\epsilon_{dba}\langle j^2\rangle\partial_a\langle v^2\rangle \\ - \lambda^2\frac{1}{90}\delta_{bd}h^v\langle j^2\rangle - \lambda^2\frac{1}{135}\delta_{bd}\langle v^2\rangle\int dk\ 8\pi k^4 N(k,\boldsymbol{R}) \\ + \epsilon_{dbm}\lambda^2\frac{22}{2025}\langle v^2\rangle\partial_m\langle j^2\rangle + \lambda^2\frac{1}{540}\epsilon_{bde}h^c\partial_e h^v + \lambda^2\frac{1}{270}\delta_{bd}h^c\langle\omega^2\rangle.\end{aligned} \tag{C30}$$

C.3. More Triple Correlators

C.3.1. $\langle v_a b_b b_c\rangle$

Using Equations (A7), (A12), (A5), and (A6), dropping the nonlocal pressure term, and keeping no large-scale derivatives, we write Equation (B4) as

$$\langle v_a b_b b_c\rangle = \mathcal{O}(\partial), \tag{C31}$$

i.e., all the local terms involve large-scale derivatives.

C.3.2. $\langle v_l(\partial_k b_j)b_m\rangle$

We need the combination $\epsilon_{klm}\langle v_l b_m\partial_k b_j\rangle$ (which appears in Equation (C26)). We do not need to keep any nonlocal terms, and we need to keep up to one large-scale derivative. We can discard $\mathcal{O}(\tau^3)$ terms. Following the same procedure as in Appendix B, we obtain

$$\epsilon_{klm}\langle v_l(\partial_k b_j)b_m\rangle = -\tau^2\frac{1}{18}\epsilon_{jdk}(\partial_d\overline{V}_k)\frac{1}{4\pi\rho}h^c h^c + \tau\frac{1}{4\pi\rho}\frac{1}{6}h^c\partial_j\langle b^2\rangle - \tau\frac{1}{4\pi\rho}\frac{1}{9}\langle b^2\rangle\partial_j h^c + \tau\frac{1}{12}h^v\partial_j\langle b^2\rangle + \tau\frac{5}{36}h^c\partial_j\langle v^2\rangle - \tau\frac{1}{9}\langle v^2\rangle\partial_j h^c + \mathcal{O}(\tau^3). \tag{C32}$$

A similar correlator also appears in Equation (C21), but this time, we can discard terms involving the mean velocity, terms with large-scale derivatives, and nonlocal terms. Following the same procedure as in Appendix B, we obtain

$$\langle v_l b_m(\partial_k b_j)\rangle = \tau\frac{1}{4\pi\rho}\frac{1}{36}h^c h^c(\delta_{kl}\delta_{jm} - \delta_{km}\delta_{jl}) + \tau\frac{1}{4\pi\rho}\frac{1}{90}(4\delta_{mk}\delta_{lj} - \delta_{ml}\delta_{kj} - \delta_{mj}\delta_{kl})\langle j^2\rangle\langle b^2\rangle - \tau\frac{1}{90}(4\delta_{mk}\delta_{lj} - \delta_{ml}\delta_{kj} - \delta_{mj}\delta_{kl})\langle\omega^2\rangle\langle b^2\rangle + \mathcal{O}(\tau^2). \tag{C33}$$

C.3.3. $\langle b_b a_c \partial_e \partial_a v_d\rangle$

This correlator appears multiplied with the mean velocity, and we do not need to keep any large-scale derivatives in it. This correlator also appears multiplied with the square of the correlation length, so we do not keep any nonlocal terms (involving the pressure and the scalar potential). Following the same procedure as in Appendix B, we write

$$\langle(\partial_e\partial_i v_d)b_b a_i\rangle = \frac{\tau}{4\pi\rho}\langle(\partial_e\partial_i b_j)b_b\rangle\epsilon_{ijd}\frac{\langle b^2\rangle}{6} + \frac{\tau}{4\pi\rho}\epsilon_{bij}\frac{h^c}{6}\langle(\partial_e\partial_j b_d)a_i\rangle + \frac{\tau}{4\pi\rho}\frac{1}{3}h^b\langle(\partial_e\nabla^2 b_d)b_b\rangle - \tau\langle(\partial_e\nabla^2 v_d)v_b\rangle\frac{1}{3}h^b + \tau\langle(\partial_e\partial_i v_d)v_j\rangle\epsilon_{ijb}\frac{\langle b^2\rangle}{6} + \mathcal{O}(\tau^2) \tag{C34}$$

and

$$\langle(\partial_e\partial_a v_d)b_i a_i\rangle = \frac{\tau}{4\pi\rho}\langle(\partial_e\partial_a b_j)b_i\rangle\epsilon_{ijd}\frac{\langle b^2\rangle}{6} + \frac{\tau}{4\pi\rho}\langle(\partial_e\partial_a b_j)a_i\rangle\epsilon_{ijd}\frac{h^c}{6} + \frac{\tau}{4\pi\rho}\epsilon_{iaj}\frac{h^c}{6}\langle(\partial_e\partial_j b_d)a_i\rangle + \frac{\tau}{4\pi\rho}\epsilon_{iaj}\frac{\langle b^2\rangle}{6}\langle(\partial_e\partial_j b_d)b_i\rangle + \frac{\tau}{4\pi\rho}\epsilon_{iej}\frac{h^c}{6}\langle(\partial_a\partial_j b_d)a_i\rangle + \frac{\tau}{4\pi\rho}\epsilon_{iej}\frac{\langle b^2\rangle}{6}\langle(\partial_a\partial_j b_d)b_i\rangle + \mathcal{O}(\tau^2). \tag{C35}$$

C.3.4. $\langle v_a b_b \partial_k \partial_c b_m\rangle$

This correlator appears in Equation (C21), multiplied by ϵ_{klm}. We need to keep up to one large-scale derivative, but we can discard terms involving the mean velocity. It is also multiplied by the square of the correlation length, so we do not keep any nonlocal terms. Following the same procedure as in Appendix B, we write

$$\epsilon_{klm}\langle v_i b_b(\partial_k\partial_i b_m)\rangle = \tau\epsilon_{blm}\frac{1}{4\pi\rho}\frac{1}{18}\langle b^2\rangle\partial_m\langle j^2\rangle - \tau\epsilon_{blj}\frac{1}{4\pi\rho}\frac{1}{72}\langle j^2\rangle\partial_j\langle b^2\rangle + \tau\epsilon_{bkl}\frac{1}{4\pi\rho}\frac{1}{18}h^c\partial_k h^c - \tau\delta_{lb}\frac{1}{4\pi\rho}\frac{1}{18}h^c\langle j^2\rangle + \tau\frac{1}{6}\epsilon_{jlb}\langle j^2\rangle\partial_j\langle v^2\rangle + \tau\epsilon_{kbl}\frac{1}{36}h^v\partial_k h^c + \tau\epsilon_{klb}\frac{1}{18}\langle v^2\rangle\partial_k\langle j^2\rangle - \tau\epsilon_{bli}\frac{1}{72}\langle\omega^2\rangle\partial_i\langle b^2\rangle + \tau\frac{1}{9}\epsilon_{bkl}h^c\partial_k h^v - \tau\delta_{lb}\frac{1}{18}h^c\langle\omega^2\rangle - \tau\epsilon_{blm}\frac{1}{18}\langle b^2\rangle\partial_m\langle\omega^2\rangle \tag{C36}$$

$$\epsilon_{klm}\langle v_a b_i(\partial_k\partial_i b_m)\rangle = -\tau\epsilon_{ila}\frac{1}{4\pi\rho}\frac{1}{18}\langle b^2\rangle\partial_i\langle j^2\rangle - \tau\delta_{la}\frac{1}{4\pi\rho}\frac{1}{9}\langle b^2\rangle\int \mathrm{d}k\ 8\pi k^4 N(k, \boldsymbol{R}) + \tau\epsilon_{kla}\frac{1}{4\pi\rho}\frac{7}{60}\langle b^2\rangle\partial_k\langle j^2\rangle - \tau\epsilon_{alj}\frac{1}{4\pi\rho}\frac{1}{24}\langle j^2\rangle\partial_j\langle b^2\rangle + \tau\epsilon_{kal}\frac{1}{4\pi\rho}\frac{1}{18}h^c\partial_k h^c + \tau\delta_{al}\frac{1}{4\pi\rho}\frac{1}{18}h^c\langle j^2\rangle - \tau\epsilon_{jla}\frac{1}{8}\langle\omega^2\rangle\partial_j\langle b^2\rangle + \tau\delta_{la}\frac{1}{9}\langle b^2\rangle\int \mathrm{d}k\ 8\pi k^4 F(k, \boldsymbol{R}) - \tau\epsilon_{kla}\frac{7}{60}\langle b^2\rangle\partial_k\langle\omega^2\rangle - \tau\epsilon_{kla}\frac{1}{36}h^c\partial_k h^v + \tau\delta_{al}\frac{1}{18}\langle\omega^2\rangle h^c \tag{C37}$$

Appendix D
The Advective Flux

Let us try to evaluate the contribution of the $-\overline{\boldsymbol{V}} \times \boldsymbol{b}$ term of $\boldsymbol{e}$ (Equation (5)). Writing $e_k = -\epsilon_{klm} \overline{V}_l b_m$, we can write the last term on the RHS of Equation (8) as

$$-\partial_j \langle \epsilon_{jki} e_k a_i \rangle_{\text{advective}} = \partial_j[\overline{V}_i \langle b_j a_i \rangle] - \partial_j[\overline{V}_j \langle b_i a_i \rangle]. \tag{D1}$$

Using Equation (A9), we can write the above as

$$-\partial_j \langle \epsilon_{jki} e_k a_i \rangle_{\text{advective}} = -\frac{2}{3}\partial_j[\overline{V}_j h^b] + \frac{1}{12}\epsilon_{ijm}(\partial_j \overline{V}_i)\partial_m \langle a^2 \rangle. \tag{D2}$$

Recall that in Appendix C.2.2, we obtained a similar advective contribution from the $\langle b_i \partial_i \varphi \rangle$ term (Equation (C27)). Adding both these terms (and identifying $\lambda^2 h^c = h^b$), we recover an advective flux given by $V_j h^b$, in agreement with the gauge-invariant calculation (Subramanian & Brandenburg 2006).

Appendix E
Full Expression without Further Approximations

$$\begin{aligned}
-\partial_j(F_j^T) = \partial_j\Bigg[& \frac{\tau}{9}\left(\langle v^2 \rangle + \frac{\langle b^2 \rangle}{4\pi\rho}\right)\partial_j h^b + \tau\frac{1}{18}\left(h^v - \frac{h^c}{4\pi\rho}\right)\partial_j \langle a^2 \rangle + \frac{\tau}{9} h^b \partial_j\left(\langle v^2 \rangle - \frac{\langle b^2 \rangle}{4\pi\rho}\right) \\
& - \tau\lambda^2 \frac{5}{54}\frac{1}{4\pi\rho} h^c \partial_j \langle b^2 \rangle + \tau\lambda^2 \frac{1}{4\pi\rho}\frac{5}{27}\langle b^2 \rangle \partial_j h^c - \tau\lambda^2 \frac{1}{9} h^v \partial_j \langle b^2 \rangle - \tau\lambda^2 \frac{1}{6} h^c \partial_j \langle v^2 \rangle \\
& + \tau\lambda^2 \frac{4}{27}\langle v^2 \rangle \partial_j h^c - \tau\frac{\lambda^2}{27}\frac{1}{4\pi\rho}\langle j^2 \rangle \partial_j h^b + \frac{\tau\lambda^2}{27}\langle b^2 \rangle \partial_j h^v + \frac{\lambda^2}{4}\langle b^2 \rangle \epsilon_{jkl}(\partial_k \overline{V}_l) \\
& - \frac{\tau^2\lambda^2}{27}(\partial_d \overline{V}_e)\frac{1}{4\pi\rho}\epsilon_{jde} h^b \int \mathrm{d}k\ 8\pi k^4 N(k, \boldsymbol{R}) + \frac{\tau^2\lambda^2}{27}(\partial_d \overline{V}_e)\epsilon_{jde} h^b \int \mathrm{d}k\ 8\pi k^4 F(k, \boldsymbol{R}) \\
& + \frac{4\tau^2\lambda^2}{27}\epsilon_{jde}(\partial_d \overline{V}_e)\frac{1}{4\pi\rho} h^c h^c - \tau^2 \frac{\lambda^2}{36}\epsilon_{jlk}\overline{V}_l \frac{1}{4\pi\rho}\langle j^2 \rangle \partial_k \langle b^2 \rangle - \tau^2 \frac{4\lambda^2}{135}\epsilon_{jlk}\overline{V}_l \frac{1}{4\pi\rho} h^c \partial_k h^c \\
& + \tau^2 \frac{7\lambda^2}{108}\epsilon_{jel}\overline{V}_l \langle j^2 \rangle \partial_e \langle v^2 \rangle - \tau^2 \frac{7\lambda^2}{180}\epsilon_{jkl}\overline{V}_l h^v \partial_k h^c + \tau^2 \frac{119\lambda^2}{4050}\epsilon_{jkl}\overline{V}_l \langle v^2 \rangle \partial_k \langle j^2 \rangle \\
& - \tau^2 \frac{\lambda^2}{18}\epsilon_{jli}\overline{V}_l \langle \omega^2 \rangle \partial_i \langle b^2 \rangle + \tau^2 \frac{\lambda^2}{60}\epsilon_{jkl}\overline{V}_l h^c \partial_k h^v - \tau^2 \frac{181\lambda^2}{2700}\epsilon_{jlm}\overline{V}_l \langle b^2 \rangle \partial_m \langle \omega^2 \rangle \\
& - \tau^2 \frac{\lambda^2}{18}\frac{1}{4\pi\rho}\epsilon_{ejd}(\partial_d \overline{V}_e)\langle j^2 \rangle \langle b^2 \rangle - \tau^2 \frac{2\lambda^2}{27}\epsilon_{edj}(\partial_d \overline{V}_e)\langle \omega^2 \rangle \langle b^2 \rangle \\
& - \tau^2 \frac{7\lambda^2}{180}\overline{V}_l \epsilon_{klj}\frac{1}{4\pi\rho}\langle b^2 \rangle \partial_k \langle j^2 \rangle - \overline{V}_l \tau^2 \epsilon_{jli}\frac{1}{18}\frac{1}{4\pi\rho}\langle b^2 \rangle \partial_i \langle b^2 \rangle + \overline{V}_l \tau^2 \epsilon_{jli}\frac{1}{6}\langle b^2 \rangle \partial_i \langle v^2 \rangle \\
& + \overline{V}_l \tau^2 \epsilon_{jli}\frac{1}{9}\langle v^2 \rangle \partial_i \langle b^2 \rangle + \overline{V}_l \tau^2 \epsilon_{jli}\frac{1}{18}\frac{h^c}{4\pi\rho}\partial_i h^b - \overline{V}_l \tau^2 \epsilon_{jli}\frac{1}{18} h^v \partial_i h^b \Bigg] \\
& - \frac{1}{3}\partial_j[\lambda^2 h^c \overline{V}_j] - \frac{2}{3}\partial_j[h^b \overline{V}_j] + \frac{1}{12}\epsilon_{ijm}(\partial_j \overline{V}_i)\partial_m \langle a^2 \rangle.
\end{aligned} \tag{E1}$$

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