22. Kinematic Dynamo Theory; Mean Field Theory

Dynamo Solutions

We seek solutions to the Induction (dynamo) equation

$$\partial \mathbf{B}/\partial t = \lambda\nabla^2\mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{22.1}$$

that do not decay with time and have no external exciting field. These are called a *dynamo.* Obviously the induction term must offset diffusion. To order of magnitude, we must have

$$\begin{aligned} &\left|\nabla\times(\vec{u}\times\vec{B})\right| \sim \frac{\pi u B}{L} \sim \left|\lambda\nabla^2 B\right| \sim \frac{\pi^2\lambda B}{L^2} \\ &\Rightarrow \frac{uL}{\lambda} \sim \pi \end{aligned} \tag{22.2}$$

where L is some characteristic size of the region in which the field is generated. The dimensionless number uL/λ is called the magnetic Reynolds number and it must be sufficiently large (about 10 or more) in order that a dynamo exist. However, the existence of a dynamo turns out to be a matter of some subtlety because it depends on the form of the flow as well as the magnitude. *Most simple flows do not produce dynamos irrespective of their magnitude.*

In this chapter we consider *kinematic* dynamos. These are solutions where we specify the velocity field (without asking where it came from). However, we will focus on physically plausible motions, especially those relevant to mean field models. *Mean field* is standard physics jargon for any situation where small scale fluctuations are averaged, yielding a large scale outcome. (In this case, it means there are small scale motions exciting a large scale field). A *fully dynamical dynamo* is one where the velocity field is determined from solution of the equation of motion (which includes the Lorentz force arising from the field). In the next chapter, we will talk about the form that convection takes in the presence of a magnetic field and discuss a little the fully dynamical dynamos (for which only numerical solutions exist).

A "Simple" Example of Dynamo Action (The α effect)

Actually, there are no really simple examples of dynamo action since they all involve 3D velocity fields and there is no "closure" of the dynamo equation when the scale length of the flow is similar to the scale length of the convection. (By lack of closure, we mean that the action of the flow on a specified field might reproduce the original field but in addition produces a field that is usually more spatially complex than the one we are trying to make.) But let's take the simplest case known, which turns out to be a case where we assume a small scale flow and attribute to it the property of helicity (defined later). Specifically, consider a flow field and magnetic field of the forms:

$$\vec{u} = \sum_{\vec{q}} \vec{u}(\vec{q}) e^{i\vec{q}.\vec{r}}; \quad \vec{B} = \vec{B}_0 e^{i\vec{k}.\vec{r}+\sigma t} + \vec{b}(\vec{r}) e^{\sigma t} \qquad (22.3)$$

where q>> k is assumed (i.e. the flow is small scale but part of the field is large scale, i.e. small wavevector). The idea is that the flow u acts on the large scale field B to produce a small scale field b (which we will compute). The flow then acts on the small scale field to reproduce the large scale field. Let's see how this works:

$$\vec{u} \times \vec{B} = e^{\sigma t} \sum_{\vec{q}} [\vec{u}(\vec{q}) \times \vec{B}_0] e^{i(\vec{q}+\vec{k}).\vec{r}}$$

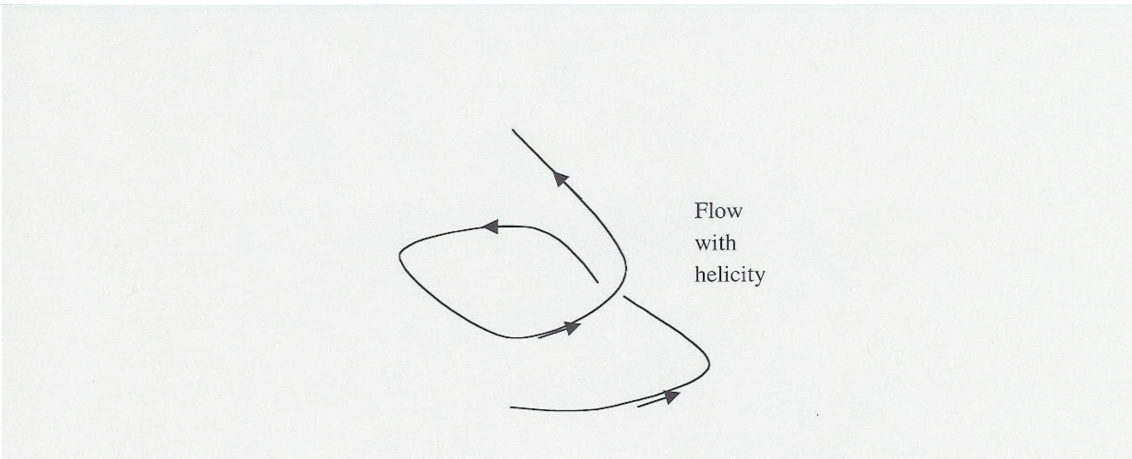
$$\therefore \vec{b}(\vec{r}) = \sum_{\vec{q}} \vec{b}(\vec{q}) e^{i(\vec{q}+\vec{k}).\vec{r}}; \; \sigma \vec{b}(\vec{q}) \approx -\lambda q^2 \vec{b}(\vec{q}) + i\vec{q} \times [\vec{u}(\vec{q}) \times \vec{B}_0] \qquad (22.4)$$

$$\therefore \vec{u} \times \vec{b} = \sum_{\vec{q},\vec{q}'} \frac{1}{(\sigma + \lambda q^2)} .i\vec{u}(\vec{q}') \times \{\vec{q} \times [u(\vec{q}) \times \vec{B}_0]\} e^{i(\vec{q}+\vec{q}'+\vec{k})}$$

(The assumption q>>k is used again in the second line.) In the spirit of *mean field theory*, we focus on those contributions that can affect the large scale field. So we choose **q′ = -q**, and we see that **u x b** can be written in the form **α.B** where α is a tensor :

$$\vec{\vec{\alpha}} = \sum_{\vec{q}} \frac{1}{(\sigma + \lambda q^2)} .i[\vec{u}(-\vec{q}) \times \vec{u}(\vec{q})]\vec{q} \qquad (22.5)$$

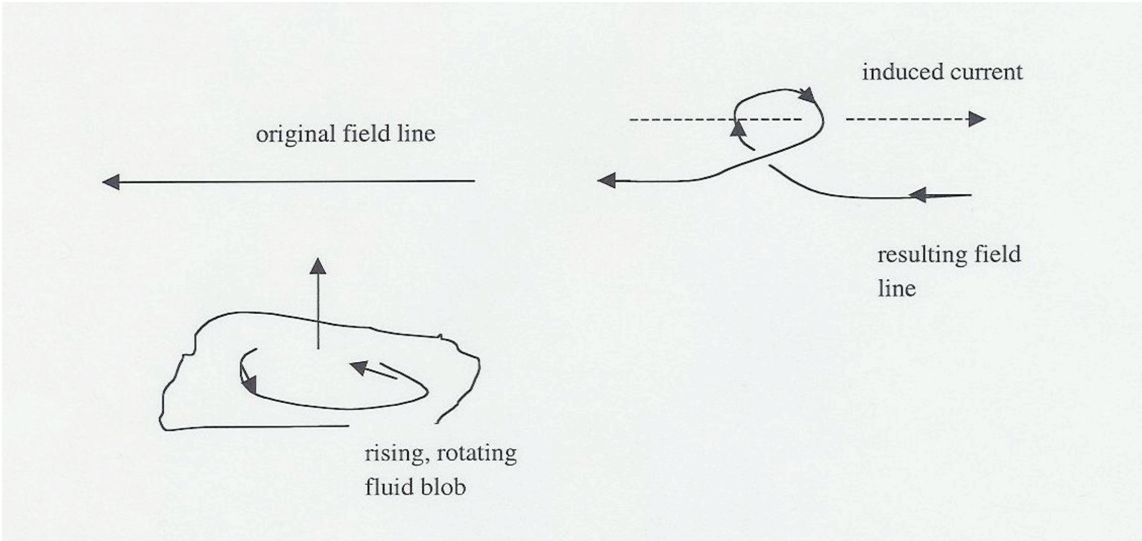
(making use of the fact that **q.u** = 0 for incompressible flow). Now **u(q).[q x u(-q)] = q.[u(-q) x u(q)]** so this tensor clearly involves a measure of the helicity, defined as **u.(∇xu)**, the dot product of vorticity and flow. The name given to this scalar quantity is self-evident if one thinks about the properties of a fluid element that follows a helical path. The crucial idea is that this can have a non-zero mean (as well as fluctuating parts) but the mean field part is most important since it can lead to a generation of large scale fields.



In general, this alpha model (as it is so called) yields an equation of the form:

$$\frac{\partial \vec{B}}{\partial t} = \lambda \nabla^2 \vec{B} + \vec{\nabla} \times (\vec{\vec{\alpha}} . \vec{B}) \qquad (22.6)$$

In the particular case where we treat alpha as a scalar (i.e., only diagonal elements, all of the same size), we can visualize the alpha effect as in the following cartoon: A current is created that is parallel (or antiparallel) to the existing field.



Mathematically, this alpha effect can by itself sustain a dynamo:

$$
\begin{aligned}
&(\sigma + \lambda k^2)\vec{B}_0 = i\alpha\vec{k} \times \vec{B}_0 \\
&\Rightarrow (\sigma + \lambda k^2)\vec{k} \times \vec{B}_0 = i\alpha\vec{k} \times (\vec{k} \times \vec{B}_0) = -i\alpha k^2 \vec{B}_0 = -i\alpha k^2 \{\frac{i\alpha\vec{k} \times \vec{B}_0}{(\sigma + \lambda k^2)}\} \\
&\Rightarrow (\sigma + \lambda k^2)^2 = \alpha^2 k^2 \\
&\therefore \sigma > 0 \Rightarrow \frac{\alpha}{\lambda k} > 1
\end{aligned}
\tag{22.7}
$$

This last requirement for dynamo growth is equivalent to exceeding a critical magnetic Reynold's number (since alpha has dimensions of a velocity and k is an inverse length). Note however that alpha is <u>not</u> the fluid velocity, in fact it is roughly fluid velocity times a small scale magnetic Reynold's number ($\alpha/\lambda q$), which may well be a small number. So in this model, at least, the criterion for a dynamo is something like (small scale Magnetic Reynold's number) x (large scale magnetic Reynold's number) > 10.

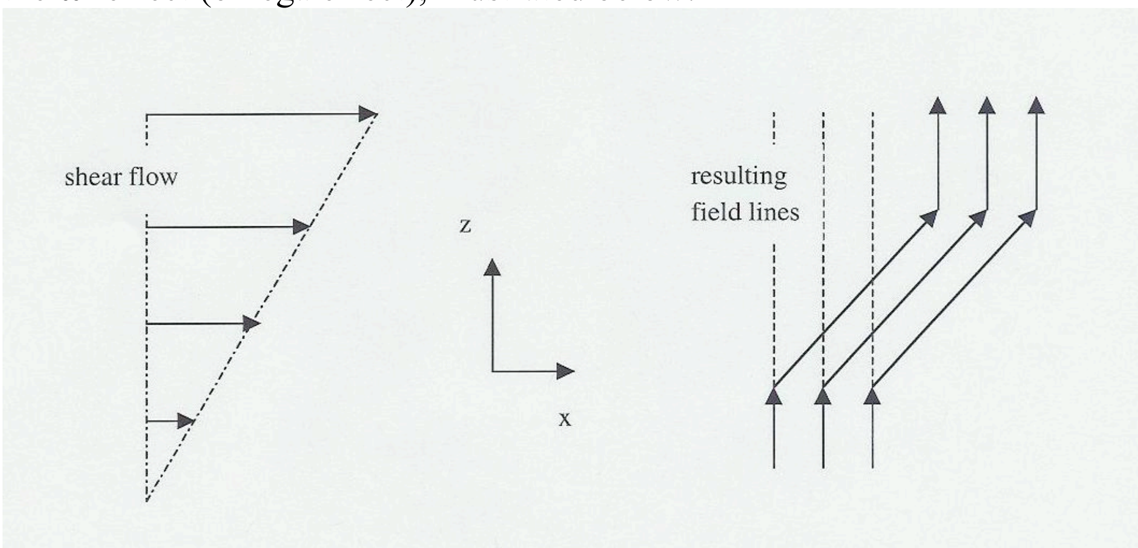
This alpha effect is popular in mathematical models. There is some doubt whether it is the dominant process in actual dynamos, at least in planets. (It is popular in stellar dynamo models). The dominant process may not be simply characterized; it is complex. But one other effect is likely to be important: Differential rotation (shear).

The ω -effect (Omega Effect).

In the context of the Cartesian model discussed above, this is a large scale flow that converts one large scale component into another. Specifically, consider the flow in the x-direction in the form ωz, and suppose the initial field (magnitude B_0) is purely in the z-direction. The induction effect is

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\omega z \hat{x} \times B_0 \hat{z}) = \omega B_0 \hat{x} \qquad (22.8)$$

and thus an x-component of the field grows linearly with time. This is called the ω -effect (omega effect), illustrated below.



It is not a dynamo by itself because it only converts one field component into another; it does not regenerate the field you started with.

A popular simple dynamo model is the αω-dynamo (alpha-omega dynamo), in which the alpha effect is used to convert one field component (e.g. the x-component) into another (e.g. the z-component) and the omega effect is used to convert the z-component back into the x-component. This is motivated by the fact that the omega effect is very powerful but cannot create a dynamo by itself, so the alpha effect is invoked to complete the regenerative cycle. (It is also true that the alpha effect is often very anisotropic and is most likely to convert horizontal field into vertical field.)

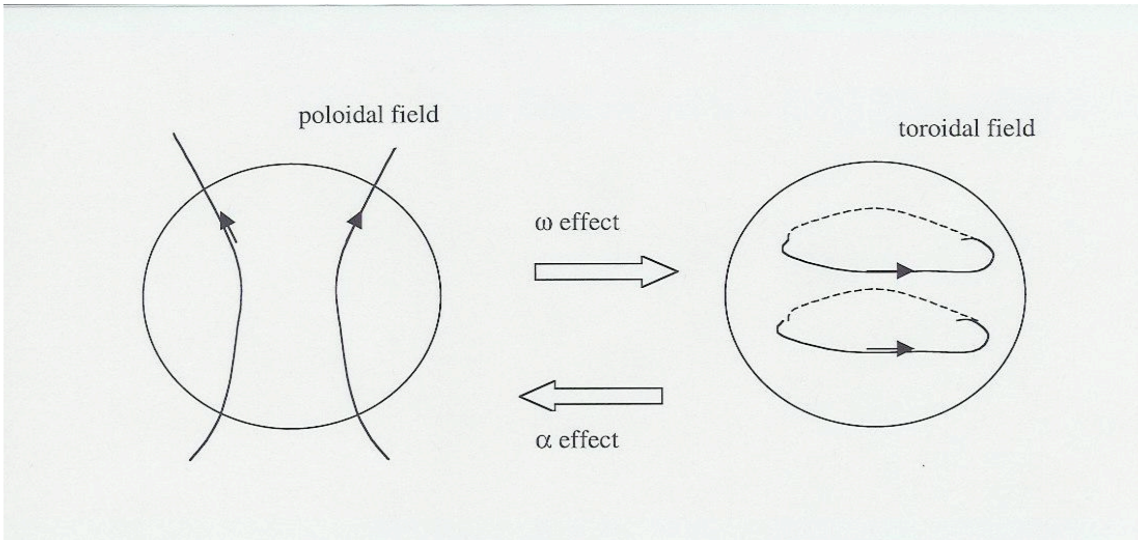
Consider a field of the form $\mathbf{B} = (B_x, 0, B_z)\exp[\sigma t+iky]$. Then the x and z components of the dynamo equation become:

$$\sigma B_x = -\lambda k^2 B_x + \omega B_z$$
$$\sigma B_z = -\lambda k^2 B_z - \alpha i k B_x$$
$$\Rightarrow B_x = \frac{\omega}{(\sigma+\lambda k^2)}.B_z = \frac{\omega}{(\sigma+\lambda k^2)}.\frac{-i\alpha k}{(\sigma+\lambda k^2)}.B_x \qquad (22.9)$$
$$\Rightarrow \sigma = -\lambda k^2 \pm \frac{(1-i)}{\sqrt{2}}.\sqrt{\alpha\omega k}$$
$$\Rightarrow \mathrm{Re}(\sigma) > 0 \quad \text{if} \quad \frac{\alpha}{\lambda k}.\frac{\omega}{\lambda k^2} > 2$$

assuming (for simplicity) that alpha and omega are positive (but it works no matter what signs they have). Notice that this is an overstability* (or, more correctly, a growing wave propagation). This solution was first found by Eugene Parker in the late 1950's and has a central role to play in the history of dynamo theory (as well as being physically sensible). In the context of a sphere, the x-component should be thought of as the toroidal field and the z-component should be thought of as the radial field. The particular simple solution above is then for a field that has spatial variation in the North-South direction only, but this can be elaborated to more realistic situations (by numerical analysis, generally). The solution is *directly applicable* to the time-varying solar magnetic field (the solar cycle). In planets, there are numerical solutions that exhibit DC behavior (i.e., the overstability is suppressed).

*Overstability is the word used in applied math and physics for any system that exhibits a growing oscillation. Of course, this is only the linear response: The finite amplitude response is not necessarily oscillatory.

This cartoon illustrates the nature of the dynamo in this instance.



Problem 22.1

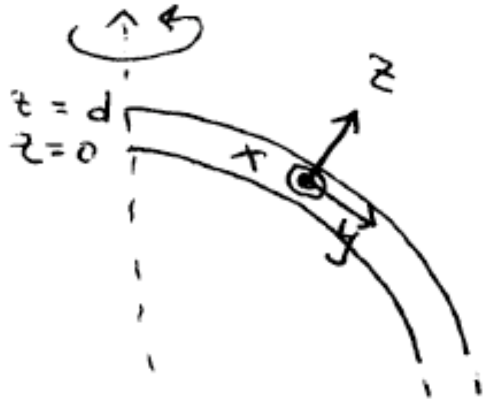
It has been suggested that small asteroidal bodies could have had dynamos early in the solar system (but only for a short time). These bodies suffered ^{26}Al heating that allowed the interior to melt and form a core. The relevant criterion is $R_m > 10$ (approximately) where the magnetic Reynolds number $R_m \equiv vL/\lambda$; v is convective velocity, L is lengthscale available for the motions and λ is the magnetic diffusivity ($1 m^2/sec$ or 10^4 cm^2/sec for liquid iron.) Assuming the mantle is also of thickness ~L (so asteroid radius is ~2L) and therefore has a heat flow $\sim k\Delta T/L$, equate this to the convective heat flux and thus estimate the convective velocity and estimate whether a dynamo is possible. Your answer will be affected by the choice of L, obviously. You can only do this calculation crudely since there are fudge parameters in mixing length theory that are poorly known. The lifespan of the dynamo will be of order L^2/κ . Why? (Use $\kappa \sim 0.01 cm^2/sec$; $k=\rho C_p\kappa$; $\Delta T \sim 1000K$).

Note: This is simpler than the real problem because you must also convince yourself that the heat flow exceeds that due to conduction along an adiabat. But the basic principle is right: Small bodies can have high heat flows and a dynamo, albeit for only a short time.

Problem 22.2

Consider a "shell" dynamo in which the field generation arises from an alpha effect. The governing equation is accordingly

$$\partial \mathbf{B}/\partial t = \lambda \nabla^2 \mathbf{B} + \alpha \nabla \mathbf{x} \mathbf{B}$$



where α, λ are constants. If the shell is thin then we can set up local Cartesian coordinates (as shown below).z=0 is the base of the shell in which the dynamo operates and z=d is at the top of the dynamo shell.

x represents a longitudinal and y a latitudinal coordinate. We seek solutions that behave like sin(ky).
Obviously, k ~1/R for a dipole (so that a field component that is zero at a pole, y=0 say, will be a maximum at the equator where $y=\pi R/2$, etc.) And k~2/R for a quadrupole, etc. We seek time-independent axisymmetric solutions (which means that there is no x-dependence or t-dependence anywhere. But there can of course be x-components of fields!)

(a) Explain why it is physically and mathematically OK to write the field in the form $\mathbf{B} = (B, \partial A/\partial z, -\partial A/\partial y)$ where B and A are scalar functions of y,z. (and the components of the vector are in the usual order x,y,z). Hence show that

$$0= \lambda\nabla^2 A + \alpha B$$
$$0= \lambda\nabla^2 B - \alpha\nabla^2 A$$

(b) Assume the domains z<0 and z>d are both insulators. Solve for B and prove that the most easily excited mode (i.e., the one with the lowest α) satisfies $(\pi/d)^2 + k^2 = (\alpha/\lambda)^2$. Hence explain why dipole and quadrupole modes are about equally likely. [Hint: This is easy but you must understand what the boundary conditions are for B. Other boundary conditions don't matter.]

(c) Repeat the analysis for the case where z<0 is a conductor but has no dynamo action (i.e., an "inner core"). z>d is still insulator or vacuum. *Warning:* The solution for this case is simple but not obvious! It will probably give you more difficulty than (b) despite the very simple answer. You need to know something about correct boundary conditions for electric field. Specifically, think about the x-component of the electric field (does it exist?) and think about the boundary condition for the x-component of the total electric field $\mathbf{E} + \mathbf{u} \times \mathbf{B}$. (In this particular case, $\mathbf{u} \times \mathbf{B}$ is replaced by $\alpha\mathbf{B}$ in the dynamo region and zero elsewhere.)