$$
\begin{array}{r}
+30 \\
-25 \\
+15 \\
\hline
+20.
\end{array}
$$

Sk Jamaluddin

Sonav Dey

Sujātā Sinha

$$F^{NV} = -\frac{\omega\tau_c^2 b^2}{3}\left[\left(\frac{2}{5}+\frac{11}{15}g\right)u^2 + \frac{13}{66}g\frac{b^2}{4\pi\rho}\right]\hat{z}$$

Let $\xi = \frac{b^2}{B_{eq}^2}$ $\qquad \eta_t = \frac{1}{3}\tau_c u^2$ $\qquad B_{eq}^2 = 4\pi\rho u^2$

then $\tau_c = \frac{3\eta_t}{u^2}$

$$\Rightarrow F^{NV} = -\eta_t\frac{b^2}{u^2}\tau_c\omega\left[\left(\frac{2}{5}+\frac{11}{15}g\right)u^2+\frac{13}{66}g\frac{b^2}{4\pi\rho}\right]\hat{z}$$

$$= -\eta_t\frac{b^2\,\omega\tau_c}{4\pi\rho u^2}\left[\left(\frac{2}{5}+\frac{11}{15}g\right)u^2 4\pi\rho+\frac{13}{66}g\,b^2\right]\hat{z}$$

$$= -\eta_t\xi\,\omega\tau_c\left[\left(\frac{2}{5}+\frac{11}{15}g\right)B_{eq}^2+\frac{13}{66}g\,b^2\right]\hat{z}$$

$$= -\eta_t\xi\,\omega B_{eq}^2\tau_c\left[\left(\frac{2}{5}+\frac{11}{15}g\right)+\frac{13}{66}g\,\xi\right]\hat{z}$$

$$B_{eq}^2 = 4\pi\rho u^2 = 4\pi\rho\frac{3\eta_t}{\tau_c}$$

$$\therefore F^{NV} = -4\pi\omega\rho\eta_t^2\xi\left(\frac{6}{5}+\frac{11}{5}g+\frac{13}{22}g\xi\right)\hat{z} \checkmark$$

We had $F^{NV} = -16\pi f \omega \rho \eta_t^2 \hat{z}$

where $f(q,\xi) = \frac{\xi}{4}\left(\frac{6}{5} + \frac{11}{5}q + \frac{13}{22}q\xi\right)$

GS 23 get

$$F_{NV} = (\vec{\nabla}\times\vec{V})\left[C_1\frac{\tau^2}{8\pi\rho}(\langle b^2\rangle)^2 + C_2\tau^2\langle v^2\rangle\langle b^2\rangle + C_4\lambda^2\langle b^2\rangle\right]$$

$$+\tau^2(C_3 - C_2)(\vec{\nabla}\times\langle v^2\rangle\nabla\langle b^2\rangle)$$

where $(C_1, C_2, C_3, C_4) = \left(\frac{7}{45}, -\frac{203}{5400}, \frac{403}{8100}, -\frac{1}{6}\right)$

$$\vec{\nabla}\times\vec{V} = \vec{\nabla}\times r\Omega\hat{\varphi} = \vec{\nabla}\times r\Omega\hat{\varphi}$$

Cylindrical coordinates: $\vec{\nabla}\times\vec{A} =$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial\varphi}\right]\hat{z}$$

$$\cancel{\therefore \vec{\nabla}\times r\Omega\hat{z} = \frac{1}{r}\frac{\partial}{\partial\varphi}(r\Omega)\hat{r} - \frac{\partial}{\partial r}(r\Omega)\hat{\varphi}}$$

$$\cancel{\text{Assume } \frac{\partial\Omega}{\partial\varphi} = 0 \Rightarrow \vec{\nabla}\times r\Omega\hat{z} = -\frac{\partial}{\partial r}(r\Omega)\hat{\varphi}}$$

$$\vec{\nabla}\times r\Omega\hat{\varphi} = -\frac{\partial}{\partial z}(r\Omega)\hat{r} + \frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega)\hat{z}$$

$$= -r\frac{\partial\Omega}{\partial z}\hat{r} + \left(\frac{1}{r}\cdot 2r\Omega + \frac{1}{r}r^2\frac{\partial\Omega}{\partial r}\right)\hat{z}$$

$$= -r\frac{\partial\Omega}{\partial z}\hat{r} + \left(2\Omega + r\frac{\partial\Omega}{\partial r}\right)\hat{z}$$

$$= -r\frac{\partial\Omega}{\partial z}\hat{r} + \Omega(2 - q)\hat{z}$$

where $q \equiv -\frac{r}{\Omega}\frac{\partial\Omega}{\partial r} = -\frac{\partial\ln\Omega}{\partial\ln r}$

$$F_r^{NV} = -r\frac{\partial \Omega}{\partial z}$$

$$F_z^{NV} = \overbrace{\Omega(2-q)\left[C_1\frac{\tau^2}{8\pi\rho}b^4 + C_2\tau^2 v^2 b^2 + C_4\lambda^2 b^2\right]}^{1}$$

$$+ \underbrace{\tau^2(C_3-C_2)\left[r\Omega\hat{\varphi}\times\langle v^2\rangle\vec{\nabla}\langle b^2\rangle\right]}_{2}$$

Focus on term 1

$$\text{term}\,1 = \Omega(2-q)b^2\tau^2\left[C_1\frac{b^2}{8\pi\rho} + C_2 v^2 + C_4\frac{\lambda^2}{\tau^2}\right]$$

$$= \frac{\Omega\tau^2 b^2}{3}\left[\frac{3C_1}{2}\frac{b^2}{4\pi\rho}(2-q) + 3C_2 v^2(2-q) + 3C_4\frac{\lambda^2}{\tau^2}(2-q)\right]$$

$$= -\frac{\Omega\tau^2 b^2}{3}\left[-\frac{3}{\cancel{2}}\cdot\frac{7}{45}\frac{b^2}{4\pi\rho}\cdot\cancel{2} + \frac{3}{2}\cdot\frac{7}{45}\frac{b^2}{4\pi\rho}q \right.$$

$$- 3\left(-\frac{203}{5400}\right)v^2\cdot 2 + 3\left(-\frac{203}{5400}\right)v^2 q$$

$$\left. - 3\left(-\frac{1}{6}\right)\frac{\lambda^2}{\tau^2}\cdot 2 + 3\left(-\frac{1}{6}\right)\frac{\lambda^2}{\tau^2}q\right]$$

$$= -\frac{\Omega\tau^2 b^2}{3}\left[\frac{7}{30}q\frac{b^2}{4\pi\rho} - \frac{7}{15}\frac{b^2}{4\pi\rho} + \frac{203}{900}v^2 - \frac{203}{1800}q v^2 + \frac{\lambda^2}{\tau^2} - \frac{1}{2}q\frac{\lambda^2}{\tau^2}\right]$$

$$= -\frac{\Omega\tau^2 b^2}{3}\left[\frac{203}{900}\left(1-\frac{1}{2}q\right)v^2 - \frac{7}{15}\left(1-\frac{1}{2}q\right)\frac{b^2}{4\pi\rho} + \left(1-\frac{1}{2}q\right)\frac{\lambda^2}{\tau^2}\right]$$

If $\lambda = v\tau$ then

$$= -\frac{\Omega\tau^2 b^2}{3}\left[\frac{1103}{900}\left(1-\frac{1}{2}q\right)v^2 - \frac{7}{15}\left(1-\frac{1}{2}q\right)\frac{b^2}{4\pi\rho}\right]$$

Redo 2017 $F^{NV} = -\frac{\Omega \tau^2 \langle b^2\rangle}{3}\left[\left(\frac{2}{5}+\frac{11}{15}q\right)\langle v^2\rangle + \frac{13}{66} q \frac{\langle b^2\rangle}{4\pi\rho}\right]\hat{z}$

GS23: $\searrow$

$$F^{NV} = (\nabla\times\bar{V})\left[C_1 \frac{\tau^2}{8\pi\rho}\langle b^2\rangle^2 + C_2\tau^2\langle v^2\rangle\langle b^2\rangle + C_4\lambda^2\langle b^2\rangle\right]$$
$$+\ \tau^2(C_3 - C_2)(\bar{V}\times\langle v^2\rangle\nabla\langle b^2\rangle)$$

where $(C_1, C_2, C_3, C_4) = \left(\frac{7}{45}, -\frac{203}{5400}, \frac{403}{8100}, -\frac{1}{6}\right)$

Let

$$\bar{V} = r\Omega\hat{\varphi} + \bar{V}_z\hat{z} \quad \text{where } \Omega = \Omega(r, z)$$

$$\nabla\times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial\varphi}\right]\hat{z}$$

$$\nabla\times\bar{V} = \left[\frac{1}{r}\frac{\partial\bar{V}_z}{\partial\varphi} - \frac{\partial}{\partial z}(r\Omega)\right]\hat{r} + \left(-\frac{\partial\bar{V}_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\Omega)\right]\hat{z}$$

$$= \left(-r\frac{\partial\Omega}{\partial z} + \frac{1}{r}\frac{\partial\bar{V}_z}{\partial\varphi}\right)\hat{r} - \frac{\partial\bar{V}_z}{\partial r}\hat{\varphi} + \left(2\Omega + r\frac{\partial\Omega}{\partial r}\right)\hat{z}$$

Let $q \equiv -\frac{\partial\ln\Omega}{\partial\ln r} = -\frac{r}{\Omega}\frac{\partial\Omega}{\partial r} \Rightarrow r\frac{\partial\Omega}{\partial r} = -q\Omega$

$$\therefore \boxed{\nabla\times\bar{V} = \left(-r\frac{\partial\Omega}{\partial z} + \frac{1}{r}\frac{\partial\bar{V}_z}{\partial\varphi}\right)\hat{r} - \frac{\partial\bar{V}_z}{\partial r}\hat{\varphi} + (2-q)\Omega\hat{z}}$$

$$\nabla a = \frac{\partial a}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial a}{\partial\varphi}\hat{\varphi} + \frac{\partial a}{\partial z}\hat{z}$$

$$\therefore \langle v^2\rangle\nabla\langle b^2\rangle = \langle v^2\rangle\left(\frac{\partial\langle b^2\rangle}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\langle b^2\rangle}{\partial\varphi}\hat{\varphi} + \frac{\partial\langle b^2\rangle}{\partial z}\hat{z}\right)$$

$$\bar{V}\times\langle v^2\rangle\nabla\langle b^2\rangle = (r\Omega\hat{\varphi} + \bar{V}_z\hat{z})\times\langle v^2\rangle\nabla\langle b^2\rangle$$

$$= \langle v^2\rangle\left[-r\Omega\frac{\partial\langle b^2\rangle}{\partial r}\hat{z} + r\Omega\frac{\partial\langle b^2\rangle}{\partial z}\hat{r} + \bar{V}_z\frac{\partial\langle b^2\rangle}{\partial r}\hat{\varphi} - \frac{\bar{V}_z}{r}\frac{\partial\langle b^2\rangle}{\partial\varphi}\hat{r}\right]$$

$$\therefore \boxed{\bar{V}\times\langle v^2\rangle\nabla\langle b^2\rangle = \langle v^2\rangle\left[\left(r\Omega\frac{\partial\langle b^2\rangle}{\partial z} - \frac{\bar{V}_z}{r}\frac{\partial\langle b^2\rangle}{\partial\varphi}\right)\hat{r} + \bar{V}_z\frac{\partial\langle b^2\rangle}{\partial r}\hat{\varphi} - r\Omega\frac{\partial\langle b^2\rangle}{\partial r}\hat{z}\right]}$$

Now consider the special case

1. $\partial\langle b^2\rangle/\partial\varphi = 0$ (axisymmetric)
2. $\bar{V}_z = 0$ (no outflow/inflow)
3. $\frac{\partial\Omega}{\partial z} = 0$ (neglect vertical shear/rotational lag)

Then, for this special case we have

$\nabla \times \bar{v} = (2-q)\Omega \hat{z}$

$\bar{\nabla} \times \langle v^2 \rangle \nabla \langle b^2 \rangle = \langle v^2 \rangle \left[r\Omega \frac{\partial \langle b^2 \rangle}{\partial z} \hat{r} - r\Omega \frac{\partial \langle b^2 \rangle}{\partial r} \hat{z} \right]$

Then $F^{NV} = (2-q)\Omega \hat{z} \left[C_1 \frac{\tau^2}{8\pi\rho} \langle b^2 \rangle^2 + C_2 \tau^2 \langle v^2 \rangle \langle b^2 \rangle + C_4 l^2 \langle b^2 \rangle \right]$

$+ \tau^2 (C_3 - C_2) \langle v^2 \rangle \left(r\Omega \frac{\partial \langle b^2 \rangle}{\partial z} \hat{r} - r\Omega \frac{\partial \langle b^2 \rangle}{\partial r} \hat{z} \right)$

$= \tau^2 (C_3 - C_2) r\Omega \langle v^2 \rangle \frac{\partial \langle b^2 \rangle}{\partial z} \hat{r}$

$+ \Big[(\underset{①}{2} - \underset{②\&③}{q}) \Omega \big(\underset{③}{\boxed{C_1 \frac{\tau^2}{8\pi\rho} \langle b^2 \rangle}} + \underset{①\&②}{\boxed{C_2 \tau^2 \langle v^2 \rangle}} + C_4 l^2 \big) \langle b^2 \rangle$

$- \tau^2 (C_3 - C_2) r\Omega \langle v^2 \rangle \frac{\partial \langle b^2 \rangle}{\partial r} \Big] \hat{z}$

Compare with Vishniac's version

He had a term $-\frac{\Omega \tau^2 \langle b^2 \rangle}{3} \frac{2}{5} \langle v^2 \rangle \hat{z}$

We have a term $2\Omega C_2 \tau^2 \langle b^2 \rangle \langle v^2 \rangle \hat{z}$

Term ① $= -\frac{203}{2700} \Omega \tau^2 \langle b^2 \rangle \langle v^2 \rangle \hat{z}$

GS coefficient : $-\frac{203}{2700} = -0.075\overline{185}$

V coefficient : $-\frac{2}{15} = -0.1\overline{3}$

~~We have~~ He had a term $-\frac{\Omega \tau^2 \langle b^2 \rangle}{3} \frac{11}{15} q \langle v^2 \rangle \hat{z}$

We have a term $-q\Omega C_2 \tau^2 \langle v^2 \rangle \langle b^2 \rangle \hat{z}$

Term ② $= \frac{203}{5400} \Omega \tau^2 q \langle v^2 \rangle \langle b^2 \rangle \hat{z}$

GS coefficient : $\frac{203}{5400} = 0.037\overline{592}$

V coefficient : $-\frac{11}{45} = -0.2\overline{4}$

He had a term $-\frac{\Omega \tau^2 \langle b^2 \rangle}{3} \frac{13}{66} q \frac{\langle b^2 \rangle}{4\pi\rho} \hat{z} = -\frac{13}{198} q \Omega \tau^2 \cdot \frac{\langle b^2 \rangle \langle b^2 \rangle}{4\pi\rho}$

We have a term $-q\Omega C_1 \frac{\tau^2}{8\pi\rho} \langle b^2 \rangle \langle b^2 \rangle \hat{z}$

Term ③ $= -\frac{7}{45} \frac{\Omega \tau^2 q \langle b^2 \rangle^2}{8\pi\rho} \hat{z} = -\frac{7}{90} q \frac{\Omega \tau^2 \langle b^2 \rangle^2}{4\pi\rho} \hat{z}$

GS coefficient : $-\frac{7}{90} = -0.0\overline{7}$

V coefficient : $-\frac{13}{198} = -0.0\overline{65}$

We had

$$F^{NV} = \left[(2-g)\Omega \left(C_1 \frac{\tau^2}{8\pi\rho}\langle b^2\rangle + C_2\tau^2\langle v^2\rangle + C_4\lambda^2 \right)\langle b^2\rangle \right.$$

① ②and③ and⑥ ③④ ①and② ⑤and⑥

minus ④and⑤

$$\left. - \tau^2(C_3-C_2)r\Omega\langle v^2\rangle \frac{\partial\langle b^2\rangle}{\partial r}\right]\hat{z}$$

⑦

$$+\tau^2(C_3-C_2)r\Omega\langle v^2\rangle\frac{\partial\langle b^2\rangle}{\partial z}\hat{r}$$

⑧

Extra terms in GS23 not in Vishniac's version:

Term ④

$$\frac{2C_1\Omega\tau^2}{8\pi\rho}\langle b^2\rangle\langle b^2\rangle\hat{z} = C_1\frac{\Omega\tau^2}{4\pi\rho}\langle b^2\rangle^2\hat{z}$$

$$= \frac{7}{45}\frac{\Omega\tau^2}{4\pi\rho}\langle b^2\rangle^2\hat{z} \quad \text{Coefficient: } \frac{7}{45} = 0.1\bar{5}$$

Term ⑤

$$2\Omega C_4\lambda^2\langle b^2\rangle\hat{z} = 2\Omega\left(-\frac{1}{6}\right)\lambda^2\langle b^2\rangle\hat{z}$$

$$= -\frac{1}{3}\Omega\lambda^2\langle b^2\rangle\hat{z} \quad \text{Coefficient: } -\frac{1}{3} = \cancel{0.3}\ -0.\bar{3}$$

Term ⑥

$$-g\Omega C_4\lambda^2\langle b^2\rangle\hat{z} = -g\Omega\left(-\frac{1}{6}\right)\lambda^2\langle b^2\rangle\hat{z}$$

$$= \frac{1}{6}g\Omega\lambda^2\langle b^2\rangle\hat{z} \quad \text{Coefficient: } \frac{1}{6} = 0.1\bar{6}$$

Term ⑦

$$-\tau^2(C_3-C_2)r\Omega\langle v^2\rangle\frac{\partial\langle b^2\rangle}{\partial r}\hat{z}$$

$$= -(C_3-C_2)\Omega\tau^2\langle v^2\rangle r\frac{\partial\langle b^2\rangle}{\partial r}\hat{z}$$

$$C_3 = \frac{403}{8100} \quad C_2 = -\frac{203}{5400} \Rightarrow C_3 - C_2 = \frac{806+609}{16{,}200}$$

$$= -\frac{1415}{16{,}200} = -\frac{\cancel{356}}{4050} = \cancel{0.0871604938}$$

$$-\frac{283}{3240}$$

$$\therefore \quad = -\frac{356}{4050}\Omega\tau^2\langle v^2\rangle r\frac{\partial\langle b^2\rangle}{\partial r}\hat{z}$$

$\cancel{0.08734567}$ $\cancel{0123}$ ≈ -0.087

Term ⑧

$$\tau^2(C_3-C_2)r\Omega\langle v^2\rangle\frac{\partial\langle b^2\rangle}{\partial z}\hat{r}$$

$$= \frac{353}{4050}\Omega\tau^2\langle v^2\rangle r\frac{\partial\langle b^2\rangle}{\partial z}\hat{r}$$

where $\cancel{\frac{353}{4050}} \approx 0.087$ $\cancel{0.0871604938\,7}$

$\frac{283}{3248}$. $\cancel{0.08}$

$\cancel{= 0.08734567}$

Consider special case $\lambda^2 = \tau^2\langle v^2\rangle$

Then $\boxed{\text{terms (1) and (5)}}$ $\boxed{\text{combine}}$ to give

$$2\Omega(C_2+C_4)\tau^2\langle v^2\rangle\langle b^2\rangle = 2\left(-\frac{203}{5400}-\frac{1}{6}\right)\Omega\tau^2\langle v^2\rangle\langle b^2\rangle$$

$$=+2\left(-\frac{1103}{5400}\right)\Omega\tau^2\langle v^2\rangle\langle b^2\rangle = -\frac{1103}{2700}\Omega\tau^2\langle v^2\rangle\langle b^2\rangle$$

$$\frac{-1103}{2700} = -0.4\overline{0851} \rightarrow \text{Vishniac had } -0.1\overline{3}$$

Likewise,

$\boxed{\text{Terms (2) and (6) combine}}$ to give

$$-g\Omega(C_2+C_4)\tau^2\langle v^2\rangle\langle b^2\rangle = \frac{1103}{5400}g\Omega\tau^2\langle v^2\rangle\langle b^2\rangle$$

where $\frac{1103}{5400} = 0.204\overline{259}$

$\rightarrow$ Vishniac had $-0.2\overline{4}$

Summary

Term	GS coeff.	V coeff
1). $\Omega\tau^2\langle v^2\rangle\langle b^2\rangle\hat{z}$	$-\frac{203}{2700} = -0.07\overline{518}$	~~203~~ $-\frac{2}{15} = -0.1\overline{3}$
2) $g\Omega\tau^2\langle v^2\rangle\langle b^2\rangle\hat{z}$	$\frac{203}{5400} = 0.037\overline{592}$	$-\frac{11}{45} = -0.2\overline{4}$
3) $g\Omega\tau^2\frac{\langle b^2\rangle}{4\pi\rho}\langle b^2\rangle\hat{z}$	$\frac{-7}{90} = -0.0\overline{7}$	$-\frac{13}{198} = -0.0\overline{65}$
4) $\Omega\tau^2\frac{\langle b^2\rangle}{4\pi\rho}\langle b^2\rangle\hat{z}$	$\frac{7}{45} = 0.1\overline{5}$	———
5) $\Omega\lambda^2\langle b^2\rangle\hat{z}$	$-\frac{1}{3} = -0.\overline{3}$	———
6) $g\Omega\lambda^2\langle b^2\rangle\hat{z}$	$\frac{1}{6} = 0.1\overline{6}$	———
7) $\Omega\tau^2\langle v^2\rangle r\frac{\partial\langle b^2\rangle}{\partial r}\hat{z}$	$-\frac{35\cancel{3}}{4050} =$ ~~-0.08716049382~~ ≈ -0.087	———
8) $\Omega\tau^2\langle v^2\rangle r\frac{\partial\langle b^2\rangle}{\partial z}\hat{r}$	$\frac{35\cancel{3}}{4050} =$ ~~0.08716049382~~ ≈ 0.087	———
1 & 5) for $\lambda^2=\tau^2\langle v^2\rangle$ $\rightarrow \Omega\tau^2\langle v^2\rangle\langle b^2\rangle\hat{z}$	$-\frac{1103}{2700} = -0.4\overline{0851}$	$-0.1\overline{3}$
2 & 6) for $\lambda^2=\tau^2\langle v^2\rangle$ $\rightarrow g\Omega\tau^2\langle v^2\rangle\langle b^2\rangle\hat{z}$	$\frac{1103}{5400} = 0.204\overline{259}$	$-0.2\overline{4}$

Now check GS expression for $-\nabla\cdot F^{NV}$.

$$F^{NV} = (\nabla\times\bar V)\left[C_1 \frac{\tau^2}{8\pi\rho}\langle b^2\rangle^2 + C_2\tau^2\langle v^2\rangle\langle b^2\rangle + C_4\lambda^2\langle b^2\rangle\right]$$
$$+\tau^2(C_3-C_2)(\bar V\times\langle v^2\rangle\nabla\langle b^2\rangle)$$

$\cancel{\nabla\cdot F^{NV}}$ $\quad \vec\nabla\cdot(\psi\vec A) = \psi\vec\nabla\cdot\vec A + \vec\nabla\psi\cdot\vec A$

$\vec\nabla\cdot(\vec\nabla\times\vec A) = 0 \qquad \vec\nabla\cdot(\vec A\times\vec B) = (\nabla\times A)\cdot B - A\cdot(\nabla\times B)$

$$-\vec\nabla\cdot\vec F^{NV} = -(\nabla\times\bar V)\cdot\nabla\left[C_1\frac{\tau^2}{8\pi\rho}\langle b^2\rangle^2 + C_2\tau^2\langle v^2\rangle\langle b^2\rangle + C_4\lambda^2\langle b^2\rangle\right]$$
$$-\tau^2(C_3-C_2)\nabla\cdot(\bar V\times\langle v^2\rangle\nabla\langle b^2\rangle)$$
$$-\nabla[\tau^2(C_3-C_2)]\cdot(\bar V\times\langle v^2\rangle\nabla\langle b^2\rangle)$$
$$= -(\nabla\times\bar V)\cdot\nabla\left[C_1\frac{\tau^2}{8\pi\rho}\langle b^2\rangle^2 + C_2\tau^2\langle v^2\rangle\langle b^2\rangle + C_4\lambda^2\langle b^2\rangle\right]$$
$$-\tau^2(C_3-C_2)\Big[(\nabla\times\bar V)\cdot(\nabla\langle b^2\rangle\langle v^2\rangle)$$
$$-\bar V\cdot\nabla\times(\nabla\langle b^2\rangle\langle v^2\rangle)\Big]$$

Last term $\rightarrow 0$ assuming $\vec\nabla\tau = 0$.

$\cancel{\nabla\times}$ $\quad \vec\nabla\times(\psi\vec A) = \psi(\vec\nabla\times\vec A) - (\vec A\times\vec\nabla)\psi$
$$= \psi(\vec\nabla\times\vec A) + \vec\nabla\psi\times\vec A$$

$\therefore -\vec\nabla\cdot\vec F^{NV} =$

$$-(\nabla\times\bar V)\cdot\nabla\left[\left(C_1\frac{\tau^2}{8\pi\rho}\langle b^2\rangle + C_2\tau^2\langle v^2\rangle + C_4\lambda^2\right)\langle b^2\rangle\right]$$
$$-\tau^2(C_3-C_2)\Big[\langle v^2\rangle(\nabla\times\bar V)\cdot\nabla\langle b^2\rangle$$
$$-\bar V\cdot(\langle v^2\rangle\overbrace{\nabla\times\nabla\langle b^2\rangle}^{0} + \nabla\langle v^2\rangle\times\nabla\langle b^2\rangle)\Big]$$

$$= -(\nabla\times\bar V)\cdot\nabla\left[\left(C_1\frac{\tau^2}{8\pi\rho}\langle b^2\rangle + C_2\tau^2\langle v^2\rangle + C_4\lambda^2\right)\langle b^2\rangle\right]$$
$$-\tau^2(C_3-C_2)\left[\langle v^2\rangle(\nabla\times\bar V)\cdot\nabla\langle b^2\rangle - \bar V\cdot(\nabla\langle v^2\rangle\times\nabla\langle b^2\rangle)\right]$$

$$= -(\nabla\times\bar V)\cdot\Big[\Big(C_1\frac{\tau^2}{8\pi\rho}\langle b^2\rangle + \cancel{C_2\tau^2\langle v^2\rangle} + C_4\lambda^2\Big)\nabla\langle b^2\rangle$$ since $\nabla\lambda^2 = 0$

$$+\left(C_1\frac{\tau^2}{8\pi\rho}\nabla\langle b^2\rangle - C_1\frac{\tau^2}{8\pi}\frac{\nabla\rho}{\rho^2}\langle b^2\rangle + C_2\tau^2\nabla\langle v^2\rangle + 0\right)\langle b^2\rangle\Big]$$

$$-\tau^2(C_3-C_2)\left[\langle v^2\rangle(\nabla\times\bar V)\cdot\nabla\langle b^2\rangle - \bar V\cdot(\nabla\langle v^2\rangle\times\nabla\langle b^2\rangle)\right]$$

$= \cancel{-(\nabla\times\bar{v})\cdot\left[\left(C_1 \frac{\tau^2}{4\pi\rho}\langle b^2\rangle + C_2\right.\right.}$

$$\therefore -\nabla\cdot F^{NV} = -(\nabla\times\bar{v})\cdot\Big[C_1 \frac{\tau^2}{4\pi\rho}\langle b^2\rangle\nabla\langle b^2\rangle + C_2\tau^2\langle b^2\rangle\nabla\langle v^2\rangle$$

$$+ C_3\tau^2\langle v^2\rangle\nabla\langle b^2\rangle + C_4 l^2\nabla\langle b^2\rangle$$

$$\boxed{-C_1\frac{\tau^2}{8\pi\rho}\langle b^2\rangle^2\frac{\nabla\rho}{\rho}}\Big]$$

extra.

$$-\tau^2(C_3-C_2)\,\bar{v}\cdot(\nabla\langle b^2\rangle\times\nabla\langle v^2\rangle)$$

✓

Agrees with GS23 eq. 17 except for the extra term, which $\to 0$ if $\vec{\nabla}\rho = \vec{0}$.

We had $\nabla\times\bar{v} = \left(-r\frac{\partial\Omega}{\partial z} + \frac{1}{r}\frac{\partial\bar{v}_z}{\partial\varphi}\right)\hat{r} - \frac{\partial\bar{v}_z}{\partial r}\hat{\varphi} + \underbrace{(2-q)\Omega\hat{z}}_{\text{Agrees with GS23}}$

If $\nabla\langle b^2\rangle \parallel \nabla\langle v^2\rangle$ then last term goes to zero.

~~$\nabla\cdot F^{NV}$~~ Incompressible so $\nabla\rho = 0$.

~~Then $\nabla\cdot F^{NV}$~~ Also, consider only last term of $\nabla\times\bar{v}$.

~~Then~~

Then

~~$\nabla\cdot F^{NV} = -(2-q)\Omega\left[C_1\tau^2 v_A^2\right.$~~

Then $-\nabla\cdot F^{NV} = -(2-q)\Omega\hat{z}\cdot\left[C_1\tau^2 v_A^2\nabla\langle b^2\rangle + C_2\tau^2\langle b^2\rangle\nabla\langle v^2\rangle\right.$

$$\left. + C_3\tau^2\langle v^2\rangle\nabla\langle b^2\rangle + C_4 l^2\nabla\langle b^2\rangle\right]\hat{z}$$

where $v_A \equiv \left(\frac{\langle b^2\rangle}{4\pi\rho}\right)^{1/2}$

If $\nabla\langle v^2\rangle \approx \frac{\langle v^2\rangle}{h}$, $\nabla\langle b^2\rangle \approx -\frac{\langle b^2\rangle}{h}$, then

$$-\nabla\cdot F^{NV} \approx \frac{(2-q)\Omega}{h}\langle b^2\rangle\left[C_1\tau^2 v_A^2 + (C_3-C_2)\tau^2\langle v^2\rangle + C_4 l^2\right]$$

✓ agrees with eq 18 of GS23.

Estimate: $q = 1$, $v_A = v$, $l = l = v\tau$.

Then $-\nabla\cdot F^{NV} \approx \frac{\Omega\tau^2 v^2}{h}\left(C_1 + (C_3-C_2) + C_4\right)\langle b^2\rangle$

Agrees with GS23.

$$\approx \left(\frac{7}{45} \cancel{-\frac{338}{4050}} - \frac{4}{6}\right)\frac{\Omega\tau^2 v^2}{h}\langle b^2\rangle = \cancel{\frac{199}{2025}}\,\cancel{\frac{\Omega\tau^2 v^2}{h}}\langle b^2\rangle$$

$$= \left(\frac{7}{45} + \frac{403}{8100} + \frac{203}{5400} - \frac{1}{6}\right)\alpha_0\langle b^2\rangle$$

$\approx 0.076\,\alpha_0\langle b^2\rangle$ ✓

$$\frac{\partial h^b}{\partial t} \simeq 0.076 \langle b^2 \rangle \alpha_0 \quad ; \alpha_0 = \frac{\Omega l^2}{H}$$

$$h^c = \frac{h}{l^2} = \frac{h^b}{l^2} \qquad \alpha_m = \frac{\tau}{3} \frac{h^c}{4\pi\rho}$$

$$\to h^b = l^2 h^c = l^2 \frac{3\alpha_m}{\tau} 4\pi\rho$$

$$\therefore \frac{\partial \alpha_m}{\partial t} \simeq 0.076 \langle b^2 \rangle \frac{\Omega l^2}{H} \frac{\tau}{3 \cdot 4\pi\rho l^2}$$

$$\simeq 0.076 \frac{v_A^2 \Omega \tau}{3H}$$

$$v_A \simeq v \simeq l\tau.$$

$$\frac{\partial \alpha_m}{\partial t} \simeq 0.025 \frac{l^2 \Omega \tau}{\tau^2 H} = 0.025 \frac{\alpha_0}{\tau}$$

$$\therefore \alpha_m \sim \overset{0.0254}{0.025}\, \alpha_0 \frac{t}{\tau} \checkmark$$

So according to this estimate, the $-\nabla \cdot$ flux term is +ve.

Thus the

2017note.pdf

$$F^{NV} = -16\pi f \Omega \rho \eta_t^2 \hat{y} \quad ; f = \frac{5}{4}\left(\frac{6}{5} + \frac{11}{5} q + \frac{13}{22} q^8\right)$$

$$= -16\pi f \Omega \rho \left(\frac{1}{3} \tau \langle u^2 \rangle\right)^2 \hat{y}$$

$$-\nabla \cdot F^{NV} = 16\pi f \Omega \rho \left(\frac{1}{3} \tau \frac{\partial \langle u^2 \rangle}{\partial z}\right)$$

assuming $\frac{\partial \Omega}{\partial y} \simeq 0$ and neglecting $\frac{\partial f}{\partial y}$ term, $\frac{\partial \tau}{\partial z}$ term.

If $\frac{\partial \langle u^2 \rangle}{\partial z} > 0$, then this term is > 0 as ~~assumed~~ estimated in GS23

Now want to put in terms of v_A and v so that we are not assuming $v_A = v$.
Also, keep q

Start w/ eq 18 of GS23

$$-\nabla \cdot F^{NV} \simeq \frac{(2-q)\Omega}{h}\langle b^2\rangle\left[C_1 \tau^2 v_A^2 + (C_3 - C_2)\tau^2\langle v^2\rangle + C_4 l^2\right]$$

$$\cancel{Note} = \frac{(2-q)\Omega\, 4\pi\rho\, v_A^2}{h}\left[C_1\tau^2 v_A^2 + (C_3 - C_2)\tau^2 v^2 + C_4 l^2\right]$$

Now put $l = \tau v$

Then

$$-\nabla \cdot F^{NV} = \frac{(2-q)\Omega\, 4\pi\rho\, v_A^2}{h}\left[C_1\tau^2 v_A^2 + (C_3 - C_2 + C_4)\tau^2 v^2\right]$$

$$= \frac{(2-q)\Omega\, 4\pi\rho\, v_A^2 \tau^2}{h}\left[C_1 v_A^2 + (C_3 - C_2 + C_4) v^2\right]$$

$$= \frac{(2-q)\Omega\, 4\pi\rho\, v_A^2 v^2 \tau^2}{h}\left[C_1 \frac{v_A^2}{v^2} + (C_3 - C_2 + C_4)\right]$$

$$\frac{v_A^2}{v^2} = \frac{b^2}{4\pi\rho v^2} = \frac{b^2}{B_{eq}^2} \equiv \xi.$$

$$-\nabla \cdot F^{NV} = \frac{(2-q)\Omega\, 4\pi\rho\, v_A^2 v^2 \tau^2}{h}\left[C_1\xi + (C_3 - C_2 + C_4)\right]$$

$$-\nabla \cdot F_\alpha = \frac{\tau}{3 \cdot 4\pi\rho \cdot l^2}(-\nabla \cdot F^{NV}) = \frac{(2-q)\Omega\tau}{3h} v_A^2\left[C_1\xi + (C_3 - C_2 + C_4)\right]$$

$\to l = v\tau.$

$$\alpha_0 = \frac{\Omega\tau^2 v^2}{h} \Rightarrow \frac{\Omega\tau}{h} = \frac{\alpha_0}{\tau v^2}$$

$$\therefore\ -\nabla \cdot F_\alpha = \frac{(2-q)\alpha_0}{3\tau}\frac{v_A^2}{v^2}\left[C_1\xi + (C_3 - C_2 + C_4)\right]$$

$$= \frac{(2-q)\alpha_0}{3\tau}\xi\left[C_1\xi + (C_3 - C_2 + C_4)\right]$$

$$= \frac{(2-q)\alpha_0}{3\tau}\xi\left[\frac{7}{45}\xi - \frac{257}{3240}\right] = \frac{(2-q)\alpha_0}{3\tau}\xi\left(\frac{7}{45}\xi - 0.079\right)$$

Check $q = 1, \xi = 1$

$$-\nabla \cdot F_\alpha = \frac{\alpha_0}{\tau} \cdot 0.0254 \checkmark$$

~~Check $q=1, \xi=1 \to -\nabla \cdot F_\alpha$~~

We had

$$-\nabla\cdot F_\alpha \simeq \frac{(2-g)\alpha_0}{3\tau}\xi\left(\frac{7}{45}\xi - \frac{257}{3240}\right)$$

$\rightarrow g=1, \xi=1 \rightarrow -\nabla\cdot F_\alpha = 0.0254\cdot\frac{\alpha_0}{\tau}$

$g=1, \xi=0.5 \Rightarrow -\nabla\cdot F_\alpha = -0.00026\,\alpha_0/\tau$

$g=1, \xi=0.4 \Rightarrow -\nabla\cdot F_\alpha = -0.00228\,\frac{\alpha_0}{\tau}$

$g=1, \xi=0.1 \Rightarrow -\nabla\cdot F_\alpha = -0.00213\frac{\alpha_0}{\tau}$

$g=1, \xi=0.01 \Rightarrow -\nabla\cdot F_\alpha = -0.00026\frac{\alpha_0}{\tau}.$

From 2017 notes I had $F^{NV} = -16\pi f\Omega\rho\frac{1}{3}\tau\langle v^2\rangle\hat{z}$

where $f = \frac{\xi}{4}\left(\frac{6}{5}+\frac{11}{5}g+\frac{13}{22}g\xi\right)$

$g=1, \xi=1 \Rightarrow f\simeq 1$

$g=1, \xi=0.1 \Rightarrow f\simeq 0.09$

~~$F_\alpha^{2014} = F_\alpha^{lit}$~~

$$-\nabla\cdot F^{NV} = +\frac{16\pi f\Omega\tau\rho}{3}\frac{\partial\langle v^2\rangle}{\partial z}$$

$$-\nabla\cdot F_\alpha^V = \frac{\tau}{3(4\pi\rho)l^2}(-\nabla\cdot F^{NV}) =$$

$$= \frac{\tau}{3(4\pi\rho)l^2}\frac{16\pi f\Omega\tau\rho}{3}\frac{\partial\langle v^2\rangle}{\partial z}$$

$$= \frac{4}{9}\frac{\Omega\tau^2}{\tau^2 v^2}f\frac{\partial\langle v^2\rangle}{\partial z} \simeq \frac{4}{9}\frac{\Omega v^2}{v^2 h}$$

2017 notes

$$F^{NV} = -16\pi f\Omega\rho\eta_t^2\hat{z} \quad ; f = \frac{\xi}{4}\left(\frac{6}{5}+\frac{11}{5}g+\frac{13}{22}g\xi\right)$$

$$= -16\pi f\Omega\rho\frac{1}{9}\tau^2\langle u^2\rangle^2\hat{z}$$

$$= -\frac{16}{9}\pi f\Omega\rho\tau^2\langle u^2\rangle^2\hat{z}$$

$$-\nabla\cdot F^{NV} = \frac{16}{9}\pi f\Omega\rho\tau^2\frac{\partial}{\partial z}(\langle u^2\rangle^2) = \frac{16}{9}\pi f\rho\Omega\tau^2\cdot 2\langle u^2\rangle\frac{\partial\langle u^2\rangle}{\partial z}$$

$$\therefore -\nabla\cdot F^{NV} = \frac{32}{9}\pi f \Omega \rho \tau^2 \langle u^2\rangle \frac{\partial\langle u^2\rangle}{\partial z}$$

$$-\nabla\cdot F_\alpha^{V} = \frac{\tau}{3(4\pi\rho)l^2}(-\nabla\cdot F^{NV}) = \frac{32}{9}\frac{\pi f \Omega\rho\tau^2\langle u^2\rangle}{3(4\pi\rho)l^2}\frac{\partial\langle u^2\rangle\tau}{\partial z}$$

$$= \frac{8}{27}\frac{f\Omega\tau\ \tau^2\langle u^2\rangle}{\cancel{3}\tau^2\langle u^2\rangle}\frac{\partial\langle u^2\rangle}{\partial z}$$

$$\simeq \frac{8}{27}\frac{f\Omega\tau}{\cancel{3}}\frac{\langle u^2\rangle}{h} = \frac{8}{27}f\frac{\alpha_0}{\tau}$$

$$f^V = \frac{\xi}{4}\left(\frac{6}{5}+\frac{11}{5}q+\frac{13}{22}q\xi^5\right) \quad \text{where } \alpha_0 = \frac{\Omega\tau^2\langle u^2\rangle}{h}$$

~~For~~

$q=1,\ \xi=1 \Rightarrow f=0.998 \Rightarrow -\nabla\cdot F_\alpha = 0.296\frac{\alpha_0}{\tau}$

$q=1,\ \xi=0.5 \Rightarrow f=\cancel{0.137}\ 0.462 \Rightarrow -\nabla\cdot F_\alpha = 0.137\frac{\alpha_0}{\tau}$

$q=1,\ \xi=0.4 \Rightarrow f=0.364 \Rightarrow -\nabla\cdot F_\alpha = 0.108\frac{\alpha_0}{\tau}$

$q=1,\ \xi=0.1 \Rightarrow f=0.086 \Rightarrow -\nabla\cdot F_\alpha = 0.026\frac{\alpha_0}{\tau}$

Note: Last term $\nabla\langle u^2\rangle\times\nabla\langle b^2\rangle\cdot\overline{V}$

<u>Models</u>: 1) $R_{\alpha,k}\neq 0.$

$R_K = 0$

$R_\nu = 0.$

$R_\Omega \neq 0$

<u>Chamin</u>

KS replacing Σ_{SFR}

$R_\alpha,\ R_x,\ R_u,\ \xi,\ R_\Omega,\ q \quad \rightarrow f(q,\xi)$

where $\xi \equiv -\nabla\cdot F_\alpha\,\tau/\alpha_0$

	Run	R_Ω	R_α	R_u	R_x	ξ	$f(q,\xi)$	Result
A	114	−20	0	0	0	1	1	Unstable
B	115	−20	0	0.45	0	1	1	Non-osc, sat
C	117	−20	0	0.45	0	1	1	" "
D	119	−20	0	0.45	0	0.1	0.1	" " 0.38$_0$

New expression for NV flux

We had derived

$$-\nabla\cdot F_{NV} = -(2-q)\Omega\left[C_1\tau^2 v_A^2\nabla\langle b^2\rangle + C_2\tau^2\langle b^2\rangle\nabla\langle v^2\rangle + C_3\tau^2\langle v^2\rangle\nabla\langle b^2\rangle + C_4 l^2\nabla\langle b^2\rangle\right]$$

where $v_A \equiv \left(\frac{\langle b^2\rangle}{4\pi\rho}\right)^{1/2}$, $q \equiv -\frac{\partial\ln\Omega}{\partial\ln r}$

and we have assumed $\bar V_z = \bar V_r = 0$, $\bar V_\varphi = r\Omega$,

$\frac{\partial\Omega}{\partial z} = 0$, $\frac{\partial}{\partial\varphi} = 0$, $\nabla\langle v^2\rangle \times \nabla\langle b^2\rangle = 0$, $\nabla\rho = 0$

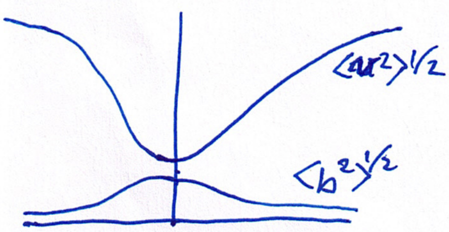
Variation of $\langle v^2\rangle$ and $\langle b^2\rangle$ with z?

Assume $\langle u^2\rangle^{1/2} = u_0 e^{+z^2/2h^2}$ $\qquad \frac{\partial}{\partial z}\langle u^2\rangle = \frac{u_0^2\cdot 2z}{h^2}e^{z^2/h^2}$

$\langle u^2\rangle = u_0^2 e^{z^2/h^2}$

$$\frac{\partial}{\partial z}\langle u^2\rangle = \frac{2u_0^2}{h}\left(\frac{z}{h}\right)e^{z^2/h^2}$$

$$= \frac{2\langle u^2\rangle}{h}\frac{z}{h}$$



$\langle b^2\rangle^{1/2} = b_0 e^{-z^2/2h^2}$

$\langle b^2\rangle = b_0^2 e^{-z^2/h^2}$

$$\frac{\partial\langle b^2\rangle}{\partial z} = -\frac{2b_0^2}{h}\frac{z}{h}e^{-z^2/h^2} = -\frac{2\langle b^2\rangle}{h}\frac{z}{h}$$

$\cancel{\langle u_0^2 \rangle = (1 - e^{-y^2/2h^2}) u_0^2}$

$\langle u^2 \rangle = u_\infty^2 - X e^{-y^2/2h^2}$ ~~$\to u_0^2$ @ $y=0$~~

$\to u_\infty^2$ as $y \to \infty$

$\to u_\infty^2 - X$ for $y = 0$

$\to u_\infty^2 - X e^{-1/2}$ for $y = h$

$$\frac{\partial \langle u^2 \rangle}{\partial y} = -X e^{-y^2/2h^2} \cdot \left(\frac{-2y}{2h^2} \right)$$

$$= \left(\frac{y}{h} \right) \frac{X}{h} e^{-y^2/2h^2}$$

$$= -\frac{y}{h} \frac{\left(\langle u^2 \rangle - u_\infty^2 \right)}{h} = \frac{y}{h} \frac{u_\infty^2 - \langle u^2 \rangle}{h}$$

$$\langle b^2 \rangle = b_0^2 e^{-y^2/2h^2}$$

$\frac{\partial \langle u^2 \rangle}{\partial y} \to 0$ @ $y=0$, $\to 0$ as $y \to \infty$

~~bad!~~

$$\frac{\partial \langle b^2 \rangle}{\partial y} = -\frac{y}{h} \frac{\langle b^2 \rangle}{h} = -\frac{y}{h} \frac{b_0^2}{h} e^{-y^2/2h^2}$$

$$\xi(y) \equiv \frac{\langle b^2 \rangle}{\langle u^2 \rangle 4\pi\rho} = \frac{b_0^2 e^{-y^2/2h}}{u_0^2 - X e^{-y^2/2h^2}}$$

$$\left. \frac{\partial \langle u^2 \rangle}{\partial y} \right|_{y=\frac{h}{2}} = \frac{1}{2} \frac{u_\infty^2 - u_\infty^2 + X e^{-1/8}}{h}$$

$$= \frac{X e^{-1/8}}{2h}$$

$$\lim_{y \to \infty} \frac{y e^{-y^2/2h^2}}{h}$$

$$\left. \frac{\partial \langle b^2 \rangle}{4\pi\rho\, \partial y} \right|_{y=\frac{h}{2}} = \frac{-1}{2} \frac{b_0^2 e^{-1/8}}{4\pi\rho h}$$

$$\lim_{x \to \infty} x e^{-x^2} = \lim_{x \to \infty} \cancel{2x e^{-x^2}}$$

Take $\frac{b_0^2}{4\pi\rho(u_\infty^2 - X)} \equiv \xi_0$

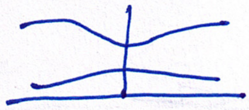
$$= \lim_{x \to \infty} \frac{x}{e^{x^2}} = \lim_{x \to \infty} \frac{1}{2x e^{x^2}} = 0.$$

e.g. $u_\infty^2 =$ ~~$10u$~~

Then: $\to X =$ ~~$\xi_0 4\pi\rho / b_0$~~ $u_\infty^2 - \frac{b_0^2}{4\pi\rho\xi_0} \simeq \frac{10 b_0^2}{4\pi\rho}$

$\Rightarrow$ ratio is $10:1$.

Case 1: Neglect $\frac{\partial\langle b^2\rangle}{\partial z\, 4\pi\rho}$ compared to $\frac{\partial\langle u^2\rangle}{\partial z}$.



$$\frac{\partial\langle u^2\rangle}{\partial z} \sim \frac{\langle u^2\rangle}{h}$$

Case 2: Set $\langle b^2\rangle = \xi\langle u^2\rangle$ with ξ fixed.

and both $\downarrow$ w/ z $\quad \langle v^2\rangle = v_0^2 e^{-z^2/2h^2}$.

$$\rightarrow \frac{\partial\langle u^2\rangle}{\partial z} \sim -\frac{\langle u^2\rangle}{h}$$

$$\text{and } \frac{\partial\langle b^2\rangle}{4\pi\rho\,\partial z} \sim -\xi\frac{\langle u^2\rangle}{h}$$

We had

$$-\nabla\cdot F_{NV} = -(2-q)\Omega\Big[C_1\tau^2 v_A^2\nabla\langle b^2\rangle + C_2\tau^2\langle b^2\rangle\nabla\langle v^2\rangle + C_3\tau^2\langle v^2\rangle\nabla\langle b^2\rangle + C_4 l^2\nabla\langle b^2\rangle\Big]$$

$$C_1 = \frac{7}{45} \quad C_2 = \frac{-203}{5400} \quad C_3 = \frac{403}{8100} \quad C_4 = \frac{-1}{6}$$

$$C_1 \approx 0.156 \quad C_2 \approx -0.038 \quad C_3 \approx 0.050 \quad C_4 \approx -0.167$$

Case 1 ~~we looked at above~~.

Case 2: $\frac{\nabla\langle b^2\rangle}{4\pi\rho} = \xi\nabla\langle v^2\rangle = -\xi\frac{\langle v^2\rangle}{h}$

$$\therefore -\nabla\cdot F_{NV} = -(2-q)\Omega\Big[-C_1\frac{\tau^2\xi^2\langle v^2\rangle^2}{h} - C_2\frac{\tau^2\xi\langle v^2\rangle^2}{h} - C_3\frac{\tau^2\xi\langle v^2\rangle^2}{h} - C_4\frac{l^2\xi\langle v^2\rangle}{h}\Big]4\pi\rho$$

$l = \text{const} \longrightarrow \; = \tau\langle v^2\rangle^{1/2}\big|_{z=0}$?

$= \tau\langle v^2\rangle^{1/2}$?

$= h$?

Case 2

Take $\lambda = \frac{\tau}{h}\int_0^h \langle v^2\rangle^{1/2}\,dz$

If $\langle v^2\rangle^{1/2} = v_0 e^{-z^2/2h^2}$ then

Let $\tilde{z} = z/h \Rightarrow z = h\tilde{z}$

$dz = h\,d\tilde{z}$

$\therefore \lambda = \frac{\tau \cancel{h}}{\cancel{h}}\int_0^1 \langle v^2\rangle^{1/2}\,d\tilde{z}$ where $\langle v^2\rangle^{1/2} = v_0 e^{-\tilde{z}^2/2}$

using Mathematica

$= \tau v_0 \int_0^1 e^{-\tilde{z}^2/2}\,d\tilde{z} = \tau v_0 \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) = 0.856\,\tau v_0$

So safe to use $\lambda \approx \tau v_0$.

$\langle v^2\rangle = \left(\langle v^2\rangle^{1/2}\right)^2 = v_0^2 e^{-z^2/h^2}$

$\frac{\partial\langle v^2\rangle}{\partial z} = -\frac{2z}{h}\frac{v_0^2}{h} e^{-z^2/h^2} = -2\left(\frac{z}{h}\right)\frac{\langle v^2\rangle}{h}$

$\therefore -\nabla\cdot F_{NV} \simeq -\frac{(2-q)\Omega}{h}\left[-C_1\tau^2\xi^2\langle v^2\rangle^2 - C_2\tau^2\xi\langle v^2\rangle^2 - C_3\tau^2\xi\langle v^2\rangle^2 - C_4\lambda^2\xi\langle v^2\rangle\right]\left(\frac{2z}{h}\right) 4\pi\rho$

$\lambda = v_0\tau$, $\langle v^2\rangle = v_0^2 e^{-z^2/h^2}$, $\langle v^2\rangle^2 = v_0^4 e^{-2z^2/h^2}$

$-\nabla\cdot F_{NV} = \frac{2(2-q)\Omega\tau^2}{h^2} v_0^4\, z\, e^{-2z^2/h^2}\left[C_1\xi^2 + C_2\xi + C_3\xi + C_4\xi\right]$

$= \frac{2(2-q)\Omega\tau^2 v_0^4 \xi}{h^2}\, z\, e^{-2z^2/h^2}\left(C_1\xi + C_2 + C_3 + C_4\right)$

$= \frac{2(2-q)\Omega\tau^2 v_0^4 \xi}{h^2}\, 4\pi\rho\, z\, e^{-2z^2/h^2}\left(\frac{7}{45}\xi \overbrace{- \frac{203}{5400} + \frac{403}{8100} - \frac{1}{6}}^{-2503/16{,}200 \,=\, -0.1545}\right)$

$= -0.309\,\frac{(2-q)\Omega\tau^2 v_0^4 \xi}{h}\, 4\pi\rho\, \frac{z}{h} e^{-2z^2/h^2}\left(1 - 1.01\xi\right)$

< 0 so oscillatory solutions.