23. Convection in the Presence of Rotation and Magnetic Field; Realistic Finite Amplitude Dynamos

Linear Stability Analysis of Convection

When you add rotation and magnetic field to a fluid, two changes must be made to the equation of motion. First, you must add the Coriolis force arising from the fact that you are describing the fluid motions relative to a rotating (i.e. non-inertial) frame of reference. Second, you must add the Lorentz body force that arises from the magnetic field. The equation of motion for a flow that is constant density except for the effect of thermal buoyancy is then:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\vec{\nabla})\vec{u} + 2\vec{\Omega} \times \vec{u} = -\frac{\nabla p}{\rho} + g\alpha\theta\hat{z} + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{\rho\mu_0} + \nu\nabla^2\vec{u} \qquad (23.1)$$

where p is the hydrodynamic pressure (the small piece of pressure left after taking account of hydrostatic equilibrium), θ is the temperature anomaly available to drive convection and z is vertically upwards. We then consider the fate of small velocity perturbations which behave as exp[i**k.r**+σt], as always. We assume incompressible flow. Very importantly, we assume that the initial field **B** is uniform (i.e., has no current associated with it), so the only source of current arises from field perturbations **b** which satisfy the dynamo equation:

$$\begin{aligned} &\frac{\partial \vec{b}}{\partial t} = \lambda\nabla^2\vec{b} + \vec{\nabla} \times (\vec{u} \times \vec{B}) \\ &\Rightarrow (\sigma + \lambda k^2)\vec{b} = i\vec{k} \times (\vec{u} \times \vec{B}) = i\vec{u}(\vec{k}.\vec{B}) \end{aligned} \qquad (23.2)$$

Warning: This is not a dynamo! We are merely looking at the fate of field disturbances superimposed on a specified field (the origin of which we do not address!) The temperature anomalies satisfy

$$(\sigma + \kappa k^2)\theta = \beta u_z \qquad (23.3)$$

in the usual way. Substituting and retaining only linear terms, we get:

$$(\sigma + \nu k^2)\vec{u} + 2\vec{\Omega}\times\vec{u} = -\nabla p + \frac{g\alpha\beta u_z\hat{\vec{z}}}{(\sigma+\kappa k^2)} - \frac{(\vec{k}.\vec{B})(\vec{k}\times\vec{u})\times\vec{B}}{\rho\mu_0(\sigma+\lambda k^2)} \qquad (23.4)$$

Note that the last term can be combined with the first (using **k.u** =0 of course). If we take the curl (which means **k x...** of course) and then take the curl again, we get successively:

$$\begin{aligned}
&[\sigma+\nu k^2 + \frac{(\vec{k}.\vec{B})^2}{\rho\mu_0(\sigma+\lambda k^2)}]\vec{k}\times\vec{u} - (2\vec{\Omega}.\vec{k})\vec{u} = \frac{g\alpha\beta u_z}{(\sigma+\kappa k^2)}.(\vec{k}\times\hat{\vec{z}}) \\
&-k^2[\sigma+\nu k^2 + \frac{(\vec{k}.\vec{B})^2}{\rho\mu_0(\sigma+\lambda k^2)}]\vec{u} - (2\vec{\Omega}.\vec{k})\vec{k}\times\vec{u} = \frac{g\alpha\beta u_z}{(\sigma+\kappa k^2)}.\vec{k}\times(\vec{k}\times\hat{\vec{z}})
\end{aligned} \qquad (23.5)$$

[Reminder: **k x [(k x u) x B] = (k.B) k x u**, using the fact that **k.u** = 0, so that's why the magnetic field term can be lumped with the other terms in **k x u**).] If you then take the z-component of the second equation (using the first equation to evaluate the z-component of **k x u**) you get:

$$-k^2[\sigma+\nu k^2 + \frac{(\vec{k}.\vec{B})^2}{\rho\mu_0(\sigma+\lambda k^2)}] - \frac{(2\vec{\Omega}.\vec{k})^2}{[\sigma+\nu k^2 + \frac{(\vec{k}.\vec{B})^2}{\rho\mu_0(\sigma+\lambda k^2)}]} = -\frac{g\alpha\beta(k_x^2+k_y^2)}{(\sigma+\kappa k^2)} \qquad (23.6)$$

Here is an interesting special case of this equation: If there is no dissipation (i.e., no viscosity or thermal diffusion or magnetic diffusion) then one of the solutions that you get is

$$\omega^2 = \frac{-g\alpha\beta(k_x{}^2+k_y{}^2)}{k^2} + \frac{(2\vec{\Omega}.\vec{k})^2}{k^2} + \frac{(\vec{k}.\vec{B})^2}{\rho\mu_o} \qquad (23.7)$$

where $\omega \equiv i\sigma$. The first term on the RHS represents the restoring force due to gravity (if beta is negative) or the acceleration (beta positive). It is identical ot what we obtained in Ch 13, as of course it must be. The second term is the restoring effect of the Coriolis force. By itself it describes Rossby waves and inertial oscillations. The first and second terms together describe internal waves in Earth's ocean and atmosphere, for example. The third term describes Alfvén waves (the restoring force due to the magnetic field). That's why one can think of a magnetic field line as being like a violin string. The three terms together describe what are often called MAC waves (M=Magnetic, A= Archimedean, C= Coriolis). There is another solution in to 23.6, called slow hydromagnetic waves (which may contribute to Westward drift of Earth's magnetic field).

It should be apparent that if the buoyancy is large enough, growing solutions are possible for equation 23.6. (This can be proved rigorously but it's messy). But let's focus on the critical state ($\sigma = 0$). We then find that

$$\begin{aligned}
&1 + Q^* + \frac{Ta^*}{(1+Q^*)} = Ra^* \\
&Ra^* \equiv \frac{g\alpha\beta(k_x^2 + k_y^2)}{\nu\kappa k^6} \\
&Ta^* \equiv \frac{(2\vec{\Omega}.\vec{k})^2}{\nu^2 k^6} \\
&Q^* \equiv \frac{(\vec{k}.\vec{B})^2}{\rho\mu_0\nu\lambda k^4}
\end{aligned} \qquad (23.8)$$

where the dimensionless numbers are a "Rayleigh" number, a "Taylor" number and a "Chandrasekhar" number respectively. (I am using quotation marks because the usual definition involves the depth of the fluid layer rather than the wavevector, hence the factors of π^4 and so on that usually arise.) For a positive growth rate it can be shown that you need Ra* (the RHS) to be larger than the LHS.

Let's look at special cases:

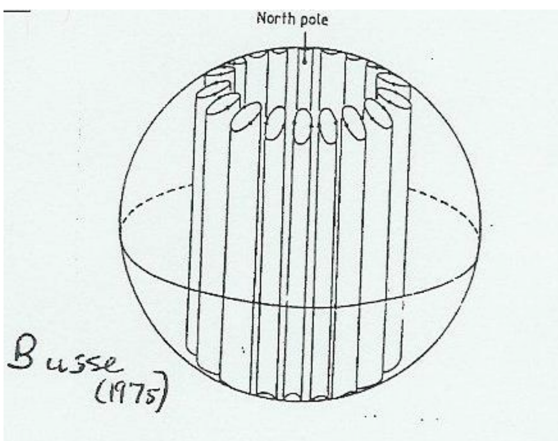
1. No rotation , no magnetic field. Convection then requires Ra* > 1, just as we found before (Ch 14)

2. Rotation but no magnetic field. Convection requires Ra*>1 + Ta*. We notice first that the behavior will depend on the Prandtl number (the ratio of viscosity to thermal diffusivity) since Ra* has $\nu\kappa$ in the denominator whereas Ta* has ν^2. If you evaluate Ta* for Earth's mantle and k~1/R (Earth radius), you get about 10^{-20}. This says that the Coriolis force is unimportant, as is apparent from the original equation of motion. But for low viscosity fluids (e.g. Earth's core) a naive evaluation for Ta* yields ~ 10^{32}. Wow! What this says is that rotation is enormously important in the core. In fact, it is so important that the fluid motions will tend to reduce it's dynamical effect by reducing the variation of the flow along the rotation axis [i.e., make $(2\mathbf{\Omega}.\mathbf{k})^2$ as small as possible.] This is called the Taylor-Proudman theorem.

Of course, it is not possible to make this component of k smaller than ~1/L, where L is the "size" of the planetary core (comparable to the radius). If we define $x = \ell/L$, where ℓ is the convective lengthscale perpendicular to the rotation axis, then the condition for onset of convection becomes something like:

$$x^4 Ra \approx \pi^4 + x^6 Ta; \quad Ra \equiv \frac{g\alpha\beta L^4}{\nu\kappa} \quad Ta \equiv \frac{4\Omega^2 L^4}{\nu^2} \qquad (23.9)$$

Minimization of Ra with respect to x predicts that $x \sim Ta^{-1/6}$ and $Ra \sim Ta^{2/3}$. So convection is strongly inhibited by rotation, and the scale of the flow perpendicular to the rotation axis is small. This is illustrated in the figure for a spherical geometry, The columnar motions are Taylor columns.

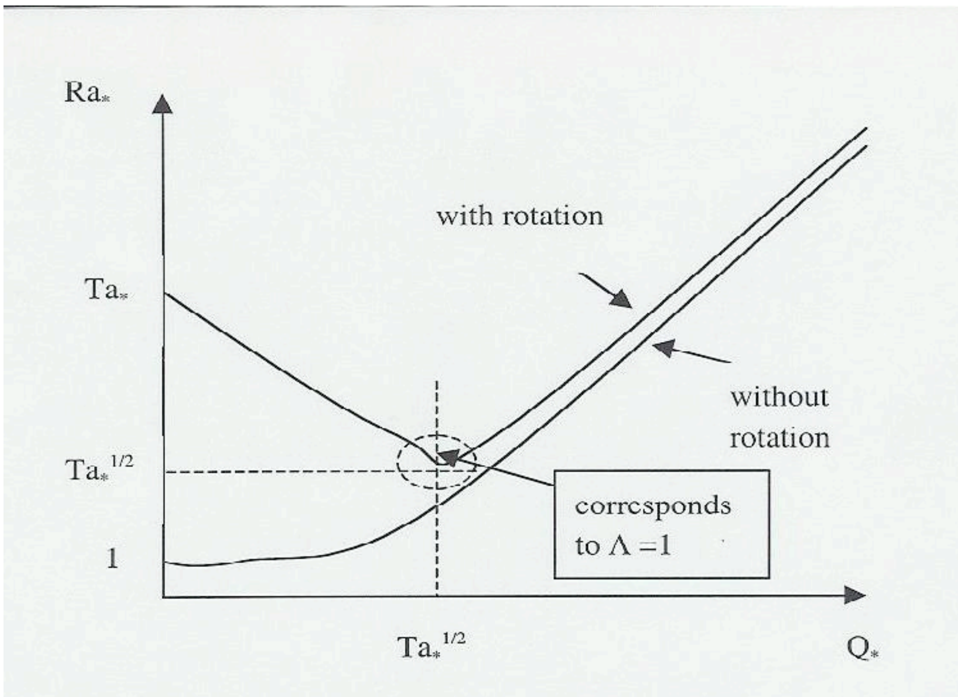


<u>3. Magnetic Field but No Rotation.</u> Convection requires $Ra^* > 1 + Q^*$. In Earth's core, with B ~ 0.001 Tesla, $Q^* \sim 10^{14}$ (assuming k ~ 1/L and L ~1000km). So the field is certainly important. If we proceed as above in defining x and minimizing the effect of the field (so that the variation along the field direction is the least it can be) then

$$x^4 Ra \approx \pi^4 + x^4 Q; \quad Q \equiv \frac{B^2 L^2}{\rho\mu_0\lambda\nu} \qquad (23.10)$$

and so Ra ~ Q at convective onset and convection is inhibited (in the sense that we only needed Ra ~ 1000 << Q in the absence of a field.) Although it's not obvious from this simple analysis, the best choice for $x \sim Q^{-1/6}$, but (unlike the rotational case) the preference for a smaller lengthscale is weak. (Nonetheless, it is observed in lab experiments).

4. Magnetic Field and Rotation. A remarkable thing happens: The combined effects enhance the convection (make it easier to excite), even though each effect by itself is an inhibition. This is immediately evident from the equation Ra* = 1+ Q* + Ta*/(1 + Q*), which shows that if you increase the field from zero, the critical Ra* will decrease from ~ Ta* to ~ $(Ta^*)^{1/2}$ at Q* ~ $(Ta^*)^{1/2}$.



At this "preferred state" the ratio of Q* to $(Ta^*)^{1/2}$ is a dimensionless number that is called the Elsasser number Λ :

$$\Lambda \equiv \frac{B^2}{2\rho\mu_0\lambda\Omega} \quad [\text{In cgs, } \Lambda \equiv \frac{B^2}{8\pi\rho\lambda\Omega}] \qquad (23.11)$$

For Earth's core, an Elsasser number of unity corresponds to a field strength of 0.001 T (i.e. ~ ten Gauss). *This is about the field that actually exists in the core.* The physical origin of this effect is clear in the equation of motion:

The Lorentz force is being balanced (in part) by the Coriolis force. This eliminates the need of the narrow, long columns that the Coriolis force demanded in the absence of a field. In fact, there is no need for highly non-equidimensional motions in this state... the convection is crudely speaking like the simple convective state in the absence of both rotation and convection (although the critical Ra is still somewhat higher, being of order $Ta^{1/2}$). The detailed form of the convection is different however (and this proves crucial, see below).

Numerical finite amplitude experiments confirm this linear stability result: the convection will be "most efficient" (i.e. deliver most heat for a given temperature gradient) if $\Lambda \sim 1$. It is plausible that a dynamo will tend to amplify the field up to this value but not beyond. (This assumes that the effect of viscosity is small). However, recent work on dynamo scaling suggests that $\Lambda \sim 1$ is at best a crude guide to field strength. This state of balancing the Lorentz and Coriolis forces is called <u>magnetostrophic balance</u> and dynamos that operate in this regime are called <u>strong field dynamos.</u> (Actually, even dynamos that are called "weak field" still tend to operate near this regime).

We can "test" $\Lambda \sim 1$, allowing for the geometric attenuation of the field (inverse cube outside the core) for the planets as follows:

Planet or Satellite	**Estimated Core field (Gauss)**	**Estimated Λ**
Mercury	0.01	$\sim 10^{-4}$
Venus	<0.001	$< 10^{-6}$
Earth	~10	~1
Jupiter	~10-100	~1
Ganymede	~1	~1

Saturn	~1-10	~1
Uranus, Neptune	~1	~0.1-1?

The estimates for the giant planets are debatable.. they are based on the conductivities in the outermost regions capable of dynamo action.

Venus does not have a dynamo so its value is included only to confirm that interpretation. Mercury is clearly anomalous and there have a number of attempts to explain this in terms of a core that is only partially convecting (Christensen) or because of a thin shell (Stanley).

In any event Elsasser number scaling may be incorrect. An alternative scaling, developed by Christensen, suggests that the heat flux should enter into the scaling and there is no dependence on rotation or magnetic diffusivity. It is in any event clear that the dependence of field on rotation is weak (not withstanding the anomalous case of Mercury). See problem 23.1.

"Realistic" Dynamos

Dynamo models can be loosely divided into four classes:

(1) *Purely kinematic, stability analysis only*. In these models, a particular flow field is prescribed and it is determined whether this flow field can lead to a growing field. The conclusion of this work is that most flow fields do not lead to a dynamo, irrespective of the magnitude of the velocity. But some plausible flows do work.

(2) *Mean field models*. These models, of the kind we discussed (e.g., alpha effect) do not attempt to describe all the field, merely the large scale field, and characterize the consequences of small scale fields and flows statistically. This is conceptually like characterizing the effect of turbulence in terms of eddy viscosity, etc. These models are "better" than purely kinematic models to the extent that the effects (such as alpha) may be derivable from a physical model, and the dynamo can be made non-linear by adding in large scale flows (e.g. flows driven by the Lorentz force).

(3) Models based on convection but highly idealized in respect of geometry or resolution or number of modes, retained, etc. These are also sometimes called mean field models even though they don't invoke small scale flow. These models suggest that convection in a rotating system is indeed able to sustain a dynamo (i.e. the conclusion given in # 1 above is not a problem.) Although limited, these kinds of models enable one to explore some fraction of parameter space.

(4) *Supercomputer models*, with realistic geometry and some realistic parameters. However, these models cannot address the onset of a dynamo (because that involves a convective state with small scales as discussed above). Moreover, some parameters are unrealistic, e.g. they use a viscosity orders of magnitude larger than what is realistic. But they get dynamos that have spectral character like Earth, occasional reversals and so forth.

For a description of current supercomputing modeling efforts, see Roberts PH, Glatzmaier GA Geodynamo theory and simulations, *Rev. Mod. Phys.* **72** 1081-1123, Oct 2000. The figure on the next page is taken from this paper.

There are also "laboratory dynamos" (e.g., large containers of liquid metal set into vigorous motion.) Although these cannot fully represent a spherical planet they enable some tests of scaling and numerical models.

What do models tell us about Field Symmetry and Spectra?

Recall that a "dipole" is a mathematical idealization. For example, a finite Helmholtz coil does not produce a dipole magnetic field (though it can certainly produce a field whose dominant multipole is the dipole). Recall also that the dipole will dominate at large distances from the source because its field decays like inverse cube of distance, while higher harmonics decay with larger inverse powers. Nonetheless we can speak of dipole-like dynamo solutions, if that term dominates.

Dipole solutions do seem to occur most readily. They have the following characteristics: (1) The largest harmonic is the axial dipole. (2) There is power in all harmonics, at least up to some high harmonic where dissipation kicks in. Crudely speaking ,there is a power law, at least over a limited range (meaning that the field energy in harmonic ℓ might decay something like $\ell^{-\alpha}$ where α is of order unity.) This is a slow decay and it means that the field at the surface of the dynamo (*not* the surface of the planet!) is harmonically rich. (3) The axial dipole is mildly enhanced and the axial quadrupole is mildly suppressed relative to this power law trend. (4) The dipole tilt is finite and neither very large nor very small (e.g., ~ten degrees).

These characteristics seem to be true of the Earth's field and Jupiter's field (though we lack sufficient data to fully test for Jupiter).

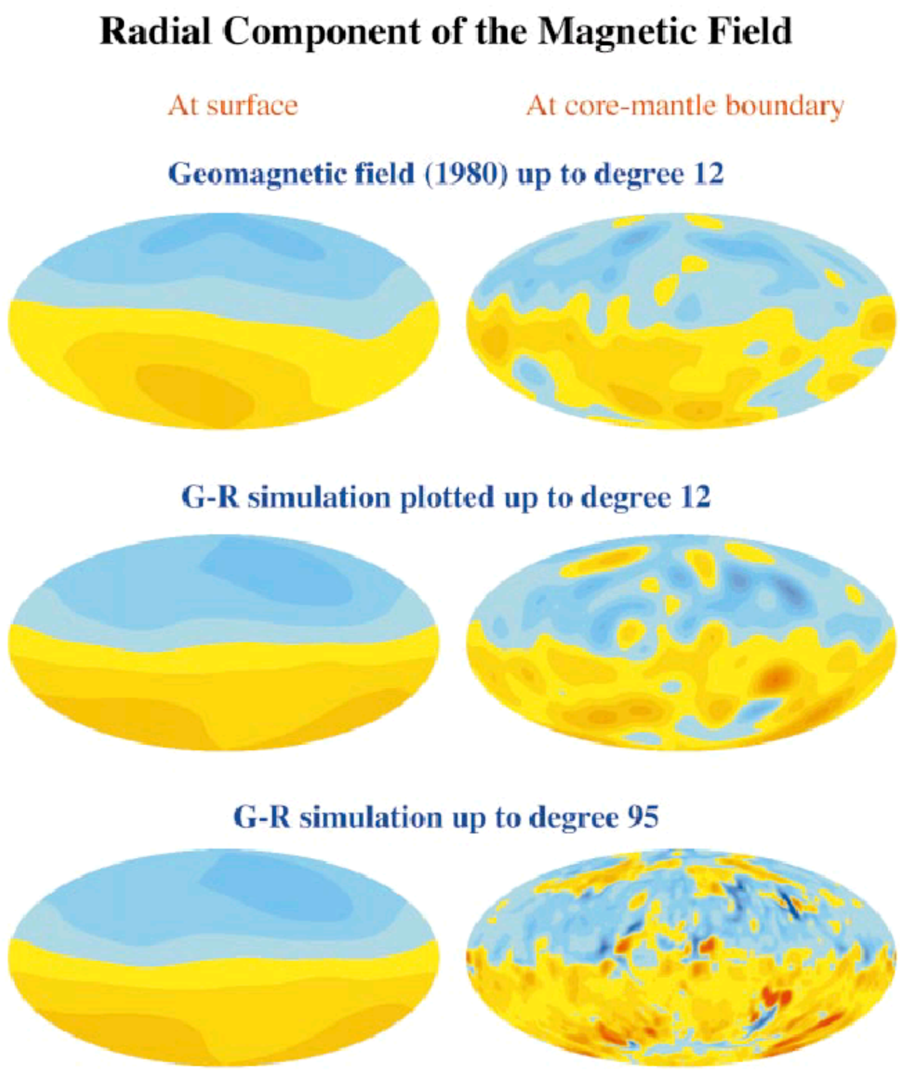


FIG. 3. Radial component of the magnetic field (reds for outward directed and blues for inward) plotted at the Earth's surface and at the core-mantle boundary. The surface fields are multiplied by 10 to obtain comparable color contrast. The Earth's field is plotted out to spherical harmonic degree 12. A snapshot from the Glatzmaier-Roberts simulation is plotted out to degree 12 (for comparison) and out to degree 95 [Color].

Quadrupole solutions have been observed but there have been insufficient examples to assess exactly what favors these over dipole solutions. It is suspected (but not proven) that dynamo generation in a thin shell may favor these.

Dynamo theory (and general symmetry arguments) tell us that it is ℓ+ m and not ℓ itself that determines symmetry properties. So a "quadrupole family" solution is one where ℓ+m is even. Such solutions will have large axial quadrupole (ℓ=2, m=0) as well as large equatorial dipole (ℓ=1, m=1). This is what we observe for Uranus and Neptune.

So Why do Some Planets have Dynamos while others do not?

The tentative conclusion is that if rotation is dynamically important, then a magnetic Reynolds number of order 10 or so is sufficient. For finite amplitude flow, importance of rotation is determined by the ratio of Coriolis force to other inertial terms, or the Rossby number defined as u/2ΩL. So perhaps what we need is

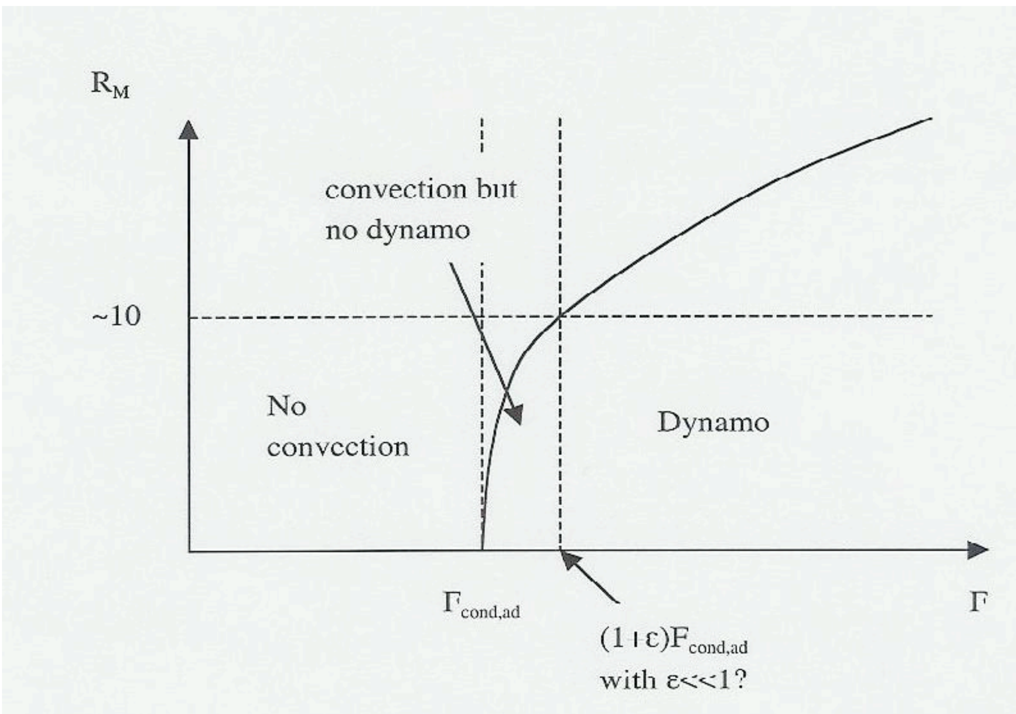
$$Dynamo \Leftrightarrow R_m \equiv \frac{u_{conv}L}{\lambda} > 10 \ \underline{\text{and}}\ Ro \equiv \frac{u}{2\Omega L} < 0.1 \quad (???) \tag{23.12}$$

Combining these two, we see that we need $\Omega\tau > 10^2$ or thereabouts, where τ is the free decay time for a magnetic field. In other words, this timescale must be large compared with the rotational period. This is satisfied easily by any planet. For example, free decay in Earth (or Venus) takes place in around 10000 years. In this context, Venus is a fast rotator. We also have our rough estimate for the convective velocity from mixing length theory, which states that

$$u_{conv} \sim 0.1\left(\frac{L[F - F_{cond,adiabat}]}{\rho H_T}\right)^{1/3} \tag{23.13}$$

There is some doubt about how this scales when there is rotation and magnetic field but the numerical models suggest that it is still roughly correct or a mild overestimate of velocity. This is the equation for thermal convection alone; in the case of compositional convection, one must add a term arising from compositional buoyancy inside the cube root. But for simplicity, lets plot what this formula suggests for Earth, assuming that the heat flow due to conduction along the adiabat is ~ 20 erg/cm^2.sec:

The important conclusion is that the range of parameters for which one has convection but no dynamo may be small. So it may be a good approximation to say that the question *will a dynamo exist?* translates to *does the core convect?*



And as we have seen, core convection may be contingent on an inner core. Perhaps even this is not enough; Earth may need an additional energy release to sustain the convection (for example, material rising up from the core and being added to the base of the mantle). So perhaps Venus and Mars do not have dynamos because they do not have inner cores. Or perhaps they are simply cooling too slowly (so the heat can be carried out by conduction alone). Or perhaps (in the case of Venus especially) they are not cooling at all. Ganymede may be aided by the stronger secular cooling arising from a CI potassium abundance.

By contrast, heat flows in giant planets are generally much higher than can be carried by conduction in regions of sufficient conductivity for a dynamo, and so a dynamo is “easy”.

What about Exoplanets?

The uncertainties with our terrestrial planets are equally valid to rocky bodies in other planetary systems. Hot Jupiters pose a different issue: They are prevented from cooling efficiently but they must have some addtionla energy source ot maintain the larger radius. Models that satisfy the observations typically predict heat fluxes as large as or larger than Jupiter and corresondinlgy similar or larger magnetic fields.

Problem 23.1

Suppose that the magnetic field B depends only on the heat flux F and fluid density ρ.

(a) Prove that the only dimensionally correct solution for the scaling law is $B \sim \mu_0^{1/2}(F/\rho)^{1/3}$.
(b) Offer a physical interpretation for this scaling.
(c) Estimate what this predicts for Earth and Jupiter and comment on whether it might plausibly be the correct scaling.