P613 HW # 2 Solutions *JM Densmore*

1. **The Free and Independent Electron Gas in Two Dimensions**

   (a) In a 2D system with periodic BC you can solve the SE with momentum $k_x = 2\pi/Ln_x$ and $k_y = 2\pi/Ln_y$. The number of allowed values of k space inside a volume Ω will be

   $$\frac{\Omega V}{4\pi^2}$$

   The volume in k space is the area of a circle with radius k$_f$. Taking into account the number of spins for the electron

   $$n = \frac{k_f^2}{2\pi} \tag{1}$$

   (b)

   $$\frac{1}{n} = \pi r_s^2 \tag{2}$$

   $$r_s = \frac{\sqrt{2}}{k_f} \tag{3}$$

   (c) In 2D the density of particles in range $k + dk$ is $2\pi k dk$

   $$n = \int \frac{2\pi k dk}{4\pi^2} f(\epsilon_k) \tag{4}$$

   Changing to an integral over energy we find

   $$n = \frac{m}{\pi\hbar^2} \int f(\epsilon) d\epsilon \tag{5}$$

   $$\tag{6}$$

   $$\begin{array}{rcl} g = & \frac{m}{\pi\hbar^2} & \epsilon > 0 \\ g = & 0 & \epsilon < 0 \end{array}$$

   (d) Since g is constant g$'$ = 0 all the terms in the BS expansion are equal to zero. So we find

   $$n = \int_0^{\epsilon_f} \frac{m}{\pi\hbar^2} dE + (\mu - \epsilon_f)\frac{m}{\pi\hbar^2} \tag{7}$$

   $$n = n(\mu - \epsilon_f)\frac{m}{\pi\hbar^2} \tag{8}$$

   $$\mu = \epsilon_f \tag{9}$$

   (e) From 2.67 we have

   $$n = \frac{m}{\pi\hbar^2} \int \frac{1}{e^{\beta(\epsilon-\mu)} + 1} d\epsilon$$

This integral can be looked up in a table. Using the electron density and the fermi energy it is easy to show

$$\epsilon_f = \mu + \frac{1}{\beta}\ln(1 + e^{-\beta\mu}) \tag{10}$$

(f) $\mu(T)$ does not have an analytic expansion about T≈ 0, which is why the BS expansion does not work in 2D.

2. **Entropy of a free electron gas**

(a) We'll start with the partition function, $\mathcal{Z}$, for a free electron gas

$$\mathcal{Z} = \sum_{n_i} e^{-\beta(n_i\epsilon_i - n_i\mu)} \tag{11}$$

$$= \sum_{n_1} e^{-\beta(n_1\epsilon_1 - n_1\mu)} * \sum_{n_N} e^{-\beta(n_N\epsilon_N - n_N\mu)} \tag{12}$$

For Fermi-Dirac statistics n can be either 0 or 1. The sums are easily evaluated.

$$\mathcal{Z} = \Pi(1 + e^{-\beta(\epsilon_i - \mu)}) \tag{13}$$

The entropy S is

$$S = -\frac{\partial\Omega}{\partial T} = k_B(\ln\mathcal{Z} + \beta\epsilon) \tag{14}$$

With the energy defined as

$$\epsilon = -\frac{\partial}{\partial\beta}\ln\mathcal{Z} \tag{15}$$

$$= \sum \frac{(\epsilon_i - \mu)e^{-\beta(\epsilon_i-\mu)}}{1 + e^{-\beta(\epsilon_i-\mu)}} \tag{16}$$

and

$$\ln(f_i) = \ln\left(\frac{e^{-\beta(\epsilon_i-\mu)}}{1 + e^{-\beta(\epsilon_i-\mu)}}\right) \tag{17}$$

$$= -\beta(\epsilon_i - \mu) - \ln(1 + e^{-\beta(\epsilon_i-\mu)}) \tag{18}$$

from (4), (6), & (8) it is easy to show that

$$S = -k_b\sum_k[f_i\ln f_i + (1 - f_i)\ln(1 - f_i)]$$

(b) We can use integration by parts to integrate

$$s = -k_B\int dEg(E)(f\ln(f) + (1 - f)\ln(1 - f))$$

$$hE = \int g(E)dE$$

$$\begin{aligned}
\frac{\partial}{\partial E}(f\ln(f) + (1-f)\ln(1-f)) &= f'\ln f + f' - f'ln(1-f) - f' \\
&= f'\ln(\frac{f}{1-f}) \qquad (19)
\end{aligned}$$

$$\ln\Big(\frac{f}{1-f}\Big) = -\beta(\epsilon-\mu) \qquad (20)$$

The entropy becomes

$$s = \beta k_B \int_0^\infty h(\epsilon)(\epsilon-\mu)\Big(\frac{-\partial f}{\partial \epsilon}\Big) \qquad (21)$$

Taylor expanding $h(\epsilon)$

$$h(\epsilon) = \sum_{l=0}^{\infty}\frac{1}{l!}h^{(l)}(\epsilon)(\epsilon-\mu)^l$$

$(l+1) = 2n \Rightarrow l = 2n-1$

$$s = \frac{1}{T}\sum_{n=1}^{\infty}h^{(2n-1)}(\mu)(2n)\int_{-\infty}^{\infty}\frac{(\epsilon-\mu)^{2n}}{2n!}(-f') \qquad (22)$$

The above integral is just $(k_BT)^{2n}C_{2n}$. For n =1, $\mu \approx \epsilon_f$.

$$s = \frac{\pi^2}{3}k_B^2Tg(\epsilon_f) \qquad (23)$$

The electron specific heat is

$$c_v = T\frac{\partial S}{\partial T}/n$$

$$c_v = \frac{\pi^2}{3}\frac{1}{n}k_B^2Tg(\epsilon_f)$$

3. **Pauli paramagnetism** This problem was solved on the quiz, but here I will use a different approach. The energy of a electron whose magnetic moment is parallel ("+") or antiparallel ("-") to H is given by

$$\epsilon_\pm = \frac{p^2}{2m} \pm \mu_BH = E_0 \pm \mu_BH \qquad (24)$$

where $E_0 = p^2/2m$. The energy levels of the system are populated according to the fermi-dirac distribution function.

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \qquad (25)$$

The density of levels is given by $\frac{4\pi V}{h^3}p^2dp$. The magnetization / volume is

$$\frac{M}{V} = \frac{4\pi\mu_B}{h^3}\int_0^\infty dpp^2[f(\epsilon_+) - f(\epsilon_-)] \tag{26}$$

Define A = -H if "+", A = H if "-". We'll change the integral in p to an integral in E.

$$\sqrt{2}m^{3/2}\int_0^\infty \frac{\sqrt{E}dE}{e^{\beta(E+\alpha)}+1} = \frac{2}{3}\alpha^{3/2}[1 + \frac{\pi^2}{8}(\frac{1}{\beta\alpha})^2 + ...] \tag{27}$$

with $\alpha = -\mu_b A + \mu$ then $f(+) \Rightarrow \alpha = \mu + \mu_B H; f(-) \Rightarrow \alpha = \mu - \mu_B H$

$$\begin{aligned} M/V &= \frac{8\pi\mu_B(2m^3)^{1/2}}{3h^3}\left((\mu+\mu_B H)^{3/2}[1+\frac{\pi^2}{8}(\frac{k_BT}{\mu+\mu_BH})^2] + \right. \\ &\left. (\mu-\mu_B H)^{3/2}[1+\frac{\pi^2}{8}(\frac{k_BT}{\mu-\mu_BH})^2]\right) \end{aligned} \tag{28}$$

After expanding in powers of H and keeping only the leading terms

$$\frac{M}{V} = \frac{8\pi\mu_B^2(2m^3\mu)^{1/2}H}{h^3}\left(1 - \frac{\pi^2}{24}(\frac{k_bT}{\mu})^2 + ...\right) \tag{29}$$

μ is the chemical potential. In the low temperature limit it is just the fermi energy ϵ_f. You should be able to convince yourself that this the same result as the quiz. $\chi = \mu_B^2 g(\epsilon_f) \approx 10^{-6}$

Using the above results, making sure to use the correct units. We find that $R_W = 1$.

7.2 Measure ΔK, then use $\Delta k = 2\pi/\Delta L$ to calculate the length of the box.
7.3 See problem 1.

7.4 Make a plot of ϵ_f vs n, which comes out to be a linear plot.

7.5 $v_{som} \approx 10^7$, $v_{dru} \approx 10^4$. $\epsilon_{som} \approx 10$ eV; $\epsilon_{dru} \approx 10^{-2}$. $\epsilon_{som}/\epsilon_{dru} \approx 10^2$.