2.1 The free and Independent Electron gas in Two dimensions.

a) $k_x = \frac{2\pi n_x}{L}$ $k_y = \frac{2\pi n_y}{L}$

area $= \left(\frac{2\pi}{L}\right)^2$.

$\Rightarrow$ A region of k space of area Ω will contain

$$\frac{\Omega}{(2\pi/L)^2} = \frac{\Omega A}{4\pi^2}$$ allowed values of k.

$\Rightarrow$ k space density is $\frac{A}{4\pi^2}$.

Consider a circles radius k_F. It contains

$$(4\pi k_F^2)\frac{A}{4\pi^2} = \frac{k_F^2 A}{\pi}$$

allowed values of k.

each k value has ~~2~~ ~~one~~ two one electron levels.

$$N = \frac{2k_F^2 A}{\pi}$$

$$n = \frac{N}{A} = \frac{2k_F^2}{\pi}$$

b) r_s is the radius of the circle whose area is ~~the~~ equal to area per conduction electrons.

$$\frac{A}{N} = \frac{1}{n} = 4\pi r_s^2.$$

$$r_s = \left[\frac{1}{4\pi n}\right]^{1/2}$$

$$r_s = \left[\frac{1}{4\pi}\frac{\pi}{2k_F^2}\right]^{1/2} = \frac{1}{2\sqrt{2}\,k_F}$$

c)

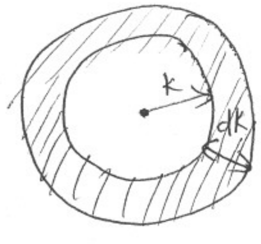
The number of k values per unit area is

$$\frac{A}{4\pi^2}$$

~~So for a function $F(k)$ and summing it over all allowed values of k,~~

$$\cancel{\sum_k F(k) = \frac{A}{4\pi^2} \sum_k F(k)\,\Delta k.}$$

or $$\cancel{\lim_{A\to\infty} \frac{1}{A} \sum_k F(k)} = \int \frac{dk}{4\pi^2} F(k).$$



Area of the shaded region. $2\pi k\, dk$

$$g(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{dN}{dk} \cdot \frac{dk}{d\varepsilon}$$

now $\frac{dN}{dk} \propto 2\pi k. = (\text{const}) \cdot 2\pi k.$

$$\frac{dk}{d\varepsilon} = \frac{m}{\hbar^2 k}.$$

$$\Rightarrow \quad g(\varepsilon) = (\text{const})\, 2\pi k \cdot \frac{m}{\hbar^2 k}$$

$$= (\text{const}) \times \frac{2\pi m}{\hbar^2}, \qquad \varepsilon > 0.$$

d) The sommerfeld expansion for n is

$$n = \int_0^{\mu} g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 g'(\mu) + O(T^4) \qquad -(2.72)$$

now since $g(\varepsilon)$ is a constant $g' = 0$.

$$\Rightarrow \quad n = \underline{\underline{(\text{const}) \times \int_0^{\mu} d\varepsilon}}$$

and from equation 2.77,

$$\mu = \varepsilon_F - \frac{\pi^2}{6}(k_B T)^2 \frac{g'(\varepsilon_F)}{g(\varepsilon_F)}$$

Since $g'(\varepsilon_F) = 0$,

$$\underline{\underline{\mu = \varepsilon_F}}$$

e) 2.67 is

$$n = \int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon)\, f(\varepsilon)$$

$g(\varepsilon) = \text{const.} = c$

$$\int^{\infty} \ldots f(\varepsilon).$$

f) μ differs from ε_F by a term
$$k_B T \ln\left(1 + e^{-\mu/k_B T}\right)$$

High temperature
the linear term dominates.

At $T=0$ we get $\mu = \varepsilon_F$. Since we consider $g(\varepsilon)$ is independent of ε, we have
$\mu = \varepsilon_F$ at all temperatures

This is wrong. The correct and real identity is
$$\lim_{T\to 0} \mu = \varepsilon_F .$$

2.2.

a) $\left(\frac{\partial u}{\partial T}\right)_n = T\left(\frac{\partial s}{\partial T}\right)_n.$

$$\partial s = \frac{1}{T}\frac{\partial u}{\partial T}dT$$

$$s = \int_0^T \frac{1}{T}\frac{\partial u}{\partial T}dT$$

$s \to 0,\ T \to 0$

$$s = \frac{1}{T}u\Big|_0^T - \int_0^T -\frac{1}{T^2}u\,dT.$$

$$= \frac{u}{T} + \cancel{\int_0^T} -\frac{u}{2T} + \int_0^T \frac{\partial u}{\partial T}\cdot\frac{1}{2T}dT.$$

$$\Rightarrow \frac{1}{2}\int_0^T \frac{1}{T}\frac{\partial u}{\partial T}dT = \frac{u}{2T}$$

$$\Rightarrow s = \frac{u}{T}.$$

$$\Rightarrow s = \int \frac{dk}{4\pi^3}\cdot\frac{\varepsilon(k)}{T}f(\varepsilon k). \quad \text{— (1)}$$

$$f = \frac{1}{e^{\varepsilon-\mu/k_B T}+1}$$

$$e^{\varepsilon-\mu/k_B T} = \frac{1}{f} - 1 = \frac{1-f}{f}$$

$$\frac{\varepsilon-\mu}{k_B T} = \ln(1-f) - \ln f.$$

$$\frac{\varepsilon-\mu}{T} = k_B[\ln(1-f) - \ln f] \quad \text{— (2)}$$

2.3

a) $$f(\varepsilon) = e^{-(\varepsilon-\mu)/k_B T}$$

now $$r_s = \left(\frac{3}{4\pi n}\right)^{1/3}.$$

$$\boxed{\varepsilon = \frac{\hbar^2 k^2}{2m}}$$

$$n = \int \frac{dk}{4\pi^3} e^{-(\varepsilon-\mu)/k_B T}.$$

$$= \frac{e^{\mu/k_B T}}{4\pi^3} \sqrt{\frac{2m}{\hbar^2}} \int_0^\infty d\varepsilon \sqrt{\varepsilon}\, e^{-\varepsilon/k_B T}$$

now let $$t = \frac{\varepsilon}{k_B T}$$

$$k_B T\, dt = d\varepsilon.$$

$$n = \frac{e^{\mu/k_B T}}{4\pi^3} \sqrt{\frac{2m}{\hbar^2}} \int_0^\infty k_B T\, dt \sqrt{k_B T} \sqrt{t}\, e^{-t}.$$

$$n = \frac{e^{\mu/k_B T}}{4\pi^3} \sqrt{\frac{2m}{\hbar^2}} (k_B T)^{3/2} \underbrace{\int_0^\infty dt \sqrt{t}\, e^{-t}}_{\text{gamma function.}}$$

$$\int_0^\infty dt\, t^{1/2} e^{-t} = \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$$

$$n = \frac{e^{\mu/k_B T}}{4\pi^3} \sqrt{\frac{2m}{\hbar^2}} (k_B T)^{3/2} \frac{1}{2}\sqrt{\pi}$$

$$n = \frac{e^{\mu/k_B T}}{8\pi^{5/2}} \sqrt{\frac{2m}{\hbar^2}} (k_B T)^{3/2}$$

$$r_s = \left[\frac{3}{4\pi} \; 8\,\pi^{5/2} \; e^{-\mu/k_BT} \sqrt{\frac{\hbar^2}{2m}} \left(k_BT\right)^{-3/2} \right]^{1/3}$$

$$r_s = e^{-\mu/3k_BT} \cdot 3^{1/3} \cdot \pi^{1/2} \cdot \hbar \left(2m k_BT\right)^{-1/2} \quad — ①$$

b) 'r_s' is defined as the radius of the sphere whose volume is equal to the volume per conduction electron. If 'r_s' increases, it means that the conduction electron density decreases. At higher temperature from ① it is clear that 'r_s' is is also large and electrons are more free.

$$r_s \gg \left(\frac{\hbar^2}{2m k_BT}\right)^{1/2} \Rightarrow r_s \gg \frac{1}{k}$$

were k is the wave vector.

This is similar as considering de Broglie wavelength. If the interparticle separation is larger than $\frac{1}{k}$ we can consider it as in classical state.