C4: Particle Physics Major Option

Passage of Particles Through Matter

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1 Passage of charged particles through matter

1.1 Energy loss by ionisation

A **charged particle** moving through different types of matter will interact with it in different ways, losing energy in various ways.

- **Solids, liquids and gases:** the charged particle will ionise the material's atoms.
- **Semiconductors:** the charged particle will produce electron-hole pairs.
- **Scintillator:** the charged particle will lead to the production of light.
- **Superheated liquid:** the charged particle will leave a trail of bubbles.

In the limit that all electrons in the material's atoms can be modelled as free, the kinetic energy (E_{KE}) loss per unit distance travelled (dx), known as the **stopping power** is:

$$\boxed{-\frac{dE}{dx} \propto \frac{1}{\beta^2}} \tag{1}$$

Here, $\beta = v/c$. This relationship can be understood by looking at the **Bethe formula**[1], derived in the following section.

1.1.1 Energy loss per unit distance: the Bethe formula

Charged particles like alpha-particles, protons and pions, scatter electrons according to **Rutherford scattering**. Therefore scattering of the charged particles by free electrons is the same process, just in the **rest frame of the electrons**.

- The Rutherford scattering formula is improved by the **Mott scattering formula** for electrons of momentum p and velocity v and particles with charge Ze:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{Z\alpha\hbar c}{2pv}\right)^2 \operatorname{cosec}^4\left(\frac{\theta}{2}\right)\left[1 - \frac{v^2}{c^2}\sin^2\left(\frac{\theta}{2}\right)\right] \tag{2}$$

- Convert the scattering angle θ into a momentum transfer q:

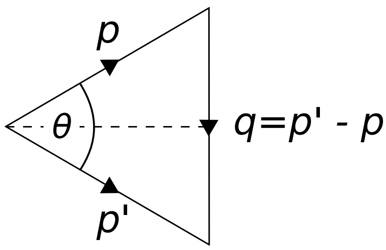


Figure 1: Momentum triangle.

$$\left(\frac{q}{2}\right) = p\sin\left(\frac{\theta}{2}\right) \tag{3}$$

$$\sin\left(\frac{\theta}{2}\right) = \left(\frac{q}{2p}\right) \tag{4}$$

$$\sin^2\left(\frac{\theta}{2}\right) = \left(\frac{q}{2p}\right)^2 \tag{5}$$

- Using (5), the $\operatorname{cosec}^4\left({}^{\theta}/_{2}\right)$ and $\sin^2\left({}^{\theta}/_{2}\right)$ terms in the Mott scattering formula, (2), can be replaced:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{Z\alpha\hbar c}{2pv}\right)^2 \frac{16p^4}{q^4}\left[1 - \frac{v^2}{c^2}\frac{q^2}{4p^2}\right] \tag{6}$$

[1]The term "Bethe-Bloch formula" actually refers to the modified Bethe formula that includes Bloch's Z^4 correction.

- Now convert the differential cross-section, $d\sigma/d\Omega$ into $d\sigma/dq^2$:

  Replace the differential solid angle, $d\Omega$, in (6) using $d\Omega = 4\pi\sin(\theta)d\theta$:

$$\frac{d\sigma}{\sin(\theta)d\theta} = 4\pi\left(\frac{Z\alpha\hbar c}{2pv}\right)^2 \frac{16p^4}{q^4}\left[1 - \frac{v^2}{c^2}\frac{q^2}{4p^2}\right] \tag{7}$$

  The operator $\sin(\theta)d\theta$ can be obtained by differentiating (5) in the form $q^2 = 4p^2\sin^2(\theta/2)$ to give:

$$\frac{dq^2}{d\theta} = 4p^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 4p^2\left[\frac{1}{2}\sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right)\right] = 2p^2\sin(\theta) \tag{8}$$

$$\sin(\theta)d\theta = \frac{dq^2}{2p^2} \tag{9}$$

  Substituting (9) into (7) and cancelling terms yields:

$$\frac{d\sigma}{dq^2} = 8\pi\left(\frac{Z\alpha\hbar c}{q^2 v}\right)^2\left[1 - \frac{v^2}{c^2}\frac{q^2}{4p^2}\right] \tag{10}$$

- Now transfer to the frame in which the **electron is at rest**, with the charged particle moving towards it:

$$\boxed{p_{\text{electron}} \rightarrow p_{\text{nucleus}} = m_e v\gamma} \tag{11}$$

$$\frac{d\sigma}{dq^2} = 8\pi\left(\frac{Z\alpha\hbar c}{q^2 v}\right)^2\left[1 - \frac{v^2}{c^2}\frac{q^2}{4(m_e v\gamma)^2}\right] \tag{12}$$

- The momentum transfer squared is the same in both frames and is related to the **energy transfer**, δ as follows:

$$\boxed{q^2 = 2m_e\,\delta} \qquad \rightarrow \qquad \boxed{dq^2 = 2m_e\,d\delta} \tag{13}$$

$$\frac{d\sigma}{2m_e\,d\delta} = 8\pi\left(\frac{Z\alpha\hbar c}{2m_e\delta v}\right)^2\left[1 - \frac{v^2}{c^2}\frac{2m_e\delta}{4(m_e v\gamma)^2}\right] \tag{14}$$

$$\frac{d\sigma}{d\delta} = 16\pi m_e\left(\frac{Z\alpha\hbar c}{2m_e\delta v}\right)^2\left[1 - \frac{m_e\delta}{2m_e^2c^2\gamma^2}\right] \tag{15}$$

- So the cross-section for energy loss in the range $\delta \rightarrow \delta + d\delta$ is:

$$\frac{d\sigma}{d\delta} = \frac{4\pi}{m_e}\left(\frac{Z\alpha\hbar c}{\delta v}\right)^2\left[1 - \frac{\delta}{2m_e c^2}\left(1 - \frac{v^2}{c^2}\right)\right] \tag{16}$$

- If the charged particle loses energy $-dE$ in a distance dx in a material containing n atoms of atomic number Z_{material} per unit volume then:

$$-dE = \begin{pmatrix}\text{number of electrons}\\ \text{per unit volume}\end{pmatrix} \times \begin{pmatrix}\text{energy loss}\\ \text{per unit volume}\end{pmatrix} \tag{17}$$

$$-dE = nZ_{\text{material}} \times dx\begin{pmatrix}\text{energy loss}\\ \text{per unit area}\end{pmatrix} \tag{18}$$

$$-\frac{dE}{dx} = nZ_{\text{material}}\int_{\delta_{\min}}^{\delta_{\max}} \delta\,\frac{d\sigma}{d\delta}\,d\delta \tag{19}$$

- Substitute (16) into (19) and integrate:

$$-\frac{dE}{dx} = nZ_{\text{material}} \int_{\delta_{\min}}^{\delta_{\max}} \delta \, \frac{4\pi}{m_e} \left(\frac{Z\alpha\hbar c}{\delta v}\right)^2 \left[1 - \frac{\delta}{2m_e c^2}\left(1 - \frac{v^2}{c^2}\right)\right] d\delta \tag{20}$$

$$-\frac{dE}{dx} = nZ_{\text{material}} \frac{4\pi}{m_e} \left(\frac{Z\alpha\hbar c}{v}\right)^2 \int_{\delta_{\min}}^{\delta_{\max}} \left[\frac{1}{\delta} - \frac{1}{2m_e c^2}\left(1 - \frac{v^2}{c^2}\right)\right] d\delta \tag{21}$$

$$-\frac{dE}{dx} = nZ_{\text{material}} \frac{4\pi}{m_e} \left(\frac{Z\alpha\hbar c}{v}\right)^2 \left[\ln\left(\frac{\delta_{\max}}{\delta_{\min}}\right) - \frac{\delta_{\max} - \delta_{\min}}{2m_e c^2}\left(1 - \frac{v^2}{c^2}\right)\right] \tag{22}$$

- At large δ, the electrons can be assumed to be free; kinematics shows that:

$$\boxed{\delta_{\max} = 2m_e v^2 \gamma^2} \tag{23}$$

- However, this is not true at small δ. At low δ the momentum and energy transfer can lead to excitation as well as ionisation: it is therefore necessary to integrate over δ and q with cross-sections that depend on the atomic structure of the material in question.
- The Bethe formula avoids this problem by defining a **mean excitation potential**, I, as follows:

$$\boxed{\delta_{\min} = I = 16Z^{0.9}} \tag{24}$$

- The final result is...

$$\boxed{-\left(\frac{dE}{dx}\right)_{\text{ionisation}} = nZ_{\text{material}} \frac{4\pi}{m_e} \left(\frac{Z\alpha\hbar c}{v}\right)^2 \left[\ln\left(\frac{2m_e v^2}{I(1 - v^2/c^2)}\right) - \frac{v^2}{c^2}\right]} \tag{25}$$

Remember: Z is the charge (in units of e) of the charged particle striking the material, which has atomic number or effective charge Z_{material} .

1.1.2 Observations on the Bethe formula

The stopping power (dE/dx) has the following general properties:

- It is independent of the mass of the incident particle.
- It depends on the charge of the incident particle squared (Z^2).
- Treating the slowly varying logarithmic term as a constant, it can be seen that $dE/dx \propto c^2/v^2$ or $dE/dx \propto 1/\beta^2$, as noted above.

1.1.3 Behaviour of the Bethe formula as a function of velocity

Low velocity regime: $\beta\gamma < 3.5$

In general the stopping power (dE/dx) falls off as $1/v^2$ in accordance with (1).

Slow moving positively charged particles can capture the material's negative electrons, reducing the incident particle's effective charge, Z, thus reducing dE/dx. Negative incident particles can suffer stripping of their electrons, reducing their value of Z.

Minimum ionisation loss: $\beta\gamma \approx 3.5$

The $1/(1 - \beta^2)$ term in the logarithm halts the general decrease in stopping power. It stops decreasing somewhere in the range of $\beta\gamma = 3 \rightarrow 4$: this is known as the **minimum ionisation loss**.

The relativistic rise: $\beta\gamma > 3.5$

Beyond the minimum ionisation loss dE/dx begins to rise again, in fact the Bethe formula predicts it should rise indefinitely.

In addition, as the incident charged particle reaches relativistic energies its transverse electric field increases, meaning more of the material's atoms are within range of the particle's electric field, leading to greater ionisation energy loss. This is known as the **relativistic rise**.

Density effect

However, the slope actually does shallow out because of the **density effect**: long distance atomic electrons are screened from the electric field of the incident particle by the dielectric effect of the intervening atoms in the material.

Stopping Power of H_2, He, C, Al, Fe, Sn and Pb

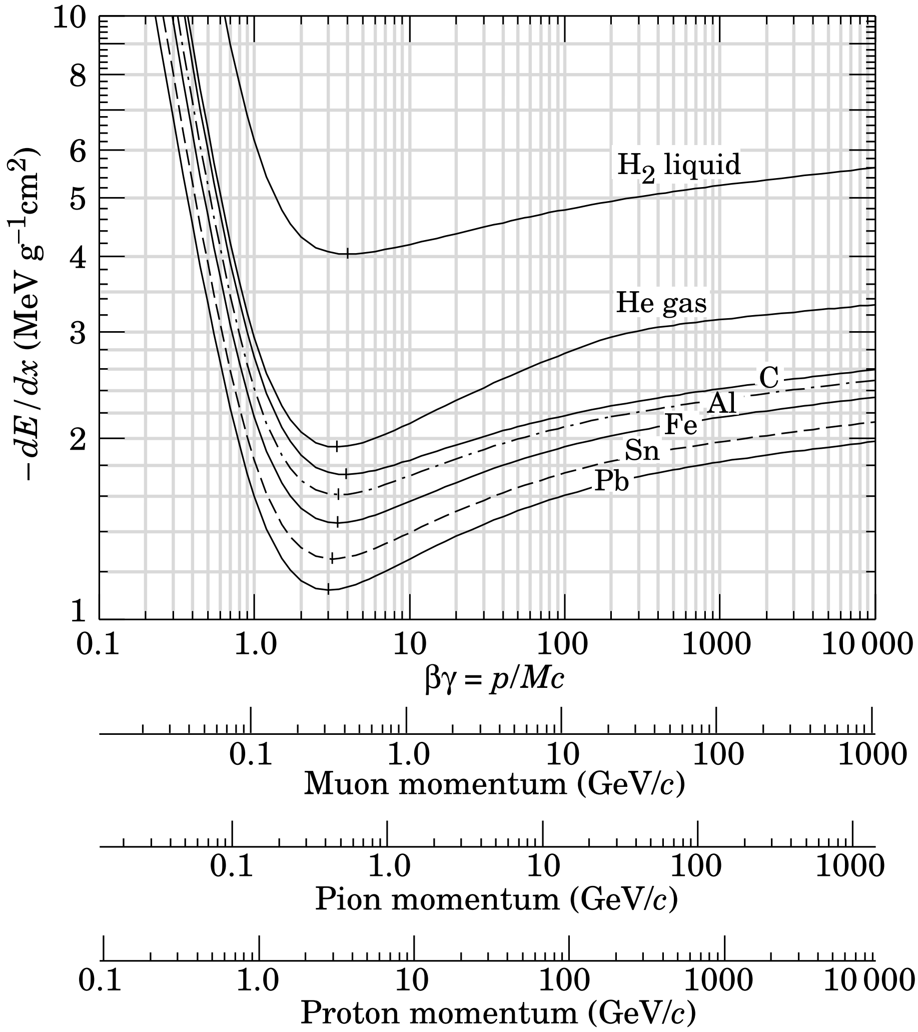


Figure 2: Mean energy loss rate divided by density in: liquid (bubble chamber) hydrogen, gaseous helium, carbon, aluminium, iron, tin, and lead. Radiative effects, relevant for muons and pions, are not included. Taken from Figure 27.3 of the Particle Data Group 2008.

1.1.4 Particle ranges

Consider an electron and a proton entering the same material. Experiment shows that:

- when an electron and a proton of the **same initial velocity** enter the material, the **electron will slow down and stop quicker** than the proton will.
- when an electron and a proton of the **same initial kinetic energy** enter the material, the **proton will lose energy more rapidly** than the electron.

The **range**, R, of a particle entering a material can be found as follows:

$$R = \int_{E_0}^{0} \frac{dE}{dE/dx} \tag{26}$$

Substituting dE/dx from the Bethe formula and ignoring the slowly varying logarithmic term:

$$R \approx \int_{E_0}^{0} \frac{1}{nZ_{\text{material}}} \frac{m_e}{4\pi} \left(\frac{v_{\text{incident}}}{Z\alpha\hbar c}\right)^2 dE \tag{27}$$

$$R \propto \int_{E_0}^{0} \frac{v^2}{Z^2}\, dE \tag{28}$$

Express the incident particle's velocity in terms of its kinetic energy; $E = {}^1/_2 mv^2$:

$$R \propto \int_{E_0}^{0} \frac{E}{mZ^2} dE \tag{29}$$

This gives an approximate range for charged particles travelling through matter:

$$\boxed{R \propto \frac{E_0^2}{mZ^2}} \qquad \rightarrow \qquad \boxed{R \propto \frac{mv^2}{Z^2}} \tag{30}$$

A more detailed calculation gives an improved formula for the range:

$$\boxed{R \propto \frac{E_0^{3/2}}{\sqrt{m}}} \qquad \rightarrow \qquad \boxed{R \propto mv^3} \tag{31}$$

How do these formulae compare to the statements made earlier?

- Using $R \propto mv^3$ shows that for two different particles with the same velocity, the heavier one will travel further. Hence the electron stops sooner than the proton because $m_e < m_p$.
- Using $R \propto E_0^{3/2}/\sqrt{m}$ shows that for two different particles with the same initial kinetic energy, the lighter one will travel further. Hence the proton stops sooner than the electron because $m_p > m_e$.

It is worth noting that since energy loss is a statistical process, particles starting with energy E will end with a spread of energies, and hence a spread of ranges: **range straggling**.

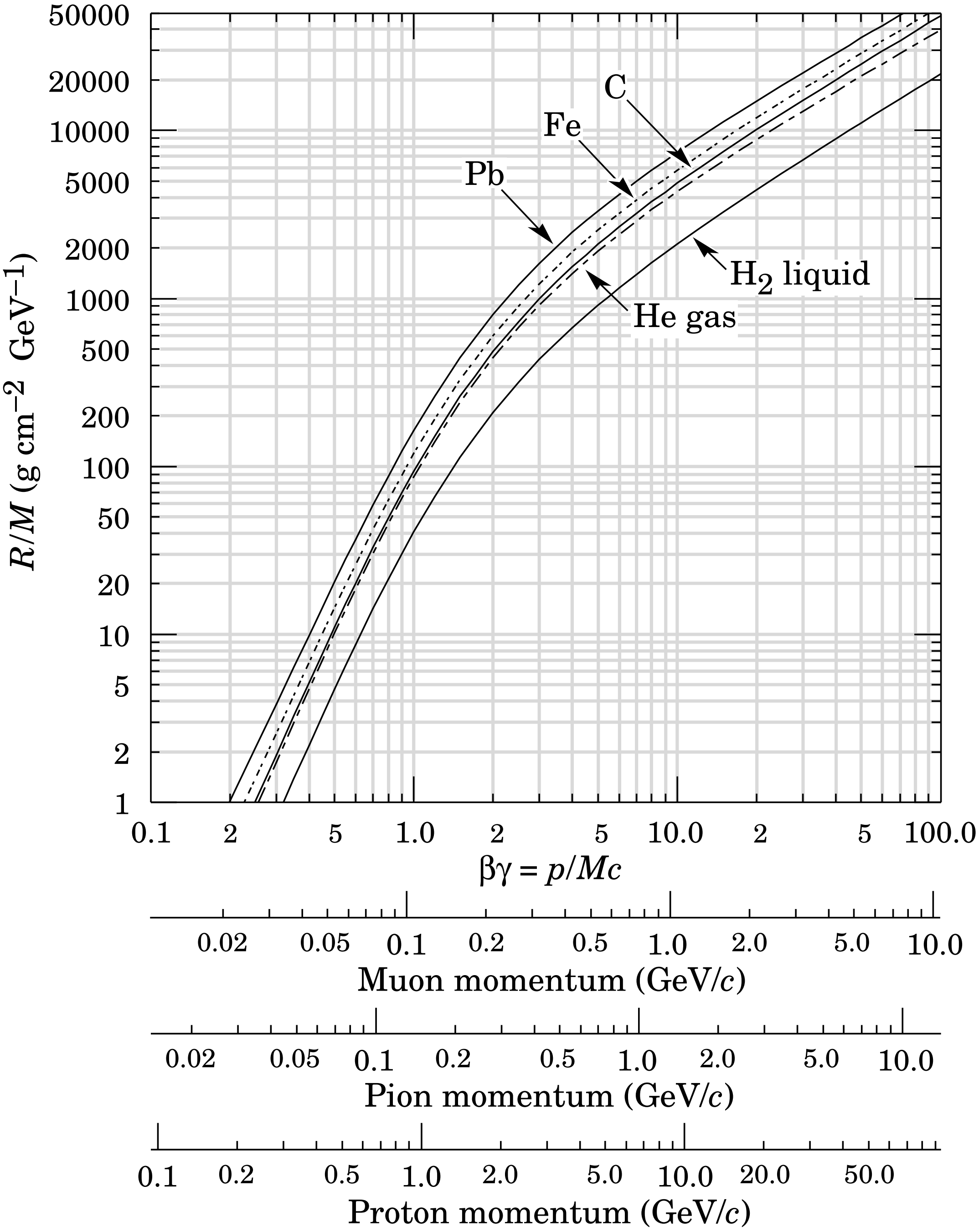


Figure 3: Range of heavy charged particles in liquid (bubble chamber) hydrogen, helium gas, carbon, iron, and lead. For example: For a K^+ whose momentum is 700 MeV/c, $\beta\gamma = 1.42$. For lead we read $R/M \approx 396$, and so the range is $195\,g\,cm^2$. Taken from Figure 27.4 of the Particle Data Group 2008.

1.1.5 Back-scattering and channelling

Low energy electrons are particularly prone to **large-angle back-scattering** and can come to rest behind where they started from.

If a charged particle is moving through a **crystal** it might take a route along a direction defined by the crystal structure that has a particularly low electron density. This is known as **channelling**.

1.1.6 The possibility of particle identification

The following table outlines in which ranges of the incident particle's velocity it can be identified:

$\beta\gamma$ **value**	**Velocity range**	**Possibility of particle identification**
$\beta\gamma < 3$	Proportional region	Since $dE/dx \propto 1/v^2$ different mass particles with the same energy will have different rates of energy loss in the same material, proportional to their different velocities, v.
$\beta\gamma \approx 3$	Minimum ionisation loss	The stopping power is now approximately energy independent - this cannot be used to identify particles.
$\beta\gamma \approx 10$	Relativistic rise	The relativistic rise has a small mass dependence - particle identification becomes possible again.

1.1.7 Final energy dissipation

Thermal excitations

Once the charged particle has insufficient energy to cause further ionisation it loses energy in collisions with the atoms of the material. This causes **thermal excitation** of the atoms - the energy lost by the incident particle heats the material.

Ionisation

If electrons generated by ionisation have sufficient energy they can cause **further ionisation**. The final state of a charged particle moving through matter will be pairs of electrons and ions, which if left uncollected, will recombine. To detect a particle these electron-ion pairs must be prevented from recombining, collected in some way and transformed into an electrical signal. Examples of such methods, to be explained fully later, include:

- Gas proportional counter;
- Liquid argon ionisation chamber;
- Solid-state germanium detector.

Scintillation

Transparent materials can become excited and radiate in the visible region - these materials are called **scintillators**. Coupling them to an array of photomultiplier tubes (PMTs) allows an electrical signal to be generated when a particle passes through the scintillator: this is the basis of the scintillation counter.

1.2 Energy loss by radiation: bremsstrahlung

When a charged particle is accelerated or decelerated it emits radiation and thus loses energy and slows down; this deceleration radiation is known as **bremsstrahlung**[2]. The two commonest occurrences of bremsstrahlung are:

- **Charged particles stopping in matter:** when charged particles enter a material they are accelerated and decelerated by the electric field of the material's atomic nuclei and atomic electrons. It is this phenomenon which is usually being refered to when the term bremsstrahlung is used.
- **Synchrotron radiation:** when ultra-relativistic particles move through magnetic fields they are forced to move along a curved path. Since their direction of motion is continually changing, they are also accelerating and so emit bremsstrahlung, except in this instance it is usually referred to as synchrotron[3] radiation.

The lighter the particle the greater the acceleration:

$$(\text{Amplitude of radiation}) \propto \frac{1}{m} \tag{32}$$

$$(\text{Intensity of radiation}) \propto \frac{1}{m^2} \tag{33}$$

1.2.1 In the E-field of atomic nuclei

Vertex factors:
① $\sqrt{\alpha}$
② $\sqrt{\alpha}$
③ $Z\sqrt{\alpha}$

Reaction rate: $W \propto (\sqrt{\alpha}\sqrt{\alpha}\,Z\sqrt{\alpha})^2$

$\boxed{W \propto Z^2\alpha^3}$

Cross-section: $\boxed{\sigma_{\text{nuclei}} \propto \frac{Z^2\alpha^3}{m^2}}$

1.2.2 In the E-field of Z atomic electrons

Vertex factors:
① $\sqrt{\alpha}$
② $\sqrt{\alpha}$
③ $\sqrt{\alpha}$

Reaction rate for 1 electron: $W \propto (\sqrt{\alpha}\sqrt{\alpha}\,\sqrt{\alpha})^2$

$W \propto \alpha^3$

Reaction rate for Z electrons: $\boxed{W \propto Z \times \alpha^3}$

Cross-section: $\boxed{\sigma_{\text{electrons}} \propto \frac{Z\alpha^3}{m^2}}$

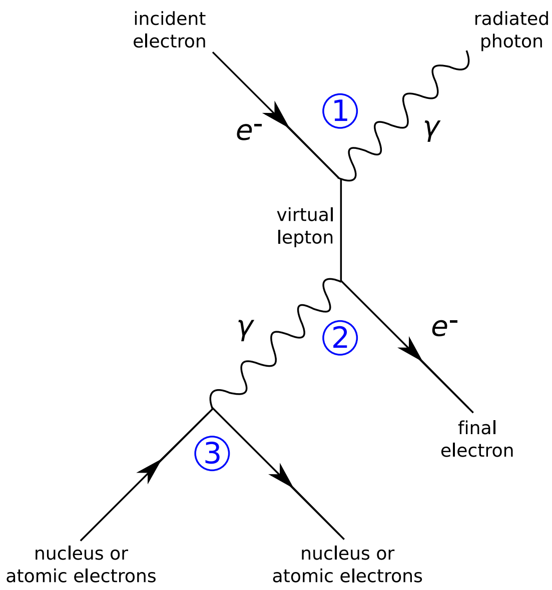


Figure 4: Feynman diagram for bremsstrahlung.

The atomic electrons do not contribute much to the bremsstrahlung process as their cross-section is Z times smaller than the cross-section of the atomic nuclei.

[2] German: *bremsen* + *Strahlung* (to brake + radiation)
[3] Greek: *syn* + *chron* + *tron* (alike + time + shortened form of electron used as a suffix for particle accelerators)

1.2.3 Energy loss per unit distance and radiation lengths

When radiative energy loss is dominant, it is useful to define a **radiation length**, X_0, the thickness of the material over which the charged particle's energy is reduced by a factor of e:

$$E = E_0\, e^{-x/X_0} \tag{34}$$

This means, an equivalent formula to the Bethe formula for energy loss per unit length due to ionisation, can be found for energy loss per unit length due to bremsstrahlung:

$$\frac{dE}{dx} = -\frac{1}{X_0} E_0\, e^{-x/X_0} \tag{35}$$

$$\boxed{-\left(\frac{dE}{dx}\right)_{\text{bremsstrahlung}} = \frac{E}{X_0}} \tag{36}$$

The radiation length, X_0, has been parametrised by Y. S. Tsai, an approximate form is:

$$\frac{1}{X_0} = \left(\frac{2\hbar}{m_e c}\right)^2 \alpha^3\, \frac{N_A}{A} \left\{ Z^2 [L_{\text{rad}} + f(Z)] + Z L'_{\text{rad}} \right\} \tag{37}$$

$$f(Z) = \alpha^2 Z^2 \left[\frac{1}{1+\alpha^2 Z^2} + 0.20206 - 0.0369\alpha^2 Z^2 + 0.0083\alpha^4 Z^4 - 0.002\alpha^6 Z^6 \right] \tag{38}$$

Tsai's values for L_{rad} and L'_{rad}, which can be used to calculate the radiation length in any element are given in the table below:

Z	**Element**	$L_{\mathbf{rad}}$	$L'_{\mathbf{rad}}$
1	H	5.31	6.144
2	He	4.79	5.621
3	Li	4.74	5.805
4	Be	4.71	5.924
$\geqslant 5$	others	$\ln(184 Z^{-1/3})$	$\ln(1194 Z^{-2/3})$

The table below shows radiation lengths for some typical materials: [4]

	Dry air	Liquid H_2	C	Al	Fe	Pb	Lead glass
Radiation length, X_0 [cm]	30,390	890.4	19.32	8.897	1.757	0.5612	1.265

1.2.4 Particle ranges

Charged particle track lengths can vary quite significantly whilst undergoing bremsstrahlung. When a particle radiates a photon, it loses a **significant fraction** of its energy quite **abruptly**, this can dramatically shorten the track length.

These abrupt energy losses can occur over a wide distance, so the total length covered by a particle (the range) can vary hugely.

[4]Source: `http://pdg.lbl.gov/2009/AtomicNuclearProperties`

1.3 Comparison between energy loss by ionisation and radiation

1.3.1 The dominant mechanism of energy loss

The cross-section for bremsstrahlung depends on the inverse square of the charged particle losing energy:

- **Electrons**: energy loss by bremsstrahlung dominates energy loss by ionisation:

$$\boxed{\left(\frac{dE}{dx}\right)^{\text{electrons}}_{\text{bremsstrahlung}} \gg \left(\frac{dE}{dx}\right)^{\text{electrons}}_{\text{ionisation}}} \tag{39}$$

- **Heavier particles**: energy loss by bremsstrahlung is much less important for muons that are 200 times heavier than electrons, and less important still for everything else. Hence energy loss by ionisation dominates bremsstrahlung:

$$\boxed{\left(\frac{dE}{dx}\right)^{\text{heavier particles}}_{\text{bremsstrahlung}} \ll \left(\frac{dE}{dx}\right)^{\text{heavier particles}}_{\text{ionisation}}} \tag{40}$$

In other words: energy loss for anything heavier than an electron is dominated by ionisation.

1.3.2 Critical energy, E_C

There are two definitions of the **critical energy**, E_C:

- **The EGS4[5] definition:** critical energy is the energy at which losses by ionisation are equal to losses by radiation.
- **The Rossi definition:** critical energy is the energy at which the ionisation loss per radiation length equals the electron's own energy.

Using equation (36) it can be shown these two definitions are the same:

Ionisation loss per radiation length equals critical energy (Rossi)...

$$\begin{pmatrix}\text{Ionisation loss}\\ \text{per unit length}\end{pmatrix} \times (1 \text{ radiation length}) = E_C \tag{41}$$

$$X_0 \left(\frac{dE}{dx}\right)_{\text{ionisation}} = E_C \tag{42}$$

$$\left(\frac{dE}{dx}\right)_{\text{ionisation}} = \frac{E_C}{X_0} \tag{43}$$

$$\left(\frac{dE}{dx}\right)_{\text{ionisation}} = \left(\frac{dE}{dx}\right)_{\text{bremsstrahlung}} \tag{44}$$

...ionisation loss equals radiation loss when at critical energy (EGS4).

Approximate empirical fits for E_C of the form $a/(Z+b)$ have been found using the Rossi definition. There is a substantial difference in ionisation between solids and liquids (s&l) and gases at the relevant energies because of the density effect:

$$E_C^{\text{s\&l}} = \frac{610\,\text{MeV}}{Z + 1.24} \tag{45}$$

$$E_C^{\text{gas}} = \frac{710\,\text{MeV}}{Z + 0.92} \tag{46}$$

[5] EGS (electron-gamma showers) is a computer program that uses Monte Carlo simulations to model EM showers

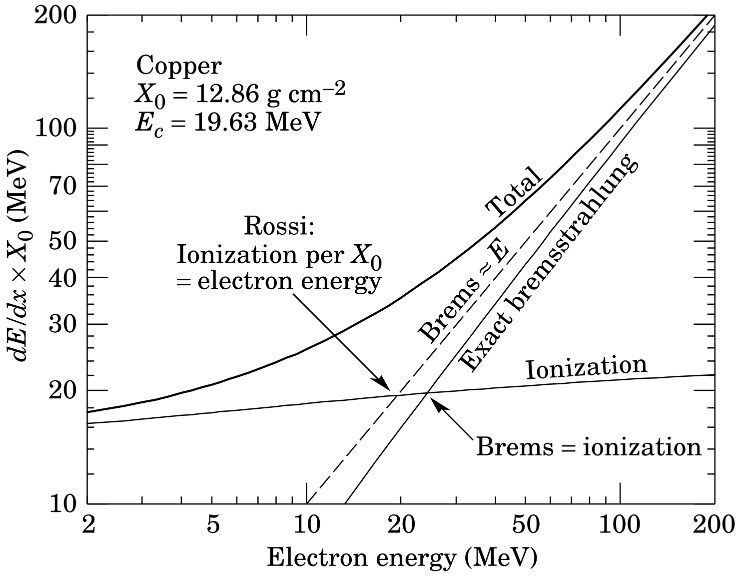


Figure 5: Comparison of the two definitions of critical energy in copper. Taken from Figure 27.12 of the Particle Data Group 2008.

The table below shows the critical energies for electrons and positrons in some typical materials: [6]

	Li	Si	Cu	Ag	Pb
EGS4 critical energy, E_C [MeV]	149.06	40.05	19.63	12.57	7.79
Rossi critical energy, E_C [MeV]	149.06	40.19	19.42	12.36	7.43

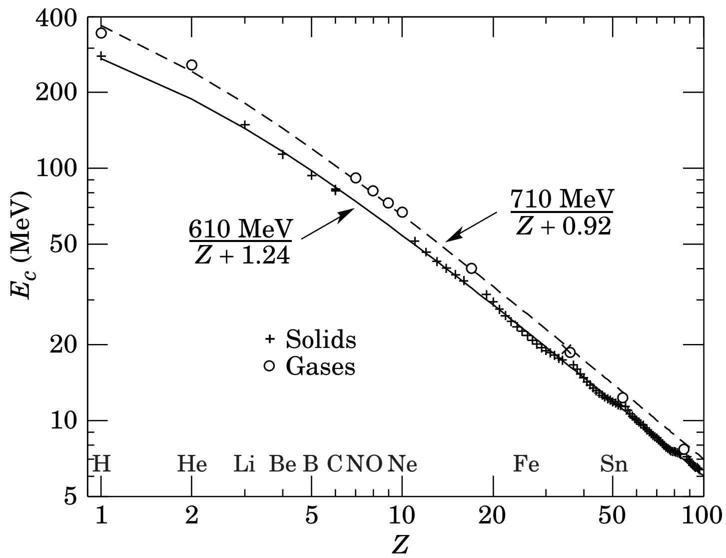
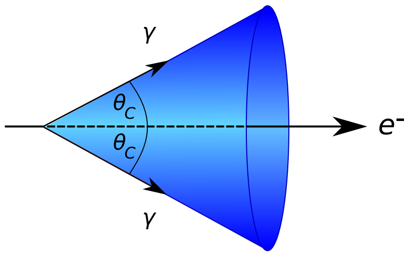


Figure 6: Electron critical energy for the chemical elements, using Rossi's definition. The fits shown are for solids and liquids (solid line) and gases (dashed line). The rms deviation is 2.2% for the solids and 4.0% for the gases. Taken from Figure 27.13 of the Particle Data Group 2008.

[6]Source: http://pdg.lbl.gov/2009/AtomicNuclearProperties/critical_energy.html

1.4 Energy loss by radiation: Čerenkov radiation

Charged particles undergo another form of energy loss in media if they travel faster than the **local** speed of light ($v = c/n$ for a medium with refractive index n) - they emit **Čerenkov radiation** at a characteristic angle, θ_C:



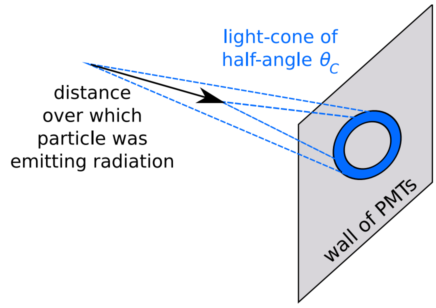


Figure 7: Left: 3D representation of a light-cone; Right: geometry that leads to a ring on a PMT wall.

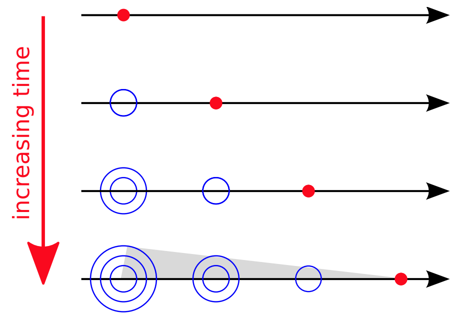
Since the particle must travel faster than the local speed of light, this imposes a **velocity threshold**:

$$\boxed{v_{\text{threshold}} = \frac{c}{n}} \qquad \rightarrow \qquad \boxed{\beta > \frac{1}{n}} \tag{47}$$

The velocity threshold is an important tool for **particle tagging** - see later.

<u>Finding the half-angle of the light-cone</u>

The moving particle creates an expanding spherical wavefront (top diagram). At the time of the next crest, the particle has moved on (second diagram) and begun a new wavelet at that location. This continues (third diagram) such that the **wavefront**, the **direction** of the ray and the particle's **trajectory** form the three sides of a right-angled triangle:



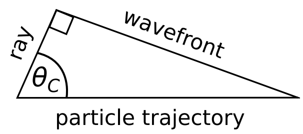


Figure 8: Left: successive rays emanating from a particle; Right: the relevant distances form a triangle.

After a time t has passed:

- The particle has travelled a distance $vt = \beta ct$, this is the length of the particle's trajectory.
- The light ray has travelled a distance $vt = \frac{c}{n}t$, this is the length of the ray.

Therefore, the angle of the ray with respect to the particle's trajectory, θ_C is:

$$\cos(\theta_C) = \frac{\text{(ray)}}{\text{(particle trajectory)}} = \frac{ct/n}{\beta ct} \qquad \rightarrow \qquad \boxed{\cos(\theta_C) = \frac{1}{\beta n}} \tag{48}$$

Bremsstrahlung originates incoherently from the incident particle itself, but Čerenkov radiation comes from the medium as a whole and is thus coherent for particle speeds above threshold.

2 Passage of photons through matter

2.1 Absorption of electromagnetic radiation

2.1.1 Linear attenuation coefficient, μ

The exponential absorption of a beam of photons may be described by:

$$\boxed{N(x) = N_0\, e^{-\mu x}} \tag{49}$$

Here, $N(x)$ is the number of photons after distance x, with N_0 the initial number of photons at $x = 0$, μ is the **linear attenuation coefficient**.

If each atomic absorbing centre is given a cross-section σ and they have a number density n then:

$$\boxed{\mu = n\sigma} \tag{50}$$

Sometimes the **mass attenuation coefficient** is used, defined as μ divided by the absorber density, ρ.

2.1.2 Partial cross-sections

The total photon interaction cross-section comprises three partial cross-sections:

$$\boxed{\sigma_{\text{total}} = \sigma_{\text{photoelectric}} + Z\sigma_{\text{Compton}} + \sigma_{\text{pair production}}} \tag{51}$$

Notice that each atomic electron contributes separately and coherently to the Compton effect.

2.2 Photoelectric effect

2.2.1 Summary

- The photoelectric effect dominates at **low-energy**.
- The photoelectric effect leads to the emission of electrons from matter after it is illuminated by light of a sufficiently high frequency; typically visible light and x-rays.
- Since light is quantised into photons of energy hf the maximum energy an electron can receive in any one interaction is hf, therefore no matter how intense the radiation is, if it is not of sufficient frequency no photoelectrons will be emitted.
- However, in a typical metal, there are both free electrons in the electron sea, and bound electrons in atomic orbitals. As a result of the need to conserve both energy and momentum in any given reaction, a free electron cannot wholly absorb a photon. Instead it is the bound atomic electrons which partake in the photoelectric effect, transfering excess energy and momentum to the nucleus.
- The $n = 1$ or K-shell electrons are tightly bound, and hence, are well coupled to the atomic nucleus making it easier for them to transfer excess momentum to the nucleus during a photoelectric interaction. For this reason, the cross-section for the emission of K-shell photoelectrons is bigger than that of L-shell photoelectrons and so forth.

2.2.2 Work function

The minimum energy an incident photon must have to liberate an electron from the surface of a particular material is the **work function** of that material:

$$E_{\text{min}} = \phi = hf_0 \tag{52}$$

Any excess energy an incident photon might have results in kinetic energy of the liberated photoelectron:

$$\boxed{E_{\text{photon}} = \phi + E_{\text{kinetic}}} \tag{53}$$

2.2.3 Cross-section

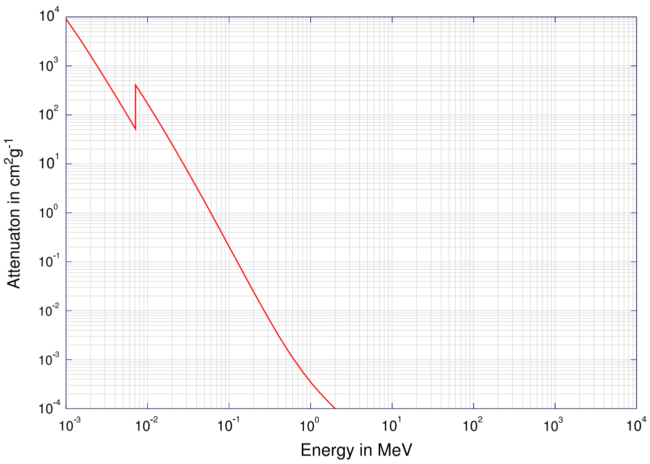


Figure 9: Photon mass attenuation coefficients for iron: photoelectric effect.

The cross-section as a function of photon energy is dominated at low energies by a **series of peaks** due to photon absorption by the bound atomic electrons.

The cross-section is a function of the atomic number of the target atom, Z, and the energy of the incident photon, E:

$$\boxed{\sigma_{\text{photoelectric}} \propto \frac{Z^n}{E^3} \qquad n = 4 \to 5} \tag{54}$$

2.3 Compton scattering

2.3.1 Summary

- Compton scattering dominates at **mid-energy**.
- Compton scattering is the scattering of photons by atomic electrons that may be regarded as essentially free.
- Comparing the photons before and after Compton scattering reveals that they undergo a **wavelength shift**, which turns out to be independent of both the initial wavelength and indeed the material they are scattered off.

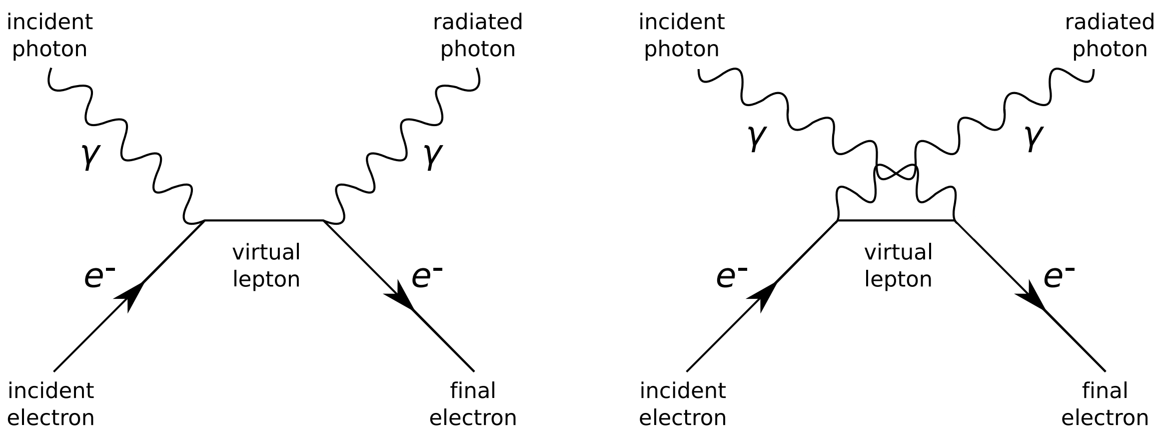


Figure 10: Feynman diagrams for Compton scattering; s-channel (left) and u-channel (right).

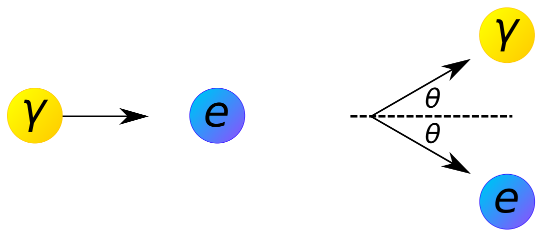


Figure 11: Compton scattering in the lab frame, before (left) and after (right).

2.3.2 Compton shift formula

- Conservation of four-vectors in the lab frame (see Figure 11):

$$\mathrm{P}_{\gamma,i} + \mathrm{P}_{e,i} = \mathrm{P}_{\gamma,f} + \mathrm{P}_{e,f} \tag{55}$$

- Rearrange so that when squared the cross-product of $\mathrm{P}_{\gamma,i}$ and $\mathrm{P}_{\gamma,f}$ arises:

$$(\mathrm{P}_{\gamma,i} - \mathrm{P}_{\gamma,f})^2 = (\mathrm{P}_{e,f} - \mathrm{P}_{e,i})^2 \tag{56}$$

$$\mathrm{P}^2_{\gamma,i} + \mathrm{P}^2_{\gamma,f} - 2\mathrm{P}_{\gamma,i} \cdot \mathrm{P}_{\gamma,f} = \mathrm{P}^2_{e,f} + \mathrm{P}^2_{e,i} - 2\mathrm{P}_{e,i} \cdot \mathrm{P}_{e,f} \tag{57}$$

- Since $\mathrm{P}^2_\gamma = 0$ and $\mathrm{P}^2_e = m_e^2 c^2$ this simplifies to:

$$\mathrm{P}_{\gamma,i} \cdot \mathrm{P}_{\gamma,f} = \mathrm{P}_{e,i} \cdot \mathrm{P}_{e,f} - m_e^2 c^2 \tag{58}$$

- This yields the angle between the initial photon's path and the final photon's path, θ:

$$\frac{E_{\gamma,i}\, E_{\gamma,f}}{c^2} - p_{\gamma,i}\, p_{\gamma,f} \cos\theta = \frac{E_{e,i}\, E_{e,f}}{c^2} - p_{e,i}\, p_{e,f} \cos\phi - m_e^2 c^2 \tag{59}$$

- Since the initial electron is stationary, $p_{e,i} = 0$ and $E_{e,i}$ is simply its rest-mass energy, $m_e c^2$:

$$E_{\gamma,i}\, E_{\gamma,f} - p_{\gamma,i}\, p_{\gamma,f} c^2 \cos\theta = m_e c^2\, E_{e,f} - m_e^2 c^4 \tag{60}$$

- For photons the relation $E = pc$ holds allowing the left hand side to be factorised:

$$E_{\gamma,i}\, E_{\gamma,f} (1 - \cos\theta) = m_e c^2\, E_{e,f} - m_e^2 c^4 \tag{61}$$

- Appeal to the conservation of energy to eliminate $E_{e,f}$:

$$E_{\gamma,i}\, E_{\gamma,f} (1 - \cos\theta) = m_e c^2 \left[E_{\gamma,i} + m_e c^2 - E_{\gamma,f}\right] - m_e^2 c^4 \tag{62}$$

$$(1 - \cos\theta) = m_e c^2 \left[\frac{1}{E_{\gamma,f}} - \frac{1}{E_{\gamma,i}}\right] \tag{63}$$

- Finally, convert the photons' energy E_γ into wavelengths using $E_\gamma = hf = hc/\lambda$:

$$(1 - \cos\theta) = m_e c^2 \left[\frac{\lambda_f}{hc} - \frac{\lambda_i}{hc}\right] \tag{64}$$

$$\boxed{\Delta\lambda = \lambda_f - \lambda_i = \frac{h}{m_e c}(1 - \cos\theta)} \tag{65}$$

- The **Compton shift formula** is usually simplified with the **Compton wavelength**, λ_C:

$$\boxed{\lambda_C = \frac{h}{m_e c}} \tag{66}$$

2.3.3 Cross-section

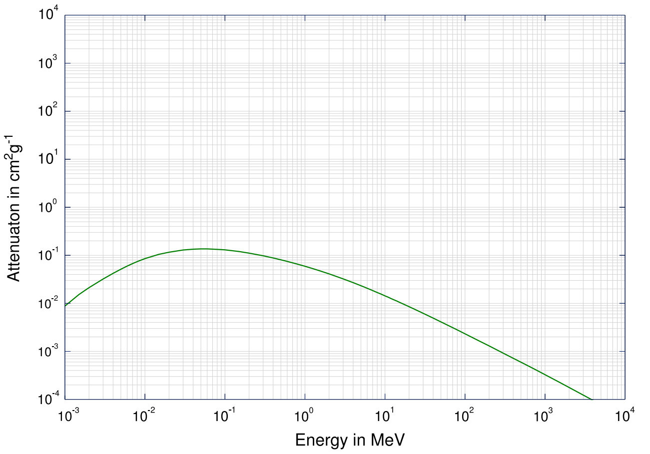


Figure 12: Photon mass attenuation coefficients for iron: Compton scattering.

The differential cross-section is given by the Klein-Nishina formula.

An empirical formula is used for the actual cross-section with parameters adjusted to fit the data. The cross-section is of the form:

$$\boxed{\sigma_{\text{Compton}} \sim \frac{\ln(E)}{E}} \tag{67}$$

2.4 Pair-production

2.4.1 Summary

- Pair production dominates at **high-energy**.
- The process is effectively gamma-ray absorption by the process: $\gamma \to e^+ \, e^-$
- Pair production cannot occur until the threshold energy of two electron masses is reached:

$$E_{\text{threshold}} = 2m_e c^2 = 2 \times 511 \text{ keV} = 1.02 \text{ MeV} \tag{68}$$

- Pair production **cannot take place in free space** because it is impossible to conserve momentum:

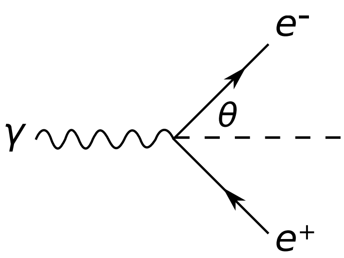


Figure 13: Would-be Feynman diagram for free space pair production, showing the geometry for equation (69).

  - ▶ Conservation of momentum:

$$p_\gamma = E_\gamma / c = 2p_e \cos\theta \tag{69}$$

  - ▶ Conservation of energy:

$$E_\gamma = 2E_e \tag{70}$$

  - ▶ Equate E_γ from both conservation laws (69) and (70):

$$E_e = p_e c \cos\theta \tag{71}$$

  - ▶ The invariant applied to the electron:

$$E_e^2 = p_e^2 c^2 + m_e^2 c^4 \tag{72}$$

  - ▶ Equate (71) to the invariant, (72):

$$p_e^2 c^2 \cos^2\theta = p_e^2 c^2 + m_e^2 c^4 \tag{73}$$

- The biggest value that the left hand side of (73) can take is $p_e^2 c^2$ when $\cos^2\theta \to 1$, so unless $m_e \to 0$ momentum can never be conserved for pair production in free space.

- Rather, like bremsstrahlung, pair production must take place in the **vicinity of an atomic nucleus** to absorb the **recoil momentum**.
- The **positron** which is produced will annihilate with an electron and produce **two 511 keV photons back-to-back**, a typical signature of pair production: $e^+e^- \to 2\gamma$.
- Positrons nearly always annihilate to two photons, instead of simply reversing the production process ($e^+e^- \to \gamma$) and annihilating to one photon. This is becuase this is also forbidden in free space, since it is simply the reverse of the forbidden free space pair-production.
- However, like pair production, it can take place in the E-field of an atom, but this rarely happens because of the ease with which free space annihilation to two photons can occur.

2.4.2 Reaction rate

Vertex factors:

1 $\sqrt{\alpha}$

2 $\sqrt{\alpha}$

3 $Z\sqrt{\alpha}$

Reaction rate:

$$W \propto (\sqrt{\alpha}\sqrt{\alpha}\,Z\sqrt{\alpha})^2$$

$$\boxed{W \propto Z^2\alpha^3}$$

Cross-section:

$$\boxed{\sigma_{\text{pair production}} \propto \frac{Z^2\alpha^3}{m^2}}$$

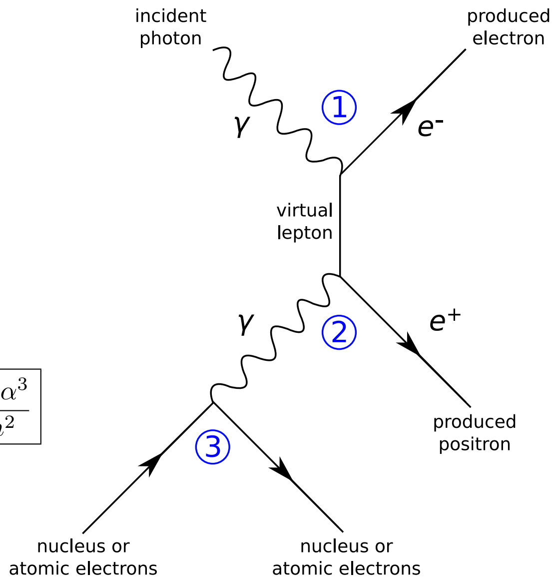


Figure 14: Feynman diagram for pair production.

2.4.3 Cross-section

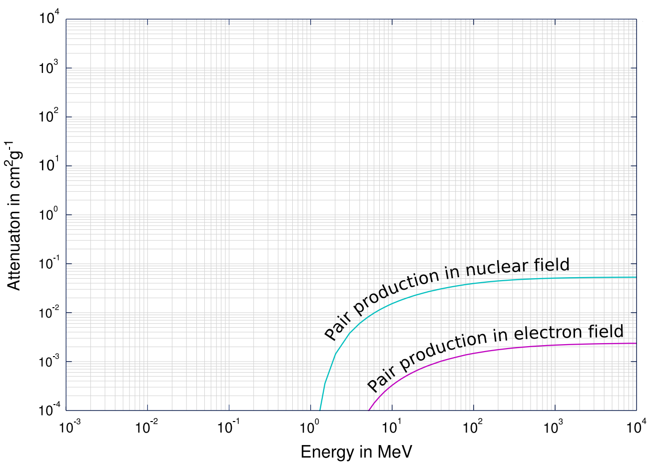


Figure 15: Photon mass attenuation coefficients for iron: pair production in nuclear field and electron field.

The cross-section for pair production obeys:

$$\boxed{\sigma_{\text{pair production}} \propto \frac{Z^2\alpha^2}{m_e^2}} \tag{74}$$

Tsai's formula for the differential cross-section for pair production is:

$$\frac{d\sigma}{dx} = \frac{A}{X_0 N_A}\left[1 - \frac{4}{3}x(1-x)\right] \tag{75}$$

Here $x = E/k$ is the fractional energy transfer to the pair produced electron and k is the incident photon energy. Integrating this between $x = 0$ and $x = 1$ gives:

$$\boxed{\sigma_{\text{pair production}} = \frac{7}{9}\left(\frac{A}{X_0 N_A}\right)} \tag{76}$$

2.5 Overall cross-section for photons

When travelling through a material, photons are attenuated by all of the aforementioned processes, each one dominating at different energies. The measured cross-section is due to the combination of all three, as shown below:[7]

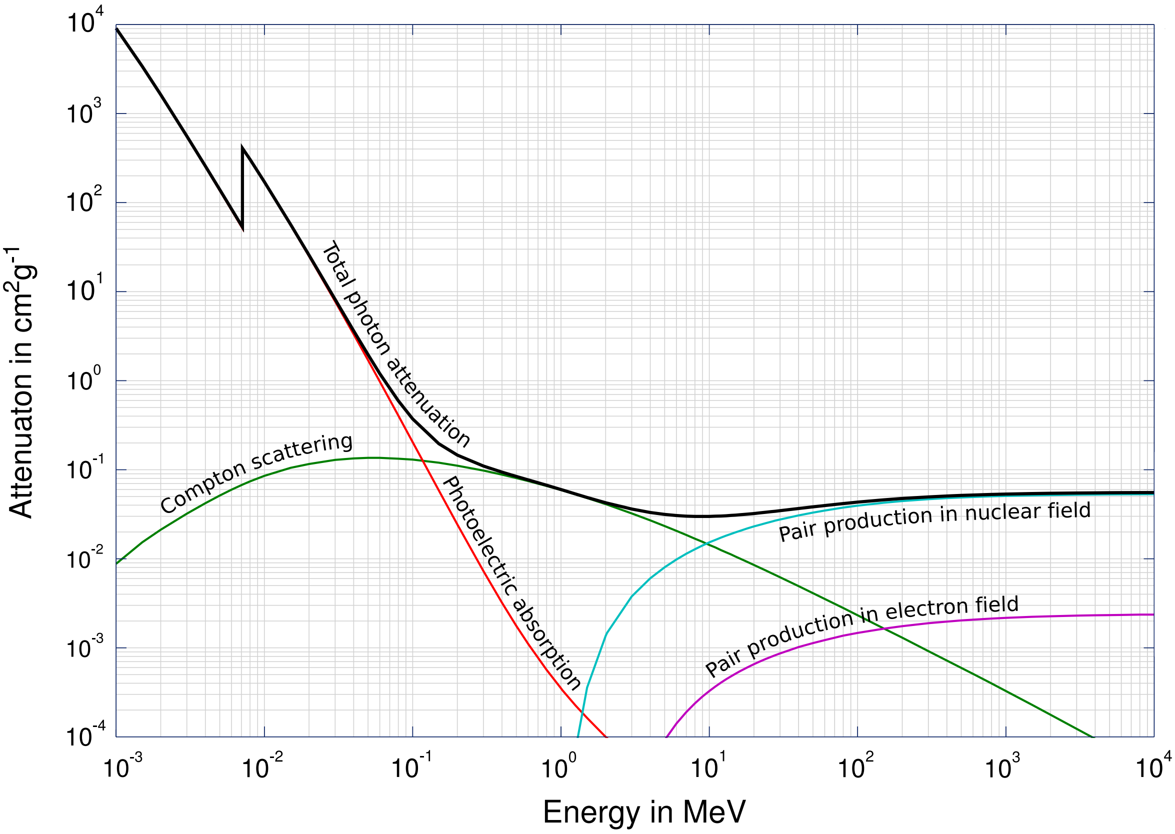


Figure 16: Photon mass attenuation coefficients for iron: total photon attenuation. Taken from Attenuation Coefficient Iron.svg at Wikimedia Commons.

2.6 Comparative summary of photon attenuation phenomena

	Photoelectric effect	Compton scattering	Pair-production
Photon energy, E_γ	Low-energy	Mid-energy	High-energy
Cross-section, σ	$\propto \frac{Z^n}{E^3}$ for $n = 4 \to 5$	$\sim \frac{\ln(E)}{E}$	$= \frac{7}{9}\left(\frac{A}{X_0 N_A}\right)$
Distinguishing features	Series of peaks	...	Threshold at 1.02 MeV

[7] All of the *Photon mass attenuation coefficients for iron* graphs are derivative works of this graph. This file is licensed under the Creative Commons Attribution ShareAlike 3.0, Attribution ShareAlike 2.5, Attribution ShareAlike 2.0 and Attribution ShareAlike 1.0 License. In short: you are free to share and make derivative works of the file under the conditions that you appropriately attribute it, and that you distribute it only under a license identical to this one: http://creativecommons.org/licenses/by-sa/3.0/.

3 Passage of hadrons through matter

3.1 Hadron-nucleus interactions

The total hadron-nucleon interaction cross-section comprises three partial cross-sections:

$$\boxed{\sigma_{\text{total}} = \sigma_{\text{elastic}} + \sigma_{\text{quasi-elastic}} + \sigma_{\text{inelastic}}} \tag{77}$$

The graph below shows total cross-sections, σ, as a function of total centre of mass energy, $\sqrt{s}$, for protons colliding with other protons, anti-protons, sigma baryons, pions, kaons and photons:[8]

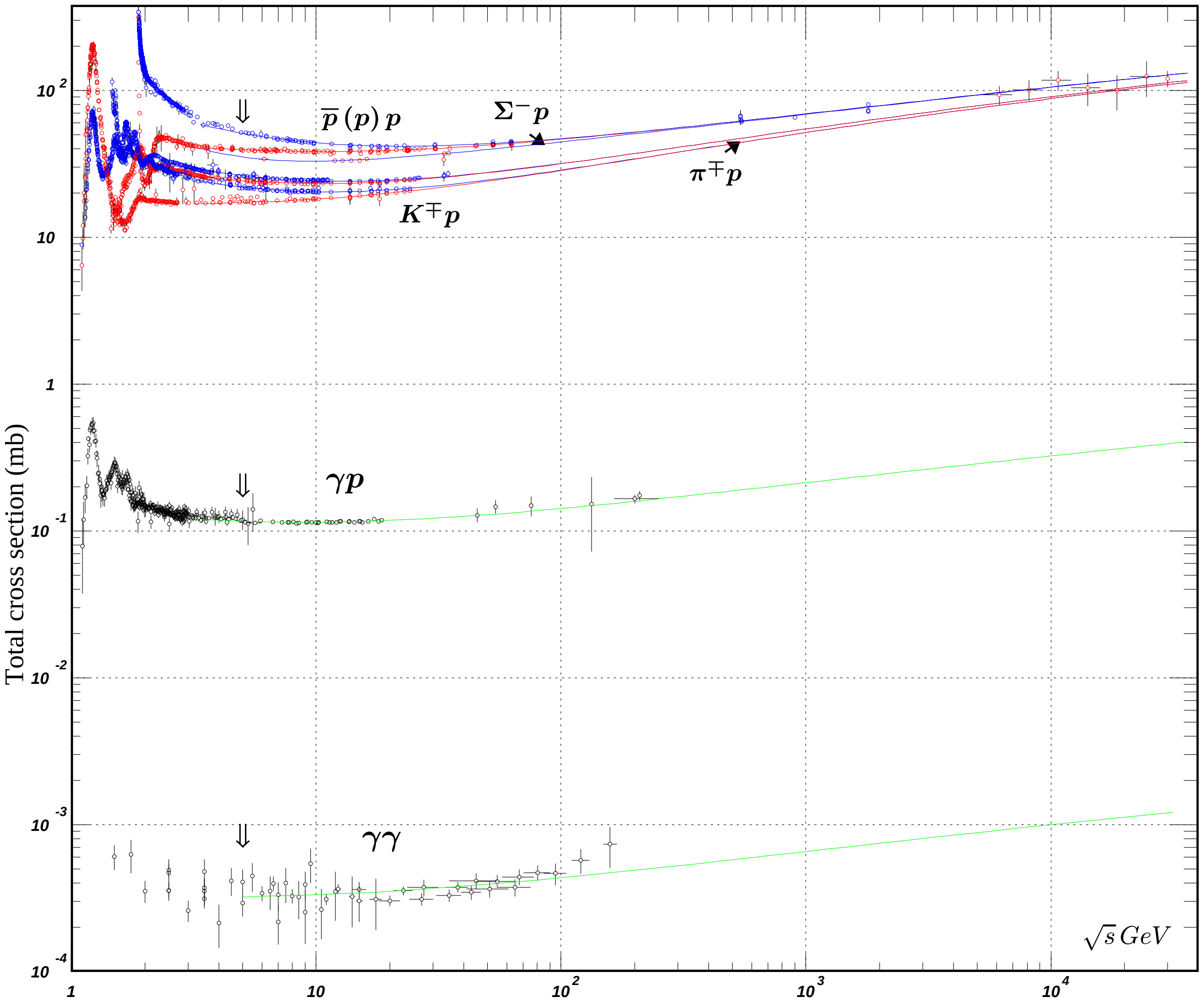


Figure 17: Summary of hadronic, γp, and $\gamma\gamma$ total cross-sections as a function of total centre of mass energy, $\sqrt{s}$. (Courtesy of the COMPAS group, IHEP, Protvino, August 2005). Taken from Figure 40.10 of the Particle Data Group 2008.

3.2 Quasi-elastic interactions

Quasi-elastic scatters occur when the hadron scatters off one of the nucleus' constituent protons or neutrons, leaving the nucleus otherwise unchanged.

[8] Corresponding computer-readable data files may be found at http://pdg.lbl.gov/2009/hadronic-xsections/.

3.3 Inelastic interactions

Inelastic interactions or **absorption interactions** are responsible for hadron **attenuation** in matter.

3.3.1 Inelastic cross-section

The inelastic cross-section can be parametrised in terms of the atomic mass A and the hadron-proton cross-section σ_0:

$$\boxed{\sigma_{\text{inelastic}} = \sigma_0 \, A^{2/3}} \tag{78}$$

So for a pion-nucleon interaction, use the pion-proton cross-section for σ_0.

3.3.2 Absorption length, λ

The mean distance travelled before an inelastic collision occurs in a material with number density n can be found using the formula for the mean free path in classical kinetic theory:

$$\boxed{\lambda = \frac{1}{n \, \sigma_{\text{inelastic}}}} \tag{79}$$

The quantity λ is known as the **absorption length** or the **interaction length**.

The table below shows absorption lengths for some typical materials: [9]

	Dry air	Liquid H_2	C	Al	Fe	Pb	Lead glass
Absorption length, λ [cm]	74,770	734.6	38.83	39.70	16.77	17.59	25.40

Hadronic showers stop developing when the hadron's energy reaches the **energy lost per absorption length**. In other words, when the hadron has insufficient energy to survive ionisation or bremsstrahlung losses along the absorption length it has to travel before it undergoes its next inelastic scatter, the shower stops developing.

This is rather like the concept of the Rossi critical energy, but instead of it being the energy loss over a radiation length it is the energy loss over an absorption length. The Rossi definition can be adapted to give a **'hadronic shower critical energy'**.

$$\boxed{E_C^{\text{hadronic}} = (\text{Energy lost per absorption length}) = \left(\frac{dE}{dx}\right) \times \lambda} \tag{80}$$

The relevant stopping power (dE/dx) must be substituted into equation (80) using the conditions in (39) and (40).

[9] Source: `http://pdg.lbl.gov/2009/AtomicNuclearProperties`

4 Development of showers in matter

4.1 Electromagnetic showers

4.1.1 Modelling an electromagnetic shower

A simple model of an **electromagnetic shower** assumes that high energy electrons or positrons undergo bremsstrahlung after one radiation length, emitting a photon. After another radiation length, the electron or positron undergoes another bremsstrahlung event whilst the photon pair produces:

$$e^{\pm} \to e^{\pm} + \gamma \tag{81}$$

$$\gamma \to e^{-} + e^{+} \tag{82}$$

Any two daughter particles have half the energy of their parent. If the initial electron has energy E_0, then after N radiation lengths a particle will have an energy:

$$\boxed{E_N = \frac{E_0}{2^N}} \tag{83}$$

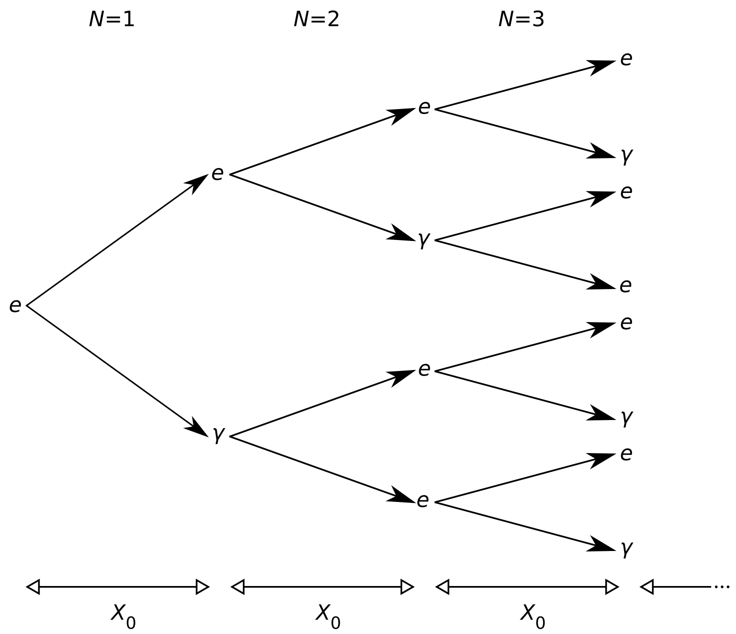


Figure 18: Development of an electromagnetic shower.

The total number of particles doubles after every radiation length, so after N radiation lengths there will be 2^N particles:

$$2^N = n_N^e + n_N^\gamma \tag{84}$$

Here, n_N^e denotes the number of electrons and positrons after N radiation lengths and n_N^γ denotes the number of photons after N radiation lengths.

Notice that at any given level, all the photons originate from an electron or positron undergoing bremsstrahlung from the previous level, and all the photons at that level have pair produced by the next level - therefore the number of photons at any given level is equal to the number of electrons and positrons at the previous level:

$$n_N^\gamma = n_{N-1}^e \tag{85}$$

The actual number of electrons at a given level can be found by making n_N^e the subject of (84) and substituting in (85) recursively. The results are:

$$n_N^e = \sum_{i=0}^{N} (-1)^{N+i}\, 2^i \qquad\qquad n_N^\gamma = \sum_{i=0}^{N-1} (-1)^{N-1+i}\, 2^i \tag{86}$$

4.1.2 Total shower depth

The shower continues to develop in this way until the energy of the particles falls below the critical energy, at which point energy loss by bremsstrahlung no-longer dominates (see Figure 5) and the shower's development is halted.

The number of radiation lengths travelled by the time E_0 has dropped to E_C can be found from (83):

$$E_{N_C} = E_C = \frac{E_0}{2^{N_C}} \qquad \rightarrow \qquad N_C = \frac{\ln(E_0/E_C)}{\ln(2)} \tag{87}$$

The total distance that the electromagnetic shower penetrates the material is the **number of radiation lengths travelled**: $N_C \times X_0$.

4.2 Hadronic showers

4.2.1 Modelling a hadronic shower

Hadronic showers are more complex than electromagnetic ones; this section outlines a simple hadronic shower model.

There are no photons involved in these showers, instead the shower propagates by hadron-nucleon **inelastic collisions**, which create more particles. This time, it is the **absorption length** (not the radiation length) that separates each level of the shower because λ is the average distance a hadron can travel before suffering another inelastic scattering event - creating the next level of the shower.

Consider a 10 GeV **sigma baryon** entering a block of **lead**; if each hadron striking a lead atom produces three more hadrons, then a hadronic shower develops as follows:

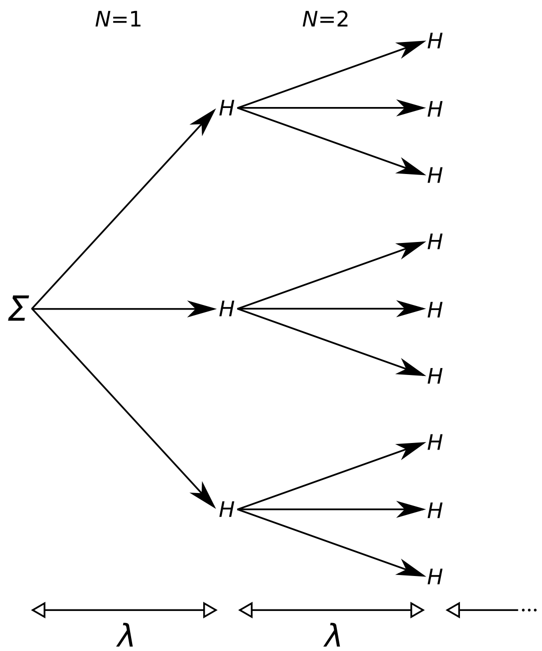


Figure 19: Development of a hadronic shower.

Any three daughter particles have one-third the energy of their parent. If the initial hadron has energy E_0, then after N absorption lengths a particle will have an energy:

$$E_N = \frac{E_0}{3^N} \tag{88}$$

4.2.2 Total shower depth

The shower continues to develop in this way until the particles have insufficient energy to travel another absorption length and the shower's development is halted. This happens when the hadrons' energy reaches the 'hadronic shower critical energy', defined in equation (80). Since the particles concerned are hadrons, not electrons, the condition in (40) means ionisation is the most important form of energy loss:

$$E_C^{\text{hadronic}} = \lambda \left(\frac{dE}{dx}\right)_{\text{ionisation}} \tag{89}$$

In order to calculate the total shower depth (the number of absorption lengths travelled) it will be necessary to find the hadronic shower critical energy. Note: this is *not* the usual critical energy concerned with radiation lengths and electromagnetic interactions: $E_C \neq E_C^{\text{hadronic}}$.

- The **stopping power** (dE/dx) due to ionisation of a 10 GeV sigma baryon in lead can be found from the Bethe formula data in Figure 2...

  First it is necessary to work out the value of $\beta\gamma$ for a 10 GeV sigma baryon:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{90}$$

  Squaring both sides and cross-multiplying the denominator yields:

$$\gamma^2 - \beta^2\gamma^2 = 1 \qquad \rightarrow \qquad \beta\gamma = \sqrt{\gamma^2 - 1} \tag{91}$$

  The total baryon energy is given by $E_{\text{total}} = \gamma mc^2$ and its rest-mass is $m_\Sigma = 1.197\,\text{GeV}/c^2$:

$$\beta\gamma = \sqrt{\left(\frac{E_{\text{total}}}{mc^2}\right)^2 - 1} = \sqrt{\left(\frac{10\,\text{GeV}}{1.197\,\text{GeV}}\right)^2 - 1} = 8.29 \tag{92}$$

  A particle with $\beta\gamma = 8.29$ has a stopping power of $1\,\text{MeV}\,\text{g}^{-1}\,\text{cm}^2$ in lead. Remember the units are such because the stopping power has been divided by the density of lead.

- The **absorption length** (λ) can be found by combining equations (78) and (79)...

$$\lambda = \frac{1}{n\,\sigma_{\text{inelastic}}} = \frac{1}{n\,\sigma_0\,A^{2/3}} \tag{93}$$

  The number density of lead, n, can be found as follows:

$$n = \frac{\text{total number of atoms}}{\text{volume of lead}} = \frac{\text{total mass} \div \text{mass of one atom}}{\text{volume of lead}} = \frac{M \div m}{V} = \frac{\rho}{m} \tag{94}$$

  The mass of one lead atom, m, can be be expressed in terms of the atomic mass number A, which is the molar mass in grams, as follows:

$$\begin{pmatrix}\text{mass of}\\ \text{one mole}\end{pmatrix} = \begin{pmatrix}\text{molar}\\ \text{mass}\end{pmatrix} = \begin{pmatrix}\text{number of particles}\\ \text{in a mole}\end{pmatrix} \times \begin{pmatrix}\text{mass of one}\\ \text{particle}\end{pmatrix} \tag{95}$$

$$A = N_A \times m \tag{96}$$

In this instance σ_0 is the sigma-proton cross-section, which can be read from the plot in Figure 17:

$$\sigma_0\,(10\,\mathrm{GeV}) = 30\,\mathrm{mb} = 30 \times 10^{-27}\,\mathrm{cm}^2 \tag{97}$$

The atomic mass number of lead is $A = 207$. Putting all this together yields:

$$\lambda = \frac{m}{\rho\,\sigma_0\,A^{2/3}} = \frac{A \div N_A}{\rho\,\sigma_0\,A^{2/3}} = \frac{A^{1/3}}{\rho\,\sigma_0\,N_A} \tag{98}$$

- Hence the **critical energy** is...

$$E_C^{\text{hadronic}} = \lambda \left(\frac{dE}{dx}\right)_{\text{ionisation}} = \frac{A^{1/3}}{\rho\,\sigma_0\,N_A}\Big(1\,\mathrm{MeV\,g^{-1}\,cm^2} \times \rho\Big) = 327\ \mathrm{MeV} \tag{99}$$

  Notice how the need to multiply the Bethe formula graph reading by the relevant density means that ρ cancels with the density factor that arises from λ. Hence to find the critical energy, it is not necessary to actually know the density of lead.

So, having found the critical energy, it now remains to calculate the the number of absorption lengths achieved, N_C, and an explicit value for the absorption length, λ in order to find their product:

- The **number of absorption lengths** reached by the shower:

$$E_{N_C} = E_C^{\text{hadronic}} = \frac{E_0}{3^{N_C}} \tag{100}$$

$$N_C = \frac{\ln(E_0/E_C)}{\ln(3)} = \frac{\ln(10\,\mathrm{GeV}/\,0.327\,\mathrm{GeV})}{\ln(3)} = 3.11 \tag{101}$$

- Since it is now necessary to calculate the value of the **absorption length** explicitly, the density of lead is required: $\rho = 11.34\,\mathrm{g\,cm^{-3}}$:

$$\lambda = \frac{A^{1/3}}{\rho\,\sigma_0\,N_A} = \frac{\sqrt[3]{207}}{11.34 \times (30 \times 10^{-27}) \times N_A} = 28.9\,\mathrm{cm} \tag{102}$$

The total distance that the hadronic shower penetrates the material is the number of absorption lengths travelled: $N_C \times \lambda$. Taking N_C from equation (101) and λ from equation (102) gives a shower depth of 90 cm.