LAB 2 Computer vision

Problem 1 : Switching from the continous to the discrete KF

Let a target displacing on a plane for what we could measure the position $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ with some uncertainty. Consider that the target mouves with a constant unknown velocity $\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$ meaning in theory that accelerations $\ddot{x}(t) = \ddot{y}(t) = 0$. In practice, there is a small variations in the acceleration such as: $\begin{matrix} \ddot{x}(t) = w_x(t) \\ \ddot{y}(t) = w_y(t) \end{matrix}$ where $w(t) = \begin{pmatrix} w_x(t) \\ w_y(t) \end{pmatrix}$ is a random vector of zero mean and variance-covariance matrix defined by :

$$E\left(w(t)w^T(t)\right) = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{pmatrix} = Q_c$$

1) Show that such a modelisation admits the following state representation :

$$\dot{X}(t) = AX(t) + Bw(t)$$

Where $X(t) = \begin{pmatrix} x(t) & \dot{x}(t) & y(t) & \dot{x}(t) \end{pmatrix}$ with A, B two matrices to be precised.

To complete the model, we consider the following observation equation :

$$Z(t) = CX(t) + b(t)$$

Where C is the measurment matrix to be precised and $b(t)$ the random measure noise of zero mean and variance-covariance matrix:

$$E\left(b(t)b^T(t)\right) = \begin{pmatrix} \theta_x & 0 \\ 0 & \theta_y \end{pmatrix}$$

2) The classical solution of the state equation is :

$$X(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-u)}Bw(u)du \quad t > t_0$$

That we descritize by puting $t_0 = nT \quad and \quad t = T + nT$:

$$\begin{aligned} X((n+1)T) &= e^{AT}X(nT) + w(n) \\ w(n) &= \int_{nT}^{(n+1)T} e^{A(nT+T-u)}Bw(u)du \end{aligned}$$

Using the matrix A found above and linearizing by the taylor development, show that

$$e^{At} = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The last step of the discretisation is to find the variance-covariance matrix of w(n). Using the fact that w(t) is a white noise such that $E\left(w(u)w(v)\right) = Q_c\delta(u-v)$ show that :

$$Q(n) = E\left(w(n)w^T(n)\right) = \int_0^T e^{Av}BQ_cB^Te^{A^Tv}dv$$

and that by the Taylor development that :

$$Q(n) = \begin{pmatrix} \sigma_x^2\frac{T^3}{3} & \sigma_x^2\frac{T^2}{2} & 0 & 0 \\ \sigma_x^2\frac{T^2}{2} & \sigma_x^2T & 0 & 0 \\ 0 & 0 & \sigma_y^2\frac{T^3}{3} & \sigma_y^2\frac{T^2}{2} \\ 0 & 0 & \sigma_y^2\frac{T^2}{2} & \sigma_y^2T \end{pmatrix}$$

Similarly, it is easy to show that

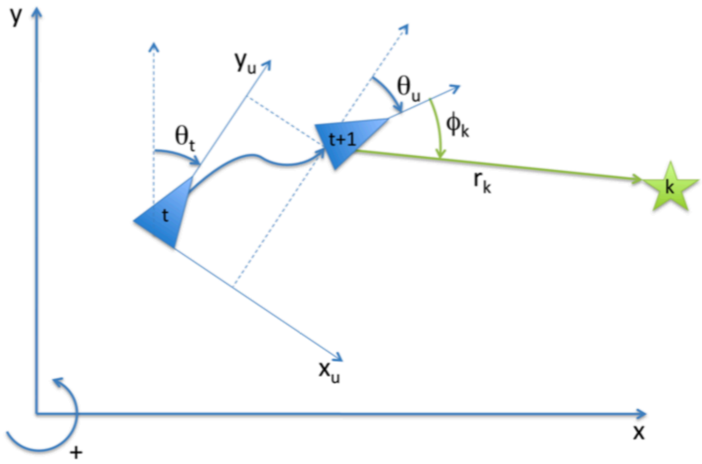
$$Z(n) = CX(n) + b(n) \quad \textit{with}$$

$$R = \begin{pmatrix} \frac{\sigma_x^2}{T} & 0 \\ 0 & \frac{\sigma_y^2}{T} \end{pmatrix}$$

R is the variance-covariance matrix of b(t).
Write the final discret state equation model of the target.

Problem 2 : EKF and SLAM

Introduction



The goal is to simulate a robot moving on a given trajectory in an environment represented by ponctual landmarks.

The modeled robot moves on a plane and perceives the direction and the distance of punctual landmarks (amers ou points d'intérêts) situated on this same plane. Its state is represented by its position and its orientation in a global reference:

$$X_t = \left[x_t, y_t, \theta_t\right]^T$$

The displacement of the robot between the instants t and t + 1 is measured thanks to the Odometry given in the reference system of the robot at time t:

$$U_t = \left[x_u, y_u, \theta_u\right]^T$$

The evolution of the state model is therefore given by:

$$X_{t+1} = f\left(X_t, U_t\right) = \begin{bmatrix} x_t + x_u \cos\left(\theta_t\right) - y_u \sin\left(\theta_t\right) \\ y_t + x_u \sin\left(\theta_t\right) + y_u \cos\left(\theta_t\right) \\ \theta_t + \theta_u \end{bmatrix}$$

With a Gaussian noise given by the covariance matrix Q

Perceptions provide measurements of the distance and direction of a landmark k supposed to be perfectly identifiable

$$Y_t = \left[r_t^k, \phi_t^k\right]^T$$

The corresponding observation model is therefore

$$Y_t^* = h^k\left(X_t^*\right) = \begin{bmatrix} \sqrt{\left(x_k - x_t\right)^2 + \left(y_k - y_t\right)^2} \\ arctg\left(\dfrac{y_k - y_t}{x_k - x_t}\right) - \theta_t \end{bmatrix}$$

where $x_k et\ y_k$ are the (known) coordinates of the landmark in the global coordinate system. This model is entached by a Gaussian noise of a covariance matrix P_y

Extended Kalman Filter

Since the models are non-linear, we will use the extended Kalman filter to integrate odometry measurements and perceptions of landmarks to produce an estimate $\hat{X}_t$ of the real robot position X_t

$$\begin{aligned} X_t^* &= f(\hat{X}_t, U_t) \\ P_t^* &= A.\hat{P}_t.A^T + B.Q.B^T \\ Y_t^* &= h(X_t^*) \\ K &= P_t^* H^T.(H.P_t^*.H^T + P_Y)^{-1} \\ \hat{X}_{t+1} &= X_t^* + K(Y - Y_t^*) \\ \hat{P}_{t+1} &= P_t^* - KHP_t^* \end{aligned}$$

where $A, B,\ et\ H$ are the Jacobians of the functions $f\ et\ h$

$$A_{ij} = \frac{\partial f_i(x,u)}{\partial x_j}$$

$$B_{ij} = \frac{\partial f_i(x,u)}{\partial u_j}$$

$$H_{ij} = \frac{\partial h_i(x)}{\partial x_j}$$

Requested Work

- Calculate the Jacobians $A, B,\ et\ H$
- Calculate the two first states X_1, X_2 giving the two first positions and orientations of the robots.
- Represent on a global Cartesian coordinate system this beginning of the Robot's trajectory.

Initial Conditions:

We considered that the measurement of odometry u is the same at each moment (failure of the IMU maybe) and therefore:

$$X_0 = [1, +4, -\frac{\pi}{2}]$$

$$Q_0 = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & \frac{\pi}{180} \end{pmatrix}, P_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3\pi}{180} & 0 \\ 0 & 0 & 1 \end{pmatrix}, Py = Identité$$

$$u = [0, 0.15, \frac{0.3\pi}{180}]$$

The first observed landmark is the point (5,12) and the second one, the point (10,19).

PS : The parameter t can be considered as discrete. Feel free to use the EKF scheme of the course if you want.