Assignment: Parameter Estimation

(1) Let $(X_1, X_2 \ldots)$ be a random sample of size n taken from a Normal population with parameters, mean $= \theta 1$ and variance $= \theta 2$. Find the maximum likelihood estimates of these 2 parameters.

- To find the maximum likelihood estimates (MLEs) of the parameters.

(1) Setup Likelihood function
The prob. density function of a normally distributed random variable X is given by:-

$$f(X \mid \theta 1, \theta 2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x-\theta 1)^2}{2\theta_2}}$$

Given the sample $X_1, X_2 \ldots X_n$, the joint density function (likelihood function) for the independent observation is:-

$$L(\theta 1, \theta 2 \mid x_1, x_2 \ldots x_n)$$

$$= \prod_{i=n} \frac{1}{\sqrt{2\pi\theta_2}} e^{\frac{-(x_i-\theta_1)^2}{2\theta_2}}$$

This can be simplified to:-

$$L(\theta 1, \theta 2 \mid x_1, x_2 \ldots x_n) = (2\pi\theta 2)^{-n/2} e^{-1/2\theta_2} \sum_{i=1}^{n} (x_i - \theta 1)^2$$

② Log Likelihood function

$$\log L(\theta 1, \theta 2) = -\frac{n}{2}\log(2\pi\theta 2) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta 1)^2$$

③ Derive MLE of $\theta 1$ (Mean)

Differentiating the log likelihood

function with respect to θ_1 and set it equal to 0.

$$\frac{d}{d\theta_1}\left[-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right]=0$$

$$\frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)=0$$

$$\sum_{i=1}^{n}x_i-n\theta_1=0$$

$$\theta_1=\frac{1}{n}\sum_{i=1}^{n}x_i$$

So, MLE of θ_1 is $\hat{\theta}_1=\bar{x}$, the mean.

(4) Derive the MLE of θ_2.

$$\frac{d}{d\theta_2}\left(-\frac{n}{2}\log(\theta_2)-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right)=0$$

$$=-\frac{n}{2\theta_2}+\frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)^2=0$$

$$\theta_2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

So, MLE of θ_2 is $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

The MLEs for the parameter of the normal distribution from the given sample are:-

$\hat{\theta}_1 = \bar{x}$ for the mean

$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$ for the variance.

Q-2 Let $X_1, X_2 - - X_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known +ve integer. Compute the value of θ using the M.L.E.

⇒ To find the maximum likelihood estimate

① Set up likelihood function

Given that each x_i follows a binomial distribution the pmf for each x_i is given by

$$P(X_i = x_i) = \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i}$$

For a random sample $X_1, X_2 \ldots X_n$ form $B(m, \theta)$; the joint pmf of observing this particular sample is :- $L(\theta \mid x_1, x_2 \ldots x_n) = \prod_{i=1}^{n} \binom{m}{x_i}$ $\theta^{x_i}(1-\theta)^{m-x_i}$

Given that $\binom{m}{x_i}$ doesn't depends on θ,

it can be treated as constant with respect to θ.

$$L(\theta \mid x_1, x_2 \ldots x_n) = \left(\prod_{i=1}^{n} \binom{m}{x_i} \right) \theta^{\sum_{i=1}^{n}} (1-\theta)$$

$$\sum_{i=1}^{n} (m - x_i)$$

Date
Page

② Log-Likelihood function.

$$\log L(\theta) = \log\left(\prod_{i=1}^{n} \binom{m}{x_i}\right) + \left(\sum_{i=1}^{m} x_i\right)\log\theta + \left(\sum_{i=1}^{n} (m - x_i)\log(1-\theta)\right)$$

Simplifying, we have

$$\log L(\theta) = C + \left(\sum_{i=1}^{n} x_i\right)\log\theta + \left[nm - \sum_{i=1}^{n} x_i\right]\log(1-\theta)$$

where C is constant (in terms of θ).

③ Differentiate with respect to θ

$$\frac{\partial}{\partial\theta}\log L(\theta) = \sum_{i=1}^{n}\left[\frac{x_i}{\theta} - \frac{m - x_i}{1-\theta}\right]$$

Set equal to 0.

$$\sum_{i=1}^{n}\left[\frac{x_i}{\theta} - \frac{m - x_i}{1-\theta}\right] = 0$$

solving for θ, we have.

$$\sum_{i=1}^{n} \left(\frac{x_i}{\theta} - \frac{m - x_i}{1-\theta} \right) = 0$$

$$\sum_{i=1}^{n} \left(\frac{x_i(1-\theta) - \theta(m - x_i)}{\theta(1-\theta)} \right) = 0$$

$$\sum_{i=1}^{n} (x_i - \theta m) = 0$$

$$\theta \sum_{i=1}^{n} x_i = nm\theta$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{m}$$

So, the max. likelihood estimate of θ is

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{m}.$$