Algorithm Analysis and Design

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October, 2018

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a. $T(n) = 3T(\frac{n}{3}) + \sqrt{n}$

$a \geq 1,\ b > 1 \quad \Rightarrow \quad$ We can use Master Theorem:
$a = 3$ and $b = 3 \quad \Rightarrow \quad \log_3 3 = 1$

$$\begin{cases} \sqrt{n} \in O(n^{1-\varepsilon}) \quad \Rightarrow \quad T(n) = \theta(n) \quad \checkmark \\ \sqrt{n} \in \theta(n) \quad \Rightarrow \quad T(n) = \theta(n \lg n) \\ \sqrt{n} \in \Omega(n^{1+\varepsilon}) \quad \Rightarrow \quad T(n) = \theta(\sqrt{n}) \end{cases}$$

b. $T(n) = 3T(\frac{n}{4}) + n \log n$

$a \geq 1,\ b > 1 \quad \Rightarrow \quad$ We can use Master Theorem:
$a = 3$ and $b = 4 \quad \Rightarrow \quad \log_4 3 < 1$

$$\begin{cases} n \log n \in O(n^{log_4 3-\varepsilon}) \\ n \log n \in \theta(n^{log_4 3}) \\ n \log n \in \Omega(n^{log_4 3+\varepsilon}) \quad \checkmark \end{cases}$$

$\Rightarrow \quad a(\frac{n}{b} \log \frac{n}{b}) < cn \log n \quad \Rightarrow \quad \frac{3}{4} \log n/4 < c \log n \Rightarrow c = \frac{3}{4}$
$T(n) = \theta(n \log n)$

c. $T(n) = 4T(\frac{n}{2}) + n^2$

$a \geq 1,\ b > 0 \quad \Rightarrow \quad$ We can use Master Theorem:
$a = 4$ and $b = 2 \quad \Rightarrow \quad \log_2 4 = 2$

$$\begin{cases} n^2 \in O(n^{2-\varepsilon}) \\ n^2 \in \theta(n^2) \quad \Rightarrow \quad T(n) = \theta(n^2 \log n) \quad \checkmark \\ n^2 \in \Omega(n^{2+\varepsilon}) \end{cases}$$

d. $T(n) = 4T(\frac{n}{2}) + n^2 \log^2 n$

$T(n) = 4(4T(\frac{n}{4}) + (\frac{n}{2}\log\frac{n}{2})^2) + (n\log n)^2$
$T(n) = 16T(\frac{n}{4}) + (n\log\frac{n}{2})^2 + (n\log n)^2$
$T(n) = 16(4T(\frac{n}{8}) + (\frac{n}{4}\log\frac{n}{4})^2) + (n\log\frac{n}{2})^2 + (n\log n)^2$
$T(n) = 64T(\frac{n}{8}) + (n\log\frac{n}{4})^2 + (n\log\frac{n}{2})^2 + (n\log n)^2$
.
.
.
$T(n) = n^2T(1) + (n\log 2)^2 + (n\log 4)^2 + (n\log 8)^2 + ... + (n\log(\frac{n}{2}))^2 + (n\log n)^2$
Assume $T(1) = 1$: $T(n) = n^2(1 + \log^2 2 + \log^2 4 + ... + \log^2\frac{n}{2} + \log^2 n)$
$T(n) = n^2(1 + 1 + 4 + 9 + 16 + ... + log^2\frac{n}{2} + \log^2 n)$
We know the sum of square numbers: $1 + 4 + 9 + 16 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$
So:
$T(n) = n^2(\frac{2\log^3 n + 3\log^2 n + \log n}{6})$
We know for Polynomial Functions that(proved in class): $P(x) = \Sigma_{i=0}^{d} a_i x^i$
$\Rightarrow \quad P(n) = \theta(n^d)$
So: $T(n) = \theta(n^2\log^3 n)$

e. $T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n}$

$T(n) = 2(2T(\frac{n}{4}) + \frac{n}{2\log\frac{n}{2}}) + \frac{n}{\log n} \quad \Rightarrow \quad T(n) = 4T(\frac{n}{4}) + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n}$
$T(n) = 4(2T(\frac{n}{8}) + \frac{n}{4\log\frac{n}{4}}) + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n} \Rightarrow T(n) = 8T(\frac{n}{8}) + \frac{n}{\log\frac{n}{4}} + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n}$
.
.
.
$T(n) = nT(1) + \frac{n}{\log 2} + \frac{n}{\log 4} + \frac{n}{\log 8} + ...$
Assume $T(1) = 1$: $T(n) = n(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n})$
We know about Harmonic Series that: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \theta(\lg n)$
$\Rightarrow \quad T(n) = \theta(n\lg\lg n)$

f. $T(n) = T(\sqrt{n}) + 1$

$T(n) = T(n^{\frac{1}{2^1}}) + 1$
$T(n) = (T(n^{\frac{1}{2^2}}) + 1) + 1$
$T(n) = ((T(n^{\frac{1}{2^3}}) + 1) + 1) + 1$
.
.
.
$T(n) = T(n^{\frac{1}{2^i}}) + i$

We assume that the first step had been $T(1)$. So:

$n^{\frac{1}{2^i}} = 1 \quad \Rightarrow \quad \frac{1}{2^i} = 0 \quad \Rightarrow \quad i = \infty$

Now we assume that the step had been $T(2)$. So:
$n^{\frac{1}{2^i}} = 2 \quad \Rightarrow \quad 2^{-i} = \log_n 2 \quad \Rightarrow \quad i = \lg\lg n$

So the cost of $T(\sqrt{n})$ is $\lg\lg n$. The cost of addition of 1s is O(1) and not important against $\lg\lg n$.
There would be a $c > 1$ and $n_0 > 1$ which for any $n \in \mathbb{N}$, $T(n) < c.\lg\lg n$.
And also,There would be a $1 > c > 0$ and $n_0 > 1$ which for any $n \in \mathbb{N}$, $c.\lg\lg n < T(n)$.

So $T(n) = \theta(\lg\lg n)$.