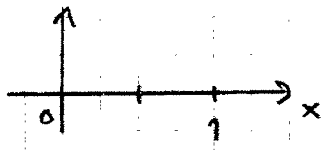
19.

The problem of a deflected cable is formulated with the differential equation

$$u'' = 1, \qquad x \in (0,1), \qquad \begin{matrix} u(0) = 0 \\ u'(1) = 0 \end{matrix}$$

We use two P1 elements (linear):



The nodes are $\quad$ nodes = [0, 0.5, 1]
and elements $\quad$ elements = [[0,1],[1,2]].

According to the Galerkin method, the residual should be orthogonal to the trial space $V = \text{span}\{\varphi_i, i = 0,1,\dots,N\}$, and we get the variational formulation

$$(u'', v) = (1, v), \qquad \forall v \in V.$$

Integrating by parts, we may write

$$(u'', v) = [u'v]_0^1 - (u', v')$$

$$= \underbrace{u'(1)}_{0}v(1) - u'(0)\underbrace{v(0)}_{0} - (u', v')$$

$\leftarrow$ since $u(0) = 0$ (Dirichlet)

$$= -(u', v')$$

The variational formulation becomes

$$-(w', v') = (1, v) \,, \qquad \forall v \in V.$$

Assume we have a linear finite element basis

$$\{\varphi_i(x),\ i = 0, 1, \ldots, N\},$$

where the basis functions $\varphi_i(x)$ are local Lagrange polynomials of the type

$$\varphi_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \,. \qquad j = \text{local index}$$

Here $n = 1$ for linear elements, and x_i, $i = 0, 1, \ldots, n$, are the nodes within an element.

The solution is approximated in the FEM basis as

$$w(x) = \sum_{j=0}^{N} c_j \varphi_j(x) \,.$$

The equation can be written in terms of basis functions:

$$\sum_{j=0}^{N} (\varphi_j', \varphi_i') c_j = -(1, \varphi_i) \,, \qquad i = 0, 1, \ldots, N \,.$$

or
$$\sum_{j=0}^{N} A_{i,j} c_j = b_i \,,$$

where
$$A_{i,j} = (\varphi_i', \varphi_j')$$
and
$$b_i = -(1, \varphi_i) \,.$$

Matrix elements of A and b can be assembled from terms calculated on a reference element.

Let us choose a reference element restricted to the interval $[-1,1]$.

The nodes are nodes $= [-1, 1]$ for linear elements P1.

The matrix element with local indices (r,s) in the element e is then

$$\tilde{A}^{(e)}_{r,s} = \int_{\Omega^{(e)}} dx\, \varphi'_{g(e,r)}(x)\, \varphi'_{g(e,s)}(x)$$

$$= \int_{-1}^{1} dX\, \tilde{\varphi}_r'(X)\, \varphi_s'(X)\, \frac{dx}{dX}$$

$$\frac{d}{dx}\tilde{\varphi}_r(X) = \frac{d}{dX}\tilde{\varphi}_r(X)\frac{dX}{dx}$$

On the reference element, $X_0 = -1$ and $X_1 = 1$, and we get

$$\tilde{\varphi}_0(X) = \frac{X-1}{-1-1} = \frac{1}{2}(1-X),$$

$$\tilde{\varphi}_1(X) = \frac{X+1}{1+1} = \frac{1}{2}(1+X).$$

$$\tilde{\varphi}_0'(X) = -\frac{1}{2}, \qquad \tilde{\varphi}_1'(X) = \frac{1}{2}$$

Here we have the affine mapping

$$x = \frac{1}{2}(x_L + x_R) + \frac{1}{2}(x_R - x_L)X$$

between the physical and reference spaces.

Then

$$\frac{dx}{dX} = \frac{1}{2}(x_R - x_L).$$

With a constant element width

$$x_R - x_L = h,$$

we get

$$\frac{dx}{dX} = \frac{h}{2}. \qquad \frac{dX}{dx} = \frac{2}{h}$$

In a calculation with constant element width, the matrix element $\tilde{A}^{(e)}_{r,s}$ is independent on e.

Let us calculate the element matrix elements:

$$\tilde{A}_{0,0} = \int_{-1}^{1} dX \, \frac{2}{h}\left(-\frac{1}{2}\right)^2 = \frac{1}{2h}\int_{-1}^{1} dX = \frac{1}{h}$$

$$\tilde{A}_{0,1} = \int_{-1}^{1} dX \, \frac{2}{h}\left(-\frac{1}{2}\right)\frac{1}{2} = -\frac{1}{2h}\int_{-1}^{1} dX = -\frac{1}{h}$$

$$\tilde{A}_{1,0} = \tilde{A}_{0,1}$$

$$\tilde{A}_{1,1} = \frac{1}{h}.$$

In a similar way, vector elements with local index 'r' in the element e are written

$$\tilde{b}^{(e)}_r = \int_{\Omega(e)} dx \, \varphi_{g(e,r)}(x) f(x)$$

$$= \int_{-1}^{1} dX \, \frac{dx}{dX} \, \tilde{\varphi}_r(X) f(x(X))$$

The explicit elements are here, when $f(x) = -1$,

$$\tilde{b}_0 = \int_{-1}^{1} dx \, \frac{h}{2} \, \frac{1}{2}(1-x)(-1)$$

$$= -\frac{h}{4} \int_{-1}^{1} dx \, (1-x)$$

$$= -\frac{h}{4} \Big/_{-1}^{1} \left(x - \frac{x^2}{2}\right)$$

$$= -\frac{h}{4} \left[1 - \cancel{\frac{1}{2}} - (-1) + \cancel{\frac{1}{2}} \right]$$

$$= -\frac{h}{2}$$

$$\tilde{b}_1 = \int_{-1}^{1} dx \, \frac{h}{2} \, \frac{1}{2}(1+x)(-1)$$

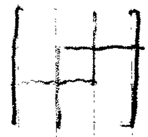
$$= -\frac{h}{4} \int_{-1}^{1} dx \, (1+x)$$

$$= -\frac{h}{4} \Big/_{-1}^{1} \left(x + \frac{x^2}{2}\right)$$

$$= -\frac{h}{4} \left[1 + \cancel{\frac{1}{2}} - (-1) - \cancel{\frac{1}{2}} \right]$$

$$= -\frac{h}{2}$$

Let us then assemble the global matrix A and vector b.



$$A_{0,0} = \tilde{A}^{(0)}_{0,0} = \frac{1}{h}$$

$$A_{1,0} = \tilde{A}^{(0)}_{1,0} = -\frac{1}{h}$$

$$A_{2,0} = 0$$

$$A_{0,1} = \tilde{A}^{(0)}_{1,0} = -\frac{1}{h}$$

$$A_{1,1} = \tilde{A}^{(0)}_{1,1} + \tilde{A}^{(1)}_{0,0} = \frac{2}{h}$$

$$A_{2,1} = \tilde{A}^{(1)}_{1,0} = -\frac{1}{h}$$

$$A_{0,2} = A_{2,0} = 0$$

$$A_{1,2} = A_{2,1} = -\frac{1}{h}$$

$$A_{2,2} = \tilde{A}^{(1)}_{1,1} = \frac{1}{h}$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{matrix} r = 1,2 \\ s = 1,2 \end{matrix}$$

$$b_0 = \tilde{b}^{(0)}_0 = -\frac{h}{2}$$

$$b_1 = \tilde{b}^{(0)}_1 + \tilde{b}^{(1)}_0 = -h$$

$$b_2 = \tilde{b}^{(1)}_1 = -\frac{h}{2}$$

$$b = \frac{h}{4}\begin{bmatrix} -4 \\ -2 \end{bmatrix} \qquad h = \frac{1}{2}$$

$$\Rightarrow c = \begin{bmatrix} -6 \\ -8 \end{bmatrix} \cdot \frac{1}{16}$$

The solution is thus

$$u(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)$$

$$\varphi_1(x) = \frac{x-0}{0.5-0} = 2x, \qquad 0 \le x \le \tfrac{1}{2}$$

$$\varphi_1(x) = \frac{x-1}{\frac{1}{2}-1} = \frac{x-1}{-\frac{1}{2}} = 2(1-x), \qquad \tfrac{1}{2} \le x \le 1$$

$$\varphi_2(x) = \frac{x-\frac{1}{2}}{1-\frac{1}{2}} = \frac{x-\frac{1}{2}}{\frac{1}{2}} = 2(x-\tfrac{1}{2})$$

$$= 2x-1, \qquad \tfrac{1}{2} \le x \le 1$$

$$u(x) = \underline{\underline{-\tfrac{3}{4}x}}, \qquad 0 \le x \le \tfrac{1}{2}$$

$$u(x) = -\tfrac{3}{4}(1-x) - x + \tfrac{1}{2}$$

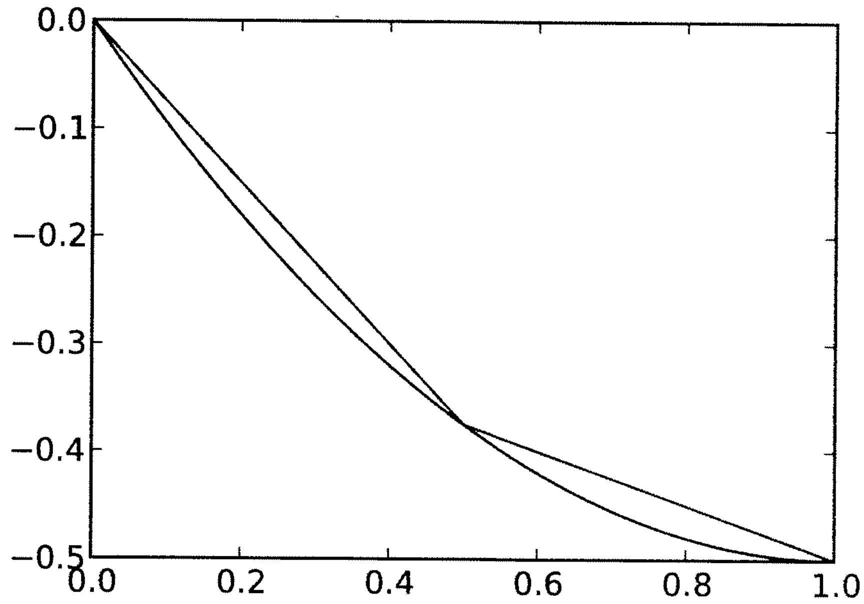
$$= \underline{\underline{-\tfrac{1}{4}x - \tfrac{1}{4}}}, \qquad \tfrac{1}{2} \le x \le 1$$

This result can be compared with the exact solution, which is

$$\underline{u(x) = \frac{x^2}{2} - x}.$$

The finite element solution is piecewise linear with the exact values at the nodal points $x = 0$, $\tfrac{1}{2}$, and 1.

Surprisingly, the FE approximation is here an interpolation solution.


0.0
−0.1
−0.2
−0.3
−0.4
−0.5
0.0
0.2
0.4
0.6
0.8
1.0