17.

$$T w''(x) = l(x)$$

$$l(x) = c = \text{constant}$$

$$w(0) = w(L) = 0$$

$$u'' = 1, \quad x \in (0,1), \quad \begin{array}{l} u(0) = 0 \\ u'(1) = 0 \end{array}$$

$$\varphi_i(x) = \sin((i+1)\pi x/2), \quad i = 1, \dots, N$$

$$u(x) = \sum_{j=0}^{N} c_j \varphi_j(x)$$

Galerkin method:

$$(u'', v) = (1, v), \quad \forall v \in V,$$

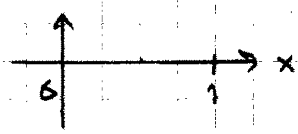
where $V = \operatorname{span}\{\varphi_i(x),\ i = 1, \dots, N\}$.

Integration by parts gives

$$\begin{aligned} (u'', v) &= [u'v]_0^1 - (u', v') \\ &= \underset{\overset{\shortparallel}{0}}{u'(1)}v(1) - u'(0)\underset{\overset{\shortparallel}{0}}{v(0)} - (u', v') \\ &= -(u', v'). \end{aligned}$$

We can now write

$$-(u', v') = (1, v), \quad \forall v \in V.$$



Inserting the expansion in terms of basic functions, we get

$$-\sum_{j=0}^{N} c_j (\varphi_j', \varphi_i') = (1, \varphi_i) , \qquad i = 0, 1, \dots, N$$

The derivative of φ_i is

$$\varphi_i'(x) = \frac{(i+1)\pi}{2} \cos((i+1)\pi x/2) .$$

$$(\varphi_0', \varphi_0') = \left(\frac{\pi}{2}\right)^2 \int_0^1 dx \, \cos^2(\pi x/2)$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos^2(\theta) = (1 + \cos 2\theta)/2$$

$$\cos^2\left(\frac{\pi x}{2}\right) = (1 + \cos(\pi x))/2$$

$$(\varphi_0', \varphi_0') = \frac{\pi^2}{8} \int_0^1 dx \, (1 + \cos(\pi x))$$

$$= \frac{\pi^2}{8} \Big/_0^1 \left[x + \frac{\sin(\pi x)}{\pi} \right]$$

$$= \frac{\pi^2}{8} \left[1 + \frac{\sin \pi}{\pi} - 0 \right] = \frac{\pi^2}{8}$$

$$(1, \varphi_0) = \int_0^1 dx \, \sin(\pi x/2)$$

$$= \Big|_0^1 - \frac{\cos(\pi x/2)}{\pi/2} = + \frac{2}{\pi}$$

$$-c_0 \frac{\pi^2}{8} = \frac{2}{\pi}$$

$$\underline{\underline{c_0 = -\frac{16}{\pi^3}}}$$

We got the approximative solution

$$u(x) \approx -\frac{16}{\pi^3}\varphi_0(x)$$

$$\underline{\underline{= -\frac{16}{\pi^3}\sin(\pi x/2)}}$$

Exact solution:

$$u'' = 1$$

$$x = \int_0^x dt \, u'' = \Big|_0^x u'(t) = u'(x) - u'(0)$$

$$u'(x) = x + u'(0)$$

$$\int_0^x dt \, u'(x) = \Big|_0^x u(t) = u(x) - u(0)$$

$$\int_0^x dt \, u'(t) = \int_0^x dt \, [t + u'(0)] = \Big|_0^x \left(\frac{t^2}{2} + t u'(0)\right)$$

$$= \frac{x^2}{2} + x u'(0)$$

$$\Rightarrow \quad u(x) = \cancel{u(0)}^{\,0} + x u'(0) + \frac{x^2}{2}$$

$$u'(x) = u'(0) + x$$

$$u'(1) = u'(0) + 1 = 0 \quad \rightarrow \quad u'(0) = -1$$

We get the exact solution

$$u(x) = \underline{\underline{\frac{x^2}{2} - x}} \; .$$

The error at $x = 1$ is

$$\varepsilon = u_e(x=1) - u(x=1)$$
$$= -\frac{1}{2} + \frac{16}{\pi^3} = \underline{\underline{0.0160\ldots}} \; .$$

In Gnuplot:

```
set xrange [0:1]
plot -16*sin(pi*x/2)/(pi**3),   x**2/2 - x
```

Next, let us do the same calculation using the least squares method.

$$\min_u \; (u'' - 1, u'' - 1)$$

We define the residual

$$R(x; c_0, c_1, \ldots, c_N) = u''(x) - 1$$
$$= \sum_{i=0}^{N} c_i \varphi_i''(x) - 1 \; .$$

The least squares method may be formulated as

$$\min_{c_0, c_1, \ldots, c_N} \; (R, R) \; .$$

A necessary condition for minimum is

$$\frac{\partial}{\partial c_i}(R,R)=0, \qquad i=0,1,\dots,N.$$

We get

$$\frac{\partial}{\partial c_i}(R,R) = 2\left(\frac{\partial}{\partial c_i}R, R\right) = 0$$

$$\Rightarrow \quad \left(\varphi_i'', \sum_{j=0}^{N} c_j \varphi_j''(x) - 1\right) = 0$$

$$\sum_{j=0}^{N} (\varphi_i'', \varphi_j'') c_j = (\varphi_i'', 1)$$

$$\varphi_0(x) = \sin(\pi x/2)$$
$$\varphi_0'(x) = \frac{\pi}{2}\cos(\pi x/2)$$
$$\varphi_0''(x) = -\left(\frac{\pi}{2}\right)^2 \sin(\pi x/2)$$

$$(\varphi_0'', \varphi_0'') = \left(\frac{\pi}{2}\right)^4 \int_0^1 dx \, \sin^2(\pi x/2)$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$\sin^2\theta = (1-\cos 2\theta)/2$$

$$(\varphi_0'', \varphi_0'') = \left(\frac{\pi}{2}\right)^4 \int_0^1 dx\, (1-\cos(\pi x))/2$$

$$= \frac{\pi^4}{32} \Big/_0^1 \left[x - \frac{\sin \pi x}{\pi}\right] = \frac{\pi^4}{32}\left[1 - \frac{\sin\pi}{\pi} - 0\right]$$
$$= \frac{\pi^4}{32}$$

$$(\varphi_0'', 1) = -\frac{\pi^2}{4}\int_0^1 dx \sin(\pi x/2)$$

$$= -\frac{\pi^2}{4}\Big|_0^1 - \frac{\cos(\pi x/2)}{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{\pi}{2}$$

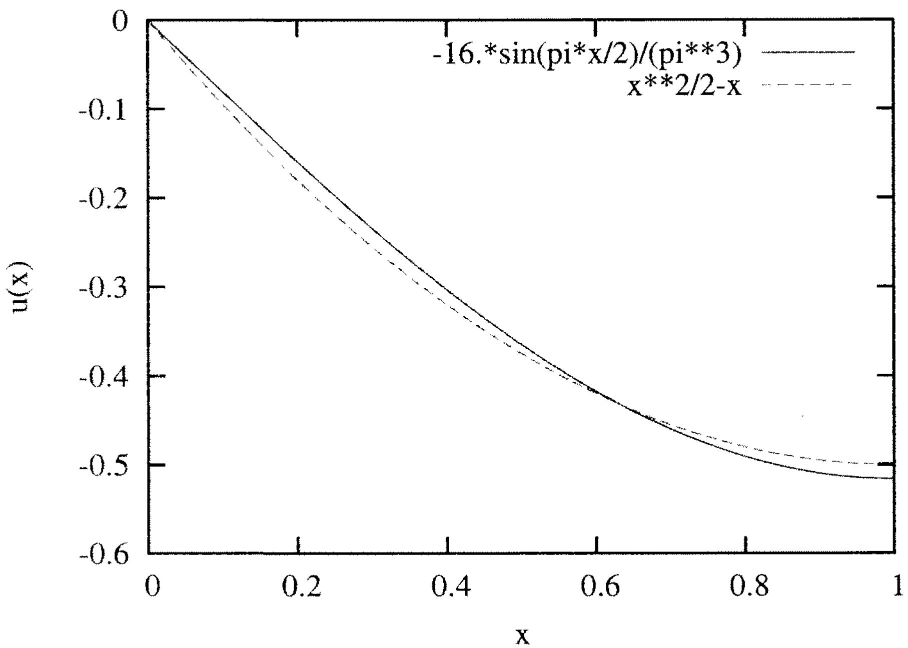
The equation becomes

$$(\varphi_0'', \varphi_0'')\, c_0 = (\varphi_0'', 1)$$

$$\frac{\pi^4}{32} c_0 = -\frac{\pi}{2}$$

$$\underline{\underline{c_0 = -\frac{16}{\pi^3}}}$$

This result is the same as that obtained using the Galerkin method.


-16.*sin(pi*x/2)/(pi**3)
x**2/2-x
u(x)
x
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0.2
0.4
0.6
0.8
1