Problem 1:

This question presents us with the case of a disease that affects 1% of the population (so 99% of the population is unaffected). When tested, the results are 95% accurate for true positives, and 90% accurate for true negatives. We are tasked with finding the probability that a person actually has the disease if they test positive. Interestingly, this question appears simple on the surface, as the answer would seem to just be 95%. However, taking all the information into account and using Bayes' Theorem tells us otherwise. Bayes' Theorem is as follows:

$$P(A|B) = (P(B|A)*P(A)) / P(B)$$

P(A|B), what we're solving for, is the probability of event A given the occurrence of event B. P(B|A) is essentially the reverse: the probability of observing event B given that A is true. P(A) is the prior probability or initial evidence (before observation) of event A. Finally, P(B) is the total probability of event B. Now all we have to do is sub in the numbers from the case. However, we don't yet have a value for P(B). Plugging in values from the problem gives us:

$$P(B) = (0.95 * 0.01) + (0.10 * 0.99)$$

$$P(B) = (0.0095) + (0.099)$$

$$P(B) = 0.1085$$

The .95 represents the 95% accuracy of true positive, the 0.01 represents the 1% of the population that gets the disease, the 0.10 represents the possible inaccuracy of a negative test, and the .99 is the portion of the population that does not have the disease. Now plugging in every value gives us this:

$$P(A|B) = (0.95 * 0.01) / 0.1085$$

$$P(A|B) = 0.087558$$

$$P(A|B) = \mathbf{8.76\%}$$

Problem 2:

The next question gives us a scenario involving spam emails. As outlined in the problem, 5% of all emails are spam (so 95% are not), the spam filter is 99% accurate (so it is inaccurate 1% of the time), and emails are mislabeled as spam 2% of the time. We are tasked with determining the probability that an email is actually spam if it is labeled as such. Finding this probability will look much like it did in problem 1, and we will again use Bayes' Theorem. As above, we don't have the total probability of the B value (an email being classified as spam). The calculation is as follows:

$$P(B) = (0.99 * 0.05) + (0.02 * 0.95)$$

$$P(B) = 0.0495 + 0.019$$

$$P(B) = 0.0685$$

Now we plug this value into Bayes' Theorem with the other appropriate information present as well (the 0.99 probability of accurate spam identification and 0.05 probability of a given email being spam).

$$P(A|B) = (0.99 * 0.05) / 0.0685$$

$$P(A|B) = 0.0495 / 0.0685$$

$$P(A|B) = 0.722628$$

$$P(A|B) = \mathbf{72.26\%}$$

Problem 3

This last problem is slightly different and slightly more involved than the last two problems. In the end, the calculations aren't much more difficult, there are just more variables at play. Now, there are two objects (Machine A and Machine B) with possible success or failure, and an additional third object (the sensor) that checks for failure (either accurately or inaccurately). Bayes' Theorem will be used eventually, and as before, we first need to find P(B). However, before we find P(B) we need to determine the probability of both machines being successful. Since Machine A makes 60% of the items at a 95% success rate and Machine B makes 40% of the items at a 90% success rate, the formula for that would look as follows:

$$P(\text{Both Succeed}) = (0.6 * 0.95) + (0.4 * 0.9)$$

$$P(\text{Both Succeed}) = 0.57 + 0.36$$

$$P(\text{Both Succeed}) = 0.93$$

Now we can find P(B). The formula is similar to the two problems above but with three separate probabilities involved:

$$P(B) = (0.05 * 0.6) + (0.1 * 0.4) + (0.05 * 0.93)$$

$$P(B) = 0.03 + 0.04 + 0.0465$$

$$P(B) = 0.1165$$

Now we substitute this in to Bayes' Formula:

$$P(A|B) = (0.6 * 0.05) / 0.1165$$

$$P(A|B) = 0.03 / 0.1165$$

$$P(A|B) = 0.257511$$

$$P(A|B) = \mathbf{25.75\%}$$