Homework 7 – Intro to Probability and Statistics

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Question 1

An urn has two white balls (W) and three red balls (R). You draw a ball from the urn. If it is white, you flip a coin. If it is red, you throw it back in the urn and draw another ball. What is the sample space of this experiment?

The sample space is defined as the group of all possible events. Since you only will get a ball if you pick up a white one, the sample space of this experiment is equal to four: White; Red-White; Red-Red-White; Red-Red-Red-White.

Question 2

Suppose you toss two dices. Consider two events:

- A: the sum of the numbers in both dices is equal to 9
- B: the number in the first die is greater than or equal 4

Define:

1. The elements of A

The four elements of event A are the following:

$$A = \{(6;3),(3;6),(5;4),(4;5)\}$$

2. The elements of B

The three elements of event B are the following:

$$B = \{(4),(5),(6)\}$$

3. The elements of $A \cap B$

The elements of $A \cap B$ is the following:

$$A \cap B = \emptyset$$

4. The elements of $A \cup B$

The elements of $A \cup B$ are the following:

$$A \cup B = \{(6;3),(3;6),(5;4),(4;5),(4),(5),(6)\}$$

5. The elements of A^C

The elements of A^C are all the 36 (thirty-six) possible combinations minus the four that are already in A.

Question 3

The probability that student A solves a given problem is $\frac{2}{3}$. The probability that student B solves the same problem is $\frac{3}{4}$. If both try to solve the problem independently, what is the chance that the problem will be solved?

As they work independently, the chance that the problem will be solved is the following:

$$P(A \cup B) = P(A) + P(B) - P(A \cap)$$

$$P(A \cup B) = P(A) + P(B) - (P(A)P(B))$$

$$P(A \cup B) = \frac{2}{3} + \frac{3}{4} - \left(\frac{2}{3} \times \frac{3}{4}\right)$$

$$P(A \cup B) = \frac{11}{12} \approx 0.90$$

Thus, cooperation has worked.

Question 4

Consider the following probability table:

	B	B^C	TOTAL
A	0.04	0.06	0.10
A^C	0.08	0.82	0.90
TOTAL	0.12	0.88	1.00

Note that $P(A) = 0.10, P(A \cap B) = \ 0.04$, and so on. Are the events A and B independent?

As it is written in the table above, $P(A) = 0.10$ and $P(B) = 0.12$. Thus, $P(A)P(B) = 0.012$. It means that the probability of A and B happen independently from each other is about 1%. Besides that, $P(A \cap B) = 0.04$. It means that the probability of A and B happen at same time is equal to 4%. Therefore, A and B are not independent.

Question 5

A company produces phones in three factories. In factory I, the company produces 40% of the phones, while in factories II and III produce 30% of the phones in each. The chance of a phone is assembled broken is 0.01 (factory I), 0.04 (factory II), and 0.03 (factory III). The phones are then taken to a warehouse.

1. If you select a phone randomly in the warehouse, what is the chance that it is broken?

The chance of anyone get a phone that is broken is a function of all chances of a phone is assembled broken. And, therefore, it is equal to: $(0.01)(0.04)(0.03) = 0.000012$

2. Suppose the phone you draw is broken. What is the probability that it was manufactured by factory I?

The chance of a phone that is broken had been manufactured by factory I is equal to the chance of be broken and come from factory I: $(0.000012)(0.4) = 0.0000048$

Question 6

Prove that if A and B are independent, then:

1. A^C and B^C are independent.

We know that:

$$P(A \cap B) = P(A)P(B)$$

So, we need to prove that:

$$P(A^C \cap B^C) = P(A^C)P(B^C)$$

We know that:

$$P(A^C) = 1 - P(A)$$

$$P(B^C) = 1 - P(B)$$

We can show that:

$$P(A^C \cap B^C) = 1 - P(A \cup B)$$

$$P(A^C \cap B^C) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$P(A^C \cap B^C) = 1 - [P(A) + P(B) - P(A \cap B)]$$

$$P(A^C \cap B^C) = 1 - [P(A) + P(B) - P(A)P(B)]$$

$$P(A^C \cap B^C) = 1 - [(1 - P(A^C)) + (1 - P(B^C)) - (1 - P(A^C))(1 - P(B^C))]$$

$$P(A^C \cap B^C) = 1 - [(1 - P(A^C)) + (1 - P(B^C)) - (1 - P(B^C) - P(A^C) + P(A^C)P(B^C)]$$

$$P(A^C \cap B^C) = 1 - [1 - P(A^C)P(B^C)]$$

$$P(A^C \cap B^C) = P(A^C)P(B^C)$$

2. A and B^C are independent.

We know that:

$$P(A \cap B) = P(A)P(B)$$

So, we need to prove that:

$$P(A \cap B^C) = P(A)P(B^C)$$

We can show that:

We know that:

$$P(A \cap B^C) = P(A) - P(A \cap B)$$

$$P(A \cap B^C) = P(A) - P(A)P(B)$$

$$P(A \cap B^C) = P(A)(1 - P(B))$$

$$P(A \cap B^C) = P(A)P(B^C)$$

3. A^C and B are independent.

We know that:

$$P(A \cap B) = P(A)P(B)$$

So, we need to prove that:

$$P(A^C \cap B) = P(A^C)P(B)$$

We can show that:

$$P(A^C \cap B) = P(B) - P(A \cap B)$$

$$P(A^C \cap B) = P(B) - P(A)P(B)$$

$$P(A^C \cap B) = P(B)(1 - P(A))$$

$$P(A^C \cap B) = P(A^C)P(B)$$

Question 7

Let $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$**.** Prove that:

1. $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$

As, $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$**,** we can rewrite as following:

$$\sum_{i=1}^{n} x_i = n\bar{x}$$

Rewriting $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$:

$$\sum_{i=1}^{n}(x_i - \bar{x}) = (x_1 - \bar{x}) + \cdots + (x_n - \bar{x})$$

$$\sum_{i=1}^{n}(x_i - \bar{x}) = (x_1 + \cdots + x_n) - (n\bar{x})$$

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x}$$

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0$$

2. $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} {x_i}^2 - n\overline{x^2}$

We already know that:

$$\sum_{i=1}^{n}(x_i - \bar{x}) = (x_1 + \cdots + x_n) - (n\bar{x})$$

Thus:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = ((x_1 + \cdots + x_n) - (n\bar{x}))^2$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = (x_1 + \cdots + x_n)^2 - 2(x_1 + \cdots + x_n)(n\bar{x}) + (n\bar{x})^2$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} {x_i}^2 - 2\left(\sum_{i=1}^{n} x_i\right)(n\bar{x}) + (n\bar{x})^2$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} {x_i}^2 - 2(n\bar{x})(n\bar{x}) + (n\bar{x})^2$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} {x_i}^2 - 2(n\bar{x})^2 + (n\bar{x})^2$$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} {x_i}^2 - n\bar{x}^2$$

Question 8

Let $P(k;\ n,p) = \binom{n}{k} p^k (1-p)^{n-k}$ the probability of k successes in a binomial distribution with n trials p probability of success. Prove that:

$$P(k+1;\ n,p) = \frac{(n-k)p}{(k+1)(1-p)} P(k;\ n,p)$$

$$P(X = k+1) = \binom{n}{k+1} p^{k+1} (1-p)^{n-k+1}$$

$$P(X = k+1) = \frac{n!}{(k+1)!\,(n-k+1)!} p^{k+1} (1-p)^{n-k+1}$$