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Forecasting multivariate realized stock market volatility

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ABSTRACT

We present a new matrix-logarithm model of the realized covariance matrix of stock returns. The model uses latent factors which are functions of lagged volatility, lagged returns and other forecasting variables. The model has several advantages: it is parsimonious; it does not require imposing parameter restrictions; and, it results in a positive-definite estimated covariance matrix. We apply the model to the covariance matrix of size-sorted stock returns and find that two factors are sufficient to capture most of the dynamics.

1. Introduction

The variances and covariances of stock returns vary over time (e.g. Andersen et al., 2005). As a result, many important financial applications require a model of the conditional covariance matrix. Three distinct categories of methods for estimating a latent conditional covariance matrix have evolved in the literature. In the first category are the various forms of the multivariate GARCH model where forecasts of future volatility depend on past volatility and shocks (e.g. Bauwens et al., 2006). In the second category, authors have modeled asset return variances and covariances as functions of a number of predetermined variables (e.g. Ferson, 1995). The third category includes multivariate stochastic volatility models (e.g. Asai et al., 2006).

In this paper, we introduce a new model of the *realized* covariance matrix.[1] We use high-frequency data to construct estimates of the daily realized variances and covariances of five size-sorted stock portfolios. By using high-frequency data we obtain an estimate of the matrix of 'quadratic variations and covariations' that differs from the true conditional covariance matrix by mean zero errors (e.g. Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2004a)). This provides greater power in determining the effects of alternative forecasting variables on equity market volatility when compared to efforts based on latent volatility models.

We transform the realized covariance matrix using the matrix logarithm function to yield a series of transformed volatilities which we term the *log-space volatilities*. The matrix logarithm is a non-linear function of all of the elements of the covariance matrix and thus the log-space volatilities do not correspond one-to-one with their counterparts in the realized covariance matrix.[2] However, modeling the time variation of the log-space volatilities is straightforward and avoids the problems that plague existing estimators of the latent volatility matrix. In particular, we do not have to impose any constraints on our estimates of the log-space volatilities.

We then model the dynamics of the log-space volatility matrix using a latent factor model. The factors consist of *both* past volatilities and other variables that can help forecast future volatility. We thus are able to model the conditional covariance matrix by combining a large number of forecasting variables into a relatively small number of factors. Indeed we show that two factors can capture the volatility dynamics of the size-sorted stock portfolios.

The factor model is estimated by GMM yielding a series of filtered estimates. We then transform these fitted values, using the matrix exponential function, back into forecasts of the realized covariance matrix. Our estimated matrix is positive

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1 Andersen et al. (2001) and Barndorff-Nielsen and Shephard (2002) formalized the notion of *realized volatility*.

2 The matrix logarithm has been used for estimators of latent volatility by Chiu et al. (1996) and Kawakatsu (2006) and was also suggested in Asai et al. (2006).

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definite by construction and does not require any parameter restrictions to be imposed. The approach can thus be viewed as a multivariate version of standard stochastic volatility models, where the variance is an exponential function of the factors and the associated parameters.

In addition to introducing our new realized covariance matrix we also test the forecasting ability of alternative variables for time-varying equity market covariances. Our motivation is that researchers have examined a number of variables for forecasting returns but there is much less evidence that the variables forecast risks. The cross-section of small- and large-firm volatility has been examined in a number of earlier papers (e.g., Kroner and Ng (1998), Chan et al. (1999), and Moskowitz (2003)). However, these papers used models of latent volatility to capture the variation in the covariances. In contrast, we construct daily measures of the realized covariance matrix of small and large firms over the 1988 to 2002 period. Our precise measures of volatility allow a more detailed examination of the drivers of conditional covariances than prior work.

Naturally all of these advantages come at a cost. The main cost is that by performing our analysis on the log-space volatilities and then using the (non-linear) matrix exponential function, the estimated volatilities are not unbiased. However, as we show below, a simple bias correction is available that greatly reduces the problem. Another cost is that direct interpretation of the effects of an instrument on expected volatility is difficult due to the non-linear nature of the model. However, using our factor model estimates, we can obtain the derivatives of the realized covariance matrix with respect to the forecasting variables. We are able to calculate the derivatives at each point in our sample, yielding a series of conditional volatility elasticities that are functions of both the level of the volatility and the factors driving the volatility. The time series allows us to determine which variables have a large impact on time-varying expected volatility.

The paper is organized as follows. In Section 2, we present our model of matrix logarithmic realized volatility. In Section 3, we outline our method for constructing the realized volatility matrices and give the sources of the data. In Section 4, we give our results. In Section 5, we conclude.

2. Model

2.1. The matrix log transformation

In this paper, we use the matrix exponential and matrix logarithm functions to model the time-varying covariance matrix. The matrix exponential function performs a power series expansion on a square matrix A

$$V = \text{expm}\,(A) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \tag{1}$$

The matrix exponential function has a number of useful properties (Chiu et al. (1996)). The most important of these is that A is a real, symmetric matrix, if and only if V is a real, positive definite matrix. The matrix logarithm function is the inverse of the matrix exponential function. Taking the matrix logarithm of a real, positive definite matrix V results in a real, symmetric matrix A:

$$A = \text{logm}\,(V).$$

The matrix logarithm and matrix exponential functions are used in our three-step procedure to obtain forecasts of the conditional covariance matrix of stock returns. In the first step, for each day t, we use high-frequency data to construct the $P \times P$ realized conditional covariance matrix V_t.[3] The V_t matrix is positive semi-definite by construction. Applying the matrix logarithm function,

$$A_t = \text{logm}\,(V_t), \tag{2}$$

yields a real, symmetric $P \times P$ matrix A_t.

In the second step, we model the dynamics of the A_t matrix. To do this, we follow Chiu et al. (1996) and apply the vech operator to the matrix A_t

$$a_t = vech\,(A_t),$$

which stacks the elements on and below the diagonal of A_t to obtain the $p \times 1$ vector a_t, where $p = \frac{1}{2}P(P+1)$. The a_t vector forms the basis for all subsequent models. Below, we present a factor model for the a_t processes which allows both lagged values of a_t and other variables to forecast the volatility.

In the third step, we transform the fitted values from the log-volatility space into fitted values in the actual volatility space. We use the inverse of the *vech* function to form a $P \times P$ symmetric matrix $\widehat{A}_t$ of the fitted values at each time t from the vector $\widehat{a}_t$. Applying the matrix exponential function

$$\widehat{V}_t = \text{expm}\,(\widehat{A}_t), \tag{3}$$

yields the positive semi-definite matrix $\widehat{V}_t$, which is our estimate of the conditional covariance matrix for day t.

2.2. Factor models of volatility

2.2.1. Forecasting variables

We will use several different groups of variables to forecast the conditional covariance matrix. Based on the existing literature, we can separate the variables into two groups. The first are matrix-log values of realized volatility $(a_t, a_{t-1}, a_{t-2}, \ldots)$ which are used to capture the autoregressive nature of the volatility. There are three potential problems in using these variables to forecast volatility. First, the existing literature shows that capturing volatility dynamics will likely require a long lag structure. To overcome this, we adapt the Heterogeneous Autoregressive model of realized volatility (HAR-RV) of Corsi (2009) and Andersen et al. (2007) to a multivariate setting. These authors show that the aggregate market realized volatility is forecast well by a (linear) combination of lagged daily, weekly and monthly realized volatility.

The second problem is that other authors have indicated that lagged realized volatility may not be the best predictor. In particular, both Andersen et al. (2007) and Ghysels et al. (2006) find that bi-power covariation – an estimate of the continuous part of the volatility diffusion – is a good predictor of the aggregate market's realized volatility.[4] We thus construct bi-power covariation matrices aggregated over the last $d = 1, 5$ and 20 days. As in (2) above, we take the matrix logarithm of the bi-power covariation matrix over the past d days to yield $A^{BP}(d)_t$. Taking the *vech* of this matrix yields the unique elements $a^{BP}(d)_t$.

The third problem is the large number of correlated predictors. It is likely that the bi-power covariation series $a^{BP}(d)_t$ is driven by a smaller number of factors. We test this by estimating the principal components of $a^{BP}(d)_t$,

$$a^{BP}(d, i), \quad i = 1, \ldots, pc, \tag{4}$$

where $a^{BP}(d, i)$ is the ith principal component of the d-day log-space bi-power covariation matrix. We find that a small number of components captures the volatility of the daily, weekly

[3] The details of how the matrix is constructed are presented below.

[4] Barndorff-Nielsen and Shephard (2004b, 2006) develop the theory of bi-power variation, and extend their results to the multivariate case (bi-power covariation) in Barndorff-Nielsen and Shephard (2005). We construct bi-power covariation measures for our portfolios using Definition 3 of Barndorff-Nielsen and Shephard (2005).

and monthly log-space bi-power covariation series.[5] In turn, these principal components are sufficient to model the realized covariance matrix. Our approach can thus be viewed as a multivariate approach to the HAR-RV model using the principal components of bi-power variation as predictors.

The second group of forecasting variables, denoted X_t, are those variables that have been shown to forecast equity market returns. In equilibrium, expected returns should be related to risk, so it is natural to question whether these variables also forecast the components of market wide volatility. Below, we use a number of variables that has been shown to predict equity market returns.

We combine the two groups of forecasting variables as

$$Z_t = (a^{BP}(1,1)_t, \ldots, a^{BP}(5,1)_t, \ldots, a^{BP}(20,1)_t, \ldots, X_t).$$

Below, we select different subsets of Z_t that correspond to existing approaches to modeling volatility.

2.2.2. Latent factors

Combining all of the forecasting variables results in the model

$$a_t = \gamma_0 + \gamma_1 Z_{t-1} + \varepsilon_t. \tag{5}$$

In this model, the number of factors driving the cross section of log-space volatility a_t is equal to the number of variables in Z_t. However, it is likely that the common variation in a_t can be explained by a much smaller number of factors, as were the bi-power covariation measures above.

To model the common variation in the realized volatility matrix, we use a latent factor approach where the factors that drive the time-varying volatility are not specified directly. Rather, we assume that our set of forecasting variables Z_t is related to the true, but unknown, volatility factors. We thus specify the k-th volatility factor $\upsilon_{k,t}$ as a linear combination of the set of N variables in Z_t:

$$\upsilon_{k,t} = \theta_k Z_{t-1}, \tag{6}$$

where the $\theta_k = \{\theta_{k,(1)}, \ldots, \theta_{k,(N)}\}$ are coefficients that aggregate the forecasting variables in Z_t. Each of the log-space volatilities a_t^i is a function of the K volatility factors:

$$a_t^i = \gamma_0^i + \beta^i \theta Z_{t-1} + \varepsilon_t^i, \quad i = 1, \ldots, p,$$

where γ_0^i is the ith element of the intercept vector γ_0, β^i is the $1 \times K$ vector of the loadings of log-space volatility i on the K factors, and the $K \times N$ matrix θ contains the coefficients on the Z_{t-1} variables for the K factors. Assembling the model for all p log-space volatilities yields

$$a_t = \gamma_0 + \beta\theta Z_{t-1} + \varepsilon_t, \tag{7}$$

where the $p \times K$ matrix β is the loading of the log-space volatilities on the time-varying factors.

We note that using latent factors to model covariance matrices has a number of advantages over existing methods.[6] First, it allows us to combine both lagged volatility measures (the principal components in (4)) as well as the X_t variables in a parsimonious manner. Previous models required each variable to be a separate factor. While the large number of variables may help forecast the covariance matrix, it is unlikely that each variable represents a specific volatility factor. Our approach can be used to weigh (via the θ coefficients) all of the variables in a way that is optimal for forecasting the covariance matrix.

A second advantage to our approach is that it avoids using expected returns in modeling the volatility matrix. Aggregating squared return or bi-power covariation data over high frequencies means that the expected return variation can be ignored. As the realized covariance matrix can be estimated more precisely than can the expected returns, we should obtain more precise measures of the determinants of the covariance matrix.

The third advantage is parsimony. For example, assume that we require 20 lags of daily log-space bi-power covariation plus 5 forecasting variables in X_t to capture volatility dynamics in our 5×5 volatility matrix. The number of parameters in a system of linear regressions (5) would be 4590 while a $K = 2$ factor version of (7) using the first three principal components of the log-space bi-power covariation matrices (for $d = 1$, 5 and 20 days) has only 69. The small number of parameters in the factor model helps in estimating and interpreting the model in-sample and should help in out-of-sample forecasting.

2.3. Estimation

Our multivariate factor model is derived from the latent factor models of expected return variation that originated with Hansen and Hodrick (1980) and Gibbons and Ferson (1985). As in these papers, we estimate our factor model of volatility in (7) by GMM with the Newey and West (1987) form of the optimal weighting matrix.[7] We use iterated GMM with a maximum number of 25 steps. The instruments are the same forecasting variables Z_t.

In its present form, (7) is unidentified due to the $\beta\theta$ combination. We thus impose the standard identification that the first K rows of the matrix β are equal to an identity matrix. The cross-equation restrictions imposed on (5):

$$H_0\colon \gamma_1 = \beta\theta \tag{8}$$

can then be tested using the standard χ^2 test statistic from a GMM system.

Our model has a potential errors-in-variables problem as the log-space bi-power covariation matrix $A^{BP}(d)_t$ is constructed with error. Using its principal components as regressors will result in biased estimates of the coefficients. Ghysels and Jacquier (2005) have noted a similar problem with estimates of time-varying beta coefficients for portfolio selection. They advocate using lagged values of the betas in an instrumental variables regression to overcome the biases. We follow that approach here and use the twice lagged values of the principal components in the GMM instrument set.

Once the coefficients have been estimated by GMM, the fitted values are reassembled into a square matrix $\widehat{A}_t$. Applying the matrix exponential function (3) yields the prediction for the covariance matrix in period t,

$$\widehat{V}_t = \operatorname{expm}\left(\widehat{A}_t\right). \tag{9}$$

We can then apply standard forecasting evaluation techniques to compare $\widehat{V}_t$ to V_t.

2.4. Bias correction

Our estimator $\widehat{V}_t$ will be biased as the estimation is done in the log-volatility space and by Jensen's inequality $E\left(\widehat{V}_t\right) \neq \operatorname{expm}\left(E\widehat{A}\right)$. An analytic bias correction exists if $\widehat{A}$ and $\widehat{\varepsilon}$ are normally distributed; however, in our data this is not the case and so we do a simple numerical bias correction on the individual volatility series.

The realized volatility matrix V_t can be decomposed into a matrix of standard deviations and correlations:

$$V_t = SD_t * C_t * SD_t',$$

where SD_t is a $P \times P$ diagonal matrix of the standard deviations and C_t is a $P \times P$ symmetric matrix of the correlations. A similar decomposition can be done for the fitted value $\widehat{V}_t$ to yield the $\widehat{SD}_t$ and $\widehat{C}_t$ matrices. We then estimate a bias correction factor as the

5 The first three principal components capture 48.5%, 85.3% and 94.2% of the variation in the 1, 5 and 20 day bi-power covariation series, respectively.

6 Examples of previous work using factor models for volatility dynamics include Engle and Lee (1999), Diebold and Nerlove (1989), King et al. (1994), and Harvey et al. (1994).

7 We use the Andrews (1991) test to determine the optimal lag length of 46 in the HAC covariance matrix.

ratio of the median values of the two standard deviation series:

$$bc = \frac{\text{med}\,(SD_t(i,i))}{\text{med}\left(\widehat{SD}_t(i,i)\right)}, \quad i = 1, \dots, 5.$$

We then bias correct the standard deviations while leaving the correlations intact. This simple method works well in that the fitted values are of the approximate magnitude of the actual realized volatility series and the statistical and economic tests presented in the paper support its use. We recognize that other more sophisticated bias-correction methods could produce better results.

2.5. Interpreting expected volatility

The matrix logarithmic volatility model has the disadvantage that the estimated coefficients cannot be interpreted directly as the effect of the variable on the specified element of the realized volatility. This results from the non-linear relationship between particular elements of $\widehat{V}_t$ and $\widehat{A}_t$. However, derivatives of the estimated covariance matrix $\widehat{V}_t$ with respect to the elements of the factor model can be easily obtained (Najfeld and Havel, 1995; Mathias, 1996).

Let $\widehat{A}(z)$ be the $P \times P$ expected conditional covariance matrix from the third step (3) of our estimation procedure, where we consider the matrix to be a function of a particular forecasting variable, say $z \in Z$. The matrix of the element-by-element derivatives of $\widehat{A}(z)$ with respect to z, $\frac{\mathrm{d}\widehat{A}(z)}{\mathrm{d}z}$, is calculated using the estimated coefficients from our factor model (7). The $P \times P$ matrix of the derivatives of the actual volatilities with respect to z,

$$\frac{\mathrm{d}}{\mathrm{d}z}\widehat{V}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\text{expm}\left(\widehat{A}(z)\right),$$

can be extracted from the upper $P \times P$ right block of the following $2P \times 2P$ matrix:

$$\begin{bmatrix} \text{expm}\left(\widehat{A}(z)\right) & \frac{\mathrm{d}}{\mathrm{d}z}\text{expm}\left(\widehat{A}(z)\right) \\ 0 & \text{expm}\left(\widehat{A}(z)\right) \end{bmatrix} = \text{expm}\begin{bmatrix} \widehat{A}(z) & \frac{\mathrm{d}\widehat{A}(z)}{\mathrm{d}z} \\ 0 & \widehat{A}(z) \end{bmatrix}, \tag{10}$$

where 0 is a $P \times P$ matrix of zeros. Eq. (10) allows one to interpret of the impact forecasting variables on the realized covariance matrix, even though the estimation occurs in the matrix-log space.

We can calculate either the average impact across the entire sample, or the conditional impact at a point in time. For example, let $\widehat{A}_t(z_t)$ be the estimated realized log-space volatilities for day t. To find the response of the expected covariance matrix to the forecasting variable z_t, we need to calculate the derivative $\frac{\mathrm{d}}{\mathrm{d}(z_t)}\text{expm}\left(\widehat{A}_t(z_t)\right)$. Given our two-factor model as defined in (7), the matrix of element-by-element derivatives, $\frac{\mathrm{d}\widehat{A}_t(z_t)}{\mathrm{d}(z_t)}$, is easily obtained. Plugging $\frac{\mathrm{d}\widehat{A}_t(z_t)}{\mathrm{d}(z_t)}$ into (10) yields the matrix $\frac{\mathrm{d}\widehat{V}_t(z_t)}{\mathrm{d}(z_t)}$. We can then calculate the time-varying elasticity

$$\varepsilon\,(i,j,z,t) \equiv \frac{\mathrm{d}\widehat{V}_t^{(i,j)}(z_t)}{\mathrm{d}(z_t)}\frac{\sigma\,(z_t)}{\widehat{V}_{t(i,j)}}, \tag{11}$$

which represents the per cent increase in the (i,j)th element of $\widehat{V}_t$ due to a one standard deviation shock in the forecasting variable, $\sigma\,(z_t)$, at time t.[8] We can therefore examine how the elasticity of a particular equity market variance or covariance changes over time in response to changes in the forecasting variables. In our results below, we show that there is a significant time variation in the elasticities.

3. Data

3.1. Realized volatility

We construct our realized covariance matrices from two data sets: the Institute for the Study of Securities Markets' (ISSM) database and the Trades and Quotes (TAQ) database. Both data sets contain continuously-recorded information on stock quotes and trades for securities listed on the New York Stock Exchange (NYSE). The ISSM database provides quotes from January 1988 through December 1992 while the TAQ database provides quotes from January 1993 through December 2002.[9]

Value-weighted portfolio returns are created by assigning stocks to one of five size-sorted portfolios based on the prior month's ending price and shares outstanding. Our choice of portfolios is partially motivated by an interest to see if the systematic components of conditional volatility are common across the size portfolios. We use the CRSP database to obtain shares outstanding and prior month ending prices.[10]

Once we have our time series of high-frequency portfolio returns, we construct our measure of realized covariance matrices using the approach of Hansen and Lunde (2006), who recommend a Newey and West (1987) type extension to the usual realized volatility construction.[11] They note the potential bias in calculating variances if the serially autocorrelated nature of the data is ignored. Our data likely suffers from this problem as the portfolios of smaller stocks will include securities that are more illiquid than stocks in the larger quintiles. The illiquidity of small stocks suggests that price and volatility responses to information shocks may take more time to be incorporated, leading to time series autocorrelation in the high-frequency returns.[12]

The summary statistics of the log-space volatility matrix $A_t = \text{logm}(V_t)$ are shown in Table 1. Many of the skewness coefficients are close to 0 while the kurtosis statistics are close to 3.00. Although all of the Jarque–Bera statistics reject the null of normally distributed data, the test statistic values (not reported) have decreased quite dramatically relative to their values for the series in V_t. Thus, taking the matrix logarithm of multivariate realized volatility results in series that are much closer to being normally distributed. This parallels the univariate finding of Andersen et al. (2001).

3.2. Forecasting variables

Our goal in this paper is to compare alternative models of the conditional covariance matrix. While all of the models use the latent factor form given in (7), they differ by the forecasting variables Z_t used. We construct four alternative models using forecasting variables that correspond to existing approaches in the literature.

The first model, labeled "MHAR-RV-BP", is a multivariate HAR model of daily realized volatility using the principal components

[8] Measuring the elasticity for the covariance elements ($i \neq j$) is problematic as the covariances can become arbitrarily small. For these elements, we therefore use $\left(\widehat{V}_{t(i,i)}\widehat{V}_{t(j,j)}\right)^{1/2}$ in the denominator of (11).

[9] The ISSM data actually begins in January 1983; however, the first four years of the data have many missing days and the necessity of a contiguous data set for our time-series analysis precludes our use of these years.

[10] We use a variety of other filters that reduces the set of securities included in our data base. For example, we exclude securities with CRSP share codes that are not 10 or 11, leading to the exclusion of preferred stocks, warrants, etc., and we only include stocks that are found in both the quotes databases (ISSM and TAQ) and CRSP.

[11] The approach of Hansen and Lunde (2006) was theoretically developed in Barndorff-Nielsen et al. (2008).

[12] We also use the procedure detailed in Hansen and Lunde (2005) to get an estimate of the covariance matrix that includes close to open, or overnight, price information.

Table 1
Summary statistics of the log-space realized covariance matrix of size-sorted stock returns. The table shows summary statistics of the log-space one-day realized covariance of stock returns on the five size- sorted NYSE, AMEX and NASDAQ portfolios. For each day in the series, the matrix logarithm of the realized volatility matrix is calculated. The summary statistics of the upper triangular elements of the resulting matrix are shown here. The table shows the: mean; median; standard deviation; the first-order autoregressive coefficient; and the skewness and kurtosis statistics. Also shown is the asymptotic marginal significance level (*P*-value) for the Jarque–Bera test of Normality. The bottom of the table presents the *QQ* test statistic of multivariate normality, multivariate skewness and multivariate kurtosis test statistics as well as their marginal significance levels.

$A(1)_{i,j}$	Mean (%)	Median (%)	Std. Dev. (%)	AR(1) statistic	Normality tests		
					Skewness statistic	Kurtosis statistic	Jarque–Bera *P*-value
$A(1)_{1,1}$	−11.880	−11.907	1.228	0.452	0.093	2.746	<0.001
$A(1)_{1,2}$	0.715	0.721	0.701	0.123	−0.064	2.894	0.110
$A(1)_{1,3}$	0.536	0.550	0.648	0.085	−0.051	2.712	0.001
$A(1)_{1,4}$	0.429	0.441	0.631	0.069	−0.125	3.047	0.006
$A(1)_{1,5}$	0.301	0.313	0.608	−0.006	−0.163	2.835	<0.001
$A(1)_{2,2}$	−12.793	−12.817	1.133	0.399	0.107	2.812	0.002
$A(1)_{2,3}$	0.987	0.994	0.737	0.172	−0.066	2.914	0.137
$A(1)_{2,4}$	0.708	0.724	0.638	0.079	−0.117	2.862	0.003
$A(1)_{2,5}$	0.454	0.477	0.598	−0.007	−0.147	2.857	<0.001
$A(1)_{3,3}$	−13.099	−13.139	1.157	0.450	0.111	2.678	<0.001
$A(1)_{3,4}$	1.160	1.191	0.684	0.111	−0.182	2.952	<0.001
$A(1)_{3,5}$	0.752	0.778	0.615	0.029	−0.185	2.802	<0.001
$A(1)_{4,4}$	−13.068	−13.114	1.109	0.415	0.245	3.062	<0.001
$A(1)_{4,5}$	1.406	1.441	0.652	0.074	−0.198	2.800	<0.001
$A(1)_{5,5}$	−11.450	−11.511	1.187	0.420	0.212	2.924	<0.001
Multivariate normality test			*QQ*	Skewness	Kurtosis		
			−6	1748	−6		
			0.001	<0.001	0.001		

of bi-power covariation as predictors. In our estimations, we find multicollinearity among all of the principal components. This is not surprising as the bi-power covariations are measuring the volatility of the equity market over different holding periods. We therefore use only the first principal component of the 5-day series ($a^{BP}(5, 1)$) and the first three principal components of the 20-day series ($a^{BP}(20, 1)$, $a^{BP}(20, 2)$, $a^{BP}(20, 3)$) as regressors. Intuitively, by including the 5-day measure we hope to capture the higher-frequency volatility variation. We found the 5-day measures were more stable than the 1-day measures, likely due to reduced estimation error. Inclusion of the 20-day measures follows from HAR models where lagged longer horizon volatility measures are included as predictors. We show below that these four series do a good job in capturing the forecastability of the 5×5 matrix. Incorporating shorter-run (e.g., $d = 1$ day) volatility measures does not improve the estimates.

The second model, "MHAR-RV-BPA", adds the asymmetric response of volatility to past return shocks to the first model. A number of authors has shown that past negative returns causes higher future equity market volatility.[13] We therefore include the returns on the small and large stock portfolios when they are negative ($R_{1,t-1} < 0$ and $R_{5,t-1} < 0$, respectively).

Our third model, "MHAR-RV-X", follows Ferson (1995) and uses variables that have been shown to forecast stock returns. These include: the interest rate on a Treasury bill (*TB*), the dividend yield (*DY*), the credit spread (*CS*), and the slope of the term structure (*TS*). We also include a variable that originates in the behavioral finance literature, the scorecard (*SC*) of McQueen and Vorkink (2004), which is a function of prior market returns.

The fourth model, "MHAR-RV-BPAX", uses all of the variables from the other models.

4. Results

4.1. Model fit

We estimated our four versions of (7) with $K = 1$ factors. However, the model was rejected for each of the variable sets used. We then estimated the models with 2 factors. Table 2 presents summary statistics for the overall model fit. We present the statistics for the non-bias corrected results to show how the model does by itself. Panel A shows the J statistics that test the over-identifying restrictions (8) from the four models. None of the models are rejected at the 10% level indicating that the restrictions of a two-factor model are not too onerous. Consequently, we report all subsequent analysis using the two-factor version of our models.

One question of interest is the reduction in the explanatory power of a variable set resulting from the imposition of the two-factor structure. While a parsimonious structure is likely to be preferred in the out-of-sample tests, the in-sample fit may be reduced by the over-identifying restrictions (8). To test this, we calculate the variance ratio statistics. In Panel B of Table 2, we measure the restrictions of the factor model on the log-space volatilities, A_t. In these ratios, the numerator is the variance of the fitted values $\widehat{A}_t$ of the log-space volatilities from the factor model (7). The denominator is the variance of the fitted values from an ordinary least squares regression of the log-space volatilities on the variables of the model as in (5). The ratio thus shows how much imposing the factor structure in (8) reduces the in-sample predictive power.

In Panel C of Table 2, we measure the restrictions of the factor model on the realized volatility matrix, V_t. In addition, the calculations for these ratios vary by whether they are for diagonal or off-diagonal elements of V_t. For the diagonal elements, the numerator is the variance of the (bias-corrected) fitted values of the realized volatility from (9). The denominator is the variance of the fitted values from an ordinary least squares regression of the realized volatilities on the variables of the model. For the off-diagonal elements, we repeat the same exercise using the Fisher transforms of the estimated correlations in the numerator and the Fisher transforms of the realized correlations in the dominator. This analysis allows us to measure the effects of the factor model on both the volatilities and the correlations contained in V_t.

While some of the ratios in Panel B are occasionally quite small (e.g. 0.029 for the (1, 4) element of the A_t matrix for the MHAR-RV-BP model), most of the ratios are above 0.7. The factor structure does not appear to impose much restrictions on the dynamics of

[13] See, for example, Pagan and Schwert (1990) and Engle and Ng (1993).

Table 2
Tests of models and variance ratios. The table presents model fit statistics as well as variance ratio measures of the statistical fit of the two latent factor MHAR-RV models of the cross section of the realized stock market volatility. Panel (A) reports the J-statistic for over-identification as well as the degrees of freedom and P-value for each model. Panels (B) and (C) report the variance ratios which show how well each latent factor model captures the variation in the data. In panel (B), the numerator is the variance of the fitted values of the (i, j)th element of the one-day log-space volatilities $A(1)$ from the latent factor model, while the denominator is the variance of the fitted values of the log-space volatilities from an OLS regression of the volatility on the variables used in that model. In panel (C), the ratio uses the realized volatilities. For this panel, the off-diagonal elements are the Fisher transforms of the realized correlations.

Model:	MHAR-RV-BP	MHAR-RV-BPA	MHAR-RV-X	MHAR-RV-BPAX
(A) J statistics from over-identifying restrictions				
χ^2 statistic	32.82	48.48	41.95	72.32
df	26	52	39	117
P-value	0.167	0.613	0.344	0.999
(B) Log-space volatilities				
$A(1)_{1,1}$	0.984	0.977	1.633	1.081
$A(1)_{1,2}$	0.874	0.625	0.675	0.567
$A(1)_{1,3}$	0.896	0.445	0.102	0.347
$A(1)_{1,4}$	0.029	0.049	0.239	0.089
$A(1)_{1,5}$	0.401	0.480	0.998	0.902
$A(1)_{2,2}$	1.108	1.067	1.482	1.085
$A(1)_{2,3}$	0.832	0.907	1.438	1.142
$A(1)_{2,4}$	1.528	0.722	0.909	0.535
$A(1)_{2,5}$	0.806	0.695	0.498	0.689
$A(1)_{3,3}$	1.049	0.981	1.595	1.198
$A(1)_{3,4}$	1.149	1.030	1.410	0.493
$A(1)_{3,5}$	2.140	1.365	1.073	0.361
$A(1)_{4,4}$	1.028	0.997	1.588	1.102
$A(1)_{4,5}$	0.524	0.391	1.102	0.164
$A(1)_{5,5}$	0.961	0.973	0.905	0.955
(C) Realized volatilities				
$V(1)_{1,1}$	1.292	1.263	1.767	1.062
$V(1)_{1,2}$	1.139	0.957	1.433	0.799
$V(1)_{1,3}$	1.037	0.862	1.161	0.715
$V(1)_{1,4}$	0.752	0.721	0.520	0.495
$V(1)_{1,5}$	0.678	0.819	0.127	0.236
$V(1)_{2,2}$	1.372	1.345	1.752	1.204
$V(1)_{2,3}$	1.186	1.299	2.172	1.123
$V(1)_{2,4}$	1.305	1.289	1.669	0.449
$V(1)_{2,5}$	1.361	1.458	0.591	0.101
$V(1)_{3,3}$	1.303	1.288	1.688	1.189
$V(1)_{3,4}$	1.305	1.320	0.772	0.240
$V(1)_{3,5}$	4.623	2.830	1.164	0.353
$V(1)_{4,4}$	1.219	1.180	1.576	1.006
$V(1)_{4,5}$	1.409	1.038	1.307	0.251
$V(1)_{5,5}$	1.067	1.079	0.990	0.795

the log-space volatilities. The results are just as strong for the realized volatilities in Panel C. Here the average ratios are above 1.00 for the volatilities in all four models suggesting that the factor model combined with the non-linear transformation in the matrix exponential function allows our model to capture more of the variation than a simple regression can. The results in Panel C do show that the models with more variables have greater difficulty in capturing the dynamics of the time-varying correlations as the off-diagonal elements of the matrix (the Fisher transforms of the realized correlations) do have lower values. Overall, the results in Table 2 suggest that the two factor representation is not too stringent and that much of the variation in expected variances and correlations is captured by our models.

4.2. Estimated coefficients

Table 3 presents the estimates of the β coefficients from the model that uses all of the forecasting variables (MHAR-RV-BPAX). The coefficients are arranged according to the elements of the A_t matrix that they correspond to. The elements that are normalized for identification are in the upper left corners of both factors.

Table 3
Beta coefficients on the implied volatility factors. The table shows the β coefficients from the two latent factor MHAR-RV models of the cross section of realized stock market volatility. The latent volatility factors are the linear combination of the variables given in Table 4. The beta coefficients for the first two elements of the log-space volatility matrix have been normalized for identification. The model is estimated separately for each set of forecasting variables by Generalized Method of Moments. The coefficients for the MHAR-RV-BPAX model described in Table 4 are presented here. The coefficients for the models using other variables are similar to the values shown here. The Newey–West standard errors (in parentheses) are asymptotically robust to general forms of heteroskedasticity and autocorrelation.

	$A(1)_{i,1}$	$A(1)_{i,2}$	$A(1)_{i,3}$	$A(1)_{i,4}$	$A(1)_{i,5}$
First factor					
$A(1)_{1,j}$	1.000				
$A(1)_{2,j}$	0.000	0.729			
		(0.027)			
$A(1)_{3,j}$	−0.075	0.042	0.699		
	(0.008)	(0.019)	(0.043)		
$A(1)_{4,j}$	−0.034	0.089	0.108	0.545	
	(0.008)	(0.008)	(0.011)	(0.042)	
$A(1)_{5,j}$	−0.071	−0.007	0.048	0.043	−0.025
	(0.007)	(0.008)	(0.011)	(0.014)	(0.058)
Second factor					
$A(1)_{1,j}$	0.000				
$A(1)_{2,j}$	1.000	2.159			
		(0.159)			
$A(1)_{3,j}$	0.449	1.962	3.719		
	(0.048)	(0.116)	(0.245)		
$A(1)_{4,j}$	0.172	0.080	0.413	3.790	
	(0.045)	(0.050)	(0.060)	(0.264)	
$A(1)_{5,j}$	−0.186	−0.286	−0.520	−0.611	5.686
	(0.041)	(0.048)	(0.059)	(0.074)	(0.353)

Each cell presents the estimated coefficient from the GMM estimation procedure and the Newey and West (1987) standard error.[14]

In the first factor, the loadings for the first four diagonal elements are significant and positive while those for the off-diagonal elements are much smaller. In the second factor, the estimated coefficients are large and significant for the diagonal elements, while the off-diagonal elements are smaller, but mostly significant. All of the log-space volatilities have a significant loading on at least one of the two factors. The overall significance of the coefficients show that the linear combinations of the forecasting variables (θZ_{t-1}) are able to forecast elements of the A_t matrix.

The estimated θ coefficients for the two factors in all of the models are shown in Table 4. The top part of Panel A presents the coefficients for the basic multivariate HAR model (MHAR-RV-BP). Most of the coefficients on the principal components of the lagged 5 and 20-day log-space bi-power covariation are significant in both factors. The difference in signs on one of them between the two factors suggests that the factors are picking up different elements of long-run volatility. However, it is difficult to say much more about the influence of the variables on realized volatility due to the highly non-linear nature of the model: the θ coefficients in the two factors interact with each other as well as the β coefficients in (7). For example, it is not necessarily true that the second principal component of the 20-day log-space bi-power covariation (with a coefficient of 0.520 in the first factor) has a larger degree of explanatory power for realized volatilities than does the first principal component of the 5-day volatility (with a coefficient of 0.165 on the first factor). The elasticities presented below show the ultimate impact of any particular forecasting variable on the volatilities.

Panel B presents the coefficients for the model that includes the (asymmetric) effects of past stock returns (MHAR-RV-BPA).

14 Only the results for the MHAR-RV-BPAX model are shown. The coefficients for the other models are similar and are available by request.

Table 4
Coefficients on and elasticities of the forecasting variables. The table shows the θ coefficients on the forecasting variables in the two latent factor MHAR-RV models of the cross section of realized stock market volatility along with the mean values of the elasticities associated with the forecasting variables. The variables for the MHAR-RV-BP model (Panel A) are: the lagged value of the first principal component of the 5 day log-space bi-power variation matrix ($a^{BP}(5, 1)$) and the lagged values of the first three principal components of the 20 day log-space bi-power variation matrices ($a^{BP}(20, 1)$, $a^{BP}(20, 2)$, and $a^{BP}(20, 3)$). In the MHAR-RV-BPA model (Panel B) the variables are those in the MHAR-RV-BP model plus the lagged values of the negative returns on the small and large stock portfolios ($R_1 < 0$ and $R_5 < 0$). The variables in the MHAR-RV-X model (Panel C) are: the lagged interest rate on a short-term Treasury bill (*TB*); the lagged credit spread (*CS*); the lagged 'scorecard' measure of the sensitivity of the market to news (*SC*); the lagged dividend yield (*DY*); and the lagged spread between the long Government bond and the 3 month Treasury bill (*TS*). The MHAR-RV-BPAX model (Panel D) includes all of the variables in the other models. The model is estimated separately for each set of forecasting variables by Generalized Method of Moments. The Newey–West standard errors (in parentheses) are asymptotically robust to general forms of heteroskedasticity and autocorrelation. The mean value of the elasticities $\varepsilon(i, j, z)$ associated with the (i, j)th element of the realized volatility matrix V and the forecasting variables z are also presented.

	$a^{BP}(5,1)$	$a^{BP}(20,1)$	$a^{BP}(20,2)$	$a^{BP}(20,3)$	$R_1 < 0$	$R_5 < 0$	*TB*	*CS*	*TS*	*SC*	*DY*
(A) MHAR-RV-BP											
θ	0.165	0.245	0.520	0.083							
(std err)	(0.027)	(0.028)	(0.038)	(0.055)							
θ	0.049	0.011	−0.138	0.045							
(std err)	(0.008)	(0.007)	(0.023)	(0.012)							
$\varepsilon(1,1,z)$	0.392	0.370	0.116	0.036							
$\varepsilon(1,5,z)$	0.305	0.185	−0.105	0.033							
$\varepsilon(5,5,z)$	0.448	0.176	−0.337	0.053							
(B) MHAR-RV-BPA											
θ	0.099	0.304	0.558	0.068	−12.229	5.586					
(std err)	(0.022)	(0.023)	(0.028)	(0.047)	(1.170)	(1.095)					
θ	0.029	0.033	−0.104	0.056	−0.111	−1.899					
(std err)	(0.005)	(0.007)	(0.017)	(0.009)	(0.312)	(0.348)					
$\varepsilon(1,1,z)$	0.242	0.519	0.148	0.041	−0.142	0.003					
$\varepsilon(1,5,z)$	0.191	0.297	−0.104	0.043	−0.052	−0.050					
$\varepsilon(5,5,z)$	0.277	0.325	−0.354	0.072	−0.018	−0.119					
(C) MHAR-RV-X											
θ							17.782	234.593	3.338	−1.293	−1.162
(std err)							(4.211)	(19.096)	(5.967)	(0.092)	(0.199)
θ							0.026	14.080	−1.412	−0.036	−0.233
(std err)							(0.628)	(3.786)	(0.703)	(0.025)	(0.038)
$\varepsilon(1,1,z)$							0.285	0.534	0.011	−0.411	−0.489
$\varepsilon(1,5,z)$							0.151	0.304	−0.005	−0.225	−0.315
$\varepsilon(5,5,z)$							0.140	0.340	−0.035	−0.230	−0.443
(D) MHAR-RV-BPAX											
θ	0.051	0.316	0.624	0.433	−17.873	10.032	4.884	21.062	2.186	−0.183	−0.195
(std err)	(0.016)	(0.017)	(0.019)	(0.043)	(0.961)	(0.908)	(1.190)	(5.218)	(1.375)	(0.041)	(0.055)
θ	0.030	0.001	−0.085	0.035	0.400	−2.906	−0.779	14.736	−1.094	0.058	−0.145
(std err)	(0.004)	(0.005)	(0.009)	(0.008)	(0.249)	(0.291)	(0.245)	(1.315)	(0.306)	(0.009)	(0.014)
$\varepsilon(1,1,z)$	0.162	0.414	0.221	0.095	−0.189	0.027	0.043	0.102	−0.003	−0.013	−0.151
$\varepsilon(1,5,z)$	0.142	0.159	−0.006	0.049	−0.062	−0.044	−0.006	0.093	−0.022	0.027	−0.139
$\varepsilon(5,5,z)$	0.239	0.123	−0.153	0.057	−0.030	−0.123	−0.041	0.159	−0.049	0.072	−0.240

The coefficients on the lagged principal components of bi-power covariation are similar in size and significance to their values in the first model. In addition an interesting pattern in the asymmetric response of volatility to past negative returns emerges. The coefficient on lagged negative returns on small stocks is negative in the first factor and not significant in the second factor. The coefficient on negative returns on large stocks has an opposite sign in the two factors. The elasticities presented below will show how the effects net out.

The coefficients on the variables usually used to forecast stock returns, model 'MHAR-RV-X', are presented in Panel C. The coefficient on the lagged short-term interest rate is significant in the first factor while those on the lagged credit spread are large and significant in both factors. The coefficients on the lagged term spread are insignificant. The coefficients on the scorecard and dividend yield variables are negative and mostly significant.

Panel D presents the coefficients for the model that includes both the lagged volatility variables as well as the standard return forecasting variables (MHAR-RV-BPAX). The results for this model illustrate an important point in determining the effects of various economic factors on volatility: including lagged volatility as a variable to determine the true influence of all variables on the volatility proves to be important. This can be seen by examining the coefficient on the credit spread which shrinks in size and becomes insignificant in the first factor. In contrast, the short-term interest rate becomes significant in the second factor. The autoregressive nature of the realized volatility implies that care must be taken when regressing volatility on lagged predictive variables which are themselves persistent.

4.3. Estimated elasticities

The lower part of each panel in Table 4 reports the average elasticities $\varepsilon(i, j, z)$ calculated using (11). We calculate the elasticities for each day t in our sample and for each forecasting variable z and report the average over the entire time period. While we calculate the elasticities for each element (i, j) of the estimated volatility matrix $\widehat{V}_t$, we present only the results for the small stock variance ($\varepsilon(1, 1, z)$, top line), the covariance between small and large stocks ($\varepsilon(1, 5, z)$, middle line), and the variance of the large stock portfolio ($\varepsilon(5, 5, z)$, bottom line) for brevity. The results for the other elements of $\widehat{V}$ are available on request.[15]

Panel A of Table 4 gives the elasticities for the MHAR-RV-BP model. The first principal components of the lagged weekly and monthly volatilities have a positive effect on all of the realized volatilities. These components appear to capture the overall level of volatility in the market. In contrast, the second principal factor of the monthly bi-power covariation has an asymmetric effects on volatility. The elasticities change sign depending on the particular element of $\widehat{V}$. For example, changing the value of the second

15 The Newey and West (1987) standard errors on the average elasticities result in P-values less than 0.002 in all cases.

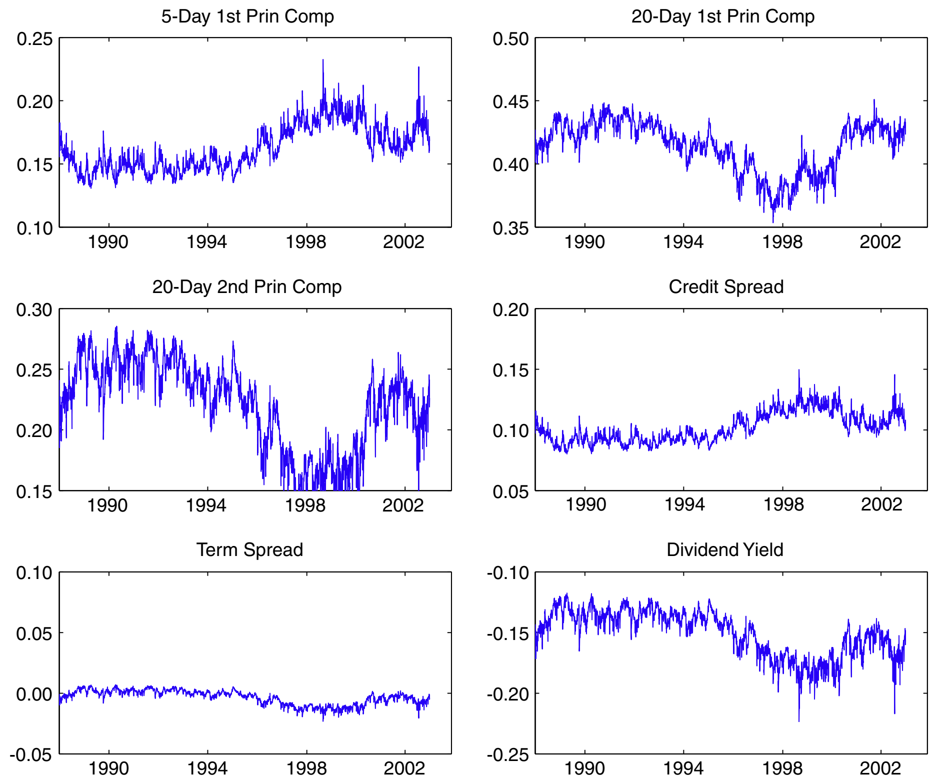

Fig. 1. Model elasticities.

principal component by a one standard deviation shock would cause tomorrow's large stock volatility to decrease by 33.7% while small stock volatility would increase by 11.6%. We find monotonic decreases in the elasticities associated with this variable as we move from the small stock volatility towards the large stock portfolio, including those portfolio results not reported. The third principal component has a small influence on realized volatilities.

Panel B presents the results for the second model which includes the principal components and the lagged stock returns (MHAR-RV-BPA). The elasticities on the principal components do not change much from their values for the first model. The elasticities on the two negative return variables ($R_{1,t-1} < 0$ and $R_{5,t-1} < 0$) are relatively large ($>|0.10|$) for their own volatility (i.e., negative shocks to the returns of small (large) stocks increase the volatility of the small (large) stock portfolio). The shocks do not have a large affect on the returns of other portfolios.

This latter finding is in contrast to some findings in the rest of the literature where shocks to large returns often have information for small firms. For example, Lo and MacKinlay (1990) show that the returns of large stocks lead those of small stocks, while Conrad et al. (1991) show that volatilities of large stocks lead those of small stocks. However, the Hansen and Lunde (2006) estimator of the realized volatility matrix V_t accounts for the interactions of the volatility between large and small stock portfolios. If new information is incorporated into the prices of large stocks first, then the returns and variances of large stocks would lead the returns and variances of small stocks. This latter effect would be captured by our estimator of the realized covariance matrix as lags of large stock volatility are used in constructing current small stock volatility. This implies that there would be little return asymmetries to be captured by any right-hand-side variable.

The elasticities on the variables typically used to model expected stock returns are shown in Panel C. An increase in the short-term interest rate or the credit spread causes the elements of the conditional covariance matrix to increase. An increase in the scorecard or dividend yield causes the elements to decrease. The slope of the term structure has a small effect.

Panel D presents the elasticities for the model that includes all of the variables. The one large change from previous smaller models is that the elasticities on the variables used in the MHAR-RV-X model greatly decrease in magnitude. For example, the impact of a change in the short-term interest rate almost completely disappears. The only variable from the third model to have a relatively large ($>|0.10|$) elasticity on all of the volatilities is the dividend yield. This, once again, shows the importance of including lagged volatility in the models. In addition, the elasticities on the return shocks are approximately the same in both the second and fourth models. This is in contrast to Kroner and Ng (1998) who find that different models give very different news impact surfaces showing the effects of past return shocks on current volatility.

While the averages of the estimated elasticities reveal a large influence for certain variables on volatility, we also find it instructive to examine the time series properties of the elasticities. Fig. 1 presents the estimated elasticities $\varepsilon(i, j, z, t)$ for six of the forecasting variables in the MHAR-RV-BPAX model. Only the elasticities for $(i, j) = (1, 1)$, the volatility of the small stock portfolio, are shown. The graphs reveal a considerable variation over time in some of the elasticities while others are more stable. For example, the elasticities on the three principal components (the first factors on the lagged 5-day and the two factors of the 20-day bi-power covariation) show some variation over the sample period. The economic variables have different paths: while the elasticities on the credit spread and term spread do not vary a lot, the elasticity on the dividend yield shows a significant time variation.

5. Conclusions

This paper has introduced a new model for the realized covariance matrix of returns. The model is parsimonious, guarantees a positive-definite covariance matrix, and does not require parameter constraints to be imposed. The model allows a number of variables to forecast the covariance matrix, yet restricts the number of factors in the estimation process. In addition, time-varying elasticities can be calculated that show the extent to which a per cent shock to the forecasting variable influences any particular element of the realized covariance matrix.

The model is applied to the covariance matrix of daily realized stock returns over the 1988 to 2002 period. Four alternative sets of forecasting variables are tested. Lagged principal components of realized weekly and monthly bi-power covariation have a strong predictive power. Some variables (e.g., the dividend yield) that forecast stock market returns also forecast the cross section of volatility. The estimated elasticities show that the influence of some of these variables changes over the sample period while others are more stable.

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