7 Three-level systems

In this section, we will extend our treatment of atom-light interactions to situations with more than one atomic energy level, and more than one (independent) coherent driving field. As in our treatment of the two-level system, it will be convenient first to address coherent effects using the Schrödinger equation, and subsequently to consider the effects of spontaneous emission using the optical Bloch equations.

7.1 Schrödinger picture

7.1.1 Level schemes

Unlike its two-level counterpart, the three-level system presents choices in the arrangement of the energy levels and driving fields. The three main classes of level scheme are shown in Fig. 5: the Λ, V, and *ladder* (or *cascade*) schemes. Of these, the Λ scheme exhibits the greatest range of interesting physics without too much mathematical complexity; we will focus on the nondegenerate case in what follows.

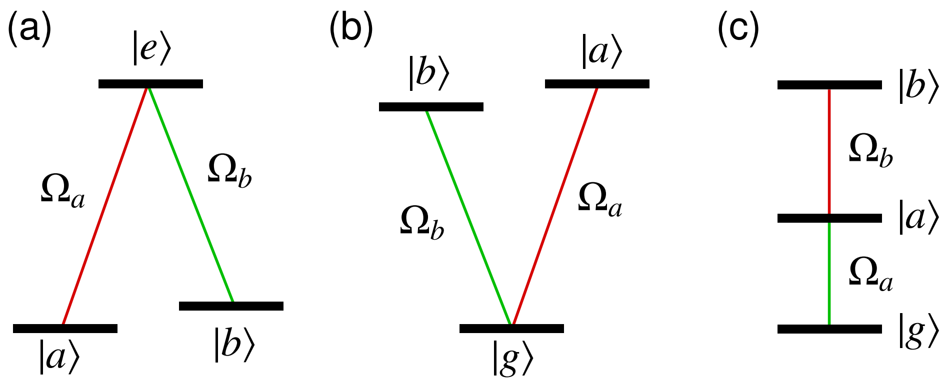

Figure 5: Common level schemes for a three-level system with two driving fields: (a) Λ, (b) V, and (c) *ladder*. Note that levels $|a\rangle$ and $|b\rangle$ in the Λ and V schemes are potentially degenerate.

7.1.2 Hamiltonian

In analogy with our previous treatment of the two-level system, we can quite straightforwardly construct the Hamiltonian for the Λ system as

$$
\begin{aligned}
\hat{H}_\Lambda = \hbar\omega_e^{(0)}|e\rangle\langle e| + \hbar\omega_a^{(0)}|a\rangle\langle a| + \hbar\omega_b^{(0)}|b\rangle\langle b| \\
+ \hbar\Omega_a \cos(\omega_a t)(|a\rangle\langle e| + |e\rangle\langle a|) \\
+ \hbar\Omega_b \cos(\omega_b t)(|b\rangle\langle e| + |e\rangle\langle b|)\,,
\end{aligned} \tag{107}
$$

where we have defined the three energy levels by $\omega^{(0)}_{\{a,b,g\}}$, and the laser frequencies by $\omega_{\{a,b\}}$, with Rabi frequencies[48] $\Omega_{\{a,b\}}$. As in the two-level case, an appropriate choice of time-dependent expansion coefficients yields a time-dependent Schrödinger equation with time-independent Hamiltonian of the form

[48] Note that, as previously, we assume a real Rabi frequency to simplify later expressions. The generalization to complex Rabi frequency is straightforward.

$$
i\frac{d}{dt}\begin{pmatrix} c_a \\ c_e \\ c_b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \Omega_a & 0 \\ \Omega_a & -2\Delta & \Omega_b \\ 0 & \Omega_b & -2\delta \end{pmatrix}\begin{pmatrix} c_a \\ c_e \\ c_b \end{pmatrix}. \tag{108}
$$

Note that here we have defined one detuning, $\Delta = \omega_a - (\omega_e^{(0)} - \omega_a^{(0)})$, similarly to the two-level case. However, we have defined the second detuning δ to *include* the first: $\delta = \Delta - \omega_b + (\omega_e^{(0)} - \omega_b^{(0)})$. This makes δ the net detuning of the two-photon transition from $|a\rangle$ to $|b\rangle$, giving a simpler matrix[49].

[49] There are *numerous* notational conventions for three-level systems. It is also common to define $\Delta_a = \omega_a - (\omega_e^{(0)} - \omega_a^{(0)})$ and $\Delta_b = \omega_b - (\omega_e^{(0)} - \omega_b^{(0)})$, in which case $\delta = \Delta_a - \Delta_b$. As we will see later, one also often sets driving field a to be the (weak) **probe** and b the (strong) **coupling** fields, leading to the notation convention Δ_p and $\Delta_c \ldots$

7.2 Far-detuned Raman processes

7.2.1 Equivalent two-level system for large detuning

An important class of physics in the three-level system arises when the detuning with respect to the excited state, Δ, is very large compared to the two-photon detuning δ and the Rabi frequencies Ω_a and Ω_b. In this case one can take the formal solution[50] for $c_e(t)$,

[50] Assuming $c_e(0) = 0$ for convenience, without loss of generality.

$$
c_e(t) = -i\int_0^t dt' e^{i\Delta(t-t')}\left[\Omega_a c_a(t') + \Omega_b c_b(t')\right]\,, \tag{109}
$$

and assume that the term in brackets varies slowly compared to the exponential and can be taken out of the integral[51] to give

[51] If we did this more carefully, it would be termed **adiabatic elimination** of the excited state.

$$
c_e(t) \approx \left(\frac{1 - e^{i\Delta t}}{\Delta}\right)\left[\Omega_a c_a(t') + \Omega_b c_b(t')\right]\,. \tag{110}
$$

Substituting this into the time-dependent Schrödinger equation [Eq. (108)] reduces it to an effective two-level form

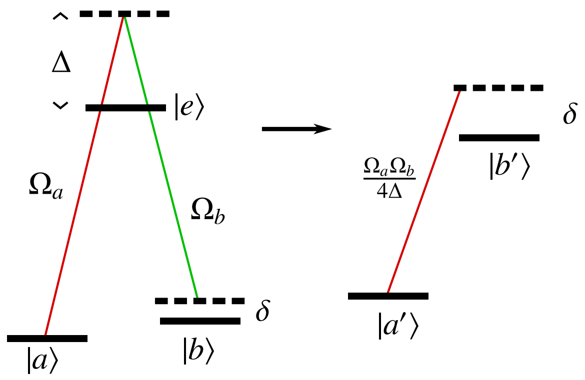

Figure 6: Effective two-level description of a far-detuned Raman process.

$$i\frac{d}{dt}\begin{pmatrix} c_a \\ c_b \end{pmatrix} = \frac{1}{4\Delta}\begin{pmatrix} \Omega_a^2 & \Omega_a\Omega_b \\ \Omega_a\Omega_b & \Omega_b^2 - 4\Delta\delta \end{pmatrix}\begin{pmatrix} c_a \\ c_b \end{pmatrix}, \tag{111}$$

with energy levels $\Omega^2_{\{a,b\}}/4\Delta$ and Rabi frequency $\Omega_a\Omega_b/4\Delta$ [See Fig. 6].

7.2.2 Dark states

For zero two-photon detuning δ, the matrix in the effective two-level Schrödinger equation, Eq. (111), can be written as

$$\frac{1}{4\Delta} = \begin{pmatrix} \Omega_a \\ \Omega_b \end{pmatrix}(\Omega_a\ \Omega_b)\ . \tag{112}$$

In this form, it is straightforward to show that the eigenvectors, and their respective eigenvalues λ, are

$$\mathbf{u}_0 = \begin{pmatrix} -\Omega_b \\ \Omega_a \end{pmatrix},\ \lambda_0 = 0\,; \quad \mathbf{u}_1 = \begin{pmatrix} \Omega_a \\ \Omega_b \end{pmatrix},\ \lambda_1 = \frac{\Omega_a^2 + \Omega_b^2}{4\Delta}\,. \tag{113}$$

Of these, the first eigenvector $\mathbf{u}_0$ is especially notable because it's zero eigenvalue makes it a **dark state**: it no longer directly couples to the light field.

This dark state is in fact preserved if we diagonalize the full three-level Hamiltonian from Eq. (108), to obtain eigenvectors

$$\mathbf{u}_0 = \frac{1}{\Omega_{\text{eff}}}\begin{pmatrix} -\Omega_b \\ 0 \\ \Omega_a \end{pmatrix}; \quad \mathbf{u}_\pm = \frac{1}{\sqrt{4\Omega_\pm^2 + \Omega_{\text{eff}}^2}}\begin{pmatrix} \Omega_a \\ 2\Omega_\pm \\ \Omega_b \end{pmatrix}, \tag{114}$$

where $\Omega_{\text{eff}} = \sqrt{\Omega_a^2 + \Omega_b^2}$ and $\Omega_\pm = [-\Delta \pm (\Omega_{\text{eff}}^2 + \Delta^2)^{1/2}]/2$.

7.2.3 STIRAP

The fact that the nature of the dark state can be manipulated using the two Rabi frequencies Ω_a and Ω_b forms the basis for coherent state manipulation techniques such as Stimulated Raman Adiabatic Passage (STIRAP). The dark state can be expressed as

$$|u_0\rangle = \frac{\Omega_b|a\rangle - \Omega_a|b\rangle}{\Omega_{\text{eff}}}. \tag{115}$$

This expression has the interesting limits of $|u_0\rangle \to |a\rangle$ as $\Omega_a \to 0$ (for $\Omega_b \neq 0$), and $|u_0\rangle \to -|b\rangle$ as $\Omega_b \to 0$ (for $\Omega_a \neq 0$).

These limits allow one to adiabatically move the dark state between $|a\rangle$ and $|b\rangle$ by sufficiently slow tuning of the coupling. A typical STIRAP pulse sequence is shown in Fig. 7. Atoms initially in state $|a\rangle$ adiabatically follow the dark state as the coupling Ω_b is increased. The coupling Ω_a is subsequently increased, and Ω_a reduced to zero, and the dark state becomes $|b\rangle$. Finally, all coupling is reduced to zero, leaving the atoms in state $|b\rangle$.

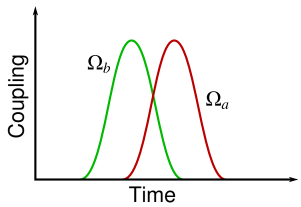

Figure 7: A typical STIRAP pulse sequence.

7.3 EIT and Autler-Townes splitting

Another interesting effect in the three-level system that is connected to the presence of dark states is electromagnetically-induced transparency (EIT). With reference to the Λ system defined by the effective Hamiltonian of Eq. (108), this effect arises when Ω_b represents a strong coupling field and Ω_a a weak probe field. You will investigate this effect, including the effects of spontaneous emission as described by the three-level optical Bloch equations, in Exercise 6.

Exercises 6

EIT and the three-level optical Bloch equations

1. Show that the dark and bright states for the Λ system given in the notes [Eq. (114)] diagonalize the effective Hamiltonian of Eq. (108).

   (*2 marks*)

2. In the Λ EIT configuration, we will use the notation "p" for the weak probe transition from $|a\rangle$ to $|e\rangle$, and "c" for the strongly coupled transition from $|b\rangle$ to $|e\rangle$. This gives effective Hamiltonian

$$H = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega_p & 0 \\ \Omega_p & -2\Delta_p & \Omega_c \\ 0 & \Omega_c & -2(\Delta_p - \Delta_c) \end{pmatrix}.$$

   Show that the optical Bloch equations for the coherences ρ_{ea}, ρ_{ba}, and ρ_{be} obey the equations

$$\begin{aligned}
\frac{d\rho_{ea}}{dt} &= \frac{i}{2}\Omega_p(\rho_{ee} - \rho_{aa}) + (i\Delta_p - \gamma_{ea})\rho_{ea} - \frac{i}{2}\Omega_c\rho_{ba}\,, \\
\frac{d\rho_{ba}}{dt} &= \frac{i}{2}\Omega_p\rho_{be} + [i(\Delta_p - \Delta_c) - \gamma_{ba}]\rho_{ba} - \frac{i}{2}\Omega_c\rho_{ea}\,, \\
\frac{d\rho_{be}}{dt} &= \frac{i}{2}\Omega_p\rho_{ba} - (i\Delta_c + \gamma_{be})\rho_{be} - \frac{i}{2}\Omega_c(\rho_{ee} - \rho_{bb})\,,
\end{aligned}$$

   where $\gamma_{ij} = (\Gamma_i + \Gamma_j)/2$. Explain the reasoning by which you arrive at the correct form for the dissipative terms.

   (*6 marks*)

3. Show that, for small Ω_p, the steady-state solution for ρ_{ba} is approximately given by

$$\rho_{ba} \approx \frac{1}{2}\left[\frac{i\Omega_c}{i(\Delta_p - \Delta_c) - \gamma_{ba}}\right]\rho_{ea}\,. \tag{116}$$

   (*2 marks*)

4. Using the above result, and assuming that the majority of the population remains in level $|a\rangle$, derive an expression for the steady-state probe coherence ρ_{ea}. Your expression should not include any other density matrix elements.

   (2 *marks*)

5. As shown in Section 6 for the two-level atom, the susceptibility of an atomic medium with respect to a coherent light source, χ, is directly proportional to the relevant coherence as determined by the optical Bloch equations (in the two-level case, $\chi \propto \rho_{ge}$). Using the same approach, derive expressions for the real and imaginary parts of the susceptibility of the three-level medium with respect to the EIT probe beam.

   (2 *marks*)

6. Using your final expression, plot graphs of the real and imaginary parts of the probe susceptibility as a function of Δ_p/Γ_e for the following coupling Rabi frequencies: $\Omega_c/\Gamma_b = 0$, 1/2, 2, 5. You should set $\Gamma_e/\Gamma_b = 10^4$ (i.e., the decay rate of the excited state is much faster than those for the lower levels), and $\Delta_c = 0$ (i.e., coupling beam on-resonance). Scale your real and imaginary susceptibilities relative to their values when $\Delta_p = \Omega_c = 0$.

   (6 *marks*)