

# 33-445 Formula Sheet

## Operator Properties

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \quad \Delta \hat{A} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} \quad \hat{U} \hat{U}^\dagger = 1 \quad [\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A} \quad \Delta \hat{A} \Delta \hat{B} \geq \langle \frac{[\hat{A}, \hat{B}]}{2i} \rangle \quad e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!}$$

## Vector Properties

$$\langle \psi | \psi \rangle = 1 \langle a_i | a_j \rangle = \delta_{ij} \hat{A} | \psi \rangle = \langle \psi |^\dagger \hat{A}^\dagger | \psi \rangle = \sum_n c_n | a_n \rangle = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \langle \psi | = \sum_n c_n^\dagger \langle a_n | = (c_1^\dagger \quad \dots \quad c_n^\dagger)$$

## All spins

$$\vec{n} = \sin \theta \cos \varphi \vec{i} + \sin \theta \sin \varphi \vec{j} + \cos \theta \vec{k}$$

$$z : \theta = 0, \varphi = 0 \quad x : \theta = \frac{\pi}{2}, \varphi = 0 \quad y : \theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2} \quad [\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \quad [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x \quad [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \quad \hat{J}_- = \hat{J}_x - i\hat{J}_y \quad \hat{J}_+ = \hat{J}_-^\dagger \quad \hat{J}_- = \hat{J}_+^\dagger$$

$$\hat{J}_+ | j, m \rangle = \hbar \sqrt{j(j+1) - m(m+1)} | j, m+1 \rangle \quad \hat{J}_- | j, m \rangle = \hbar \sqrt{j(j+1) - m(m-1)} | j, m-1 \rangle$$

## Spin $\frac{1}{2}$ (Electrons)

$$| +n \rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\varphi} \end{pmatrix} \quad | -n \rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{-i\varphi} \end{pmatrix} \quad \hat{S}_n = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix}$$

## Spin 1 (Photons)

$$| X' \rangle = \cos \phi | X \rangle + \sin \phi | Y \rangle \quad | R \rangle = \frac{1}{\sqrt{2}}(| X \rangle + i | Y \rangle) \quad \hat{R} \phi \vec{k} | R \rangle = e^{i\phi} | R \rangle \quad \hat{J}_z | R \rangle = \hbar | R \rangle$$

$$| Y' \rangle = -\sin \phi | X \rangle + \cos \phi | Y \rangle \quad | L \rangle = \frac{1}{\sqrt{2}}(| X \rangle - i | Y \rangle) \quad \hat{R} \phi \vec{k} | L \rangle = e^{i\phi} | L \rangle \quad \hat{J}_z | L \rangle = -\hbar | L \rangle$$

## Spin 1 (General)

$$| 1, 1 \rangle_n = \begin{pmatrix} e^{-i\varphi} \frac{1+\cos \theta}{2} \\ \frac{\sin \theta}{\sqrt{2}} \\ e^{i\varphi} \frac{1-\cos \theta}{2} \end{pmatrix} \quad | 1, 0 \rangle_n = \begin{pmatrix} -e^{-i\varphi} \frac{\sin \theta}{\sqrt{2}} \\ \cos \theta \\ e^{i\varphi} \frac{\sin \theta}{\sqrt{2}} \end{pmatrix} \quad | 1, -1 \rangle_n = \begin{pmatrix} e^{-i\varphi} \frac{1-\cos \theta}{2} \\ -\frac{\sin \theta}{\sqrt{2}} \\ e^{i\varphi} \frac{1+\cos \theta}{2} \end{pmatrix} \quad \hat{S}_n = \hbar \begin{pmatrix} \cos \theta & \frac{e^{-i\varphi} \sin \theta}{\sqrt{2}} & 0 \\ \frac{e^{i\varphi} \sin \theta}{\sqrt{2}} & 0 & \frac{e^{-i\varphi} \sin \theta}{\sqrt{2}} \\ 0 & \frac{e^{i\varphi} \sin \theta}{\sqrt{2}} & -\cos \theta \end{pmatrix}$$

## Hamiltonian

$$i\hbar \frac{d}{dt} | \psi(t) \rangle = \hat{H} | \psi(t) \rangle \quad \hat{H}^\dagger = \hat{H} \quad \frac{\partial \hat{A}}{\partial t} = 0 \Rightarrow \frac{d\langle \hat{A} \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle \quad \hat{U}(t) | \psi(0) \rangle = | \psi(t) \rangle$$

$$\frac{\partial \hat{H}}{\partial t} = 0 \Rightarrow \hat{U}(t) = e^{-i\hat{H} \frac{t}{\hbar}}$$