Dirac freed from $\mathbb{C}$

Guy Barrand ()

April 16, 2023

Abstract

By exploiting the Majorana representation of the gamma matrices, we first rewrite the free Dirac equation without complex numbers and discuss real solutions for it. Then, we show that the minimal complex electromagnetic coupling could be handled also without complex numbers by just considering two coupled real fields instead of one complex field. When done, we show how to get pure real solutions for the hydrogen atom. In particular without any usage of Pauli matrices and complex spherical harmonics, as is done traditionally by using the Dirac representation of the gamma matrices. It leads to the same conclusions concerning the quantization of the energy of the hydrogen atom, but done with much less complicated mathematics. Beside being more close to intuition, this approach may interest teachers seeking a way to present the school case of the hydrogen atom spectrum to an audience satisfied by having only a good knowledge of real matrix algebra and real continous functions calculus.

1 The Dirac equation

Over a $\psi^\alpha(x)$ tuple, it reads traditionally:

$$i(\gamma^\mu)^\alpha_\beta \partial_\mu \psi^\beta(x) = \frac{mc}{\hbar}\psi^\alpha(x)$$

or in a more compact matrix notation:

$$i\gamma^\mu \partial_\mu \psi(x) = \frac{mc}{\hbar}\psi(x) \tag{1}$$

with the γ^μ being four 4x4 complex matrices verifying:

$$\{\gamma^\mu, \gamma^\nu\} = 2(\eta^{-1})^\mu_\nu I \tag{2}$$

$$\eta \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \overset{\text{dem}}{=} \eta^{-1}$$

Written in this way, it compells that the $\psi(x)$ tuple is a priori complex.

2 Solutions

By exploiting (2) it is easy to show that the solutions of the complex equation (1) can be presented in the form:

$$\psi[p,u](x) \stackrel{\text{def}}{=} e^{i\frac{p_\mu x^\mu}{\hbar}}(I - \frac{1}{mc}p_\alpha\gamma^\alpha)u \tag{3}$$

where p_μ is a constant "four-momentum" real tuple that verifies:

$$l(p) \stackrel{\text{def}}{=} p\eta^{-1}\,{}^{\mathrm{t}}p \stackrel{\text{def}}{=} p_\mu(\eta^{-1})^\mu_\nu({}^{\mathrm{t}}p)^\nu \stackrel{\text{def}}{=} (mc)^2$$

and u is a not null constant complex tuple. (${}^{\mathrm{t}}$ is for the transposition operation).

If interpreting $\rho[p,u](x) \stackrel{\text{def}}{=} \overline{\psi[p,u]}\gamma^0\psi[p,u](x)$ as a probability density, then u should have the dimension of the invert square root of a volume:

$$[u] \stackrel{\text{def}}{=} \frac{1}{\sqrt{L^3}}$$

It is interesting to note also that $\rho[p,u](x)$ is in fact constant in x^μ since the leading exponential in (3) cancels in this quantity. (The Dirac conjugate is, as usual, $\overline{\psi} \stackrel{\text{def}}{=} {}^{\mathrm{t}}\psi^*\gamma^0$, which induces that $\overline{\psi}\gamma^0\psi \stackrel{\text{dem}}{=} {}^{\mathrm{t}}\psi^*\psi$ is always a pure real positive quantity).

3 Notation

For functions/fields depending of parameters, we use:

$$name[parameters](arguments)$$

For example as in the upper (3) for a ψ field over the space-time x argument and depending of the p and u tuple parameters.

We use also $\stackrel{\text{def}}{=}$ in case an equality is a definition (left side is defined by the right side), and $\stackrel{\text{dem}}{=}$ when an equality comes from a demonstration (left side is demonstrated to be the right side). We have also $\stackrel{\text{cas}}{=}$ in case an equality is demonstrated by using a computer algebra system (CAS).

We definitely avoid the practice of setting $\hbar = c = 1$ which complicates the reading of the dimensionality of quantities.

4 The $\tilde{S}_\alpha \xi$ Dirac equation

We introduce now the four real symmetric matrices:

$$\tilde{S}_0 \overset{\text{def}}{=} I \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \tilde{S}_1 \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{S}_2 \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \tilde{S}_3 \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the real antisymmetric one:

$$\xi \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

We can show that the γ^μ matrices in the Majorana representations (see [1] p.694):

$$\gamma^0 \overset{\text{def}}{=} i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^1 \overset{\text{def}}{=} i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^2 \overset{\text{def}}{=} i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 \overset{\text{def}}{=} i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

can be written:

$$\gamma^\mu \overset{\text{dem}}{=} -i\eta^{\mu\alpha} \tilde{S}_\alpha \xi$$

which induces that we have:

$$\{\tilde{S}_\mu \xi, \tilde{S}_\nu \xi\} \overset{\text{dem}}{=} -2\eta^\mu_\nu I \tag{4}$$

Equipped withe these, we can rewrite now the Dirac equation as:

$$h^{\mu\alpha}\, \tilde{S}_\alpha \xi\, \partial_\mu \psi(x) = \frac{mc}{\hbar} \psi(x) \tag{5}$$

$$h^{\mu\alpha} \overset{\text{def}}{=} \eta^{\mu\alpha} \overset{\text{def}}{=} (\eta^{-1})^\mu_\alpha \overset{\text{dem}}{=} \eta^\mu_\alpha$$

Since all matrices are real, the $\psi(x)$ tuple is now a priori real (and then containing only four real fields instead of eight if complex). See [2] for a detailed presentation on this way to write the Dirac equation and about its Lorentz invariance.

5 Real solutions

Despite that (5) is real, as in the section (2), we can look for complex solutions and show, by exploiting (4), that:

$$\psi[p,u](x) \stackrel{\text{def}}{=} e^{i\frac{p_\mu x^\mu}{\hbar}}(I + i\frac{1}{mc}p_\alpha h^{\alpha\beta}\tilde{S}_\beta\xi)u$$

verifies (5). These solutions involve also eight real fields.

But it is interesting to search for pure real solutions involving only four real fields. We can show easily, also by exploiting (4), that:

$$\mathscr{V}[p,\mathscr{U}](x) \stackrel{\text{def}}{=} [\cos(\omega[p](x))I - \sin(\omega[p](x))\frac{1}{mc}p_\alpha h^{\alpha\beta}\tilde{S}_\beta\xi]\mathscr{U} \tag{6}$$

are solutions of (5), where $\mathscr{U}$ is a not null constant real tuple and:

$$\omega[p](x) \stackrel{\text{def}}{=} \frac{p_\mu x^\mu}{\hbar}$$

About density, $\overline{\psi}\gamma^0\psi(x)$ on a real $\mathscr{V}(x)$ is now:

$$\overline{\mathscr{V}}\gamma^0\mathscr{V}(x) \stackrel{\text{dem}}{=} {}^{\text{t}}\mathscr{V}(x)\mathscr{V}(x)$$

and it is interesting to note that $\rho[p,\mathscr{U}](x) \stackrel{\text{def}}{=} \overline{\mathscr{V}[p,\mathscr{U}]}\gamma^0\mathscr{V}[p,\mathscr{U}](x)$ is constant in x^μ only when the three momentum $\vec{p}$ is null. Else, with $\vec{p} \neq \vec{0}$, we show below that this density is strictly positive and that it oscillates according a $2\omega[p](x)$ phase on the whole space-time. If interpreted as a probability density, it means that "finding the particle at x" is never null and oscillates with such a phase on the whole space-time. This is a drastic difference compared with the upper complex solution.

5.1 The density is strictly positive

We have:

$$\rho[p,\mathscr{U}](x) \stackrel{\text{def}}{=} \overline{\mathscr{V}[p,\mathscr{U}]}\gamma^0\mathscr{V}[p,\mathscr{U}](x) \stackrel{\text{dem}}{=} {}^{\text{t}}\mathscr{U}\Sigma(x)\mathscr{U}$$

where $\Sigma(x)$ is:

$$\Sigma(x) \stackrel{\text{def}}{=} [\cos(\omega(x))I + \sin(\omega(x))q^\alpha\xi\tilde{S}_\alpha][\cos(\omega(x))I - \sin(\omega(x))q^\beta\tilde{S}_\beta\xi]$$

and $q^\alpha \stackrel{\text{def}}{=} \frac{1}{mc}h^{\alpha\beta}p_\beta$. This matrix is real symmetrix and then its eigenvalues $d_{\mu=0,1,2,3}(x)$ are real. It can be written:

$$\Sigma(x) \stackrel{\text{def}}{=} PD\,{}^{\text{t}}P$$

with $D(x)$ diagonal filled with the eigenvalues and $P(x)$ a matrix, verifying $P\,{}^{\mathrm{t}}P = I$, for which the columns are the eigenvectors $\mathscr{D}_\mu(x)$. By using a computer algebra system, it can be shown that the four real eigenvalues are two d_+ and two d_- with:

$$d_+ \overset{\text{cas}}{=} 1 + 2y + 2\sqrt{y(y+1)} \qquad d_- \overset{\text{cas}}{=} 1 + 2y - 2\sqrt{y(y+1)}$$

$$y \overset{\text{def}}{=} \sin^2(\omega)\frac{\vec{p}^2}{(mc)^2} \qquad \omega[p](x) \overset{\text{def}}{=} \frac{p_\mu x^\mu}{\hbar}$$

We have $d_+(y) \geqslant 1 \geqslant d_-(y) > 0$ for $y \geqslant 0$ which induces that:

$$\rho[p, \mathscr{U}](x) \overset{\text{dem}}{=} d_\mu(\mathscr{D}^\mu_\alpha \mathscr{U}^\alpha)^2$$

is strictly positive.

5.2 The density oscillates in space-time if $\vec{p} \neq \vec{0}$

With a little bit of (real) algebra, we can show that:

$$\rho[p, \mathscr{U}](x) \overset{\text{dem}}{=} A[p, \mathscr{U}] + B[p, \mathscr{U}] - \cos(2\omega[p](x))B[p, \mathscr{U}] - \sin(2\omega[p](x))C[p, \mathscr{U}]$$

with:

$$A[p, \mathscr{U}] \overset{\text{def}}{=} {}^{\mathrm{t}}\mathscr{U}\mathscr{U}$$

$$B[p, \mathscr{U}] \overset{\text{def}}{=} (\vec{q}.\vec{q})A - q^0 q^{k=1,2,3}\,{}^{\mathrm{t}}\mathscr{U}\tilde{S}_k\mathscr{U}$$

$$C[p, \mathscr{U}] \overset{\text{def}}{=} q^k\,{}^{\mathrm{t}}\mathscr{U}\tilde{S}_k\xi\mathscr{U}$$

which demonstrates that the density $\rho[p, \mathscr{U}](x)$ oscillates in space-time if $\vec{p} \neq \vec{0}$.

6 Electromagnetism

Thanks to the Majorana representation of the gamma matrices, we have been able to get rid of complex numbers in the Dirac equation, but traditionally the sticky "i" appears also when handling electromagnetism by writing:

$$i\gamma^\mu\{\partial_\mu\psi(x) + i\frac{q}{\hbar c}\Phi_\mu(x)\psi(x)\} = \frac{mc}{\hbar}\psi(x) \tag{7}$$

with q being the electromagnetic charge and $\Phi_\mu(x)$ being the electromagnetic potential related to the three dimensional Maxwell real $U(t, \vec{x})$ and $\vec{A}(t, \vec{x})$ with:

$$\Phi_\mu(x \overset{\text{def}}{=} (ct, \vec{x})) \overset{\text{def}}{=} (U(t, \vec{x}), -\vec{A}(t, \vec{x}))$$

By using our $\tilde{S}_\alpha \xi$ representation, the upper equation becomes:

$$h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,\{\partial_\mu\psi(x) + i\frac{q}{\hbar c}\Phi_\mu(x)\psi(x)\} = \frac{mc}{\hbar}\psi(x) \tag{8}$$

which exhibits the fact that "i" appears now only in the electromagnetic coupling to $\Phi_\mu(x)$. The complex coupling induces that $\psi(x)$ has to be complex, but if writing:

$$\psi[\mathscr{V},\mathscr{W}](x) \stackrel{\text{def}}{=} \mathscr{V}(x) + i\mathscr{W}(x)$$

we see that (8) can be written as two equations without complex numbers on two real coupled fields $\mathscr{V}(x)$ and $\mathscr{W}(x)$:

$$h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,\{\partial_\mu\mathscr{V}(x) - \frac{q}{\hbar c}\Phi_\mu(x)\mathscr{W}(x)\} = \frac{mc}{\hbar}\mathscr{V}(x)$$
$$h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,\{\partial_\mu\mathscr{W}(x) + \frac{q}{\hbar c}\Phi_\mu(x)\mathscr{V}(x)\} = \frac{mc}{\hbar}\mathscr{W}(x)$$

or:

$$[h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,\partial_\mu - \frac{mc}{\hbar}I]\mathscr{V}(x) = [\frac{q}{\hbar c}\Phi_\mu(x)h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,]\mathscr{W}(x) \tag{9}$$

$$[h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,\partial_\mu - \frac{mc}{\hbar}I]\mathscr{W}(x) = -[\frac{q}{\hbar c}\Phi_\mu(x)h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,]\mathscr{V}(x) \tag{10}$$

We see now that complex numbers appear for electromagnetism in (7) and (8) mainly to write in a more compact form two coupled real quantities.

About density, $\rho(x) \stackrel{\text{def}}{=} \overline{\psi}\gamma^0\psi(x)$ is now here:

$$\rho[\mathscr{V},\mathscr{W}](x) \stackrel{\text{dem}}{=} {}^{\text{t}}\mathscr{V}(x)\mathscr{V}(x) + {}^{\text{t}}\mathscr{W}(x)\mathscr{W}(x) \tag{11}$$

which is clearly positive or null.

7 A word on charge conjugation

With (7), charge conjugation transformation would consist to find a transformation $\mathscr{C}(\psi)$ such that:

$$i\gamma^\mu\{\partial_\mu - i\frac{q}{\hbar c}\Phi_\mu(x)\}\mathscr{C}(\psi)(x) = \frac{mc}{\hbar}\mathscr{C}(\psi)(x) \tag{12}$$

By using the Dirac (or Chiral) representation of the gamma matrices, we get that:

$$\mathscr{C}(\psi)(x) \stackrel{\text{dem}}{=} i\gamma^2\psi^*(x) \tag{13}$$

but by working with $(9, 10)$, we see that:

$$\mathscr{C}(\mathscr{V},\mathscr{W})(x) \stackrel{\text{dem}}{=} (\mathscr{W},\mathscr{V})(x) \tag{14}$$

that is to say the charge conjugation is reduced to just a swapping of the two coupled real fields! We find this much more appealing than the much more algebraic (13). (On $\psi(x)$, (14) would be written $\mathscr{C}(\psi)(x) \stackrel{\text{dem}}{=} i\psi^*(x)$).

8 The hydrogen atom

By taking:

$$\vec{A}(t,\vec{x}) \overset{\text{def}}{=} \vec{0} \qquad U(t,\vec{x}) \overset{\text{def}}{=} \frac{e}{4\pi\varepsilon_0\|\vec{x}\|} \qquad q \overset{\text{def}}{=} -e \qquad e > 0$$

inducing:

$$\frac{q}{\hbar c}U(t,\vec{x}) \overset{\text{dem}}{=} -\frac{\alpha}{\|\vec{x}\|} \qquad \alpha \overset{\text{def}}{=} \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$$

we are going to show that the two upper real coupled equations can be solved without using any complex numbers at all. To reach this we need first to introduce some maths related to the $h^{\mu\alpha}\,\tilde{S}_\alpha\xi\,\partial_\mu\{\}$ operator in spherical coordinates.

9 Spherical coordinates

Passing from cartesian to spherical coordinates in three dimensions is a question of a transformation $\mathscr{S}$ such that:

$$\mathscr{S}_1(x,y,z) \overset{\text{def}}{=} \sqrt{x^2+y^2+z^2} \overset{\text{def}}{=} \tilde{r}(x,y,z) \quad 0 \leqslant \tilde{r}(x,y,z)$$

$$\mathscr{S}_2(x,y,z) \overset{\text{def}}{=} \arctan(\frac{\sqrt{x^2+y^2}}{z}) \overset{\text{def}}{=} \tilde{\theta}(x,y,z) \quad 0 \leqslant \tilde{\theta}(x,y,z) \leqslant \pi$$

$$\mathscr{S}_3(x,y,z) \overset{\text{def}}{=} \arctan(\frac{y}{x}) \overset{\text{def}}{=} \tilde{\varphi}(x,y,z) \quad 0 \leqslant \tilde{\varphi}(x,y,z) \leqslant 2\pi$$

with the invert transformation reading:

$$(\mathscr{S}^{-1})_1(r,\theta,\varphi) \overset{\text{dem}}{=} r\sin\theta\cos\varphi$$

$$(\mathscr{S}^{-1})_2(r,\theta,\varphi) \overset{\text{dem}}{=} r\sin\theta\sin\varphi$$

$$(\mathscr{S}^{-1})_3(r,\theta,\varphi) \overset{\text{dem}}{=} r\cos\theta$$

A function $f(x,y,z)$ is transformed in $\mathscr{S}(f)(r,\theta,\varphi) \overset{\text{def}}{=} \tilde{f}(r,\theta,\varphi)$ with:

$$\mathscr{S}(f)(\mathscr{S}(x)) \overset{\text{def}}{=} f(x) \overset{\text{def}}{=} \tilde{f}(\tilde{r}(x,y,z),\tilde{\theta}(x,y,z),\tilde{\varphi}(x,y,z))$$

$$\Leftrightarrow \quad \mathscr{S}(f)(r,\theta,\varphi) \overset{\text{def}}{=} \tilde{f}(r,\theta,\varphi) \overset{\text{dem}}{=} f(\mathscr{S}^{-1}(r,\theta,\varphi))$$

An issue is around derivatives, we have:

$$\partial_j f(x,y,z) \overset{\text{dem}}{=} \partial_k[\mathscr{S}(f)](\mathscr{S}(x,y,z))\partial_j\mathscr{S}^k(x,y,z)$$

$$\Leftrightarrow \quad \partial_j f(\mathscr{S}^{-1}(r,\theta,\varphi)) \overset{\text{dem}}{=} \partial_k \tilde{f}(r,\theta,\varphi)\partial_j \mathscr{S}^k(\mathscr{S}^{-1}(r,\theta,\varphi))$$

$$\Leftrightarrow \quad \partial_j f(\mathscr{S}^{-1}(r,\theta,\varphi)) \overset{\text{dem}}{=} \partial_k \tilde{f}(r,\theta,\varphi)\mathscr{S}_j^k(r,\theta,\varphi)$$

with the matrix:

$$\mathscr{S}_j^k(r,\theta,\varphi) \overset{\text{def}}{=} \partial_j \mathscr{S}^k(\mathscr{S}^{-1}(r,\theta,\varphi)) \overset{\text{dem}}{=} \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi/r & \cos\theta\sin\varphi/r & -\sin\theta/r \\ -\sin\varphi/(r\sin\theta) & \cos\varphi/(r\sin\theta) & 0 \end{pmatrix}$$

10 Notation, 3D indices

In all the below, latin letters (j,k,l) are used for 3D indices. They go from one to three. In expressions where the same letter appear, the summation is assumed and this whatever their up or down position. Then for example:

$$A^j B_j = A_j B^j = A^j B^j = A_j B_j = A_1 B_1 + A_2 B_2 + A_3 B_3$$

11 Laplace operators

On the Laplace operator defined as:

$$\Delta\{f\}(x,y,z) \overset{\text{def}}{=} (\partial_1^2 f + \partial_2^2 f + \partial_3^2 f)(x,y,z)$$

we can show that:

$$\Delta\{f\}(x,y,z) \overset{\text{dem}}{=} \tilde{\Delta}\{\tilde{f}\}(\tilde{r}(x,y,z),\tilde{\theta}(x,y,z),\tilde{\varphi}(x,y,z))$$

with:

$$\tilde{\Delta}\{\tilde{f}\}(r,\theta,\varphi) \overset{\text{dem}}{=} [\frac{1}{r^2}\partial_r(r^2\partial_r\tilde{f}) + \frac{1}{r^2}\hat{\Theta}\{\tilde{f}\}](r,\theta,\varphi)$$

and the Laplace angular operator $\hat{\Theta}$, acting only on the angular part of an angular function $a(\theta,\varphi)$, being defined as:

$$\hat{\Theta}\{a\}(\theta,\varphi) \overset{\text{def}}{=} [\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta a) + \frac{1}{\sin^2\theta}\partial_\varphi^2 a](\theta,\varphi)$$

12 $\tilde{n}^j(\theta,\varphi)$ functions, $\hat{N}_j$ and $\hat{A}_j$ operators

We introduce the functions $\tilde{n}^j(\theta,\varphi)$:

$$\tilde{n}^1(\theta,\varphi) \overset{\text{def}}{=} \sin\theta\cos\varphi \qquad \tilde{n}^2(\theta,\varphi) \overset{\text{def}}{=} \sin\theta\sin\varphi \qquad \tilde{n}^3(\theta,\varphi) \overset{\text{def}}{=} \cos\theta$$

the operators $\hat{N}_j$ on any angular function $a(\theta,\varphi)$:

$$\hat{N}_1\{a\}(\theta,\varphi) \stackrel{\text{def}}{=} \cos\theta\cos\varphi\partial_\theta a - \frac{\sin\varphi}{\sin\theta}\partial_\varphi a$$
$$\hat{N}_2\{a\}(\theta,\varphi) \stackrel{\text{def}}{=} \cos\theta\sin\varphi\partial_\theta a + \frac{\cos\varphi}{\sin\theta}\partial_\varphi a$$
$$\hat{N}_3\{a\}(\theta,\varphi) \stackrel{\text{def}}{=} -\sin\theta\partial_\theta a$$

and the operators $\hat{A}_j$:

$$\hat{A}_l\{a\}(\theta,\varphi) \stackrel{\text{def}}{=} \varepsilon_{ljk}\tilde{n}^j(\theta,\varphi)\hat{N}_k\{a\}(\theta,\varphi) \stackrel{\text{dem}}{=} \begin{pmatrix} -\sin\varphi\partial_\theta a - \cot\theta\cos\varphi\partial_\varphi a \\ \cos\varphi\partial_\theta a - \cot\theta\sin\varphi\partial_\varphi a \\ \partial_\varphi a \end{pmatrix}$$

(ε_{ljk} is the traditional Levi-Civita symbol with $\varepsilon_{123} \stackrel{\text{def}}{=} 1$).

We have various nice relations:

$$\tilde{n}^j\tilde{n}^j \stackrel{\text{dem}}{=} 1 \qquad \hat{N}_j\{\tilde{n}^j\} \stackrel{\text{dem}}{=} 2 \qquad \varepsilon^{ljk}\hat{N}_j\{\tilde{n}^k\} \stackrel{\text{dem}}{=} 0$$
$$\tilde{n}^j\hat{N}_j \stackrel{\text{dem}}{=} 0 \qquad \varepsilon^{ljk}\hat{N}_j\circ\hat{N}_k \stackrel{\text{dem}}{=} \hat{A}_l$$
$$\varepsilon^{ljk}\hat{A}_j\circ\hat{A}_k \stackrel{\text{dem}}{=} -\hat{A}_l \qquad \hat{A}_j\circ\hat{A}_j \stackrel{\text{dem}}{=} \hat{\Theta} \tag{15}$$

If having an expression with matrices M^j of the type:

$$M^j\partial_j\psi(x,y,z) = \psi(x,y,z)$$

it becomes in spherical:

$$M^j\mathscr{S}_j^k(r,\theta,\varphi)\partial_k\tilde{\psi}(r,\theta,\varphi) = \tilde{\psi}(r,\theta,\varphi)$$
$$M^j\mathscr{S}_j^k(r,\theta,\varphi)\partial_k\tilde{\psi}(r,\theta,\varphi) \stackrel{\text{dem}}{=} \frac{1}{r}M^j\hat{N}_j\{\tilde{\psi}\}(r,\theta,\varphi) + \tilde{n}^j(\theta,\varphi)M^j\partial_r\tilde{\psi}(r,\theta,\varphi) \tag{16}$$

13 Properties of the three $\tilde{S}_{j=1,2,3}$ and $S_{j=1,2,3}$ matrices

Let us introduce the extra symmetric matrices:

$$S_1 \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad S_2 \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

If considering only the $\tilde{S}_{j=1,2,3}$ matrices of the section (4) (then ignoring the $\tilde{S}_0$), we can show that:

$$\{\tilde{S}_j,\xi\} \stackrel{\text{dem}}{=} 0 \qquad \xi\tilde{S}_j\xi \stackrel{\text{dem}}{=} \tilde{S}_j$$
$$\{\tilde{S}_j,\tilde{S}_k\} \stackrel{\text{dem}}{=} 2\delta_{jk}I \qquad [\tilde{S}_j,\tilde{S}_k] \stackrel{\text{dem}}{=} -2\varepsilon_{jkl}(2S_l\xi) \qquad \Rightarrow \tilde{S}_j\tilde{S}_k \stackrel{\text{dem}}{=} \delta_{jk}I - \varepsilon_{jkl}(2S_l\xi)$$
$$(2S_j\xi)(2S_k\xi) \stackrel{\text{dem}}{=} -\delta_{jk}I + \varepsilon_{jkl}(2S_l\xi) \tag{17}$$

14 $\tilde{S}_j$, $\hat{N}_j$ and their relation to $\hat{\Theta}$

By introducing the angular matrix function:

$$\Sigma(\theta,\varphi) \overset{\text{def}}{=} \tilde{S}_j \tilde{n}^j(\theta,\varphi) \qquad (\Rightarrow \{\Sigma,\xi\} \overset{\text{dem}}{=} 0 \quad \Sigma^2(\theta,\varphi) \overset{\text{dem}}{=} I)$$

we can show that:

$$\tilde{S}_j \Sigma \hat{N}_j \overset{\text{dem}}{=} -\Sigma \tilde{S}_j \hat{N}_j \qquad \tilde{S}_j \tilde{S}_k \hat{N}_k \{\tilde{n}^j\} \overset{\text{dem}}{=} \tilde{S}_k \tilde{S}_j \hat{N}_k \{\tilde{n}^j\} \overset{\text{dem}}{=} \tilde{S}_k \hat{N}_k \{\Sigma\} \overset{\text{dem}}{=} 2I$$

An important relation is:

$$\tilde{S}_j \Sigma \hat{N}_j \overset{\text{dem}}{=} 2\, S_l \xi\, \hat{A}_l$$

so that if $\mathscr{Z}(\theta,\phi)$ is a four angular fields such that:

$$\tilde{S}_j \Sigma \hat{N}_j \{\mathscr{Z}\} \overset{\text{def}}{=} \lambda \mathscr{Z} \qquad (\Leftrightarrow \tilde{S}_j \hat{N}_j \{\Sigma \mathscr{Z}\} \overset{\text{dem}}{=} (\lambda+2)\mathscr{Z}\ )$$

we have:

$$\tilde{S}_j \Sigma \hat{N}_j \{\tilde{S}_k \Sigma \hat{N}_k \{\mathscr{Z}\}\} \overset{\text{dem}}{=} (\lambda)^2 \mathscr{Z} \quad \Leftrightarrow \quad 2\, S_j \xi\, \hat{A}_j \{2\, S_k \xi\, \hat{A}_k \{\mathscr{Z}\}\} \overset{\text{dem}}{=} (\lambda)^2 \mathscr{Z}$$

which leads, by using (17) and (15), to:

$$\hat{\Theta}\{\mathscr{Z}\} \overset{\text{def}}{=} \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta \mathscr{Z}) + \frac{1}{\sin^2\theta}\partial^2_\varphi \mathscr{Z} \overset{\text{dem}}{=} -\lambda(\lambda+1)\mathscr{Z}$$

It permits to connect λ to the l eigenvalues of the Laplace angular operator.

15 Eigenvalues and vectors of $\tilde{S}_j \Sigma \hat{N}_j$

For the hydrogen atom, an important step in finding solutions for the coupled (9, 10) equations is to find the eigenvalues and eigenvectors of the operator $\tilde{S}_j \Sigma \hat{N}_j$, that is to say to find the couples $(\lambda, \mathscr{Z}(\theta,\varphi))$ such that:

$$\tilde{S}_j \Sigma \hat{N}_j \{\mathscr{Z}\} \overset{\text{def}}{=} \lambda \mathscr{Z} \quad \Rightarrow \hat{\Theta}\{\mathscr{Z}^\alpha\} \overset{\text{dem}}{=} -\lambda(\lambda+1)\mathscr{Z}^\alpha$$

This second equation pushes to look for solutions of the form:

$$\mathscr{Z}[l,m]^\alpha(\theta,\varphi) \overset{\text{def}}{=} z^\alpha(l,m) C_l^m(\theta,\varphi)$$

or:

$$\mathscr{Z}[l,m]^\alpha(\theta,\varphi) \overset{\text{def}}{=} z^\alpha(l,m) S_l^m(\theta,\varphi)$$

with C_l^m, S_l^m being the cosine and sine spherical harmonics defined as:

$$C_l^m(\theta,\varphi) \overset{\text{def}}{=} y(l,m) P_l^m(\cos\theta)\cos(m\phi)$$

$$S_l^m(\theta,\varphi) \stackrel{\text{def}}{=} y(l,m)P_l^m(\cos\theta)\sin(m\phi)$$
$$y(l,m) \stackrel{\text{def}}{=} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$$
with $l \in \mathbb{N}$, $m \in \mathbb{Z}$ such that $-l \leqslant m \leqslant l$ and $P_l^m(x)$ being the associated Legendre functions. The C_l^m, S_l^m angular functions verify:

$$\hat{\Theta}\{C_l^m\}(\theta,\varphi) \stackrel{\text{dem}}{=} -l(l+1)C_l^m(\theta,\varphi)$$
$$\hat{\Theta}\{S_l^m\}(\theta,\varphi) \stackrel{\text{dem}}{=} -l(l+1)S_l^m(\theta,\varphi)$$

By using the recursive property of the $P_l^m(x)$ functions:

$$2(m+1)\cot\theta P_l^{m+1}(\cos\theta) + P_l^{m+2}(\cos\theta) \stackrel{\text{dem}}{=} -(l+m+1)(l-m)P_l^m(\cos\theta)$$

we can show, with rather lengthy calculations, that for $1 \leqslant l$ and $-l \leqslant m \leqslant l-2$:

$$\lambda_-(l) \stackrel{\text{def}}{=} -(l+1) \leqslant -2$$

$$\mathscr{Z}[\lambda_-(l),m](\theta,\varphi) \stackrel{\text{def}}{=} \mathscr{Z}_-[l,m](\theta,\varphi) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2l+1}}\begin{pmatrix} \sqrt{l+m+1} & C_l^{m+1}(\theta,\varphi) \\ \sqrt{l-m} & C_l^m(\theta,\varphi) \\ -\sqrt{l-m} & S_l^m(\theta,\varphi) \\ \sqrt{l+m+1} & S_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

verifies:

$$\tilde{S}_j\Sigma\hat{N}_j\{\mathscr{Z}[\lambda_-(l),m]\} \stackrel{\text{dem}}{=} \lambda_-(l)\,\mathscr{Z}[\lambda_-(l),m] \tag{18}$$
$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, {}^{\mathrm{t}}\mathscr{Z}_-[l,m](\theta,\varphi)\mathscr{Z}_-[l,m](\theta,\varphi) \stackrel{\text{dem}}{=} 1$$

and:

$$\lambda_+(l) \stackrel{\text{def}}{=} l \geqslant 1$$

$$\mathscr{Z}[\lambda_+(l),m](\theta,\varphi) \stackrel{\text{def}}{=} \mathscr{Z}_+[l,m](\theta,\varphi) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2l+1}}\begin{pmatrix} -\sqrt{l-m} & C_l^{m+1}(\theta,\varphi) \\ \sqrt{l+m+1} & C_l^m(\theta,\varphi) \\ -\sqrt{l+m+1} & S_l^m(\theta,\varphi) \\ -\sqrt{l-m} & S_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

verifies:

$$\tilde{S}_j\Sigma\hat{N}_j\{\mathscr{Z}[\lambda_+(l),m]\} \stackrel{\text{dem}}{=} \lambda_+(l)\,\mathscr{Z}[\lambda_+(l),m] \tag{19}$$
$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, {}^{\mathrm{t}}\mathscr{Z}_+[l,m](\theta,\varphi)\mathscr{Z}_+[l,m](\theta,\varphi) \stackrel{\text{dem}}{=} 1$$

The relations (18, 19) had been checked numerically by computer with various values of l, m, θ, φ. See also Appendix A for a proof that they correspond to what is done traditionally in textbooks by using angular operators defined with Pauli matrices.

Noting that any constant $\mathscr{Z}(\theta,\varphi)$ function is an eigenvector with the 0 eigenvalue, then we have eigens for all λ in $\mathbb{Z}$ except for -1. (Not so clear to us if -1 is an eigenvalue or not).

We point out that we do not use any half integer numbers, as found in most textbooks. These are not needed, and introducing them in expressions as in "$j \pm \frac{1}{2}$ where $j = l \pm \frac{1}{2}$" complicates awfully the reading and the understanding of the angular solutions.

16 Unfolding the real Dirac equation

For the hydrogen atom, the coupled (9, 10) equations become:

$$[h^{\mu\alpha}\tilde{S}_\alpha\xi\partial_\mu - \frac{mc}{\hbar}I]\mathscr{V}(x) = -\frac{Z\alpha}{\tilde{r}(\vec{x})}\xi\mathscr{W}(x) \quad \Rightarrow [\xi\partial_0 - \tilde{S}_j\xi\partial_j - \frac{mc}{\hbar}I]\mathscr{V}(x) = -\frac{Z\alpha}{\tilde{r}(\vec{x})}\xi\mathscr{W}(x)$$

$$[h^{\mu\alpha}\tilde{S}_\alpha\xi\partial_\mu - \frac{mc}{\hbar}I]\mathscr{W}(x) = \frac{Z\alpha}{\tilde{r}(\vec{x})}\xi\mathscr{V}(x) \quad \Rightarrow [\xi\partial_0 - \tilde{S}_j\xi\partial_j - \frac{mc}{\hbar}I]\mathscr{W}(x) = \frac{Z\alpha}{\tilde{r}(\vec{x})}\xi\mathscr{V}(x)$$

We look for $(\mathscr{V}(x),\mathscr{W}(x))$ of the form:

$$\mathscr{V}(ct,\vec{x}) \stackrel{\text{def}}{=} \cos(\frac{E}{\hbar}t)a(\vec{x}) + \sin(\frac{E}{\hbar}t)b(\vec{x})$$

$$\mathscr{W}(ct,\vec{x}) \stackrel{\text{def}}{=} -\sin(\frac{E}{\hbar}t)a(\vec{x}) + \cos(\frac{E}{\hbar}t)b(\vec{x})$$

By injecting these in the upper right equations and isolating the $\sin(\frac{E}{\hbar}t)$ and $\cos(\frac{E}{\hbar}t)$ parts, we get the two equations on the spatial $a(\vec{x})$, $b(\vec{x})$ fields:

$$-\frac{E}{\hbar c}\xi a - \tilde{S}_j\xi\partial_j b - \frac{mc}{\hbar}b = \frac{Z\alpha}{\tilde{r}(\vec{x})}\xi a \quad \Rightarrow \frac{E}{\hbar c}a - \tilde{S}_j\partial_j b - \frac{mc}{\hbar}\xi b = -\frac{Z\alpha}{\tilde{r}(\vec{x})}a$$

$$\frac{E}{\hbar c}\xi b - \tilde{S}_j\xi\partial_j a - \frac{mc}{\hbar}a = -\frac{Z\alpha}{\tilde{r}(\vec{x})}\xi b \quad \Rightarrow -\frac{E}{\hbar c}b - \tilde{S}_j\partial_j a - \frac{mc}{\hbar}\xi a = \frac{Z\alpha}{\tilde{r}(\vec{x})}b$$

By passing to spherical coordinates and exploiting (16) with $M^j = \tilde{S}_j$, we have:

$$\frac{1}{r}\tilde{S}_j\hat{N}_j\{\tilde{b}\} + \Sigma\partial_r\tilde{b} + \frac{mc}{\hbar}\xi\tilde{b} = \frac{E}{\hbar c}\tilde{a} + \frac{Z\alpha}{r}\tilde{a}$$

$$\frac{1}{r}\tilde{S}_j\hat{N}_j\{\tilde{a}\} + \Sigma\partial_r\tilde{a} + \frac{mc}{\hbar}\xi\tilde{a} = -\frac{E}{\hbar c}\tilde{b} - \frac{Z\alpha}{r}\tilde{b}$$

Now we look for separable solutions in $r, (\theta,\varphi)$ with:

$$\tilde{a}(r,\theta,\varphi) \stackrel{\text{def}}{=} -f(r)\mathscr{Y}(\theta,\varphi) - g(r)\xi\mathscr{Z}(\theta,\varphi)$$

$$\tilde{b}(r,\theta,\varphi) \stackrel{\text{def}}{=} f(r)\xi\mathscr{Y}(\theta,\varphi) + g(r)\mathscr{Z}(\theta,\varphi)$$

It leads to (with $'$ for first derivative):

$$f'\Sigma\xi\mathscr{Y} + \frac{f}{r}\tilde{S}_j\hat{N}_j\{\xi\mathscr{Y}\} + g'\Sigma\mathscr{Z} + \frac{g}{r}\tilde{S}_k\hat{N}_k\{\mathscr{Z}\} = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\mathscr{Y} - (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\xi\mathscr{Z}$$

$$-f'\Sigma\mathscr{Y} - \frac{f}{r}\tilde{S}_j\hat{N}_j\{\mathscr{Y}\} - g'\Sigma\xi\mathscr{Z} - \frac{g}{r}\tilde{S}_k\hat{N}_k\{\xi\mathscr{Z}\} = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\xi\mathscr{Y} - (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\mathscr{Z}$$

We look now for $\mathscr{Z}(\theta,\varphi)$ and $\mathscr{Y}(\theta,\varphi)$ such that:

$$\tilde{S}_j\Sigma\hat{N}_j\{\mathscr{Z}\} \stackrel{\text{def}}{=} \lambda\mathscr{Z} \qquad \mathscr{Y}(\theta,\varphi) \stackrel{\text{def}}{=} \Sigma(\theta,\varphi)\mathscr{Z}(\theta,\varphi)$$

We can show that we have:

$$\tilde{S}_j\hat{N}_j\{\Sigma\mathscr{Z}\} \stackrel{\text{dem}}{=} (\lambda+2)\mathscr{Z}$$

$$\tilde{S}_j\hat{N}_j\{\Sigma\mathscr{Y}\} \stackrel{\text{dem}}{=} -\lambda\mathscr{Y} \qquad \tilde{S}_j\Sigma\hat{N}_j\{\mathscr{Y}\} \stackrel{\text{dem}}{=} -(\lambda+2)\mathscr{Y}$$

$$\tilde{S}_j\hat{N}_j\{\mathscr{Z}\} \stackrel{\text{dem}}{=} -\lambda\mathscr{Y} \qquad \tilde{S}_j\hat{N}_j\{\mathscr{Y}\} \stackrel{\text{dem}}{=} (\lambda+2)\mathscr{Z}$$

By using all these in the two upper equations, we have:

$$-f'\xi\mathscr{Z} - \frac{\lambda+2}{r}f\xi\mathscr{Z} + g'\mathscr{Y} - \frac{\lambda}{r}g\mathscr{Y} = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\mathscr{Y} - (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\xi\mathscr{Z}$$

$$-f'\mathscr{Z} - \frac{\lambda+2}{r}f\mathscr{Z} + g'\xi\mathscr{Y} - \frac{\lambda}{r}g\xi\mathscr{Y} = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\xi\mathscr{Y} - (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\mathscr{Z}$$

Then we multiply the first one by -1 and the second one by $-\xi$:

$$f'\xi\mathscr{Z} + \frac{\lambda+2}{r}f\xi\mathscr{Z} - g'\mathscr{Y} + \frac{\lambda}{r}g\mathscr{Y} = (\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\mathscr{Y} + (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\xi\mathscr{Z}$$

$$f'\xi\mathscr{Z} + \frac{\lambda+2}{r}f\xi\mathscr{Z} + g'\mathscr{Y} - \frac{\lambda}{r}g\mathscr{Y} = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\mathscr{Y} + (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\xi\mathscr{Z}$$

Adding and subtracting the two upper equations, we get:

$$f'\xi\mathscr{Z} + \frac{\lambda+2}{r}f\xi\mathscr{Z} = (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g\xi\mathscr{Z}$$

$$g'\mathscr{Y} - \frac{\lambda}{r}g\mathscr{Y} = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f\mathscr{Y}$$

Finally, by multiplying the first upper one by $-\xi$, and assuming that $\mathscr{Y},\mathscr{Z}$ are not null, we get the two radial equations on $f(r), g(r)$:

$$f' + \frac{\lambda+2}{r}f = (\frac{E}{\hbar c} + \frac{Z\alpha}{r} + \frac{mc}{\hbar})g$$

$$g' - \frac{\lambda}{r}g = -(\frac{E}{\hbar c} + \frac{Z\alpha}{r} - \frac{mc}{\hbar})f$$

We reach here a standard situation concerning the radial part. In textbooks, there are various ways to solve these. We are going to follow the "Landau, Lifshitz" way which is rather clean and "compact". We sumup in the next section what is done in the paragraph \$36 (Motion in a Coulomb field) of [3] for the "Discrete spectrum ($\varepsilon < m$)" case, which leads to the quantization of the energy for the hydrogen atom.

17 Solving the radial part for $|E| < mc^2$

To simplify the readability of formulas, we introduce:

$$\chi \stackrel{\text{def}}{=} \lambda + 1 \qquad u \stackrel{\text{def}}{=} \frac{E}{mc} \qquad v \stackrel{\text{def}}{=} \frac{mc}{\hbar} \qquad \tau \stackrel{\text{def}}{=} Z\alpha$$

which lead to:

$$f' + \frac{1+\chi}{r} f = (\frac{\tau}{r} + u + v) g \tag{20}$$

$$g' + \frac{1-\chi}{r} g = -(\frac{\tau}{r} + u - v) f \tag{21}$$

If looking to $|u| < v$, we can introduce:

$$\beta \stackrel{\text{def}}{=} \sqrt{v^2 - u^2} \qquad \gamma \stackrel{\text{def}}{=} \sqrt{\chi^2 - \tau^2}$$

and look for solutions of the form:

$$f(r) \stackrel{\text{def}}{=} \sqrt{v+u}\, e^{-\beta r} (2\beta r)^{\gamma - 1} (Q_1 + Q_2)(2\beta r)$$

$$g(r) \stackrel{\text{def}}{=} -\sqrt{v-u}\, e^{-\beta r} (2\beta r)^{\gamma - 1} (Q_1 - Q_2)(2\beta r)$$

By injecting these in (20), (21) and doing the change of variable $x \stackrel{\text{def}}{=} 2\beta r$, we reach:

$$x(Q_1' + Q_2')(x) + (\gamma + \chi)(Q_1 + Q_2)(x) - xQ_2(x) + \tau\sqrt{\frac{v-u}{v+u}}(Q_1 - Q_2)(x) = 0$$

$$x(Q_1' - Q_2')(x) + (\gamma - \chi)(Q_1 - Q_2)(x) + xQ_2(x) - \tau\sqrt{\frac{v+u}{v-u}}(Q_1 + Q_2)(x) = 0$$

Adding and subtracting the two upper equations give:

$$xQ_1'(x) + (\gamma - \frac{\tau u}{\beta}) Q_1(x) + (\chi - \frac{\tau v}{\beta}) Q_2(x) = 0 \tag{22}$$

$$xQ_2'(x) + (\gamma + \frac{\tau u}{\beta} - x) Q_2(x) + (\chi + \frac{\tau v}{\beta}) Q_1(x) = 0 \tag{23}$$

By taking $Q_2(x)$, depending of $Q_1(x), Q_1'(x)$, from the first one and injecting it in the second one (and the same for $Q_1(x)$ from the second one injected in the first one), we get two second order equations only on $Q_1(x)$ and $Q_2(x)$ respectively:

$$xQ_1''(x) + (2\gamma + 1 - x) Q_1'(x) - (\gamma - \frac{\tau u}{\beta}) Q_1(x) = 0$$

$$xQ_2''(x) + (2\gamma + 1 - x) Q_2'(x) - (\gamma + 1 - \frac{\tau u}{\beta}) Q_2(x) = 0$$

These are nothing more than two Kummer equations!

$$xf''(x) + (b - x)f'(x) - af(x) = 0$$

It is well known that such equation has the solution ${}_1F_1[a, b](x)$ (the confluent hypergeometric function). We get then:

$$Q_1(x) \stackrel{\text{dem}}{=} {}_1F_1[\gamma - \frac{\tau u}{\beta}, 2\gamma + 1](x)\, A$$

$$Q_2(x) \stackrel{\text{dem}}{=} {}_1F_1[\gamma + 1 - \frac{\tau u}{\beta}, 2\gamma + 1](x)\, B$$

By setting $x = 0$ in (22) and (23) (and noting that ${}_1F_1[a, b](0) = 1$), we get:

$$B \stackrel{\text{dem}}{=} -\frac{\gamma - \frac{\tau u}{\beta}}{\chi - \frac{\tau v}{\beta}} A$$

The function ${}_1F_1[a, b](x)$ is "well-behaved" only if $-a \in \mathbb{N}$. Such condition leads here to:

$$\gamma - \frac{\tau u}{\beta} \stackrel{\text{def}}{=} -p \quad p \in \mathbb{N}$$

$$\gamma + 1 - \frac{\tau u}{\beta} \stackrel{\text{def}}{=} -q \quad q \in \mathbb{N}$$

$$\Rightarrow \quad \gamma - \frac{\tau u}{\beta} \stackrel{\text{dem}}{=} \gamma - \frac{\tau u}{\sqrt{v^2 - u^2}} \stackrel{\text{def}}{=} -n_r \quad n_r \in \mathbb{N}^*$$

Which gives:

$$|u| \stackrel{\text{dem}}{=} \frac{v}{\sqrt{1 + \frac{\tau^2}{(n_r+\gamma)^2}}} \qquad n_r \in \mathbb{N}^*$$

By returning to the $(E, \lambda, Z\alpha, m)$ variables, we reach the well known, and rather impressive, quantization formula:

$$|E[n_r, \lambda]| = \frac{mc^2}{\sqrt{1 + \frac{(Z\alpha)^2}{(n_r + \sqrt{(1+\lambda)^2 - (Z\alpha)^2})^2}}} \tag{24}$$

with $n_r \in \mathbb{N}^*$ and $\lambda \in \mathbb{Z} - \{-1, 0\}$ being the eigenvalues of the $\tilde{S}_j \Sigma \hat{N}_j$ operator defined in the previous section. (Strangely, despite that $\lambda = 0$ is an eigenvalue, it is not considered in [3]).

17.1 Normalization of $f(r), g(r)$

In order to have:

$$\int_0^\infty dr\, r^2\, (f^2 + g^2)(r) \stackrel{\text{dem}}{=} 1$$

A should be:

$$A \stackrel{\text{dem}}{=} \frac{\beta^2}{v} \frac{1}{\Gamma[2\gamma+1]} \sqrt{\frac{2\Gamma[2\gamma+1+n_r]|\chi - \tau v/\beta|}{n_r!\tau}}$$

$$\Rightarrow \quad [A] \stackrel{\text{dem}}{=} \frac{1}{L} \quad \Rightarrow [f] = [g] \stackrel{\text{dem}}{=} \frac{1}{\sqrt{L^3}}$$

See [3] for the proof. We checked it numerically by computer.

18 Put all together

By gathering backward the various pieces, we write, at last, the full coupled solutions $(\mathscr{V}(x), \mathscr{W}(x))$:

$$\lambda_-(l) \stackrel{\text{def}}{=} -(l+1) \leqslant -2 \qquad \lambda_+(l) \stackrel{\text{def}}{=} l \geqslant 1 \qquad l \in \mathbb{N}^*$$

$$\gamma[\lambda] \stackrel{\text{def}}{=} \sqrt{(1+\lambda)^2 - (Z\alpha)^2}$$

$$E[n_r, \lambda] \stackrel{\text{def}}{=} \pm \frac{mc^2}{\sqrt{1 + \frac{(Z\alpha)^2}{(n_r+\gamma)^2}}} \qquad n_r \in \mathbb{N}^*$$

$$\beta \stackrel{\text{def}}{=} \sqrt{(\frac{mc}{\hbar})^2 - (\frac{E}{mc})^2} \qquad v \stackrel{\text{def}}{=} \frac{mc}{\hbar}$$

$$A \stackrel{\text{def}}{=} \frac{\beta^2}{v} \frac{1}{\Gamma[2\gamma+1]} \sqrt{\frac{2\Gamma[2\gamma+1+n_r]|\lambda + 1 - Z\alpha v/\beta|}{n_r!Z\alpha}}$$

$$Q_1(x) \stackrel{\text{def}}{=} {}_1F_1[\gamma - \frac{Z\alpha E}{mc\beta}, 2\gamma+1](x)\, A$$

$$B \stackrel{\text{def}}{=} -\frac{\gamma - \frac{Z\alpha E}{mc\beta}}{\lambda + 1 - \frac{Z\alpha mc}{\hbar\beta}} A$$

$$Q_2(x) \stackrel{\text{def}}{=} {}_1F_1[\gamma + 1 - \frac{Z\alpha E}{mc\beta}, 2\gamma+1](x)\, B$$

$$f(r) \stackrel{\text{def}}{=} \sqrt{\frac{mc}{\hbar} + \frac{E}{mc}}\, e^{-\beta r} (2\beta r)^{\gamma-1} (Q_1 + Q_2)(2\beta r)$$

$$g(r) \stackrel{\text{def}}{=} -\sqrt{\frac{mc}{\hbar} - \frac{E}{mc}}\, e^{-\beta r}\,(2\beta r)^{\gamma-1}\,(Q_1 - Q_2)(2\beta r)$$

$$-l \leqslant p \leqslant l-2 \qquad p \in \mathbb{Z}$$

$$\mathscr{Z}[\lambda_-, p](\theta,\varphi) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+p+1} & C_l^{p+1}(\theta,\varphi) \\ \sqrt{l-p} & C_l^{p}(\theta,\varphi) \\ -\sqrt{l-p} & S_l^{p}(\theta,\varphi) \\ \sqrt{l+p+1} & S_l^{p+1}(\theta,\varphi) \end{pmatrix}$$

$$\mathscr{Z}[\lambda_+, p](\theta,\varphi) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} -\sqrt{l-p} & C_l^{p+1}(\theta,\varphi) \\ \sqrt{l+p+1} & C_l^{p}(\theta,\varphi) \\ -\sqrt{l+p+1} & S_l^{p}(\theta,\varphi) \\ -\sqrt{l-p} & S_l^{p+1}(\theta,\varphi) \end{pmatrix}$$

$$\tilde{a}(r,\theta,\varphi) \stackrel{\text{def}}{=} -[f(r)\Sigma(\theta,\varphi) + g(r)\xi]\,\mathscr{Z}(\theta,\varphi)$$

$$\tilde{b}(r,\theta,\varphi) \stackrel{\text{def}}{=} [f(r)\xi\Sigma(\theta,\varphi) + g(r)I]\,\mathscr{Z}(\theta,\varphi)$$

$$\tilde{r}(x,y,z) \stackrel{\text{def}}{=} \sqrt{x^2+y^2+z^2} \quad \tilde{\theta}(x,y,z) \stackrel{\text{def}}{=} \arctan(\frac{\sqrt{x^2+y^2}}{z})$$

$$\tilde{\varphi}(x,y,z) \stackrel{\text{def}}{=} \arctan(\frac{y}{x})$$

$$\mathscr{V}(ct,x,y,z) \stackrel{\text{def}}{=} [\cos(\frac{E}{\hbar}t)\tilde{a} + \sin\frac{E}{\hbar}t)\tilde{b}]\,(\tilde{r}(x,y,z),\tilde{\theta}(x,y,z),\tilde{\varphi}(x,y,z))$$

$$\mathscr{W}(ct,x,y,z) \stackrel{\text{def}}{=} [-\sin(\frac{E}{\hbar}t)\tilde{a} + \cos(\frac{E}{\hbar}t)\tilde{b}]\,(\tilde{r}(x,y,z),\tilde{\theta}(x,y,z),\tilde{\varphi}(x,y,z))$$

We enforce that no complex numbers appear here. The found functions $f(r), g(r)$ had been tested numerically in (20), (21) by computer for various values of n_r, λ, r. The same with the two upper $(\mathscr{V}(x), \mathscr{W}(x))$ in (9, 10). The open source C++ programs are available at the author GitHub gbarrand in the repository "papers" under the directory "Dirac_freed_from_C".

19 Conclusions

We wrote the Dirac equation without complex numbers and showed that we can solve the hydrogen discrete spectrum by using only real numbers. Put all together, had been used only 4x4 real matrix algebra, taking first and second derivatie of continuous real functions and a set of them as cosine, sine, exponential, Legendre and the confluent hypergeometric functions. In particular we did not need to introduce Hilbert spaces, "impulse", "orbital" and "spin" operators using complex numbers in their definitions. All these much more complicated mathematics, beside having the bad taste to move away micro-physics from intuition, appeared to be not needed to get the (24) formula. Incidentally, this fully real approach may intereset teachers wanting to present the school case of the hydrogen atom spectrum to students having just in hands a good knowledge of real matrix algebra and real functions calculus.

It is rather interesting to note that to reach the energy quantization (24), which does the connection with what is observed by doing the spectroscopy of an hydrogen gaz, it had not been needed to attach to the couple $(\mathscr{V}(x), \mathscr{W}(x))$ the concept/idea of probability, in particular because the density (11) is not used to get the energy spectrum. Someone may argue that we do not come here with an interpretation at all of $(\mathscr{V}, \mathscr{W})$; that's right, and this article let this question open. But the fact to be able to avoid good part of the formalism developed in "Quantum Mechanics" textbooks on such a school case as the hydrogen energy spectrum may help those interested in the problem of knowing which ideas (if any!) to attach to this formalism and then to attach to micro-physics.

Appendix A Eigens of $\tilde{S}_j \Sigma \hat{N}_j$ versus $\sigma_j(\sigma_k \tilde{n}^k)\hat{N}_j$

A.1 Eigens of $\sigma_j(\sigma_k \tilde{n}^k)\hat{N}_j$

With the standard Pauli 2x2 matrices being:

$$\sigma_1 \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 \overset{\text{def}}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the Y_l^m being the complex spherical harmonics:

$$Y_l^m(\theta, \varphi) \overset{\text{def}}{=} y(l,m) P_l^m(\cos\theta) e^{im\phi}$$

we can show that for $1 \leqslant l$ and $-l \leqslant m \leqslant l-2$:

$$\lambda_-(l) \overset{\text{def}}{=} -(l+1) \leqslant -2$$

$$\Omega[\lambda_-(l), m](\theta, \varphi) \overset{\text{def}}{=} \Omega_-[l,m](\theta,\varphi) \overset{\text{def}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l-m} & Y_l^m(\theta,\varphi) \\ -\sqrt{l+m+1} & Y_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

verifies:

$$\sigma_j(\sigma_k \tilde{n}^k)\hat{N}_j\{\Omega[\lambda_-(l), m]\} \overset{\text{def}}{=} \lambda_-(l)\, \Omega[\lambda_-(l), m]$$

and:

$$\lambda_+(l) \overset{\text{def}}{=} l \geqslant 1$$

$$\Omega[\lambda_+(l), m](\theta, \varphi) \overset{\text{def}}{=} \Omega_+[l,m](\theta,\varphi) \overset{\text{def}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m+1} & Y_l^m(\theta,\varphi) \\ \sqrt{l-m} & Y_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

verifies:

$$\sigma_j(\sigma_k \tilde{n}^k)\hat{N}_j\{\Omega[\lambda_+(l), m]\} \overset{\text{def}}{=} \lambda_+(l)\, \Omega[\lambda_+(l), m]$$

These results correspond to what is found in textbooks up to a $\pm$ multiplication factor. The proofs are rather lengthy, but still manageable by hand. They could be checked numerically by computer with any software able to handle complex 2x2 matrices and the first partial derivative of a real function. These eigenvectors correspond to the $\varphi_{j,m}^{(-)}$ and $\varphi_{j,m}^{+}$ of [1] p77 (but presented with half integers in their definitions). They correspond also to the "spherical harmonic spinors" found in the paragraph \$ 24 of [3] (also defined with half integers).

A.2 dex()

When having a complex square matrix $X + iY$, with X, Y reals, we define the matrix $\mathrm{dex}(X + iY)$ with:

$$\mathrm{dex}(X + iY) \stackrel{\mathrm{def}}{=} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

On a complex column $V + iW$, with V, W reals, we define the column $\mathrm{dex}(V + iW)$ with:

$$\mathrm{dex}(V + iW) \stackrel{\mathrm{def}}{=} \begin{pmatrix} V \\ W \end{pmatrix}$$

For example on a $\psi \stackrel{\mathrm{def}}{=} \mathscr{V} + i\mathscr{W}$, we have:

$$\mathrm{dex}(\psi) \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathscr{V} \\ \mathscr{W} \end{pmatrix}$$

A lot of properties can be shown about dex(), the most useful one being:

$$\mathrm{dex}((X + iY)(X' + iY')) \stackrel{\mathrm{dem}}{=} \mathrm{dex}(X + iY)\mathrm{dex}(X' + iY')$$

$$\mathrm{dex}((X + iY)(V + iW)) \stackrel{\mathrm{dem}}{=} \mathrm{dex}(X + iY)\mathrm{dex}(V + iW)$$

With Pauli matrices we have:

$$\mathrm{dex}(i\sigma_1) \stackrel{\mathrm{dem}}{=} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{dex}(i\sigma_2) \stackrel{\mathrm{dem}}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathrm{dex}(i\sigma_3) \stackrel{\mathrm{dem}}{=} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

which are related to the S_{123} defined in section (13) with:

$$\mathrm{dex}(i\sigma_1) \stackrel{\text{dem}}{=} 2S_3\xi \qquad \mathrm{dex}(i\sigma_2) \stackrel{\text{dem}}{=} -2S_2\xi \qquad \mathrm{dex}(i\sigma_3) \stackrel{\text{dem}}{=} -2S_1\xi \tag{25}$$

dex() is a very useful tool to help decomplexify some equation. We are going to use dex() to show the relation between the eigens of our real $\tilde{S}_j\Sigma\hat{N}_j$ operator and the eigens of the complex operator $\sigma_j(\sigma_k\tilde{n}^k(\theta,\varphi))\hat{N}_j$ described in the previous section.

Indeed, we can show that if $a, b(\theta,\varphi)$ are two two real columns such that:

$$\sigma_j(\sigma_k\tilde{n}^k)\hat{N}_j\{a+ib\} \stackrel{\text{def}}{=} \lambda(a+ib)$$

we have:

$$\mathrm{dex}[\quad \sigma_j(\sigma_k\tilde{n}^k)\hat{N}_j\{a+ib\} \stackrel{\text{def}}{=} \lambda(a+ib) \quad]$$

which is equivalent to:

$$\tilde{S}_j\Sigma\hat{N}_j\{\begin{pmatrix} -\tilde{I}a \\ -b \end{pmatrix}\} \stackrel{\text{dem}}{=} \lambda\begin{pmatrix} -\tilde{I}a \\ -b \end{pmatrix}$$

with the 2x2 real matrix $\tilde{I}$ being:

$$\tilde{I} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The proof uses:

$$\sigma_j\sigma_k \stackrel{\text{dem}}{=} \delta_{jk}I + i\varepsilon_{jkl}\sigma_l$$

$$\tilde{S}_j\tilde{S}_k \stackrel{\text{dem}}{=} \delta_{jk}I - \varepsilon_{jkl}(2S_l\xi)$$

$$\begin{pmatrix} \tilde{I} & 0 \\ 0 & -I \end{pmatrix} 2S_{j=123}\xi \begin{pmatrix} -\tilde{I} & 0 \\ 0 & -I \end{pmatrix} \stackrel{\text{dem}}{=} 2(-S_3, S_2, S_1)\xi \quad \Leftrightarrow \quad (25)$$

A.3 Recovering $\mathscr{L}_-[l,m]$ and $\mathscr{L}_+[l,m]$ of section (15)

Noting that:

$$Y_l^m \stackrel{\text{dem}}{=} C_l^m + iS_l^m$$

with:

$$a_- \stackrel{\text{def}}{=} \frac{1}{\sqrt{2l+1}}\begin{pmatrix} \sqrt{l-m} \;\; C_l^m(\theta,\varphi) \\ -\sqrt{l+m+1} \;\; C_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

$$b_- \stackrel{\text{def}}{=} \frac{1}{\sqrt{2l+1}}\begin{pmatrix} \sqrt{l-m} \;\; S_l^m(\theta,\varphi) \\ -\sqrt{l+m+1} \;\; S_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

$$a_+ \overset{\text{def}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m+1} & C_l^m(\theta,\varphi) \\ \sqrt{l-m} & C_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

$$b_+ \overset{\text{def}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m+1} & S_l^m(\theta,\varphi) \\ \sqrt{l-m} & S_l^{m+1}(\theta,\varphi) \end{pmatrix}$$

we have:

$$\begin{pmatrix} -\tilde{I} & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} a_- \\ b_- \end{pmatrix} \overset{\text{dem}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m+1} & C_l^{m+1}(\theta,\varphi) \\ \sqrt{l-m} & C_l^m(\theta,\varphi) \\ -\sqrt{l-m} & S_l^m(\theta,\varphi) \\ \sqrt{l+m+1} & S_l^{m+1}(\theta,\varphi) \end{pmatrix} \overset{\text{def}}{=} \mathscr{Z}_-[l,m]$$

$$\begin{pmatrix} -\tilde{I} & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} \overset{\text{dem}}{=} \frac{1}{\sqrt{2l+1}} \begin{pmatrix} -\sqrt{l-m} & C_l^{m+1}(\theta,\varphi) \\ \sqrt{l+m+1} & C_l^m(\theta,\varphi) \\ -\sqrt{l+m+1} & S_l^m(\theta,\varphi) \\ -\sqrt{l-m} & S_l^{m+1}(\theta,\varphi) \end{pmatrix} \overset{\text{def}}{=} \mathscr{Z}_+[l,m]$$

which do the connection of what is found in textbooks with our real way of doing.

References

[1] C.Itzykson, J.B.Zuber *Quantum Field Theory* (McGraw-Hill International Book Company, 1980)

[2] G.Barrand *Symmetric Lorentz-Minkowski, Antisymmetric Dirac-Majorana* (`https://doi.org/10.48550/arXiv.2304.03701`)

[3] V.B.Berestetskii, E.M.Lifshitz, L.P.Pitaevskii *Relativistic Quantum Theory. Part 1.* (Pergamon Press)