Financial Engineering Formulas

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1 Integration by parts

$u = u(x)$ and $du = u'(x)dx$, while $v = v(x)$ and $dv = v'(x)dx$

$$\int u\, v' dx = u v - \int v\, u' dx. \tag{1}$$

2 First fundamental theorem of asset pricing

No arbitrage $\Leftrightarrow$ there exists a martingale measure $\mathbb{Q}$ equivalent to the historical measure $\mathbb{P}$.

European style contingent claim is a financial product which can only be exercised at maturity.

3 Binomial Model

3.1 One Period Binomial Model

No arbitrage $\Leftrightarrow 0 < d < 1 + r < u$

Solving using risk neutral probabilities:

1. Use the evolution of the stock S, i.e., $u = \frac{S_{1u}}{S_0}$, $d = \frac{S_{1d}}{S_0}$ to calculate q_u, q_d:

$$q_u = \frac{(1+r)-d}{u-d}, \quad q_d = \frac{u-(1+r)}{u-d}, \quad 1+r = q_d\, d + q_u\, u, \quad q_u, q_d \in (0,1), \quad q_u + q_d = 1 \tag{2}$$

2. In the one-period binomial model, no arbitrage implies that the price H_0 of any contingent claim H_1 is given by

$$H_0 = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[H_1] \tag{3}$$

where $\mathbb{Q}$ is the martingale measure given by q_u and q_d.

3.2 Second fundamental theorem of asset pricing

In the absence of arbitrage, the binomial model is complete iff there exists a *unique* martingale measure $\mathbb{Q}$.

Binomial option pricing formula

$$H_0 = \frac{1}{(1+\tilde{r})^N} \sum_{i=0}^{N} \binom{N}{i} q_u^i q_d^{N-i} H\left(S_0 u^i d^{N-i}\right), \quad \tilde{r} = \frac{rT}{N} \tag{4}$$

$N =$ number of steps.

4 Probability Density Function

The standard normal distribution has probability density

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}. \quad \text{with } \mathrm{E}[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx. \tag{5}$$

or with μ, σ:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{6}$$

$X \sim N(\mu, \sigma^2)$ can be written as $X = \mu + \sigma Y$ with $Y \sim N(0,1)$

5 Itô

$$df(t, X_t, Y_t) = \frac{\partial}{\partial t} f(t, X_t, Y_t)\, dt \tag{7}$$

$$+ \frac{\partial}{\partial X_t} f(t, X_t, Y_t)\, dX_t + \frac{1}{2}\frac{\partial^2}{\partial X_t^2} f(t, X_t, Y_t)\, dX_t^2 \tag{8}$$

$$+ \frac{\partial}{\partial Y_t} f(t, X_t, Y_t)\, dY_t + \frac{1}{2}\frac{\partial^2}{\partial Y_t^2} f(t, X_t, Y_t)\, dY_t^2 \tag{9}$$

$$+ \frac{\partial}{\partial X_t \partial Y_t} f(t, X_t, Y_t)\, dX_t dY_t \tag{10}$$

$\times$	dW_i	dt
dW_j	$\rho_{ij} dt$	0
dt	0	0

Product rule Given two Brownian motions X_t and Y_t, then:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t \tag{11}$$

6 Black Scholes Model

Black-Scholes PDE with dividends

$$\frac{\partial C_t}{\partial t} + (r-q)\, S_t \frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S^2} = rC_t \tag{12}$$

6.1 Assumptions

6.1.1 "Usual assumptions":

1. no arbitrage
2. no market frictions

6.1.2 Additional assumptions:

1. Stock follows a geometric Brownian motion: Log-normal stock prices → **Normal instantaneous returns**,
2. Price increments $S_{t+\Delta t} - S_t$ are **independent of the past** F_t,
3. Constant **drift** and **volatility** (standard deviation of returns),
4. Constant and known **risk-free interest rates**,
5. **One single random factor**, the stock price which evolves **continuously**.
6. **No dividends**.

7 Options

Write like $(S_t - 100)$

Butterfly Long 2 calls, short 2 calls: low volatility expected

Straddle Long 1 ATM call, long 1 ATM put: high volatility expected

Strangle Long 1 call, long 1 put: high volatility expected

The Black-Scholes option pricing formula (relates the price of a call option C_t (or a put option P_t respectively) to the strike price K, time to maturity $\tau = T - t$, risk-free interest rate r, a constant volatility of returns of the underlying asset σ, dividend yield q and current spot price S_t:

$$C_t = S_t e^{-q\tau}\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \quad \text{where } \tau = T - t \tag{13}$$

$$P_t = Ke^{-r\tau}\Phi(-d_2) - S_t e^{-q\tau}\Phi(-d_1), \tag{14}$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau} \tag{15}$$

Call Delta:

$$\Delta_{C_t} = e^{-q\tau}\Phi(d_1), \quad \text{where } \tau = T - t \tag{16}$$

Φ is the cumulative density function.

Proof: The key is to replace d_2 with $d_1 - \sigma\sqrt{\tau}$ in the right moment:

$$\text{with } \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{\tau}} \tag{17}$$

$$\frac{\partial C}{\partial S} = \Phi(d_1) + \underbrace{S\phi(d_1)\frac{1}{S\sigma\sqrt{\tau}}}_{=\phi(d_1)\frac{1}{\sigma\sqrt{\tau}}} - Ke^{-r\tau}\Phi(d_2)\frac{1}{S\sigma\sqrt{\tau}}, \quad \text{(product rule)} \tag{18}$$

$$\text{continue with } \Phi(d_2) = \Phi\left(d_1 - \sigma\sqrt{\tau}\right) \tag{19}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1-\sigma\sqrt{\tau})^2}{2}} \tag{20}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(d_1^2 - 2d_1\sigma\sqrt{\tau} + \sigma^2\tau\right)} \tag{21}$$

$$= \phi(d_1) e^{-\frac{1}{2}\left(-2d_1\sigma\sqrt{\tau} + \sigma^2\tau\right)} \tag{22}$$

$$= \phi(d_1) e^{d_1\sigma\sqrt{\tau} - \frac{\sigma^2\tau}{2}} \tag{23}$$

$$= \phi(d_1) e^{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau - \frac{\sigma^2\tau}{2}} \tag{24}$$

$$= \phi(d_1)\frac{S}{K} e^{r\tau} \tag{25}$$

$$\text{thus } \frac{\partial C}{\partial S} = \Phi(d_1) + \phi(d_1)\frac{1}{\sigma\sqrt{\tau}} - \underbrace{Ke^{-r\tau}\phi(d_1)\frac{S}{K}e^{r\tau}\frac{1}{S\sigma\sqrt{\tau}}}_{=\phi(d_1)\frac{1}{\sigma\sqrt{\tau}}} \tag{26}$$

$$\text{hence } \frac{\partial C}{\partial S} = \Phi(d_1) \tag{27}$$

Call Gamma:

$$\Gamma_{C_t} = \frac{e^{-q\tau}\phi(d_1)}{S_t\sigma\sqrt{\tau}} \tag{28}$$

ϕ is the probability density function.

8 Put-Call Parity

Both options have same maturity T. Market is free of arbitrage ($\rightarrow$ law of one price).

$$C_t - P_t = S_t e^{-q(T-t)} - Ke^{-r(T-t)}, \quad \text{remember: similarity to BS-Formula} \tag{29}$$

$$\tag{30}$$

$Ke^{-r(T-t)}$ can be written as $B(t,T)$, i.e., bond pays 1 at maturity T.

8.1 Forward Contract

enter for free at time t_0, strike $K = S_{t_0} e^{r(T-t_0)}$
value of a forward:

$$F_t = S_t - Ke^{-r(T-t)} \tag{31}$$

Geometric Brownian Motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{32}$$

$$\tag{33}$$

Change of measure from $\mathbb{P}$ to $\mathbb{Q}$:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \quad \widetilde{W} = W_t + \frac{\mu - r}{\sigma} t \tag{34}$$

$\mathbb{E}[W_t] = 0, \quad Var(W_t) = t, \quad \sigma^2_{W_t} = t$

Gaussian Shift Theorem $\mathbb{E}[e^{cZ} f(Z)] = e^{\frac{c^2}{2}} \mathbb{E}[f(Z+c)]$

9 Implied Volatility

No arbitrage conditions

$$-e^{r_{t,T}(T-t)} \leq \frac{\partial C\left(S_t, t, K, T, r_{t,T}, q_{t,T}\right)}{\partial K} \leq 0, \tag{35}$$

$$\frac{\partial^2 C\left(S_t, t, K, T, r_{t,T}, q_{t,T}\right)}{\partial K^2} \geq 0, \tag{36}$$

$$\max\left(e^{-q_{t,T}(T-t)} S_t - e^{-r_{t,T}(T-t)} K, 0\right) \leq C\left(S_t, t, K, T, r_{t,T}, q_{t,T}\right) \leq e^{q_{t,T}(T-t)} S_t, \tag{37}$$

$$\frac{\partial C\left(S_t, t, K, T, r_{t,T}, q_{t,T}\right)}{\partial T} \geq 0, \tag{38}$$

35: $K1 < K2$, then $(S_T - K1)^+ > (S_T - K2)^+$, i.e., call with lower strike has a higher payoff in all future states.

36: positive risk-neutral density, also corresponds to infinitesimal butterfly spread.

37: Buying a call and selling another call with a shorter maturity (but same strike) than the first option defines a *calendar spread strategy*. All calendar spreads need to have positive value.

10 Breeden-Litzenberger formula

$$f_T^S(K, \tau) := f_{S_T|S_t}(S_T = K) = e^{r\tau} \frac{\partial^2 C_t}{\partial K^2}. \tag{39}$$

11 Local Volatility

$$\sigma(K,T)^2 = \frac{\frac{\partial C}{\partial T}(K,T) + q_T C(K,T) + \left(r_T - q_T\right) K \frac{\partial C}{\partial K}(K,T)}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2} C(K,T)} \tag{40}$$

12 Jump Diffusion

12.1 Merton Model

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma d\tilde{W}_t + \left(e^{J_t} - 1\right) d\tilde{N}_t \tag{41}$$

12.2 Poisson Process

$$\mathbb{E}^{Q}[e^{J_t}] = \int_0^{\infty} \frac{1}{\mu} e^{-\frac{1}{\mu} \pm 1} x \, dx - 1, \quad \pm \text{ is the jump direction} \tag{42}$$