1 a)

Steady state:

$$\frac{d\left[R^*\right]}{dt} = 0 = \frac{1}{\kappa_D} - \frac{\theta_B}{\kappa_D} - \theta_B$$

$$\frac{d[x^*]}{dt} = 0 = \frac{\frac{V_1}{V_2}(1-x^*)}{\kappa_1+1-x^*} - \frac{x^*}{\kappa_2+x^*}$$

$$\frac{d[y^*]}{dt} = 0 = \frac{\frac{V_3}{V_4}(1-y^*)}{\kappa_3+1-y^*} - \frac{y^*}{\kappa_4+y^*}$$

b)

In[1]:= $\text{Solve}\left[0 == \frac{\text{V12}\ (1-x)}{\kappa 1+1-x} - \frac{x}{\kappa 2+x}, x\right]$ // FullSimplify

Out[1]= $$\left\{\left\{x \to -\frac{1+\kappa 1+\text{V12}\ (-1+\kappa 2)+\sqrt{(1+\kappa 1+\text{V12}\ (-1+\kappa 2))^2+4\ (-1+\text{V12})\ \text{V12}\ \kappa 2}}{2\ (-1+\text{V12})}\right\},\right.$$
$$\left.\left\{x \to \frac{-1+\text{V12}-\kappa 1-\text{V12}\ \kappa 2+\sqrt{(1+\kappa 1+\text{V12}\ (-1+\kappa 2))^2+4\ (-1+\text{V12})\ \text{V12}\ \kappa 2}}{2\ (-1+\text{V12})}\right\}\right\}$$

In[2]:= $\text{Solve}\left[0 == \frac{\text{V34}\ (1-y)}{\kappa 3+1-y} - \frac{y}{\kappa 4+y}, y\right]$ // FullSimplify

Out[2]= $$\left\{\left\{y \to -\frac{1+\kappa 3+\text{V34}\ (-1+\kappa 4)+\sqrt{(1+\kappa 3+\text{V34}\ (-1+\kappa 4))^2+4\ (-1+\text{V34})\ \text{V34}\ \kappa 4}}{2\ (-1+\text{V34})}\right\},\right.$$
$$\left.\left\{y \to \frac{-1+\text{V34}-\kappa 3-\text{V34}\ \kappa 4+\sqrt{(1+\kappa 3+\text{V34}\ (-1+\kappa 4))^2+4\ (-1+\text{V34})\ \text{V34}\ \kappa 4}}{2\ (-1+\text{V34})}\right\}\right\}$$

In[3]:= $\theta\text{B}[\kappa\text{Dinv\_}] = \kappa\text{Dinv} * (1+\kappa\text{Dinv})^{-1};$

In[4]:= `V12 = 5 * θB[κDinv];`

In[5]:= `κ1 = κ2 = κ3 = κ4 = κ;`

In[6]:= $x[\kappa\text{Dinv\_}] = \frac{-1+\text{V12}-\kappa 1-\text{V12}\ \kappa 2+\sqrt{(1+\kappa 1+\text{V12}\ (-1+\kappa 2))^2+4\ (-1+\text{V12})\ \text{V12}\ \kappa 2}}{2\ (-1+\text{V12})};$

In[7]:= `V34 = 10 * x[κDinv];`

In[8]:= $y[\kappa\text{Dinv\_}] = \frac{-1+\text{V34}-\kappa 3-\text{V34}\ \kappa 4+\sqrt{(1+\kappa 3+\text{V34}\ (-1+\kappa 4))^2+4\ (-1+\text{V34})\ \text{V34}\ \kappa 4}}{2\ (-1+\text{V34})};$

In[9]:=
```
Plot1 = Plot[θB[κDinv], {κDinv, .01, 100},
   PlotRange → All, PlotStyle → Black, PlotLegends → {"θB"}];
```

In[10]:=
```
Plot2 = Plot[x[κDinv] /. κ → .1, {κDinv, .01, 100},
   PlotRange → All, PlotStyle → Blue, PlotLegends → {"x*,κ=.1"}];
```

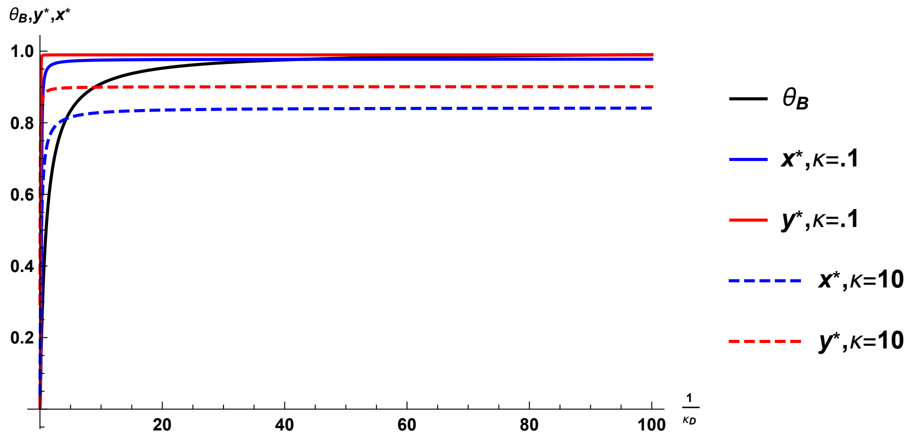
In[11]:=
```
Plot3 = Plot[y[ϰDinv] /. ϰ → .1, {ϰDinv, .01, 100},
    PlotRange → All, PlotStyle → Red, PlotLegends → {"y*,ϰ=.1"}];
```

In[12]:=
```
Plot4 = Plot[x[ϰDinv] /. ϰ → 10, {ϰDinv, .01, 100},
    PlotRange → All, PlotStyle → {Blue, Dashed}, PlotLegends → {"x*,ϰ=10"}];
```

In[13]:=
```
Plot5 = Plot[y[ϰDinv] /. ϰ → 10, {ϰDinv, .01, 100},
    PlotRange → All, PlotStyle → {Red, Dashed}, PlotLegends → {"y*,ϰ=10"}];
```

In[14]:= Show[Plot1, Plot2, Plot3, Plot4, Plot5, AxesLabel → {"$\frac{1}{\varkappa_D}$", "θ_B,y*,x*"}]

Out[14]=



c)

In[15]:= `Limit[θB[ϰDinv], ϰDinv → ∞]`

Out[15]= 1

In[16]:= Solve[ϰDinv * $(1 + \varkappa\text{Dinv})^{-1}$ == .9, ϰDinv]

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[16]= {{ϰDinv → 9.}}

In[17]:= Solve[ϰDinv * $(1 + \varkappa\text{Dinv})^{-1}$ == .1, ϰDinv]

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[17]= {{ϰDinv → 0.111111}}

In[18]:= nH_θB = $\frac{\text{Log[81]}}{\text{Log}[9/0.1111111111111111]}$

Out[18]= 1.

In[19]:= `Limit[x[ϰDinv] /. ϰ → .1, ϰDinv → ∞] // N`

Out[19]= 0.977834

```
In[20]:= Solve[x[κDinv] == .9 (0.9778336096873995) /. κ → .1, κDinv]
Out[20]= {{κDinv → 0.491016}}

In[21]:= Solve[x[κDinv] == .1 (0.9778336096873995) /. κ → .1, κDinv]
Out[21]= {{κDinv → 0.123392}}
```

$$\text{In[22]:= nH\_x} = \frac{\text{Log[81]}}{\text{Log}\left[0.4910158410173033 / 0.12339209680565799\right]}$$

```
Out[22]= 3.18183

In[23]:= Limit[x[κDinv] /. κ → 10, κDinv → ∞] // N
Out[23]= 0.842194

In[24]:= Solve[x[κDinv] == .9 (0.84219357067906130) /. κ → 10, κDinv]
Out[24]= {{κDinv → 1.4772}}

In[25]:= Solve[x[κDinv] == .1 (0.84219357067906130) /. κ → 10, κDinv]
Out[25]= {{κDinv → 0.0203141}}
```

$$\text{In[26]:= nH\_x10} = \frac{\text{Log[81]}}{\text{Log}\left[1.4771993697773982 / 0.020314065438098716\right]}$$

```
Out[26]= 1.02516

In[27]:= Limit[y[κDinv] /. κ → .1, κDinv → ∞] // N
Out[27]= 0.989761

In[28]:= Solve[y[κDinv] == .9 (0.9897606926690938) /. κ → .1, κDinv]
Out[28]= {{κDinv → 0.165254}}

In[29]:= Solve[y[κDinv] == .1 (0.9897606926690938) /. κ → .1, κDinv]
Out[29]= {{κDinv → 0.0854483}}
```

$$\text{In[30]:= nH\_y} = \frac{\text{Log[81]}}{\text{Log}\left[0.1652539651084165 / 0.0854483462457909\right]}$$

```
Out[30]= 6.66258

In[31]:= Limit[y[κDinv] /. κ → 10, κDinv → ∞] // N
Out[31]= 0.900898

In[32]:= Solve[y[κDinv] == .9 (0.9008976309688341) /. κ → 10, κDinv]
Out[32]= {{κDinv → 0.160132}}

In[33]:= Solve[y[κDinv] == .1 (0.9008976309688341) /. κ → 10, κDinv]
Out[33]= {{κDinv → 0.00238147}}
```

In[34]:=
```
nH_y10 = Log[81] / Log[0.16013204731505948 / 0.002381472471587167]
```
$$nH\_y10 = \frac{\text{Log}[81]}{\text{Log}[0.16013204731505948 / 0.002381472471587167]}$$

Out[34]= 1.04424

For θ_B, n_H =1

When κ=0.1,

x^*, n_H =3.18183

y^*, n_H =6.66258

When κ=10,

x^*, n_H =1.02516

y^*, n_H =1.04424

d)

In[35]:= $\frac{\theta B[.15] - \theta B[.1]}{\theta B[.1]}$

Out[35]= 0.434783

In[36]:= $\frac{x[.15] - x[.1]}{x[.1]}$ /. $\kappa \to .1$

Out[36]= 1.01826

In[37]:= $\frac{x[.15] - x[.1]}{x[.1]}$ /. $\kappa \to 10$

Out[37]= 0.279659

In[38]:= $\frac{y[.15] - y[.1]}{y[.1]}$ /. $\kappa \to .1$

Out[38]= 4.0258

In[39]:= $\frac{y[.15] - y[.1]}{y[.1]}$ /. $\kappa \to 10$

Out[39]= 0.0565662

e)

Due to the zero-order sensitivity, the signals produced are very switch-like responses as demonstrated above for low values of κ; when κ=0.1, a small change in input leads to a drastic percent change in output, however, when κ is increased to 10, a small change in input yields a small percent change in output. In agreement, as κ increases, the hill coefficients for both y^* and x^* decrease and the responses for both become less switch-like.