1.

Adding the recycle terms, the species balances were determined as

$\frac{\partial R_s}{\partial t} = -k_f L R_s + k_r R_s^* - k_e^* R_s + k_{rec} R_i + v_s$

$\frac{\partial R_s^*}{\partial t} = k_f L R_s - k_r R_s^* - k_e^* R_s^* + k_{rec} R_i^*$

$\frac{\partial R_i}{\partial t} = k_e R_s - k_{deg} R_i - k_{rec} R_i$

$\frac{\partial R_i^*}{\partial t} = k_e^* R_s^* - k_{deg} R_i^* - k_{rec} R_i^*$

Set the species balances for surface active, internal active, surface free, and internal free receptors equal to zero and solve for the concentrations.

In[1]:=
```
Solve[{0 == -kf * L * Rs + kr * Rsst - ke * Rs + krec * Ri + vs,
  0 == kf * L * Rs - kr * Rsst - kest * Rsst + krec * Rist, 0 == kest * Rsst - kdeg * Rist - krec * Rist,
  0 == ke * Rs - kdeg * Ri - krec * Ri}, {Ri, Rs, Rist, Rsst}]
```

Out[1]=
$$\left\{\left\{\text{Ri} \to \frac{\text{ke}\ (\text{kdeg kest} + \text{kdeg kr} + \text{kr krec})\ \text{vs}}{\text{kdeg}\ (\text{kdeg ke kest} + \text{kdeg ke kr} + \text{ke kr krec} + \text{kdeg kest kf L} + \text{kest kf krec L})},\right.\right.$$
$$\text{Rs} \to \frac{(\text{kdeg} + \text{krec})\ (\text{kdeg kest} + \text{kdeg kr} + \text{kr krec})\ \text{vs}}{\text{kdeg}\ (\text{kdeg ke kest} + \text{kdeg ke kr} + \text{ke kr krec} + \text{kdeg kest kf L} + \text{kest kf krec L})},$$
$$\text{Rist} \to \frac{(\text{kdeg kest kf L} + \text{kest kf krec L})\ \text{vs}}{\text{kdeg}\ (\text{kdeg ke kest} + \text{kdeg ke kr} + \text{ke kr krec} + \text{kdeg kest kf L} + \text{kest kf krec L})},$$
$$\left.\left.\text{Rsst} \to \frac{(\text{kdeg}^2\ \text{kf L} + 2\ \text{kdeg kf krec L} + \text{kf krec}^2\ \text{L})\ \text{vs}}{\text{kdeg}\ (\text{kdeg ke kest} + \text{kdeg ke kr} + \text{ke kr krec} + \text{kdeg kest kf L} + \text{kest kf krec L})}\right\}\right\}$$

The total concentration of active receptor is the sum of Rist and Rsst.

In[2]:=
```
RactT = (kdeg kest kf L + kest kf krec L) vs / (kdeg (kdeg ke kest + kdeg ke kr + ke kr krec + kdeg kest kf L + kest kf krec L)) +
  (kdeg^2 kf L + 2 kdeg kf krec L + kf krec^2 L) vs / (kdeg (kdeg ke kest + kdeg ke kr + ke kr krec + kdeg kest kf L + kest kf krec L)) // FullSimplify
```

Out[2]=
$$\frac{\text{kf}\ (\text{kdeg} + \text{krec})\ (\text{kdeg} + \text{kest} + \text{krec})\ \text{L vs}}{\text{kdeg ke}\ (\text{kdeg}\ (\text{kest} + \text{kr}) + \text{kr krec}) + \text{kdeg kest kf}\ (\text{kdeg} + \text{krec})\ \text{L}}$$

Next, set the species balances for surface active, internal active, surface free, and internal free receptors with no recycle equal to zero and solve for the concentrations.

In[3]:=
```
Solve[{0 == -kf * L * Rs + kr * Rsst - ke * Rs + vs, 0 == kf * L * Rs - kr * Rsst - kest * Rsst,
  0 == kest * Rsst - kdeg * Rist, 0 == ke * Rs - kdeg * Ri}, {Ri, Rs, Rist, Rsst}]
```

Out[3]=
$$\left\{\left\{\text{Ri} \to \frac{\text{ke}\ (\text{kest} + \text{kr})\ \text{vs}}{\text{kdeg}\ (\text{ke kest} + \text{ke kr} + \text{kest kf L})}, \text{Rs} \to \frac{(\text{kest} + \text{kr})\ \text{vs}}{\text{ke kest} + \text{ke kr} + \text{kest kf L}},\right.\right.$$
$$\left.\left.\text{Rist} \to \frac{\text{kest kf L vs}}{\text{kdeg}\ (\text{ke kest} + \text{ke kr} + \text{kest kf L})}, \text{Rsst} \to \frac{\text{kf L vs}}{\text{ke kest} + \text{ke kr} + \text{kest kf L}}\right\}\right\}$$

The total concentration of active receptor with no recycle is the sum of Rist and Rsst.

In[4]:=
$$\text{RactTnore} = \frac{\text{kest kf L vs}}{\text{kdeg}\left(\text{ke kest} + \text{ke kr} + \text{kest kf L}\right)} + \frac{\text{kf L vs}}{\text{ke kest} + \text{ke kr} + \text{kest kf L}} \;//\; \text{FullSimplify}$$

Out[4]=
$$\frac{\left(\text{kdeg} + \text{kest}\right)\ \text{kf L vs}}{\text{kdeg ke}\ \left(\text{kest} + \text{kr}\right) + \text{kdeg kest kf L}}$$

The maximum concentration of active receptor is attained as L increases infinitely. Adding a recycle increases the maximum concentration of active receptor.

In[5]:=
$$\text{MaxRec} = \text{Limit}\left[\left(\text{kf}\ \left(\text{kdeg} + \text{krec}\right)\ \left(\text{kdeg} + \text{kest} + \text{krec}\right)\ \text{L vs}\right) \Big/ \left(\text{kdeg ke}\ \left(\text{kdeg}\ \left(\text{kest} + \text{kr}\right) + \text{kr krec}\right) + \text{kdeg kest kf}\ \left(\text{kdeg} + \text{krec}\right)\ \text{L}\right),\ \text{L} \to \text{Infinity}\right] \;//\; \text{FullSimplify}$$

Out[5]=
$$\frac{\left(\text{kdeg} + \text{kest} + \text{krec}\right)\ \text{vs}}{\text{kdeg kest}}$$

In[6]:=
$$\text{MaxNoRec} = \text{Limit}\left[\frac{\left(\text{kdeg} + \text{kest}\right)\ \text{kf L vs}}{\text{kdeg ke}\ \left(\text{kest} + \text{kr}\right) + \text{kdeg kest kf L}},\ \text{L} \to \text{Infinity}\right]$$

Out[6]=
$$\frac{\text{kdeg kf vs} + \text{kest kf vs}}{\text{kdeg kest kf}}$$

$\frac{\left(\text{kdeg}+\text{kest}+\text{krec}\right)\ \text{vs}}{\text{kdeg kest}} > \frac{\left(\text{kdeg}+\text{kest}\right)\ \text{vs}}{\text{kdeg kest}}$, therefore adding a recycle increases the maximum concentration of active receptor.